A STUDY OF FUZZY LOGIC AND FUZZY SEQUENCE WITH THEIR APPLICATION TO REAL WORLD

A THESIS SUBMITTED TO THE

CENTRAL DEPARTMENT OF MATHEMATICS
INSTITUTE OF SCIENCE AND TECHNOLOGY
TRIBHUVAN UNIVERSITY, NEPAL

FOR THE AWARD OF
DOCTOR OF PHILOSOPHY
IN MATHEMATICS

BY

GYAN PRASAD PAUDEL

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DECLARATION

The thesis entitled **"A Study of Fuzzy Logic and Fuzzy Sequence with Their Application to the Real World",** which is being submitted to the Central Department of Mathematics, Institute of Science and Technology(IOST), Tribhuvan University, Nepal for the award of the degree of Doctor of Philosophy (Ph.D.), is a research work carried out by me under the supervision of Prof. Dr. Narayan Prasad Pahari, Central Department of Mathematics, Tribhuvan University and Co-supervision Prof. Dr. Sanjeev Kumar, Department of Mathematics, Dr. Bhimrao Ambedkar University, Agra, India.
This research is original and has not been submitted earlier in part or full in this or any other form to any university or institute, here or elsewhere, for the award of any degree.

....
Gyan Prasad Paudel

RECOMMENDATION

This is to recommend that **Gyan Prasad Paudel** has carried out research entitled **“A Study of Fuzzy Logic and Fuzzy Sequence with Their Application to the Real World”** for the award of Doctor of Philosophy (PhD) in **Mathematics** under my supervision. To my knowledge, this work has not been submitted for any other degree.
He has fulfilled all the requirements laid down by the Institute of Science and Technology (IOST), Tribhuvan University, Kirtipur for the submission of the thesis for the award of Ph.D. degree.

....

Prof. Dr. Narayan Prasad Pahari
Supervisor
Central Department of Mathematics
Tribhuvan University, Kirtipur, Kathmandu, Nepal

....

Prof. Dr. Sanjeev Kumar
Co-supervisor
Dr.Bhimrao Ambedker University
Khandari Campus, Agra, India

June 14, 2023

TRIBHUVAN UNIVERSITY
INSTITUTE OF SCIENCE AND TECHNOLOGY
CENTRAL DEPARTMENT OF MATHEMATICS

LETTER OF APPROVAL

June 14, 2024

On the recommendation of **Prof. Dr. Narayan Prasad Pahari and Prof. Dr. Sanjeev Kumar**, this Ph. D. thesis submitted by **Mr. Gyan Prasad Paudel**, entitled "**A Study of Fuzzy Logic and Fuzzy Sequence with Their Application to the Real World**" is forwarded by Central Department Research Committee (CDRC) to the Dean, IOST, T.U.

....
Prof. Dr. Chet Raj Bhatta
Head,
Central Department of Mathematics
Tribhuvan University
Kirtipur, Kathmandu
Nepal

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सोधसार

अनुक्रम स्थान र भिन्नता अनुक्रमले गणित विषयको धेरै क्षेत्रहरु जस्तै Schauder आधार, Summability सिद्धान्त, निश्चित विन्दु सिद्धान्त, गैर-रैखिक विश्लेषण, टोपोलोजिकल भेक्टर स्पेसको संरचनात्मक विश्लेषणात्मक महत्पूर्ण भूमिका खेल्दछ । फजि तर्क गणितमा देखिने एउटा अनिश्चितता र अस्पष्टताको अध्ययन हो । यो तर्क संगत निर्णय गर्न लचिलो र अनिश्चिततामा आधारित तर्कविधि हो । जसले अस्पष्ट वा अपूर्ण जानकारीलाई सम्बोधन गर्दछ र विशेष समस्याहरु समाधान गर्दछ फजि समूहलाई गणित क्षेत्रको एक विस्तृत क्षेत्रमा सफलतापूर्वक प्रयोग गरिरहेको छ । अनुक्रमका आधारित तथ्याङ्कहरुको अनिश्चितता र अशुद्धतालाई फजि अनुक्रमले बलियो रुपरेखामा रहेर विश्लेषण गर्दछ । यो सोधपत्रले फजि वास्तविक नम्बरहरुको आधारभुत टोपोलोजिकल विशेषताहरुको बारेमा जानकारी गराउदछ । यसमा सोधपत्रमा हामीले क्लासहरु $l_F(X,\bar{\lambda},\bar{p})$र $l_F(X,\bar{\lambda},\bar{p},L)$को आधारभुत टोपोलोजिकल विशेषताहरु अध्ययन गर्न Orlicz (अरलिच) र Parnorm (पारानर्म) फलनको प्रयोग गर्दछौं । यसका साथै यस सोधपत्रको अनुक्रम भिन्नता क्लास $S(X,M,P,A)$ को रेखिय, पूर्णता, सोलिडिटि जस्ता विशेषताहरुको अध्ययन गर्दछ भने $F_\infty(\rho,M,p,A)$, $F_c(\rho,M,p,A)$ र $F_o(\rho,M,p,A)$जस्ता क्लासहरुको सामान्यिकृत स्वरुपको पनि अध्ययन गर्दछ । साथै हामीले यस सोधपत्रमा फजि वास्तविक संख्याहरुको दोहोरो अनुक्रम क्लास $\mathrm{Z}_F\ (M,\lambda,\xi)\ ; Z_F = l_\infty^F, C^F, C_o^F$ का केही टोपोलोजिकल गुणहरुको वारेमा पनि अध्ययन गर्दछौं । थप रुपमा सोधपत्रले फजि संख्याहरुको p-bounded विविधताका अनुक्रम भिन्नताको क्लासको सामान्यीकृत स्वरुपलाई पनि समावेश गर्दछ । यस बाहेक यस सोधपत्रले फजि वास्तविक संख्याहरुको वास्तविक विश्वपरिवेशको विभिन्न व्यवहारिक कार्यान्वयनहरुको समेत अन्वेषण गर्दछ । विशेष गरि फजि सेट र फजि लजिकलाई कसरी निर्णय प्रक्रियामा प्रयोग गरिन्छ भन्ने देखाउनका लागि Bellmen-Zadeh को max-min विधिको प्रयोगलाई समेत यस सोधपत्रले देखाएको छ । यस बाहेक यस सोधपत्रमा स्वास्थ्य सेवा क्षेत्रमा समेत फजि संख्याहरुको प्रयोग हुने र Sanchez को मेडिकल अवस्थालाई सम्बोधन गर्दै कुनै विशेष क्षैँत्रको घटना अध्ययनमा देखिएको मुद्धाको पहिचान र मूल्याङ्कनमा समेत फजि अङ्कगणितिय आधारित विधिहरुको प्रयोग भएको देखाउछ । साथै यस सोधपत्रले फजि सेट र फजि तर्कलाई प्रयोग गरि बिमा भुक्तानिको समयमा गरिने दाविमा हुनसक्ने धोखाधडीलाई पहिचान गर्न र आन्तरिक लेखापरिक्षकहरुलाई सहयोग गर्न तयार गरिएको फजि मोडेलको प्रस्तुत गरिएको छ ।

ABSTRACT

Sequence space and difference sequence spaces play an important role in many areas of analysis, such as the Schauder basis, summability theory, fixed point theory, non-linear analysis, and structural theory of topological vector space. Fuzzy logic is the study of uncertainty and vagueness. It is a flexible, uncertainty-based reasoning method for rational decision-making that addresses vague or incomplete information and solves specific problems. The fuzzy set theory has been successfully applied in a wide range of mathematical fields. Fuzzy sequence analysis offers a robust framework for handling uncertainty and imprecision in sequence-based data, enhancing practicality and effectiveness. This dissertation deals with the fundamental topological properties of sequence spaces and the difference sequence spaces of fuzzy real numbers. To study the basic topological properties of the classes $\ell_F(X, \bar{\lambda}, \bar{p})$ and $\ell_F(X, \bar{\lambda}, \bar{p}, L)$ we use the Orlicz and paranorm function. Moreover, linearity, completeness, solidity, and some inclusion properties of a class $\mathcal{S}(X, M, \alpha, P)$ of difference sequences and classes $F_\infty\left(\overline{\rho}, M, \overline{p}, A\right), F\left(\overline{\rho}, M, \overline{p}, A\right)$ and $F_o\left(\overline{\rho}, M, \overline{p}, A\right)$ of generalized difference sequences are also studied. We also study some topological properties of classes $Z^F(M, \lambda, \xi)$ where $Z^F = \ell_\infty^F, C^F, C_o^F$ of double sequences of fuzzy real numbers. Additionally, this thesis also includes the generalized form of the p-bounded variation bV_p^F of fuzzy real numbers. Besides, this thesis further explores the practical implementation of fuzzy real numbers in various real-world scenarios. Specifically, it examines how fuzzy sets and fuzzy logic are employed in decision-making processes, particularly in selecting the best option using the Bellmen-Zadeh max-min method. Furthermore, this thesis delves into the field of healthcare and addresses Sanchez's medical condition, utilizing a case study to illustrate the application of fuzzy arithmetic-based methods in identifying and assessing medical issues with a case study. Moreover, the thesis extends its exploration to the domain of insurance fraud detection. It presents a fuzzy model designed to assist internal auditors in identifying potentially fraudulent claims during the claim-settlement process.

LIST OF SYMBOLS

$\mathbb{N}$	set of positive integers
$\mathbb{R}$	set of real numbers or real line
$\mathbb{C}$	set of complex numbers
ℓ_∞	spaces of bounded sequences over complex numbers
C	spaces of convergent sequences over complex numbers
C_o	spaces of null sequences over complex numbers
ℓ_∞^F	space of bounded sequences over fuzzy real numbers
C^F	space of convergent sequences over fuzzy real numbers
C_o^F	space of null sequences over fuzzy real numbers
ω	space of all real or complex sequences
ω^F	space of all sequences over fuzzy real numbers
μ_X	membership function
$\overline{lim}$	limit superior
$\underline{lim}$	limit inferior

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Chapter 1

Introduction

1.1 Introduction

In real life, there may be instances, where we are unable to determine whether a statement is true or false. In real life, we also come across many situations, where the boundaries of the sets are vague or uncertain. To model such a set with ambiguous, imprecise, or vague concepts, mathematicians, logicians, and computer scientists are working together. Fuzzy set theory is a mathematical framework for dealing with uncertainty and vagueness in various fields including engineering, economics, medicine, and decision-making [116]. Fuzzy mathematical tools are effective at addressing the inherent ambiguity and uncertainty present in the parameters and variables in the model. Fuzzy logic can be considered a generalization of Boolean logic [116]. The fuzzy logic technique is based on the Fuzzy set theory, which was first introduced by the American mathematician Lotfi A. Zadeh [117] in 1965. According to the traditional view, science should strive for certainty in all its manifestations, and uncertainty is regarded as unscientific [116]. But fuzzy logic deals with such problems which have no clear answer with vague and unfocused information. Thus fuzzy logic is the method of reasoning that resembles human reasoning and approach of computing based degree of truth than true or false. The fuzzy set extends the traditional crisp set by allowing a degree of membership between 0 and 1, instead of binary membership 0 and 1 in the classical set[118]. The fuzzy set provides a flexible and powerful approach to the model and represents uncertain or imprecise information which is often encountered in real-life applications. Especially, fuzzy logic may be viewed as an attempt at the formation of two remarkable capabilities [116]. First, the capability to converse the reason and make a rational decision in an environment of vagueness, uncertainty, incompleteness of information, conflicting information, the partiality of truth, and partiality of possibility[116].

Second, the capability to perform a wide variety of physical and mental tasks without any measurement and any computation [116].

In the past 50+ years, there has been growing interest in studying the properties and applications of fuzzy numbers, which are defined as fuzzy sets defined on the real number line. Fuzzy numbers provide a powerful tool for representing and manipulating uncertain or vague quantities in real-life problems, where precise numerical value may not be available or meaningful [103]. In this dissertation, we use fuzzy logic and fuzzy set theory in studying various types of sequence spaces with their topological properties and applying them to solve some real-life decision-making problems.

The study of sequence space is in fact a special case of the more general study of function space if the domain is restricted to the set of natural numbers[71]. Let ω be the set of all functions from the set of natural numbers $\mathbb{N}$ to the set of real or complex numbers. Then ω can be turned into a vector space and the subspace of X of ω is called a sequence space [71]. So that the sequence space is a vector space whose elements are infinite scalars of real or complex numbers and is closed under coordinate-wise addition and scalar multiplication. Numerous mathematical disciplines have used the fuzzy set concept with great success. The fuzzy set and fuzzy real numbers have been extensively studied in various types of sequence spaces by large numbers of researchers. In 1986, Motloka [66] analyzed bounded and convergent sequences of fuzzy numbers and proved that every convergent sequence of fuzzy is bounded. Additionally, Nanda[69] defined a new metric to demonstrate the completeness of a convergent and bounded sequence of fuzzy real numbers. An attractive area of research in mathematics and its application is the study of fuzzy sequence space. So numerous researchers made a significant contribution, and several new concepts and findings have been developed in this field. Savas[92] proposed certain classes of sequences of fuzzy numbers, investigated them, and analyzed some of their properties including completeness, solidity, symmetry, and convergence free, and also looked at various inclusion relations that were relevant to these classes.

In the dissertation, we shall analyze the fundamental topological features of the class $\ell_F(X, \bar{\lambda}, \bar{p})$ defined as

$$\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}\right) = \left\{ X = (X_k) : X_k \in \omega\left(F\right), k \geq 1 \text{ and} \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) < \infty,\ for\ some\ \rho > 0 \right\};$$

where, $\overline{p} = (p_k)$ be a sequence of strictly positive real numbers

$\overline{\lambda} = (\lambda_k)$ and $\overline{\mu} = (\mu_k)$ be non-zero sequences of real numbers, and $t_k = \left|\frac{\lambda_k}{\mu_k}\right|,\ k \geq 1.$

Further, if $\inf_k p_k = l > 0$, we show the class $\ell_{\mathcal{F}}\left(X,\ \overline{\lambda},\ \overline{p},\ L\right)$ defined by

$$\ell_{\mathcal{F}}\left(X,\overline{\lambda},\overline{p},L\right)=\left\{X=(X_k):X_k\in\omega\left(F\right),k\geq 1\&\sum_{k=1}^{\infty}M\left(\frac{||\lambda_k X_k|\ |^{p_k/L}}{\rho}\right)<\infty, for\ some\ \rho>0\right\}$$

forms a paranormed space with respect to the function defined as

$$g\left(X\right)=\inf\left\{\ \rho>0:\ \sum_{k=1}^{\infty}M\left(\frac{\|\lambda_k\ \ X_k\|^{p_k/L}}{\rho}\right)\leq 1\right\},\ where\ L=sup_k\ p_k.$$

will be covered in this dissertation.

Moreover, linearity, completeness, solidity, and some inclusion properties of a class $\mathcal{S}(X, M, \alpha, P)$ of difference sequences and classes $F_\infty\left(\overline{\rho}, M{,}\overline{p}, A\right), F\left(\overline{\rho}, M{,}\overline{p}, A\right)$ and $F_o\left(\overline{\rho}, M{,}\overline{p}, A\right)$ of generalized difference sequences are also studied, which are respectively defined as

$$\mathcal{S}\left(\ X\ ,M,\alpha,\ P\right)=\ \left\{X=(X_k)\in\omega^F\ :\ \sum_{k=1}^{\infty}M\left(\frac{\|\alpha_k\ \Delta X_k\|^{p_k}}{\rho}\right)<\infty,\ for\ some\ \rho>0\right\}$$

where, $P=(p_k)$ and $Q=(q_k)$ be any two sequences of strictly positive real numbers and (α_k) be a sequence of non-zero real numbers.

$$F_\infty\left(M{,}\overline{p},A\right)=\left\{X=(X_k):X_k\in\omega\left(F\right):\sup_k\ \left[M\left(\frac{\overline{d}\left(a_k(\Delta_m\ X_k),\overline{0}\right)}{\rho}\right)\right]^{p_k}<\infty, for\ some\ \rho>0\right\}$$

$$F\left(M,\overline{p},A\right)=\left\{X=(X_k):X_k\in\omega\left(F\right):\lim_k\ \left[M\left(\frac{\overline{d}\ \left(a_k(\Delta_m\ X_k),X_o\right)}{\rho}\right)\right]^{p_k}=0, for\ some\ \rho>0\right\}$$

$$F_o\left(M,\overline{p},A\right)=\left\{X=(X_k):X_k\in\omega\left(F\right):\lim_k\left[M\left(\frac{\overline{d}\ \left(a_k(\Delta_m X_k),\ X_o\right)}{\rho}\right)\right]^{p_k}=0, for\ some\ \rho>0\right\}$$

And the classes of double sequences $Z^F\left(\ M,\ \lambda,\ \xi\ \right)$ for $Z^F\ =\ \ell_\infty^F\ ,\ C^F,\ C_o^F$ are defined by

$$\ell_\infty^F\left(M,\lambda,\xi\right)=\left\{(X_{nk})\in\omega^F\ :\ sup_{n,k}\ \mathrm{M}\left(\frac{\lambda(X_{nk},\overline{0})}{\rho}\right)<\infty\ ;\ sup_{n,k}\ \mathrm{M}\left(\frac{\xi(X_{nk},\overline{0})}{\rho}\right)<\infty\right\},$$

$$\mathrm{C}^F\left(M,\lambda,\xi\right)=\left\{(X_{nk})\in\omega^F\ :\ lim_{n,k}\ \mathrm{M}\left(\frac{\lambda\left(X_{nk},L\right)}{\rho}\right)=0\ ;\ lim_{n,k}\ \mathrm{M}\left(\frac{\xi\left(X_{nk},L\right)}{\rho}\right)=0\ \right\},$$

$$C_o^F\,(\,M,\ \lambda,\xi) = \left\{ (X_{nk}) \in \omega^F : lim_{n,k}\ \mathrm{M}\left(\frac{\lambda\,(X_{nk},\ \bar{0})}{\rho}\right) = 0\ ;\ lim_{n,k}\ \mathrm{M}\left(\frac{\xi\,(X_{nk},\ \bar{0})}{\rho}\right) = 0 \right\},$$

for some $\rho > 0, L \in \mathbb{R}\,(\mathrm{I})$.

Before we discuss the various topological properties of the classes mentioned above, we will first discuss the relevant works done previously. Kizmaz [51] pioneered the concept of difference sequence space for the first time in 1981, $\ell_\infty\,(\Delta)$, $C\,(\Delta)$, $C_o(\Delta)$ as follow:

$$\ell_\infty\,(\Delta) = \{\ X = (X_k\ ) \in \omega\ :\ (\Delta X_k) \in \ell_\infty\}$$

$$C\,(\Delta) = \{\ X = (X_k\ ) \in \omega\ :\ (\Delta X_k) \in C\}$$

$$C_o(\Delta) = \{\ X = (X_k\ ) \in \omega\ :\ (\Delta X_k) \in C_o\}.$$

where, $\ell_\infty,\ C,\ C_o$ are the sequence spaces of bounded, convergent, and null sequences respectively with ΔX_k = $\mathrm{X}_k - \mathrm{X}_{k+1}$ and showed that these classes form a Banach Space with respect to the norm $\|X\|_\Delta =\ |X_1| + \|\Delta X\|_\infty$.

This concept of difference sequence spaces is generalized in the forms $\ell_\infty\,(\Delta^m)$, $C\,(\Delta^m)$, $C_o(\Delta^m)$ by Et and Cloka [32] in 1995 and defined as follows:

$$\ell_\infty\,(\Delta^m) = \{\ X = (X_k\ )\ \in \omega\ :\ (\Delta^m X_k) \in \ell_\infty\}$$

$$C\,(\Delta^m) = \{\ X = (X_k\ ) \in \omega\ :\ (\Delta^m X_k) \in C\}$$

$$C_o(\Delta^m) = \{\ X = (X_k\ ) \in \omega\ :\ (\Delta^m X_k) \in C_o\}.$$

and, showed that these classes are Banach Spaces with respect to the norm

$$\|X\|_\Delta = \sum_i^m |X_i| + \|\Delta^m X_k\|_\infty$$

where, $m \in \mathbb{N}$, $\Delta^o X = X_k$, $\Delta \mathrm{X}_k = X_k - X_{k+1}$ and $\Delta^m X_k = \Delta^{m-1}\mathrm{X}_k - \Delta^{m-1}\mathrm{X}_{k+1}$.

A generalization of the idea of difference sequence of real numbers is the notion of difference sequence of fuzzy real numbers. Baurch and Tripathy [10] introduced the sequence spaces $\ell_\infty^F(\Delta_m)$, $\mathrm{C}^F(\Delta_m)$ and $C_o^F(\Delta_m)$ of fuzzy real numbers in 2009 defined as follow:

$$\mathrm{Z}\,(\Delta_m) = \{\ X = (X_k\ )\ \in \omega^F\ :\ (\Delta_m X_k) \in Z\}$$

for $Z = \mathrm{C}^F,\ C_o^F,\ \ell_\infty^F$ where $\Delta_m X_k = X_k - X_{k+m}$ for all $k \in \mathbb{N}$ and studied various topological properties of the spaces with the help of metric defined as

$$\rho(\,X,\,Y) = \sum_{i=1}^{m} \overline{d}\,(X_k,\,Y_k) + \,sup_k\,\overline{d}\,(\Delta_m X_k,\,\Delta_m Y_k)$$

Using the concept of difference sequence space, Tripathy and Das [108] introduced the class of p–bounded variation of difference sequence space of fuzzy real numbers denoted by bV_p^F , $1 \le p < \infty$ as follows :

$$bV_p^F \;= \left\{\, X = (X_k) \in \omega^F : \left[\sum_{k=1}^{\infty} \{\overline{d}\,(\Delta X_k\,,\,\overline{0})\}^p\right] < \infty \right\}$$

where, $\Delta X_k \;=\; X_k - X_{k+1}$ for all $k \in \mathbb{N}$, and have proved that the class of sequences bV_p^F , $1 \le p < \infty$ forms a complete metric space with the metric defined by

$$\rho\,(X, Y) = \overline{d}\,(X_1, Y_1) + \left[\sum_{k=1}^{\infty} \{\overline{d}\,(\Delta X_k, \Delta Y_k)\}^p\right]^{1/p}, for X = (X_k)\,, Y = (Y_k) \in bV_p^F\;.$$

Moreover, Paudel and et al. [77] generalized the class of p – bounded variation denoted by $bV_p^F\,(\Delta_m)$ and defined by

$$bV_p^F\,(\Delta_m) = \{X = (X_k) \in \omega\,(F) : \sum_{k=1}^{\infty} \{\bar{d}(\Delta_m X_k, \bar{0})\}^p < \infty$$

with $\Delta_m X_k = X_k - X_{k+m}$.

They defined a new metric to characterize the topological structure of generalized class of p-bounded variation of difference sequence space $bV_p^F\,(\Delta_m)$ for $1 \le p < \infty$, as follow:

$$\bar{d}(X, Y) = [\sum_{k=1}^{\infty} \{\bar{d}(\triangle_m X_k, \triangle_m Y_k)\}^p]^{\frac{1}{p}}.$$

Various researchers used the Orlilcz function to study the different topological properties of the various classes of sequence spaces of fuzzy real numbers. In 2008, Tripathy and Sarma [111] used the Orlicz function M to define the following classes:

$$(\ell_\infty)_F\,(M\,) = \left\{X = (X_k) \in \omega\,(F)\ : sup_k\ M\left(\frac{\overline{d}(\,X_k\,,\overline{0}\,)}{\rho}\right) < \infty,\;\; for\ some\ \rho > 0\right\}$$

$$\mathrm{C}_F\,(M) = \left\{X = (X_k) \in \omega\,(F)\ : lim_k\ M\left(\frac{\overline{d}(\,X_\mathrm{k}\,, L\,)}{\rho}\right) = 0\;\;, \text{for some } \rho > 0\ and\ L \in \mathbb{R}\right\}$$

$$(\mathrm{C_o})_F\,(M\,) = \left\{X = (X_\mathrm{k}) \in \omega\,(F)\ : lim_k\;\ M\left(\frac{\overline{d}(\,X_\mathrm{k}\,,\;\bar{0})}{\rho}\right)\; = \;0\,, for\ some\ \rho > 0\right\}.$$

and studied various topological properties by introducing a new metric defined as

$$f\,(X,\ Y) = \inf\left\{\rho > 0\ :\ \sup\nolimits_{\mathrm{k}} M\left(\frac{\bar{d}(\ X_{\mathrm{k}}\ ,Y_k)}{\rho}\right)\ \leq 1\right\}.$$

To study the various properties of difference sequence spaces using the difference operator Δ^m, where m is a fixed positive integer and Orlicz function M, Subramanian and et al.[99] 2012 introduced a class of sequence spaces. Similarly, Paudel and Pahari [79] used the Orlicz function to introduce the generalized the classes of difference sequences of fuzzy real numbers as follows:

$$F_\infty\left(\bar{d}, M, p, A\right) = \left\{X = (X_k) \in \omega^F : sup_k \left[M\left(\frac{\bar{d}(a_k(\Delta_m X_k), \bar{0})}{\rho}\right)\right]^{p_k} < \infty, for\ some\ \rho > 0\right\}$$
$$F\left(\bar{d}, M, p, A\right) = \left\{X = (X_k) \in \omega^F : lim_k \left[M\left(\frac{\bar{d}\left(a_k(\Delta_m X_k), X_o\right)}{\rho}\right)\right]^{p_k} = 0, for\ some\ \rho > 0, X_o \in \mathbb{R}\right\}$$

$$F_o\left(\bar{d}, M, p, A\right) = \left\{X = (X_k) \in \omega^F : lim_k \left[M\left(\frac{\bar{d}\,(a_k(\Delta_m X_k),\ \bar{0})}{\rho}\right)\right]^{p_k} = 0, for\ some\ \rho > 0\right\}$$

where $p = (p_k)$, a sequence of strictly positive real number $A = (a_k)$, a sequence of real numbers and $\Delta_m X_k = X_k - X_{k+m}$ and studied some topological properties of the classes.

Additionally, Paudel et al [80] used the Orlicz function to define the following classes of double sequences of fuzzy real numbers as follows:

$$\ell_\infty^F\,(\ M,\ \lambda, \xi) = \left\{(X_{nk}) \in \omega^F\ \ : sup_{n,k}\ M\left(\frac{\lambda\,(X_{nk},\ \bar{0})}{\rho}\right) {<}\infty\ \ ;\ sup_{n,k}\ M\left(\frac{\xi\,(X_{nk},\ \bar{0})}{\rho}\right) {<}\infty\ \right\}.$$

$$\mathrm{C}^F\,(M, \lambda, \xi) = \left\{(X_{nk}) \in \omega^F : lim_{n,k}\ M\left(\frac{\lambda\,(X_{nk}, L)}{\rho}\right) = 0\ ;\ lim_{n,k}\ \ M\left(\frac{\xi\,(X_{nk}, L)}{\rho}\right) = 0, L \in \mathbb{R}\,(\mathrm{I})\right\}.$$

$$C_o^F\,(M, \lambda, \xi) = \left\{(X_{nk}) \in \omega^F : lim_{n,k}\ M\left(\frac{\lambda(X_{nk}, \bar{0})}{\rho}\right) = 0\ ; lim_{n,k}\ M\left(\frac{\xi(X_{nk}, \bar{0})}{\rho}\right) = 0, \text{as } n,k{\to}\infty\right\}.$$

for some $\rho > 0$, and studied various properties like linearity, completeness, solidity etc.

Also, in 2023, Paudel et al [82] studied the classes of fuzzy sequence and introduced a Paranorm to study some of the properties of the classes.

An outstanding contribution and plenty of works have been done in real-life applications with the help of fuzzy set theory. In the fields of economics, medicine, engineering, agriculture, decision-making, numerous researchers worked using fuzzy set theory. Stojic [98] presented a model for evaluating the level of economic development of countries in a particular regions using fuzzy logic.

Moreover, Kumar et al [57] have presented a model based on a fuzzy expert system that will help insurance companies to evaluate the risk cancellation policies in the presence of diabetes.

In our daily lives, we may face numerous encounters in selecting the best one for actual outcomes. Paudel et al [78] addressed the difficulties encountered when deciding on the best candidate from a group of people in the same group environment using the fuzzy maximum-minimum composition.

In our daily lives, decision-making problems may have a membership and non-membership degrees with the possibility of hesitation. Using these concepts, Atanassav [7] introduced the concept of intuitionistic and fuzzy set theory, which is capable of capturing information that includes membership and non-membership values with some possible degree of hesitation. The intuitionistic fuzzy set has been applied in various fields of research in making a decision and medical diagnosis. Ejewa and Onasanya [30] showed how an intuitionistic fuzzy set could be used to solve real-world decision-making issues, like medical diagnostics and bioinformatics. Similarly, in 2020, Ejewa and Onyek [31] tested a new method's applicability by conducting hypothetical medical diagnoses and a few patients and determined their respective diagnoses based on the correlation coefficient values between each patient and each disease. Fuzzy soft theory is gaining importance for finding a coherent and logical solution to various real-life problems. The concept of fuzzy soft set and intuitionistic fuzzy soft set was used by Hooda et al [42] to study medical diagnosis using Sanchez's methodology. Moreover, Paudel et al [84] explored how Sanchez's medical theory could be used in medical diagnosis and provide a fuzzy-arithmetic–based algorithm for identifying medical conditions. A description of fuzzy pattern reorganization techniques to be used in the cluster analysis of risk and classification of claims was developed by Derrig et al [27] , while Deshmukh et al [28] suggested a fuzzy reasoning approach based on rules to quantify the threat of management fraud. Paudel et al [to be appear] demonstrate how a fuzzy set can be intuitively used to calculate red-flag on a propositional or duration scale, how ambiguous law could be used to combine several red-flags, and how a particular measure of the threat of administrative fraud can be created.

1.2 Rationale

Fuzzy describes things that are ambiguous or unclear. Everyone encounters the situations in real life where they are unsure of whether a statement is true or false. When such a situation arises, fuzzy logic offers useful flexibility for reasoning by taking the situation's uncertainties into account. Especially, fuzzy logic may be viewed as an attempt at the formation of two remarkable capabilities: rational decision-making in vagueness and uncertainty, and performing various tasks without measurement or computation in a wide range of situations. After carefully analyzing all the variable's information and making the best decision, fuzzy logic assists in solving specific problems. The fuzzy logic approach simulates human decision-making by taking into account all the potential outcomes between true and false as a digital value. Fuzzy logic and fuzzy set theory provide a framework to deal with situations where the boundaries between categories are not well defined and the information is vague or incomplete.

The fuzzy set theory has been successfully applied in a wide range of mathematical fields. Numerous researchers have studied the fuzzy set and fuzzy real numbers in depth in various types of sequence spaces. Fuzzy set and fuzzy logic provide a way to generalize the idea of convergence, continuity, and other basic properties connected with conversational sequence space in the context of sequence space. The concept of a sequence is expanded to include sequences with fuzzy elements in fuzzy sequence spaces, where each element is represented by a membership degree that indicates how much it belongs to the set. In comparison to conventional sequence space, fuzzy sequence space offers a more reliable and adaptable representation. They are able to capture variations and fluctuations in data that crisp sequences might not be able to adequately represent. Also, the approximation function and data are the topics covered in approximation theory, which has connections with fuzzy sequence spaces. A fuzzy approximation of uncertain or imprecise data can be produced using fuzzy sequences, which can also be used to approximate fuzzy functions. Moreover, the development of the mathematical foundation for fuzzy set theory is aided by the study of fuzzy sequence spaces. It facilitates the development of formal frameworks and properties of fuzzy sequence spaces, opening up new theoretical and conceptual avenues in mathematics.

A lot of works can be done in the fields of sequence space through fuzzy set theory. So for the new researcher in the field of sequence space, fuzzy set theory is a good platform. Generally, studying fuzzy logic and fuzzy set theory provide a helpful framework for handling uncertainty, simulating human reasoning, processing linguistic variables, designing control systems, aiding decision-making, and solving complex problems in various domains. Overall, studying fuzzy space provides a useful mathematical tool for dealing with ambiguity and vagueness. It makes possible to develop a trustworthy model, algorithm, and framework for making a decision, all of which are better equipped to handle the challenging real-world problems brought on by ambiguous or incomplete data.

1.3 Objective of the Study

This thesis is based on our research work. The objectives are as follows:

1. To expand the notion of sequence space of real and complex numbers by employing various operators in fuzzy real numbers.

2. Fuzzifying the spaces $\ell_M(X, \bar{\lambda}, \bar{p})$ and $\ell_M(X, \bar{\lambda}, \bar{p}, L)$ studied by Pahari and Srivastava [72] and its conversion in difference sequence spaces of fuzzy real numbers.

3. To study the p -bounded variation of sequence space of fuzzy real numbers in generalized

form.

4. To study the double sequence space of fuzzy real numbers using the Orlicz function.
5. To address the challenge seen in making-decision to select the best one using Bellmen- Zadeh method
6. To study the real-life applications of fuzzy logic and fuzzy set theory with the case study.

1.4 Organization of the Study

Uncertainty and ambiguity are the subjects of fuzzy logic. It is a versatile approach to uncertainty-based reasoning for making rational decisions that deal with ambiguous or insufficient data and offer solutions to particular issues. In a variety of mathematical disciplines, the fuzzy set theory has been effectively employed. Pure and application of fuzzy set theory are the two main divisions of this thesis. The sequence spaces for fuzzy real numbers are studied in the thesis's first section. On the other hand, fuzzy sets have been employed in several real-world applications in the applied section. With the use of the membership function specified on the fuzzy set, this thesis addresses the issues with decision-making in picking the best one. There are five primary chapters that make up this thesis.

Fuzzy logic, fuzzy sets, and fuzzy difference sequence spaces of fuzzy real numbers are all introduced in Chapter 1. In this chapter, basic definitions, fuzzy set theory, and fuzzy logic ideas have all been covered. Besides of these, chapter one also contains some introductions to sequence spaces of fuzzy real numbers. In the same chapter, it will also emphasize why fuzzy logic should be studied. Second chapter contains the fundamental concepts of fuzzy set theory, which are essential to our study.

Some well-known fuzzy number sequence spaces will be studied in chapter three. With the aid of the Orlicz function and Paranorm function, the fundamental topological characteristics of classes $\ell_F(X, \bar{\lambda}, \bar{p})$ and $\ell_F(X, \bar{\lambda}, \bar{p}, L)$ of fuzzy real numbers will be examined in this chapter. The linearity, completeness, and solidity characteristics of the classes of double sequences $\ell_\infty^F(M, \lambda, \xi)$, $C_\infty^F(M, \lambda, \xi)$, and $C_o^F(M, \lambda, \xi)$ will be covered in detailed in the same chapter together with double sequence spaces of fuzzy real numbers. Chapter four covers the basic topological characteristics of the generalized difference sequence spaces of fuzzy real numbers. This chapter further investigates the generalized completeness feature of the class $bV_p^F$of p -bounded variation. In addition, various classes inclusion connections will be studied and investigated.

Fuzzy sets will be employed in real-world applications in Chapter 5. The membership function constructed on the fuzzy set using the Bellmen-Zadeh approach will be used in this chapter to solve the issues with decision-making when picking the best. A case study is offered to validate our work, and Sanchez's medical theory will also be utilized to offer a fuzzy arithmetic-based strategy for locating medical concerns. The issue of fraud in the claims-settlement procedure is becoming quite problematic for insurance companies. The fuzzy model we create in this chapter will help internal auditors spot claims that could be the result of fraudulent activity. A case study will be provided to demonstrate the effectiveness of the method.

Chapter 2

Fundamental Concepts and Preliminaries

2.1 Introduction

Before the introduction of Fuzzy Logic, mathematics was limited only to two conclusions that were true and false (1 and 0) only. Lotfi Zadeh [117] developed the concept of fuzzy set theory as an extension of classical set theory the 1965. It allows for the formal representation and manipulation of information's uncertainty and ambiguity. Fuzzy logic is a strong tool for control systems, expert systems, and other applications that deal with ambiguous or incomplete information because it enables the modeling of human-like decision-making processes through the use of linguistic variables and membership functions [4]. Fuzzy sets are defined by a membership function that characterizes the degree of membership for each element of the universal set. The membership degrees provides greater flexibility and expressiveness in representing and reasoning with uncertain or imprecise information, making fuzzy set theory a powerful tool for dealing with real-world problems characterized by ambiguity and fuzziness. So the core concept of fuzzy set theory is degree of membership. Instead of using a binary value of 0 or 1 to indicate membership or non-membership, fuzzy sets assign a degree of membership between 0 and 1. This degree of membership represents the extent to which an element belongs to a fuzzy set.

2.2 Preliminaries

Definition 1. ***(Fuzzy set and membership function)****[3]: Let U be the universe of discourse and* $X \subseteq U$*. A fuzzy set X in U is defined as the collection of order pair (* x, $\mu_X(x)$*) where* $\mu_X : U \rightarrow [0, 1]$ *and* $x \in U$ *. Here* $\mu_X(x)$ *is called the degree of membership of* x *and* μ_X *is called the membership*

function.
A membership function maps the element of the universe of discourse to a value between 0 and 1, indicating the degree to which the element belongs to the fuzzy set. This value represents the "fuzziness" or uncertainty associated with the membership of an element in the set. It can take various shapes depending on the specific application and the linguistic variables being considered. Common shapes include triangular, trapezoidal, Gaussian, and sigmoidal, among others. These shapes are chosen based on the characteristics of the problem domain and the requirements of the fuzzy logic system.

Membership functions are used in the fuzzification and defuzzification steps of a fuzzy logic system to map the non-fuzzy input values to fuzzy linguistic terms and vice versa. A membership function is also used to quantify a linguistic term.

Definition 2. *(**Fuzzification**)[41]:Fuzzification is the process of assigning the numerical input of a system to fuzzy sets with some degree of membership. This degree of membership may be anywhere within the interval [0,1]. If it is 0 then the value does not belong to the given fuzzy set, and if it is 1 then the value completely belongs within the fuzzy set. Any value between 0 and 1 represents the degree of uncertainty that the value belongs in the set. Because it enables the depiction of the ambiguity and uncertainty found in many real-world situations, the degree of membership function is a key idea in fuzzy logic. By assigning degrees of membership rather than strict binary classifications, fuzzy logic provides a more flexible and expressive framework for reasoning and decision-making in situations where precise boundaries are not applicable or are difficult to define.*

Definition 3. *(**Defuzzification**)[41]: Defuzzification is the process of producing a quantifiable result in Crisp logic for given fuzzy sets and corresponding membership degrees. To represent and handle imprecise or uncertain data, fuzzy logic systems make use of fuzzy sets and linguistic variables. To produce a single numerical result, defuzzification is required after fuzzy inference, which entails applying fuzzy logic rules to the fuzzy input values to generate fuzzy output values. There are several methods for defuzzification and the choice of defuzzification method depends on the specific application and the desired behavior of the fuzzy logic system.*

The defuzzification process, which yields a single crisp value with regard to the fuzzy set, is depicted in the following figure.

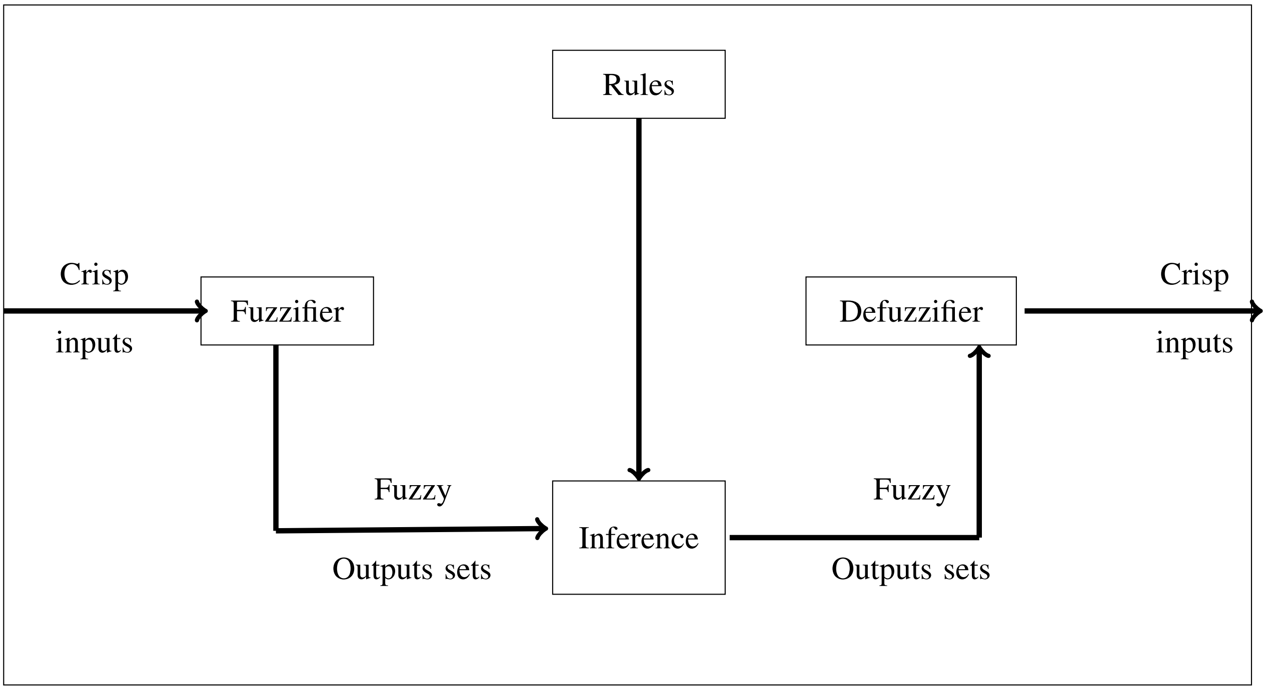


Figure 2.1: Fuzzy inference system [41]

Definition 4. *(**Support of fuzzy set**)[6]: The support of a fuzzy set X on U is the collection of all those elements of U having a membership value greater than* 0.

$$i.e. \quad support(X) = \{x \in U : \mu_X(x) > 0\}.$$

Definition 5. *(**α-level Set**)[6]: Let α be a number in $[0, 1]$,then α-cut is defined as the set of all those elements of the fuzzy set X whose membership values are not less than α.*
Symbolically;

$$X^{\alpha} = \{x \ : \mu_X(x) \geq \alpha, \alpha \in [0,\ 1]\}.$$

In particular, if $U = \{$ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$\}$ be a universal set and let
$X = \{$ (1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)$\}$.
Then $\alpha - cut$ of the set X is a crisp set and is $X^{\alpha} = \{$ 1, 2, 3, 4, 5, 6$\}$.
We note that the elements 7, 8, 9, and 10 are not part of the support of X.
The maximum value of the membership degree is known as the height of the fuzzy set. The fuzzy set having height one is called the Normalized fuzzy set.

Definition 6. *(**Convexity**)[5]: A fuzzy set X on U is convex if and only if for all x_1, $x_2 \in X$ and all, $\lambda \in$[0, 1] , $\mu_X(\lambda x_1 + (1-\lambda)x_2) \geq Minimum[\mu_X(x_1), \mu_X(x_2)]$.*

Definition 7. *(**Subset and proper subset**)[116]: The fuzzy set A is called a subset of a fuzzy set or included in the fuzzy set B denoted by* A⊆B *if for each $x \in U$, $\mu_A(x) \leq \mu_B(x)$.*
A fuzzy set A is called a proper subset of the fuzzy set B denoted by A⊂B *if for each $x \in U$, $\mu_A(x) < \mu_B(x)$.*

Definition 8. ***(Core of fuzzy set)**[116]: The core of a fuzzy set X is denoted by $core(X)$ and is the set of all points $x \in U$ such that $\mu_X\,(x) = 1$. So,*

$$core(X) = \{\ x \in U :\ \mu_X\,(x) = 1\}.$$

The figure below clearly depicts the core set of fuzzy sets.

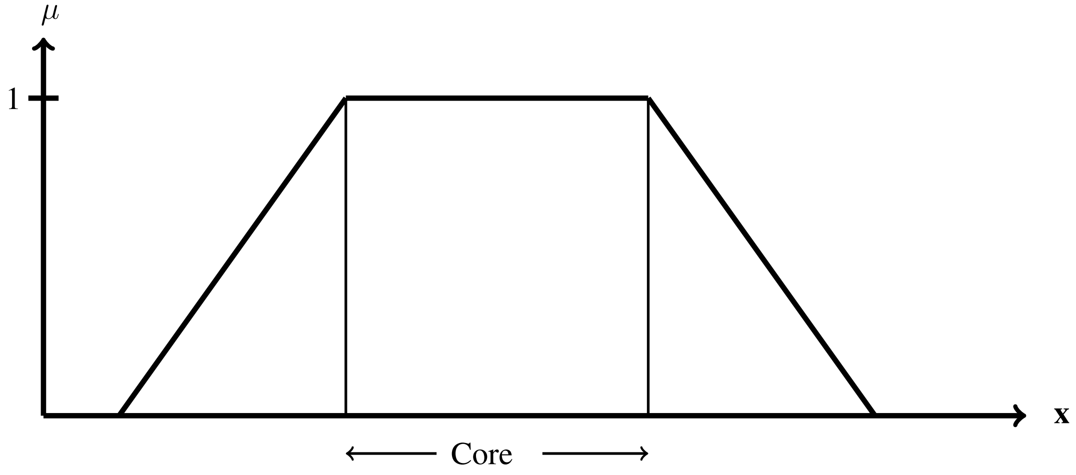


Figure 2.2: Core of fuzzy set

Definition 9. ***(Fuzzy number)**[9] Fuzzy numbers are subsets of real lines with additional properties and thus generalize the classical real numbers. Fuzzy numbers are essential for fuzzy analysis, differential equations, and various applications of fuzzy sets and fuzzy logic.*

A fuzzy number is a fuzzy set X in the real line $\mathbb{R}$ satisfying

1. *X is normal, i.e. there exists a real number x such that $\mu_X\,(x) = 1$*
2. *X is fuzzy convex i.e. for any pair of x, y in support of (X) such that*

   $\mu_X\,(\theta x + (1-\theta)\,y) \geq \min\{\mu_X\,(x)\,,\ \mu_X\,(y)\}$ *for all* $\theta \in [0,1]$
3. *X is upper semi-continuous i.e. for $\alpha \in [0,1]$ the α-level set $X^\alpha = \{x \in \mathbb{R} : \mu_X(x) \geq \alpha\}$ is closed.*

In other words, we have
A fuzzy real number X is a fuzzy set, or a mapping between each real number $\mathbb{R}$ and its membership value $\mu_X(x)$, where $\mu_X : \mathbb{R} \to I = [\,0,\ 1]$ such that the fuzzy number X is

1. *normal if there exists $x\ \in \mathbb{R}$ such that $\mu_X(x)$ = 1*
2. *convex if for x , $y \in \mathbb{R}$ and 0 $\leq \theta \leq$ 1 , $\mu_X(\theta x + (1-\theta y) \geq$ min $\{\mu_X(x), \mu_X(y)\}$*

3. *upper semi-continuous if for* $\varepsilon > 0, X^{-1}([0,\ a+\varepsilon))$*, for all* $a \in I$*, is open in the usual topology of* $\mathbb{R}$*.*

Definition 10. ***(Triangular fuzzy number)****[19] Let a, b, and c be three real numbers with* $a\ < b <$ c*. Then a fuzzy number of the form X = (a, b, c) is a triangular fuzzy number with the membership function defined by*

$$\mu_X\,(x) = \begin{cases} \frac{x-a}{b-a}\ ,\ if\ x \in [a,\ b] \\ \frac{c-x}{c-b}\ ,\quad if\ x \in [b,\ c] \\ 0\ ,\qquad otherwise \end{cases}$$

We note that, $\mu_X\,(b) = 1$ *means b need not be a mid-point of a and c.*

Graphically, the triangular fuzzy number has the triangualr shape with three vertices a, b, and c is shown below:

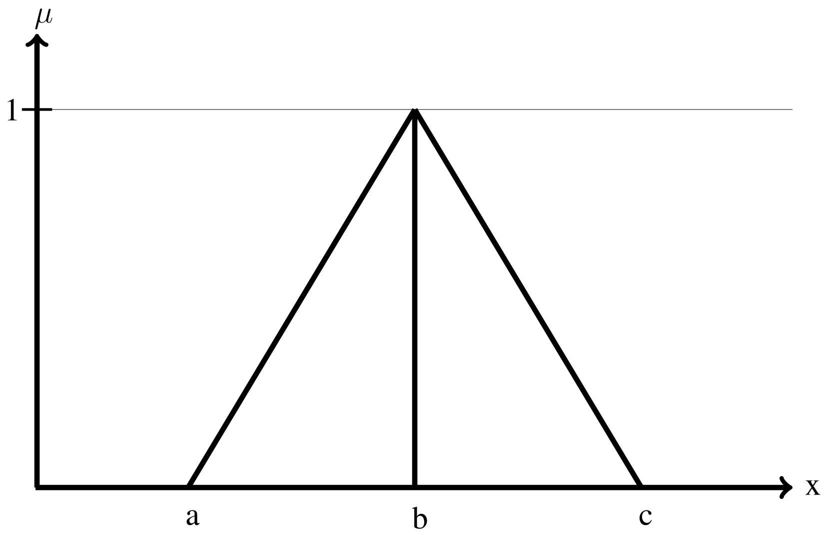


Figure 2.3: Triangular fuzzy number[14]

Definition 11. ***(Trapezoidal fuzzy number)****[98] Let* a, b, c *and* d *be real numbers with* $a < b <$ $c < d$*. Then a fuzzy number of the form A = (a, b, c, d) is a trapezoidal fuzzy number with the membership function defined by*

$$\mu_A(x) = \begin{cases} 0\ , & for\ x\ \leq a \\ \frac{x-a}{b-a}\ , & for\ a \leq x \leq b \\ 1\ , & for\ \ b \leq x \leq c \\ \frac{d-x}{d-c}\ , & for\ c \leq x \leq d \\ 0\ , & for\ x \geq d \end{cases}$$

Graphically, the trapezoidal fuzzy number has a trapezoidal shape with four vertices
(*a*, *b*, *c* ,*d*), as depicted in figure.

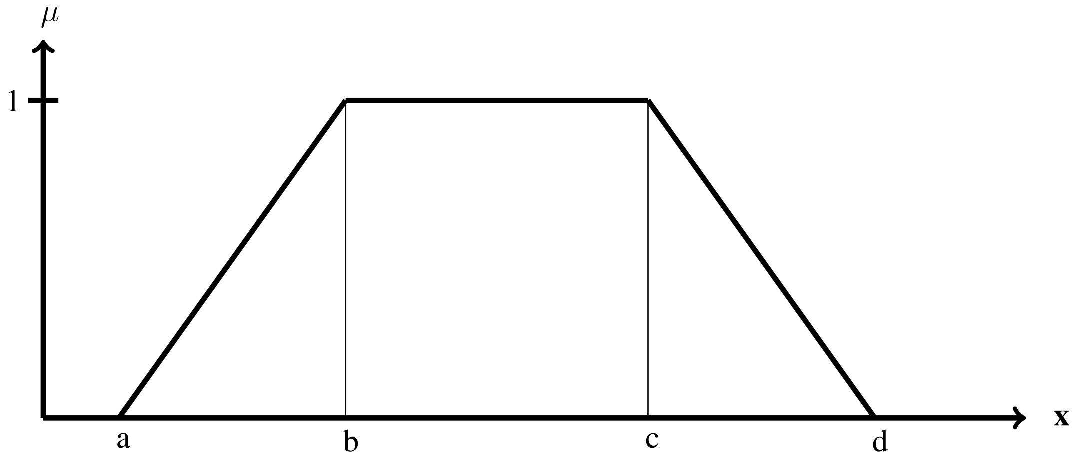


Figure 2.4: Trapezoidal fuzzy number [14]

Definition 12. *[9] Let D be the set of all bounded intervals $A = [a, b]$ on the real line $\mathbb{R}$. For any $A, B \in D$ with $A = [a_1, b_1]$ and $B = [a_2, b_2]$, $A \leq B$ if $a_2 \leq a_1$ and $b_1 \leq b_2$. Define a relation d on D by*

$$d(A, B) = max\{\mid a_1 - a_2 \mid, \mid b_1 - b_2 \mid\}$$

Then clearly , d defines a metric on D and obviously (D, d) is a complete metric space. Suppose $\mathbb{R}(I)$ denotes the set of all fuzzy numbers which are upper semi-continuous and have compact support. In other words , if $X \in \mathbb{R}(I)$ then for any $\alpha \in [0, 1]$

$$X^{\alpha} = \begin{cases} t : X(t) \geq \alpha & for\ \alpha \in (0, 1] \\ t : X(t) > \alpha & for\ \alpha = 0 \end{cases} \tag{2.1}$$

The addition and scalar multiplication on $\mathbb{R}(I)$ are defined as

$$[X + Y]^{\alpha} = X^{\alpha} + Y^{\alpha} \text{ and } (aX)^{\alpha} = a(X)^{\alpha} \text{ for all } \alpha \in [0, 1].$$

Consider a mapping $\bar{d}$: $\mathbb{R}(I) \times \mathbb{R}(I) \longrightarrow \mathbb{R}$ by the relation

$$\bar{d}(X, Y) = \sup d(X^{\alpha}, Y^{\alpha}) \text{ for } 0 \leq \alpha \leq 1.$$

Clearly $\bar{d}$ defines a metric on $\mathbb{R}(I)$ and $(\mathbb{R}(I), \bar{d})$ forms a complete metric space. Also , for any $X, Y \in \mathbb{R}(I), X \leq Y$ if and only if $[X^{\alpha}] \leq [Y^{\alpha}]$ for $\alpha \in [0, 1]$ and $X^{\alpha} = [x_1^{\alpha}, x_2^{\alpha}]$ and $Y^{\alpha} = [y_1^{\alpha}, y_2^{\alpha}]$.

2.3 Operations on Fuzzy Sets

Let *U* be the universal set, and *A* and *B* are two fuzzy sets in *U*. Let $\mu_A(x)$ and $\mu_B(x)$ be the membership values of the element $x \in U$ in the fuzzy sets *A* and *B* respectively. Then

1. The **union** of A and B is denoted by $A\cup B$ and its membership value is defined by

   $$\mu_{\mathrm{A\cup\ B}}(x) = \max\{\mu_A(x),\ \mu_B(x)\},$$

   and we write $\mu_{\mathrm{A\cup\ B}} = \mu_{\mathrm{A}} \vee \mu_{\mathrm{B}}$.

   Similarly, the **intersection** of A and B is denoted by $A\cap B$ and its membership value is denoted by$\mu_{\mathrm{A\cap\ B}}(x)$ and defined as

   $$\mu_{\mathrm{A\cap\ B}}(x) = \min\{\mu_A(x),\ \mu_B(x)\},$$

   and we write $\mu_{\mathrm{A\cap\ B}} = \mu_{\mathrm{A}} \wedge \mu_{\mathrm{B}}$.

   In particular, let $A = \{(a, 0.1), (b, 0.2), (c, 0.6)\}$,

   and $B = \{(a, 0.5), (b, 0.1), (c, 0.3), (d, 0.9)\}$. Then

   $A\cup B = \{(a, 0.5), (b, 0.2), (c, 0.6), (d, 0.9)\}$, and

   $\mathrm{A\cap B} = \{(a, 0.1), (b, 0.1), (c, 0.3), (d, 0.0)\}$.

2. The **complement** of A is denoted by A^c and is a fuzzy set. The membership function of A^c is denoted by μ_{A^c} and is defined by

   $$\mu_{\mathrm{A}^c}(x) = 1 - \mu_{\mathrm{A}}(x) \quad \text{for all } x \in U.$$

   Thus, if $A = \{(a, 0.1), (b, 0.2), (\mathrm{c}, 0.6)\}$ then, $\mathrm{A}^c = \{(a,\ 0.9),\ (b,\ 0.8),\ (\mathrm{c},\ 0.4)\}$.

3. The **difference** of A and B is a fuzzy set denoted by A/B or $A - B$. The membership function ofA/B is denoted by $\mu_{\mathrm{A/B}}$ and is defined by

   $$\mu_{A/B}(x) = \mu_{A\cap B^c}(x) = \min\{\mu_A(x),\ \mu_{B^c}(x)\} \quad \text{for all } x \in U.$$

   In the above example, $B^c = \{(a,\ 0.5),\ (b,\ 0.9),\ (c,\ 0.7),\ (d,\ 0.1)\}$.

   Then,

   $$\mu_{A/B} = \mu_{A\cap B^c} = \{(a,\ 0.1), (b,\ 0.2), (c,\ 0.6),\ (d,\ 0.0)\}.$$

4. The **algebraic product** $A(x)$ and $B(x)$ for all $x \in U$, is denoted by $A(x)$**.** $B(x)$ and defined as follows

   $$A(x).B(x) = \{(x, \mu_A(x).\mu_B(x)) : x \in U\}.$$

5. The **algebraic sum** $A(x)$ and $B(x)$ for all $x \in U$, is denoted by $A(x) + B(x)$ and defined as follows

   $$A(x) + B(x) = \{(x, \mu_{A+B}(x), x \in \mathrm{U}\}, \text{where}$$

   $$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x).$$

In particular, if $A(x) = \{(x_1, 0.1), (x_2, 0.2), (x_3, 0.3), (x_4, 0.4)\}$

$B(x) = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0.8), (x_4, 0.9)\}$.

Then, $A(x).B(x) = \{(x_1, 0.05), (x_2, 0.14), (x_3, 0.24), (x_4, 0.36)\}$, and

$A(x) + B(y) = \{(x_1, 0.55), (x_2, 0.76), (x_3, 0.86), (x_4, 0.94)\}$.

2.4 Cartesian Product of Two Fuzzy Sets

Let A and B be two fuzzy sets of the universal set *X* and *Y* respectively. Then the Cartesian product of A and B is denoted by $A \times B$ and is defined by

$$A \times B = \{(x, y, \mu_{A \times B}(x, y)) : x \in X,\ y \in Y\} \text{ where,}$$

$$\mu_{A \times B}(x, y) = \min\{\mu_A(x),\ \mu_B(y)\}\ .$$

If $A = \{(x_1,$ 0.2), $(x_2,$ 0.3), $(x_3,$ 0.5), $(x_4,$ 0.6)$\}$ and $B = \{(y_1,$ 0.8), $(y_2,$ 0.6), $(x_3,$ 0.3). Then,

$$\min\{\mu_A(x_1),\ \mu_B(y_1)\} = \min\{0.2,\ 0.8\} = 0.2$$

$$\min\{\mu_A(x_1),\ \mu_B(y_2)\} = \min\{0.2,\ 0.6\} = 0.2$$

$$\min\{\mu_A(x_1),\ \mu_B(y_3)\} = \min\{0.2,\ 0.3\} = 0.2$$

$$\min\{\mu_A(x_2),\ \mu_B(y_1)\} = \min\{0.3,\ 0.8\} = 0.3$$

$$\min\{\mu_A(x_2),\ \mu_B(y_2)\} = \min\{0.3,\ 0.6\} = 0.3$$

$$\min\{\mu_A(x_2),\ \mu_B(y_3)\} = \min\{0.3,\ 0.3\} = 0.3$$

$$\min\{\mu_A(x_3),\ \mu_B(y_1)\} = \min\{0.5,\ 0.8\} = 0.5$$

$$\min\{\mu_A(x_3),\ \mu_B(y_2)\} = \min\{0.5,\ 0.6\} = 0.5$$

$$\min\{\mu_A(x_3),\ \mu_B(y_3)\} = \min\{0.5,\ 0.3\} = 0.3$$

$$\min\{\mu_A(x_4),\ \mu_B(y_1)\} = \min\{0.6,\ 0.8\} = 0.6$$

$$\min\{\mu_A(x_4),\ \mu_B(y_2)\} = \min\{0.6,\ 0.6\} = 0.6$$

$$\min\{\mu_A(x_4),\ \mu_B(y_3)\} = \min\{0.6,\ 0.3\} = 0.3\ .$$

Then, $A \times B = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix}$

2.5 Operations on Fuzzy Matrices

1. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two fuzzy matrices, then the **sum** of A and B is a fuzzy matrix defined as

   $C = A + B$ =(c_{ij})= (max $\{a_{ij},\ b_{ij}\}$)

   As an illustration, if $A = \begin{bmatrix} 0.2 & 0.3 \\ 0.5 & 0.8 \end{bmatrix}$ and $B = \begin{bmatrix} 0.6 & 0.7 \\ 0.5 & 0.3 \end{bmatrix}$, then $C = A + B = \begin{bmatrix} 0.6 & 0.7 \\ 0.5 & 0.8 \end{bmatrix}$

2. The product of A and B is denoted by AB or $A.B$ and is defined by C =AB, whose elements are given by

   C = (c$_{ij}$)= max[min $\{a_{ik},\ b_{kj}\}$]

   In particular, if $A = \begin{bmatrix} 0.2 & 0.50 & 0.0 \\ 0.4 & 1.0 & 0.1 \\ 0.0 & 1.0 & 0.0 \end{bmatrix}$ and $B = \begin{bmatrix} 1.0 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.5 \\ 0.0 & 1.0 & 0.1 \end{bmatrix}$.

   Then, for C =AB

   c_{11}= max[min $\{(0.2, 1.0)\}$, min$\{$ $(0.5,\ 0.0)\}$, min$\{$ $(0.0,\ 0.0)\}\}$] =max $\{0.2,\ 0.0,\ 0.0\} = 0.2$

   c_{12}= max[min $\{(0.2, 0.1)\}$, min$\{$ $(0.5,\ 0.0)\}$, min$\{$ $(0.0,\ 0.5)\}\}$] =max $\{0.1,\ 0.0,\ 0.0\} = 0.1$

   c_{13}= max[min $\{(0.2, 0.0)\}$, min$\{$ $(0.5,\ 0.5)\}$, min$\{$ $(0.0,\ 0.1)\}\}$] =max $\{0.0,\ 0.5,\ 0.0\} = 0.5$

   Similarly, c_{21}= 0.4 c_{22}= 0.1 c_{23}= 0.5 c_{31}= 0.0 c_{32}= 0.0 c_{33}= 0.4

   Then, $C = AB = \begin{bmatrix} 0.2 & 0.1 & 0.5 \\ 0.4 & 0.1 & 0.5 \\ 0.0 & 0.0 & 0.5 \end{bmatrix}$.

3. Let A, B, and C be three fuzzy sets, $R \subseteq A \times B$ and $S \subseteq B \times C$. Then the composition of the relation R followed by S is denoted by SoR is the relation from A to C and its membership function is defined by

   $$\mu_{SoR}(x, z) = \max\left\{ \min_{y} \{\mu_R(x, y),\ \mu_S(y, z)\} \right\}$$

   Note $SoR \subseteq A \times C$

   If the matrices M_R and M_S represent the relations R and S respectively, then the composite relation SoR is represented by the matrix M_{SoR}, which is defined as $M_{SoR} = M_R.M_S$.

   In particular, if A = $\{1, 2, 3\}$, B = $\{a, b, c, d\}$ and C = $\{\alpha, \beta, \gamma\}$ are three sets. Let us consider a relation $R \subseteq A \times B$ and $S \subseteq B \times C$ such that

$$R = A \times B = \begin{bmatrix} 0.1 & 0.2 & 0.0 & 1.0 \\ 0.3 & 0.3 & 0.0 & 0.2 \\ 0.8 & 0.9 & 1.0 & 0.4 \end{bmatrix}, \quad S = B \times C = \begin{bmatrix} 0.9 & 0.0 & 0.3 \\ 0.2 & 1.0 & 0.8 \\ 0.8 & 0.0 & 0.7 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}.$$

Then $$SoR = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.3 & 0.3 & 0.3 \\ 0.8 & 0.9 & 0.8 \end{bmatrix}.$$

Chapter 3

Topological Properties of Sequence Space and Double Sequence Space of Fuzzy Real Numbers

3.1 Introduction

A sequence space is a vector space that contains infinite sequences of real or complex numbers. Suppose, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of all natural numbers, real numbers, and complex numbers respectively. Consider a set $\omega = \{x = (x_k) \in \mathbb{R} \text{ or } \mathbb{C}\}$ with the operations defined as

$(x_n) + (y_n) = (x_n + y_n)$ and $\alpha\,(x_n) = (\alpha x_n)$ for all $n \in \mathbb{N}$ and scalar α.

Then the set ω forms a vector space with respect to the operation. Any vector subspace X of ω is then called a sequence space [71].

3.2 Topological Properties of the Classes $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}\right)$ and $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$.

Let F be the universe of discourse and $X \subseteq F$. Then X is said to be fuzzy set [20] in F if X is the collection of order pair $(x,\ \mu_X)$ where, $\mu_X : F \to [0,1]$. So that the fuzzy set X in F is defined as the collection $X = \{(x,\ \mu_X(x)) : x \in F \text{ and } \mu_X : F \to [0,1]\}$, and the function μ_X is called the membership function and $\mu_X(x)$ is called the degree of the element x. A fuzzy real number X is a fuzzy set, or a mapping between each real number $\mathbb{R}$ and its membership value $X(t)$, where $X \colon \mathbb{R} \to I = [\,0,\ 1]$ such that

The fuzzy number *X* is

i. normal if there exists $t \ \in \mathbb{R}$ such that *X*(t) = 1

ii. convex if for t , $s \in \mathbb{R}$ and $0 \leq \theta \leq 1$, X (θ t + (1 – θ) s) ≥ min { *X*(t) , *X*(s) }

iii. X is upper semi-continuous if for $\varepsilon > 0, X^{-1}([0,\ a+\varepsilon))$, for all $a \in I$, is open in the usual topology of $\mathbb{R}$.

The α- level set on *X* is denoted by X^{α} and defined by $X^{\alpha} = \{\ t\, \epsilon\, \mathbb{R} : X\ (t) \geq \alpha\ \}$.

The collection of all fuzzy numbers with membership values greater than zero is referred to as support of fuzzy a number.

Let $\mathbb{R}$ (I) be the collection of all upper semi-continuous fuzzy real numbers with compact support. A sequence X $= (X_k)$ of fuzzy real numbers is a function from the set of natural numbers to the set $\mathbb{R}$ (I). The fuzzy number X_k is the k^{th} value of the function and is known as the k^{th} term of the sequence.

Let (x_n) be any sequence. Define

$$M_n = \sup_n \{x_n,\ x_{n+1},\ x_{n+2},\ x_{n+3},\ .\quad .\quad .\}$$

$$M_{n+1} = \sup_n \{\ x_{n+1},\ x_{n+2},\ x_{n+3},\ .\quad .\quad .\}$$

Then, clearly, $M_n \geq M_{n+1}$ and so (M_n) is decreasing sequence. If (M_n) is convergent then it converges to its infimum. Whatever the limit of(M_n) is the lim sup (x_n) and we write $\overline{lim}_{n\to\infty}\, x_n$.

Similarly, we define

$$m_n = \inf_n\ \{x_n,\ x_{n+1},\ x_{n+2},\ x_{n+3},\ .\quad .\quad .\}$$

$$m_{n+1} = \inf_n \{\ x_{n+1},\ x_{n+2},\ x_{n+3},\ .\quad .\quad .\}$$

Then, clearly, $m_n \leq m_{n+1}$ and so (m_n) is increasing sequence. If (m_n) is convergent then it converges to its supremum. Whatever the limit of (m_n) is the lim inf (x_n) and we write $\underline{lim}_{n\to\infty} x_n$

Definition 13. ***(Limit supremum and limit infimum)****[101]: Let $X = (X_k)$ be a bounded sequence of fuzzy numbers. The limit infimum and limit supremum of the sequence are defined as*

$$\liminf X_k = \lim_{n\to\infty} \inf_{k\geq n} X_k \quad ;$$

$$\limsup X_k = \lim_{n\to\infty} \sup_{k\geq n} X_k \quad .$$

We note that the limit infimum or limit supremum of the bounded sequence of fuzzy numbers may not exist.

Let $(X_n) = ((-1)^n) = \{-1,\ 1,\ -1\ ,\ 1,\ \ .\ \ .\ \ .\quad \}$.

Here, $M_n = 1,\ M_{n+1} = 1,$ $M_{n+2} = 1,$ $M_{n+3}=1, \ldots$

Then, $\overline{\lim}_{n\to\infty} X_n = \ \inf\{M_n,\ \ M_{n+1},\ M_{n+2,}\ \ M_{n+3,}\ \ .\ .\ .\}$

$$= \inf\{1,\ \ 1,\ \ 1,\ \ \ 1.\ .\ .\}$$

$$= 1.$$

And, $\underline{\lim}_{n\to\infty} X_n\ = \sup\{m_n,\ \ m_{n+1},\ m_{n+2,}\ \ m_{n+3,}\ \ .\ .\ .\}$

$$=\ \sup\{-1,\ -1,\ -1,\ -1\ .\ .\ .\}$$

$$= -1.$$

The idea of the Orlicz space was first put forth by Wladyslaw in 1932. The concept of Orlicz function was then used by Lindenstrauss and Tzafriri [59] to construct the sequence space as follows

$$\ell_M = \{\ x\ \in \omega :\ \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right)<\infty,\ \text{for some } \rho > 0\}.$$

The space ℓ_M with the norm $\|x\|$ defined by

$$\|x\| = \inf\left\{\ \rho > 0\ :\ \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right)<1\right\}$$

becomes a Banach Space and is called Orlicz sequence space.

Definition 14. ***(Orlicz function)*** *[59]: A function* $M\ \ : [\,0,\infty) \to [0,\infty)$*, which is continuous, non-decreasing, and convex with the properties that*

$$M(0) = 0,\ \ M(t) > 0 \ \mathit{and}\ M(t) \to 0 \text{ as } t \to \infty$$

Based on the Orlicz function and fuzzy numbers we now introduce a class of sequence of fuzzy sequences as follows:

Let $\overline{p} = (p_k)$ be a sequence of strictly positive real numbers and $\overline{\lambda} = (\lambda_k)$ and $\overline{\mu} = (\mu_k)$ be non-zero sequences of real numbers and if $t_k = \left|\frac{\lambda_k}{\mu_k}\right|,\ k \geq 1$. Let

$$\ell_{\mathcal{F}}\left(X,\overline{\lambda},\overline{p}\right) = \left\{X = (X_k) : X_k \in \omega\,(F)\,, k \geq 1 \text{ and} \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho}\right) < \infty,\ for\ some\ \rho > 0\right\}.$$

Theorem 3.2.1. *[82]The relation* $\ell_{\mathcal{F}}\left(X,\overline{\lambda},\overline{p}\right) \subseteq \ell_M\left(X,\overline{\mu},\overline{p}\right)$ *holds if and only if* $\liminf_k t_k > 0$.

Proof. Suppose that $\liminf_k\ t_k > 0$ holds and let $X = (X_k) \in \ell_{\mathcal{F}}\left(X,\ \overline{\lambda},\ \overline{p}\right)$. Then there exists

$\rho >$ such that

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, X_k\|^{p_k}}{\rho}\right) < \infty.$$

Here, $\liminf_k\, t_k > 0$, so there exists *m* and a positive integer *K* such that $m|\mu_k|^{p_k} \le |\lambda_k\,|^{p_k}$ for $k\ \ge K$. Let us choose $\rho_1 > 0$ such that $\rho\ \le m\ \rho_1$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k\, X_k\|^{p_k}}{\rho_1}\right) &= \sum_{k=1}^{\infty} M\left(\frac{|\mu_k|^{p_k}\|X_k\|^{p_k}}{\rho_1}\right)\\ &\le \sum_{k=1}^{\infty} M\left(\frac{|\lambda_k\,|^{p_k}\|X_k\|^{p_k}}{m\ \rho_1}\right)\\ &= \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, X_k\|^{p_k}}{m\rho_1}\right)\\ &\le \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, X_k\|^{p_k}}{\rho}\right) < \infty.\end{aligned}$$

and therefore, $\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k\, X_k\|^{p_k}}{\rho_1}\right) < \infty \Longrightarrow X = (X_k) \in \ell_{\mathcal{F}}\,(\,X\,,\,\overline{\mu},\,\overline{p})\,.$

Hence $\ell_{\mathcal{F}}\left(\,X\,,\,\overline{\lambda},\,\overline{p}\right)\ \subseteq\ \ell_{\mathcal{F}}\,(\,X\,,\,\overline{\mu},\,\overline{p})$.

Conversely, suppose that $\ell_{\mathcal{F}}\,\left(\,X,\,\overline{\lambda},\,\overline{p}\right)\ \subseteq\ \ell_{\mathcal{F}}\,(\,X,\,\overline{\mu},\,\overline{p})$. Then we show that, $\liminf_k\, t_k>0$. But we assume that $\liminf_k t_k = 0$, then we can find a sequence $(k(n))$ of integers such that $k\,(n+1) > k\,(n) \ge 1\,,\ n \ge 1$ for which $n^2|\lambda_{k(n)}|^{p_{k(n)}} \le |\mu_k|^{p_{k(n)}}$.
Let $Z\ \in X$ with $\|Z\| = 1$ and define $X = (X_k)$ by the relation

$$X_k = \begin{cases} \lambda_{k(n)}^{-1}\, n^{-2/p_{k(n)}}\ \ Z\,, & \text{for } k = k\,(n)\,,\ n \ge 1\\ 0\,, & \text{otherwise}\end{cases}$$

Let $\rho > 0$. Then using the convexity of M, we have

$$\begin{aligned}\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, X_k\|^{p_k}}{\rho}\right) &= \sum_{k=1}^{\infty} M\left(\frac{\left\|\ n^{-2/p_{k(n)}}\ \ Z\right\|^{p_{k(n)}}}{\rho}\right)\\ &= \sum_{k=1}^{\infty} M\left(\frac{\|Z\|^{p_{k(n)}}}{n^2\rho}\right)\\ &= \sum_{k=1}^{\infty} M\left(\frac{1}{n^2\rho}\right)\end{aligned}$$

$$\leq M\left(\frac{1}{\rho}\right)\sum_{k=1}^{\infty}\frac{1}{n^2}\ < \infty$$

$$\therefore\ \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ X_k\|^{p_k}}{\rho}\right) < \infty\ \Longrightarrow X \in\ \ell_{\mathcal{F}}\left(\,X,\ \overline{\lambda},\ \overline{p}\right).$$

But on the other hand, we have

$$\begin{aligned}\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k\ X_k\|^{p_k}}{\rho}\right) &= \sum_{n=1}^{\infty} M\left(\frac{\left\|\mu_{k(n)}\ \ \lambda_{k(n)}^{-1}\ n^{-2/p_{k(n)}}\ \ Z\right\|^{p_{k(n)}}}{\rho}\right)\\ &= \sum_{n=1}^{\infty} M\left(\frac{\left\|\frac{\mu_{k(n)}}{\lambda_{k(n)}}\right\|^{p_{k(n)}}}{n^2\rho}\right)\\ &\geq \sum_{n=1}^{\infty} M\left(\frac{1}{\rho}\right) = \infty.\end{aligned}$$

Therefore, $\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k\ X_k\|^{p_k}}{\rho}\right)$diverges and so $X \notin \ell_{\mathcal{F}}\left(\,X,\ \overline{\mu},\ \overline{p}\right)$.

This contradicts that $\ell_{\mathcal{F}}\left(\,X,\ \overline{\lambda},\ \overline{p}\right) \subseteq\ \ell_{\mathcal{F}}\left(\,X,\ \overline{\mu},\ \overline{p}\right)$. So we must have $\liminf_k\ t_k{>}0$. □

Theorem 3.2.2. *[82]* $\ell_{\mathcal{F}}\left(\,X\ ,\ \overline{\mu},\ \overline{p}\right) \subseteq \ell_{\mathcal{F}}\left(\,X\ ,\ \overline{\lambda},\ \overline{p}\right)$ *if and only if* $\limsup_k\ t_k\ < \infty$.

Proof. Suppose that $\limsup_k t_k{<}\infty$ holds and let $X = (X_k) \in \ell_{\mathcal{F}}\left(\,X\ ,\ \overline{\mu},\ \overline{p}\right)$. Then there exists $\rho > 0$ such that $\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k\ X_k\|^{p_k}}{\rho}\right) < \infty$.

Here, $\limsup_k t_k{>}0$, so there exists $N > 0$ and a positive integer *K* such that
$N\ |\mu_k|^{p_k} > |\lambda_k\ |^{p_k}$ for $k\ \geq K$.

Let us choose $\rho_2 > 0$ such that $N\ \rho\ \leq\ \rho_2$. Then, we have

$$\begin{aligned}\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ X_k\|^{p_k}}{\rho_2}\right) &= \sum_{n=1}^{\infty} M\left(\frac{|\lambda_k\ |^{p_k}\|X_k\|^{p_k}}{\rho_2}\right)\\ &\leq \sum_{n=1}^{\infty} M\left(\frac{N|\mu_k\ |^{p_k}\|X_k\|^{p_k}}{\rho_2}\right)\\ &\leq \sum_{n=1}^{\infty} M\left(\frac{\|\mu_k\ X_k\|^{p_k}}{\rho}\right)\ < \infty.\end{aligned}$$

Therefore, $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ X_k\|^{p_k}}{\rho_2}\right) < \infty \Longrightarrow X = (X_k) \in \ell_{\mathcal{F}}\left(\,X, \overline{\lambda},\ \overline{p}\right).$

Hence, $\ell_{\mathcal{F}}\,(\,X,\,\overline{\mu},\,\overline{p}\,) \subseteq \ell_{\mathcal{F}}\,\big(X,\,\overline{\lambda},\,\overline{p}\big)\,.$

Conversely, suppose that $\ell_{\mathcal{F}}\,(\,X,\,\overline{\mu},\,\overline{p}\,) \subseteq \ell_{\mathcal{F}}\,\big(\,X,\,\overline{\lambda},\,\overline{p}\big)$. We show that $\lim\,\sup_k t_k < \infty$. Assume that $\lim\,\sup_k t_k = \infty$. Then we can find a sequence $(k(n))$ of integers such that $k\,(n+1) \geq k\,(n) \geq 1\,,\ n \geq 1$ for which $\left|\frac{\lambda_{k(n)}}{\mu_{k(n)}}\right|^{p_{k(n)}} > n^2$.

Let $Z\ \in X$ with $\|Z\| = 1$ and define $X = (X_k)$ by the relation

$$X_k = \begin{cases} \mu_{k(n)}^{-1}\ n^{-2/p_{k(n)}} Z, & for\ k = k\,(n)\,,\ n \geq 1 \\ 0\,, & \text{otherwise} \end{cases}$$

Let $\rho > 0$. Then using the convexity of M, we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k\ X_k\|^{p_k}}{\rho}\right) = \sum_{n=1}^{\infty} M\left(\frac{\left\|\ n^{-2/p_{k(n)}}\quad Z\right\|^{p_{k(n)}}}{\rho}\right)$$

$$= \sum_{n=1}^{\infty} M\left(\frac{\|Z\|^{p_{k(n)}}}{n^2 \rho}\right)$$

$$= \sum_{n=1}^{\infty} M\left(\frac{1}{n^2 \rho}\right)$$

$$\leq M\left(\frac{1}{\rho}\right) \sum_{n=1}^{\infty} \frac{1}{n^2}\ < \infty.$$

Therefore, $\displaystyle\sum_{k=1}^{\infty} M\left(\frac{\|\mu_k\ X_k\|^{p_k}}{\rho}\right) < \infty \implies X \in\ \ell_{\mathcal{F}}\,(\,X\,,\,\overline{\mu},\,\overline{p}\,)\,.$

But on the other hand,

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ X_k\|^{p_k}}{\rho}\right) = \ \sum_{n=1}^{\infty} M\left(\frac{\left\|\lambda_{k(n)}\ \mu_{k(n)}^{-1}\ n^{-2/p_{k(n)}}\quad Z\right\|^{p_{k(n)}}}{\rho}\right)$$

$$= \sum_{n=1}^{\infty} M\left(\frac{\left|\frac{\mu_{k(n)}}{\lambda_{k(n)}}\right|^{p_{k(n)}}}{n^2 \rho}\right)$$

$$\geq \sum_{n=1}^{\infty} M\left(\frac{1}{\rho}\right) = \infty$$

Therefore, $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, X_k\|^{p_k}}{\rho}\right)$ diverges $\Longrightarrow X \notin \ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{p}\right)$.

This contradicts that $\ell_{\mathcal{F}}\,(X,\, \overline{\mu},\, \overline{p}) \subseteq \ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{p}\right)$. So we must have $\limsup_k t_k < \infty$. □

Theorem 3.2.3. *[82] If $0 < p_k \leq q_k < \infty$ for all but finitely many values of k, then $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}\right) \subseteq \ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{q}\right)$.*

Proof. Let $X = (X_k) \in \ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}\right)$ then there exists $\rho > 0$ such that

$$\sum_{k=1}^{\infty} M\left(\frac{||\lambda_k X_k|\,|^{p_k}}{\rho}\right) < \infty.$$

This relation shows that there exists an integer $K \geq 1$ such that $\|\lambda_k\, X_k\| < 1$. So that

$$\|\lambda_k X_k\|^{q_k} \leq \|\lambda_k X_k\|^{p_k}\ ,\ \forall k \geq K$$

$$\Longrightarrow \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, X_k\|^{q_k}}{\rho}\right) \leq \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, X_k\|^{p_k}}{\rho}\right) < \infty.$$

This shows that $X = (X_k) \in\ \ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{q}\right)$ and hence $\ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{p}\right) \subseteq \ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{q}\right)$. □

3.3 Some Topological Structures of the Class $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}\right)$.

In this section, we will discuss the linear topological structures of $\ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{p}\right)$.

Theorem 3.3.1. *[82] The class $\ell_{\mathcal{F}}\left(X\, ,\, \overline{\lambda},\, \overline{p}\right)$ forms a linear space.*

Proof. Suppose $X = (X_k)$ and $Y = (Y_k)$ be two elements of $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}\right)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k}}{\rho_1}\right) < \infty\ \ \&\ \ \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, Y_k\|^{p_k}}{\rho_2}\right) < \infty.$$

Then for any scalars α and β, let us choose $\rho_3 > 0$ such that

$$2D\rho_1 \max(1,\ |\alpha|)\ \leq \rho_3 \text{ and } 2D\rho_2 \max(1,\ |\beta|)\ \leq \rho_3$$

Now, using the inequality $|X + Y|^{p_k}\ \leq\ D\left\{|X|^{p_k} + |Y|^{p_k}\right\}$ where, $\sup_k p_k = L$ and $D = \max\left\{1\, ,\, 2^{L-1}\right\}$, we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_{\ k}(\alpha X_k + \beta Y_k)\|^{p_k}}{\rho_3}\right) = \sum_{k=1}^{\infty} M\left(\frac{\|\alpha\lambda_{\ k} X_k\ +\ \beta\lambda_k\, Y_k\|^{p_k}}{\rho_3}\right)$$

$$\begin{aligned}
&\leq \sum_{k=1}^{\infty} M\left(\frac{D\,\|\alpha\lambda_{\,k}\,X_k\|^{p_k} \;+\; D\,\|\beta\lambda_{\,k}\,X_k\|^{p_k}}{\rho_3}\right)\\
&= \sum_{k=1}^{\infty} M\left(\frac{D\,|\alpha|^{p_k}\,\|\lambda_{\,k}\,X_k\|^{p_k}}{\rho_3} + \frac{D\,|\beta|^{p_k}\,\|\lambda_{\,k}\,Y_k\|^{p_k}}{\rho_3}\right)\\
&\leq \;\sum_{k=1}^{\infty} M\left(\frac{1}{2\rho_1}\,\|\lambda_{\,k}X_k\|^{p_k} + \frac{1}{2\rho_2}\,\|\lambda_{\,k}Y_k\|^{p_k}\right)\\
&= \frac{1}{2}\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\,X_k\|^{p_k}}{\rho_1}\right) + \frac{1}{2}\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\,Y_k\|^{p_k}}{\rho_2}\right)\\
&< \infty.
\end{aligned}$$

Therefore $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda\,(\alpha X_k + \beta Y_k)\|^{p_k}}{\rho_3}\right) < \infty \Longrightarrow \alpha X + \beta Y \in \ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{p}\right)$ for $X = (X_k)$, $Y = (Y_k) \in \ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{p}\right)$ and for any scalars α and β. Hence $\ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{p}\right)$ is linear space.

□

Theorem 3.3.2. *[82] If* $\ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{p}\right)$ *is a linear space then* $lim\ sup_k t_k < \infty$.

Proof. Suppose $\ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{p}\right)$ is a linear space, but we assume that $lim\ sup_k t_k = \infty$. Then there exists a sequence $(k(n))$ of positive integers such that

$$k(n+1) > k(n) \geq 1, n \geq 1, \text{for which } p_{k(n)} > 0 \text{ for each } n \geq 1.$$

Let us choose $Z \in X$ such that $\|\mathrm{Z}\| = 1$ and define a sequence $X = (X_{\mathrm{k}})$ by the relation

$$X_k = \begin{cases} \lambda_{k(n)}^{-1}\, n^{-2/p_{k(n)}} \quad Z\,,\ for\ k = k\,(n)\,,\ n \geq 1 \\ \qquad 0\,, \qquad\qquad\qquad otherwise \end{cases}$$

Then for $k = k\,(n)\,,\ n \geq 1$, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\,X_k\|^{p_k}}{\rho}\right) &= \sum_{k=1}^{\infty} M\left(\frac{\left\|n^{-2/p_{k(n)}}Z\right\|^{p_{k(n)}}}{\rho}\right)\\
&= \sum_{k=1}^{\infty} M\left(\frac{1}{n^2\rho}\right)\\
&\leq M\left(\frac{1}{\rho}\right)\sum_{k=1}^{\infty}\frac{1}{n^2} \;< \infty.
\end{aligned}$$

This shows that $X \in \ell_{\mathcal{F}}\left(X,\, \overline{\lambda},\, \overline{p}\right)$.

But for any $\rho > 0$ and $p_{k(n)} > n$ for $n \geq 1$ and $\alpha = 4$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\, \alpha X_k\|^{p_k}}{\rho}\right) &= \sum_{n=1}^{\infty} M\left(\frac{\left\|4n^{-2/p_{k(n)}} Z\right\|^{p_{k(n)}}}{\rho}\right)\\ &= \sum_{n=1}^{\infty} M\left(\frac{4^{p_{k(n)}}}{n^2\,\rho}\right)\\ &\geq \sum_{n=1}^{\infty} M\left(\frac{4^n}{n^2\,\rho}\right)\\ &\geq \sum_{n=1}^{\infty} M\left(\frac{1}{\rho}\right) = \infty\end{aligned}$$

Thus, $\alpha\, X_k \notin \ell_{\mathcal{F}}\left(X,\ \overline{\lambda},\ \overline{p}\right)$, which contradicts the linearity of $\ell_{\mathcal{F}}\left(X,\ \overline{\lambda},\ \overline{p}\right)$. So that $lim\ sup_k t_k < \infty$.

□

Theorem 3.3.3. *(i)* $\ell_{\mathcal{F}}\left(X,\ \overline{\lambda},\ \overline{p}\right) \subseteq \ell_{\mathcal{F}}\left(X,\ \overline{p}\right)$ *if and only if* $\lim\inf_k |\lambda_k| > 0$

(ii) $\ell_{\mathcal{F}}\left(X, \overline{p}\right) \subseteq \ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}\right)$ *if and only if* $\lim\sup_k |\lambda_k|^{p_k} < \infty$;

(iii) $\ell_{\mathcal{F}}\left(X,\ \overline{p}\right) = \ell_{\mathcal{F}}\left(X,\ \overline{\lambda},\ \overline{p}\right)$ *if and only if* $0 < \lim\inf_k |\lambda_k|^{p_k} \leq \lim\sup_k |\lambda_k|^{p_k} < \infty$

Definition 15. ***(Paranormed space)****[71]: Let* X *be a vector space. A function* g: $X \to \mathbb{R}$ *is called a* ***Paranorm*** *on* X *if it satisfies the following conditions:*

1. $g(0) = 0$
2. $g(x) \geq 0$, *for all* $x \in X$.
3. $g(-x) = g(x)$, *for all* $x \in X$.
4. $g(x + y) \leq g(x) + g(y)$, *for all* $x,\ y \in X$
5. *if* (a_n) *is a sequence of scalars with* $a_n \to a$ as $n \to \infty$ *and* $\{x_n\}$ *is a sequence of vectors such that* $g(x_n - x) \to 0$ as $n \to \infty$ then $g(a_n x_n - ax) \to 0$ as $n \to \infty$ *(Continuity of scalar multiplication).*

   Then the pair $(X,\ g)$ *is called a paranormed space.*

Theorem 3.3.4. *[82] If* $\inf_k p_k = l > 0$, *then the class* $\ell_{\mathcal{F}}\left(X,\ \overline{\lambda},\ \overline{p},\ L\right)$ *defined by*

$$\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right) = \left\{X = (X_k) : X_k \in \omega(F), k \geq 1\ \&\ \sum_{k=1}^{\infty} M\left(\frac{||\lambda_k X_k||^{p_k/L}}{\rho}\right) < \infty, for\ some\ \rho > 0\right\}$$

forms a paranormed space with respect to the function defined as

$$g(X) = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ X_k\|^{p_k/L}}{\rho}\right) \leq 1\right\},\ where\ L = sup_k\, p_k.$$

Proof. Let us consider a set

$$A\,(X\,) = \left\{ \rho > 0\ :\ \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ \ X_k\|^{p_k/L}}{\rho}\right) \leq 1 \right\}.$$

We assume that $\inf_{\mathrm{k}} p_k = l > 0$ and

$$g\,(X) = \inf\left\{\ \rho > 0\ :\ \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ \ X_k\|^{p_k/L}}{\rho}\right) \leq 1\right\}$$

Then,

1. $$g\,(0) = \inf\left\{\ \rho > 0\ :\ \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ \ 0\|^{p_k/L}}{\rho}\right) \leq 1\right\}$$

$$= \inf\left\{\ \rho > 0\ :\ \sum_{k=1}^{\infty} M\left(\frac{\|0\|^{p_k/L}}{\rho}\right) \leq 1\right\}\ = 0.$$

2. $$g\,(-X\,) = \inf\left\{\rho > 0\ :\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ (-X_k)\|^{p_k/L}}{\rho}\right) \leq 1\right\}$$

$$= \inf\left\{\ \rho > 0\ :\ \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ \ X_k\|^{p_k/L}}{\rho}\right) \leq 1\right\}$$

$$= g(X).$$

3. Let $X\ ,\ Y\ \in \ell_{\mathcal{F}}\left(\,X\ ,\ \overline{\lambda},\ \overline{p}\,,\ L\right)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ \ X_k\|^{p_k/L}}{\rho_1}\right) < \infty\ \ and\ \ \sum_{\mathrm{k}=1}^{\infty}\ M\left(\frac{\|\lambda_k\ \ X_k\|^{p_k/L}}{\rho_2}\right) < \infty$$

Let $\rho_3 =\ \rho_1 + \rho_2$, where $\rho_1\ ,\rho_2 \in A\,(X\,)$. Then,

$$\sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}}(X_{\mathrm{k}}{+}Y_k)\|^{\frac{\mathrm{p_k}}{\mathrm{L}}}}{\rho_3}\right) = \sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}}(X_k + Y_k)\|^{\frac{\mathrm{p_k}}{\mathrm{L}}}}{\rho_1 + \rho_2}\right)$$

$$\leq \frac{\rho_1}{\rho_1 + \rho_2}\sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}} X_{\mathrm{k}}\|^{\frac{\mathrm{p_k}}{\mathrm{L}}}}{\rho_1}\right) + \frac{\rho_2}{\rho_1 + \rho_2}\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}} Y_k\|^{\frac{\mathrm{p_k}}{\mathrm{L}}}}{\rho_2}\right)$$

$$\leq \frac{\rho_1}{\rho_1 + \rho_2}.1 + \frac{\rho_2}{\rho_1 + \rho_2}.1 = 1\ .$$

Therefore $\sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}}(X_{\mathrm{k}}+Y_k)\|^{\frac{\mathrm{p_k}}{\mathrm{L}}}}{\rho_3}\right) \leq 1.$

This relation shows that $\rho_3 = \rho_1 + \rho_2 \in A(X)$. Thus

$g(X+Y) \leq \rho_3 = \rho_1 + \rho_2$ for $\rho_1 \in A(X)$ and $\rho_2 \in A(Y)$.

i.e, $g(X+Y) \leq \rho_1 + \rho_2$

i.e, $g(X+Y) \leq g(X) + g(Y)$.

Hence the triangle inequality holds.

4. Suppose $X^n = (X_k^n)$ be a sequence in $\ell_{\mathcal{F}}(X, \overline{\lambda}, \overline{p}, L)$ such that $g(X^n) \to 0$ as $n \to \infty$ and (α_{n}) be a sequence of scalars such that $\alpha_{\mathrm{n}} \to \alpha$. Then

$$g(\alpha_{\mathrm{n}} X^n) = \inf\left\{ \rho \ : \ \sum_{k=1}^{\infty} M\left(\|\lambda_k \alpha_n X_k^n\|^{\frac{p_k}{L}}\right) \leq 1 \right\}$$

$$= \inf\left\{ \rho \ : \ \sum_{k=1}^{\infty} M\left(|\alpha_n|^{\frac{p_k}{L}} \ \|\lambda_k X_k^n\|^{\frac{p_k}{L}}\right) \leq 1 \right\}$$

$$\leq \inf\left\{ \rho : \sum_{k=1}^{\infty} M\left(G^{\frac{p_k}{L}} \ \|\lambda_k X_k^n\|^{\frac{p_k}{L}}\right) \leq 1 \right\}, \textit{where } G = sup_n |\alpha_{\mathrm{n}}|.$$

Then for $\beta = max\{1, G\}$, we have

$$g(\alpha_{\mathrm{n}} X^n) \leq \inf\left\{ \rho \ : \ \sum_{k=1}^{\infty} M\left(\frac{\beta \|\lambda_k X_k^n\|^{\frac{p_k}{L}}}{\rho}\right) \leq 1 \right\}.$$

Taking $\frac{\rho}{\beta} = \gamma$ then $\gamma > 0$ and we have

$$g(\alpha_{\mathrm{n}} X^n) \leq \inf\left\{ \gamma : \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k^n\|^{\frac{p_k}{L}}}{\gamma}\right) \leq 1 \right\}$$

$$= g(X^{\mathrm{n}}) \to 0 \ \ as \ \ n \to \infty.$$

Therefore, $g(\alpha_{\mathrm{n}} X^n) \to 0$ as $n \to \infty$.

Now, let $\alpha_{\mathrm{n}} \to 0$ as $n \to \infty$ then for $0 < \varepsilon < 1$, we can find a positive integer N such that $|\alpha_{\mathrm{n}}| \leq \varepsilon$ for all $n \geq N$ and let $X \in \ell_{\mathcal{F}}(X, \overline{\lambda}, \overline{p}, L)$. Since $\inf_k p_k = l > 0$, so that $|\alpha_n|^{\frac{p_k}{L}} \leq (\varepsilon)^{\frac{l}{\mathrm{L}}}$ for all $n \geq N$.

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_n \lambda_k X_k\|^{\frac{p_k}{L}}}{\rho}\right) = \sum_{k=1}^{\infty} M\left(\frac{|\alpha_n|^{\frac{p_k}{L}} \ \|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho}\right)$$

$$\leq \sum_{k=1}^{\infty} M\left(\frac{\varepsilon^{\frac{l}{L}}\ \|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho}\right)$$

Thus, if $\rho\ \in A(\varepsilon^l\ X\ )$ then, $\rho \in A\left(\alpha_n X\right)\ i.e\ A(\varepsilon^l\ X\ ) \subseteq A\left(\alpha_n X\right)$.

Taking infimum over such $\rho' s$, then we get

$$\inf\left\{\ \rho\ :\ \rho \in A\left(\alpha_n X\right)\ \right\}\ \leq \inf\left\{\ \rho\ :\ \rho \in A\left(\varepsilon^{\frac{l}{L}}\ X\right)\right\} = \varepsilon^{\frac{l}{L}}\ \inf\left\{\ \rho\ :\ \rho \in A\left(X\right)\right\}\ .$$

This shows that, $g\left(\alpha_n X\right) \leq g(\varepsilon^{\frac{l}{L}} X\ )$ for all $n \geq N$. That is $g\left(\alpha_n X\right) \to 0\ as\ n \to \infty$.

Hence, $\ell_{\mathcal{F}}\left(\ X\ ,\ \overline{\lambda},\ \overline{p},\ L\right)$ is paranormed space. □

Theorem 3.3.5. *[82] The paranormed space* $\left(\ell_{\mathcal{F}}\left(\ X,\ \overline{\lambda},\ \overline{p}\ ,\ L\right),\ \ g\right)$ *is complete.*

Proof. Let $\left({X_k}^i\right)$ be a Cauchy sequence in $\ell_{\mathcal{F}}\left(\ X,\ \overline{\lambda},\ \overline{p},\ L\right)$. Let $r > 0$ be a fixed positive real number such that $M(r) \geq 1$. Then for every $\frac{\varepsilon}{\mathrm{r}} > 0$, there exists an integer $N \geq 1$, such that

$$g\left(X_k^i - X_k^j\right) < \tfrac{\varepsilon}{r} \quad \text{for all } i,\ j\ \geq N \ldots (*)$$

By the definition of paranorm, we see that

$$\sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k X_k^i{-}\lambda_k X_k^j\right\|^{\frac{p_k}{L}}}{g\left(X_k^i\right) - g(X_k^j)}\right) \leq 1 \ \ \textit{for all} \ \ i,\ j\ \geq N$$

.

So that, $M\left(\frac{\left\|\lambda_k X_k^i{-}\lambda_k X_k^j\right\|^{\frac{p_k}{L}}}{g(X^i){-}g(X_k^j)}\right) \leq 1 \leq M(r)$ for all $i,\ j\ \geq N$ and for all $k \geq 1$.

Since *M* non-decreasing , we have $\frac{\left\|\lambda_k X_k^i{-}\lambda_k X_k^j\right\|^{\frac{p_k}{L}}}{g\left(X_k^i\right){-}g(X_k^j)} < r$.

Then from (*), we have

$$\left\|\lambda_k X_k^i{-}\lambda_k X_k^j\right\|^{\frac{p_k}{L}} <\ \varepsilon.$$

This shows that $\left({X_k}^i\right)$ is a Cauchy sequence in $\mathbb{R}(I)$. Since $\mathbb{R}(I)$ is complete, so there exists X_k in $\mathbb{R}(I)$ for all $k \geq 1$ such that $X_k^i \longrightarrow X_k$ as $i \longrightarrow \infty$.
To complete the proof of the theorem , we show that X= $(X_k) \in \ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$.

Let us choose $\rho > 0$ such that $g\left(X^i{-}X^j\right) {<}\rho < \varepsilon$ for all $i,\ j\ \geq N$

$$\sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k X_k^i{-}\lambda_k X_k\right\|^{\frac{p_k}{L}}}{\rho}\right) \leq \sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k X_k^i{-}\lambda_k X_k^j\right\|^{\frac{p_k}{L}}}{g\left(X_k^i\right) - g(X_k^j)}\right) \ \leq 1, \ \text{for all} \ \ i\ ,\ j\ \geq N$$

Since, M is continuous, taking limit as $j \longrightarrow \infty$ we see that

$$\sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k X_k^i - \lambda_k X_k\right\|^{\frac{p_k}{L}}}{\rho}\right) \leq 1, \text{ for all } i \geq N.$$

Taking infimum of such ρ's, we get

$$\inf \sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k X_k^i - \lambda_k X_k\right\|^{\frac{p_k}{L}}}{\rho}\right) \leq 1$$

$$\Longrightarrow g\left(X^i - X\right) = \inf\left\{\rho : \sum_{k=1}^{\infty} M\left(\frac{\left\|\lambda_k X_k^i - \lambda_k X_k\right\|^{\frac{p_k}{L}}}{\rho}\right) \leq 1\right\} \leq \rho < \varepsilon \text{ for all } i \geq N.$$

$\Longrightarrow g\left(X^i - X\right) < \varepsilon$ for all $i \geq N$.

This shows that $X^i \longrightarrow X$ as $i \longrightarrow \infty$ and so,

$X^i - X \in \ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$, for all $i \geq N$.

Hence, X^i and $X^i - X$ are the elements of $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$.
So that, $X = X^i - (X^i - X) \in \ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$. That is $X \in \ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$.

□

Theorem 3.3.6. *[82] The space $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$ is normal.*

Proof. Let $X = (X_k) \in \ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$. So that $\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{\frac{p_k}{L}}}{\rho}\right) < \infty$ for some $\rho > 0$. Let (α_{k}) be a sequence of scalars such that $|\alpha_{\mathrm{k}}| \leq 1$ for all $k \geq 1$. Since M is non-decreasing, we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k\ \alpha_k X_k\|^{\frac{p_k}{L}}}{\rho}\right) = \sum_{\mathrm{k}=1}^{\infty} M\left(\frac{|\alpha_{\mathrm{k}}|^{\frac{\mathrm{p_k}}{L}}\ \|\lambda_{\mathrm{k}}\ \mathrm{X_k}\|^{\frac{\mathrm{p_k}}{L}}}{\rho}\right)$$

$$\leq \sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}}\ \mathrm{X_k}\|^{\frac{\mathrm{p_k}}{L}}}{\rho}\right) < \infty$$

Thus, $(\alpha_k X_k) \in \ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$. So $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$ is normal. □

Definition 16. *($\triangle_2$ **- condition**)[5] An Orlicz function M is said to satisfy $\triangle_2$ - condition for all values of t if there exists $\gamma > 0$ such that $M(2t) \leq \gamma M(t)$*

Let us introduce a sub-class $\overline{\ell}_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$ of $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$ as follows :

$$\overline{\ell}_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right) = \left\{X = (X_k) : X_k \in \omega\left(F\right), k \geq 1, \text{and} \sum_{k=1}^{\infty} M\left(\frac{\|\lambda_k X_k\|^{p_k/L}}{\rho}\right) < \infty \textit{ for all } \rho > 0\right\}.$$

Theorem 3.3.7. *If M satisfies $\mathbf{\Delta_2}$ − condition, then, $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right) = \overline{\ell}_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$.*

Proof. Suppose $X = (X_k) \in \ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$ then for $\rho > 0$, $\sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}}\ \mathrm{X}_{\mathrm{k}}\|^{\frac{\mathrm{p}_{\mathrm{k}}}{L}}}{\rho}\right) < \infty$. Let us consider an arbitrary $\rho_1 > 0$.

Case I: If $\rho \leq \rho_1$, then clearly we have

$$\sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}}\ \mathrm{X}_{\mathrm{k}}\|^{\frac{\mathrm{p}_{\mathrm{k}}}{L}}}{\rho_1}\right) \leq \sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}}\ \mathrm{X}_{\mathrm{k}}\|^{\frac{\mathrm{p}_{\mathrm{k}}}{L}}}{\rho}\right) < \infty$$

and hence we get $X = (X_k) \in \overline{\ell}_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$.

Case II : If $\rho > \rho_1$, then by using $\mathbf{\Delta_2} - condition$, we get

$$\sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}}\mathrm{X}_{\mathrm{k}}\|^{\frac{\mathrm{p}_{\mathrm{k}}}{L}}}{\rho_1}\right) = \sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\frac{\rho}{\rho_1}\|\lambda_{\mathrm{k}}\mathrm{X}_{\mathrm{k}}\|^{\frac{\mathrm{p}_{\mathrm{k}}}{L}}}{\rho}\right)$$

$$\leq K\frac{\rho}{\rho_1}\sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\lambda_{\mathrm{k}}\mathrm{X}_{\mathrm{k}}\|^{\frac{\mathrm{p}_{\mathrm{k}}}{L}}}{\rho}\right) < \infty$$

where, *K* is the number involved in $\Delta_2 - condition$ of *M*.

Hence $\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right) \subseteq \overline{\ell}_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$, and hence

$\ell_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right) = \overline{\ell}_{\mathcal{F}}\left(X, \overline{\lambda}, \overline{p}, L\right)$. □

3.4 Double Sequence Space of Fuzzy Real Numbers Defined by Orlicz Function and its Topological Properties

3.4.1 Introduction

The study of double sequences began in the late 19th century with the work of mathematicians such as Georg Cantor[17], Camille Jordan [47], and Charles Méray[67]. Cantor[17] introduced the concept of a double infinite sequence in his work on set theory, while Jordan and Méray independently studied double sequences in their work on Fourier series and series of functions [47]. In the early 20th century, the study of double sequences was further developed by mathematicians such as Hardy, Littlewood , and Mandelbrojt [40], who used double sequences to study the convergence of a series of functions. In recent years, double sequences have been studied extensively in the context of operator theory, where the researchers are used to study the convergence of operator sequences. They also have applications in other areas of mathematics, such as combinatorics, graph theory, and dynamical system. [40] In 1996, Savas [94] introduced some new double sequence spaces of fuzzy numbers to study the double statistical convergence of a sequence of fuzzy numbers. Additionally, in 2012, Das [22] introduced a certain vector-valued difference double sequences that were defined by the Orlicz function in order to explore their various properties. In the same way, Savas and Patterson [96] in 2007 created some new double sequence spaces and looked into some of their characteristics . Additionally, double sequence spaces were introduced by Tripathy and Sarma [110] in 2009 and some of their topological and algebraic characteristics were investigated. In order to explore their various characteristics, Tripathy and Sarma [104] proposed specific double sequence spaces of fuzzy real numbers defined by the Orlicz function in 2011. In 1996, Savas [95] used the Orlicz function to examine whether a sequence of fuzzy numbers is double strongly p-convergent with respect to an Orlicz function and to develop a novel concept for double lacunary sequence spaces of fuzzy numbers. In 2014, Klnç, and Solak [49] created new modulus-defined double sequence spaces, and they studied the topological and algebraic properties of these spaces. Double sequence space of fuzzy real numbers is being good platform for new researchers in the field of pure mathematics of fuzzy real numbers. It has been studied with new concepts, for instances, we refer a few Altay et al.[6] ,Dabbas et al.[21], Mansoor et al. [65], et al.[88], Sarma [91], Savas[95], Savas[96] , and many more.

This section introduces and studies a new double sequence space after introducing the idea of fuzzy real numbers. The double sequence space here is concerned with sequences of fuzzy real numbers, each element of which is a fuzzy real number represented by a membership function. We employ the Orlicz function, whose integration enables a more subtle and flexible characterization of the double sequence space.

Definition 17. *[100],[102] Let D be the set of all bounded intervals $A = [a, b]$ on the real line $\mathbb{R}$. For any $A, B \in D$ with $A = [a_1, b_1]$ and $B = [a_2, b_2]$, $A \leq B$ if $a_2 \leq a_1$ and $b_1 \leq b_2$. Define a relation d on D by*

$$d(A, B) = max\{| \ a_1 - a_2 \ |, | \ b_1 - b_2 \ |\}$$

Clearly, d defines a metric on D. Then (D, d) is a complete metric space.

Consider a mapping $\overline{d} \ : \ \mathbb{R}\,(I) \ \times \mathbb{R}(I) \to \mathbb{R}$ by the relation

$$\overline{d}\,(X, \ Y) = \quad sup_{\alpha \in [0,1]} \ d(X^{\alpha} \ , \ Y^{\alpha}), \text{where}$$

$$X^{\alpha} = \begin{cases} t \ : \ \ X\,(t) \geq \alpha \ for \ \alpha\epsilon \ (\ 0, \ 1] \\ \ t \ : \ \ X\,(t) > \alpha \ for \ \alpha = 0 \end{cases}$$

The addition and scalar multiplication on $\mathbb{R}(I)$ are defined as:

$$[X + Y]^{\alpha} = \ X^{\alpha} + \ Y^{\ \alpha} \ \ and \ \ (aX)^{\alpha} = a \ X^{\alpha}, \ \ \forall \ \alpha\epsilon \ [\ 0, \ 1]$$

Then $\overline{d}$ defines a metric on $\mathbb{R}(I)$ and ($\mathbb{R}(I)$, $\overline{d}$) is a complete metric space.

Then for any $X,\ Y \in \mathbb{R}(I),\ \ X \leq Y$ *if and only if* $[X^{\alpha}] \leq [Y^{\alpha}]$ *for* $\alpha\epsilon[\,0,\ 1]$ *and*

$X^{\alpha} = [\ \mathrm{x}_1^{\alpha}\ ,\ \mathrm{x}_2^{\alpha}\ ]$ *and* $Y^{\alpha} = [\ \mathrm{y}_1^{\alpha}\ ,\ y_2^{\alpha}\ ]$.

Let $\lambda : \mathbb{R}\,(I) \times \mathbb{R}(I) \rightarrow \mathbb{R}$ *be defined by*

$$\lambda\,(X, Y) = sup_{\alpha\in[0,1]}\ \lambda_{\alpha}\,(X^{\alpha}\ ,\ Y^{\alpha})\,,$$

where $\lambda_{\alpha} : \mathbb{R}\ \times\mathbb{R} \rightarrow \mathbb{R}$ *is defined by*

$$\lambda_{\alpha}\,(X^{\alpha}\ ,\ Y^{\alpha})=\min\{\ |X_1^{\alpha} -\ Y_1^{\alpha}|\ ,\ |X_2^{\alpha} -\ Y_2^{\alpha}|\}\ .$$

Similarly, $\xi : \mathbb{R}\,(I) \times\ \mathbb{R}(I) \rightarrow \mathbb{R}$*, is defined by*

$$\xi\,(X, Y) = sup_{\alpha\in[0,1]}\ \xi_{\alpha}\,(X^{\alpha}, Y^{\alpha})\,,$$

where $\xi_{\alpha} : \mathbb{R}\ \times\ \mathbb{R} \rightarrow \mathbb{R}$ *is defined by*

$$\xi_{\alpha}\,(X^{\alpha}\ ,\ Y^{\alpha})=\max\{\ |X_1^{\alpha} -\ Y_1^{\alpha}|\ ,\ |X_2^{\alpha} -\ Y_2^{\alpha}|\ \}\ .$$

A sequence of fuzzy numbers X = (X_k *) is a function* $X\ :\ \mathbb{N} \rightarrow \mathbb{R}(\mathrm{I})$*, where* $\mathbb{N} = \{\ 0,\ 1,\ 2,\ \ldots, \ldots, \ldots\}$*. The number* X_k *is the* k^{th} *value of the function at* $k\in\ \mathbb{N}$ *and is the* k^{th} *term of the sequence.*

Definition 18. ***(Double sequence of fuzzy numbers****)[93] A double sequence of fuzzy numbers* $X = (X_{nk})$ *is a function* $X : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathrm{I})$*, the set of fuzzy real numbers. The fuzzy number* X_{nk} *is the value of the function at the point* $(n, k) \in \mathbb{N}\times\ \mathbb{N}$ *and is called* $(n, k)^{th}$ *term of the double sequence.*

The function X: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathrm{I})$ *defined by X* $(n, k) = \frac{n}{n+k}$ *is a double sequence[38].*

The function X: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathrm{I})$ *defined by X* $(n, k) = \frac{1}{n} + \frac{1}{k}$ *is a double sequence [38].*

Definition 19. ***(Bounded fuzzy sequence****) [93]A double sequence space X = (*X_{nk} *) of fuzzy numbers is said to be bounded if there exist fuzzy numbers M and m such that* $m \leq X_{nk}\ \ \leq\ M$ *for all* $n,\ k \in \mathbb{N}$.

Definition 20. ***(Cauchy sequence****) [93] A double sequence* $\mathrm{X} =(\ X_{nk})$ *of fuzzy numbers is said to be a Cauchy double sequence if*

$$\forall\,\varepsilon > 0,\ \exists\, n_o \in\ \mathbb{N} : d\left(X_{nk}^{i}\ , X_{nk}^{j}\right) <\ \varepsilon\, for\ \min\,(i, j)\ \geq n_o$$

We also say that the double sequence $\mathrm{X} =(\ X_{nk})$ *of fuzzy numbers converges to a fuzzy number* X_o *if* X_{nk} *tends to* $\ X_o$ *as both* n *and* k *tend to* ∞ *independently of one another.*

Let ω be the set of all real or complex-valued double sequences which is a vector space with coordinate-wise addition and scalar multiplication. Then any vector subspace of ω is called double sequence space.

A double sequence space ω^F of fuzzy numbers is said to be solid [93] if
$(Y_{nk}) \in \omega^F$, whenever $|Y_{nk}| \leq |X_{nk}|$ for all n, k $\in \mathbb{N}$ for some $(X_{nk}) \in \omega^F$.

Theorem 3.4.1. *[38] A monotonic double sequence of real numbers is convergent if and only if it is bounded.*

Theorem 3.4.2. *[94] Let C_d denotes the set of all double convergent sequences of fuzzy real numbers. Then C_d is a complete metric space with the metric defined by*

$\rho(\mathrm{X}, \mathrm{Y}) = sup_{m,n}\, \overline{d}(\mathrm{X}_{mn}, Y_{mn})$, *where* $X = X_{mn}$, $Y = Y_{mn}$
are double convergent sequences of fuzzy real numbers.

Theorem 3.4.3. *[105] The classes of double sequence spaces $(2^{\ell_\infty})_F(M)$, $(2^{C_\infty})_F(M)$ and $(2^{C_o})_F(M)$ of fuzzy numbers defined by Orlicz function as*

$$(2^{\ell_\infty})_F(M) = \left\{ X = (X_{nk}) \ : sup_{n,k}\ \mathrm{M}\left(\frac{\overline{d}\,(X_{nk},\ \overline{0})}{\rho}\right) < \infty,\ \text{for some } \rho > 0 \text{ and } (X_{nk}) \in \mathbb{R}(\mathrm{I}) \right\};$$

$$(2^{C_\infty})_F(M) = \left\{ X = (X_{nk}) \ : lim_{n,k}\ \mathrm{M}\left(\frac{\overline{d}\,(X_{nk},\ \ X)}{\rho}\right) = 0,\ \text{for some } \rho > 0 \text{ and } X \in \mathbb{R}(\mathrm{I}) \right\};$$

$$(2^{C_o})_F(M) = \left\{ X = (X_{nk}) \ : sup_{n,k}\ \mathrm{M}\left(\frac{\overline{d}\,(X_{nk},\ \overline{0})}{\rho}\right) = 0,\ \text{for some } \rho > 0 \right\}.$$

form complete metric spaces with respect to the metric defined by

$$f\,(X,\ Y) = \inf\left\{\rho > 0\ :\ sup_{n,k}\ M\left(\frac{\overline{d}(\ X_{\mathrm{nk}}\ , Y_{nk})}{\rho}\right)\ \leq 1\right\}.$$

Theorem 3.4.4. *[105] The class of double sequence spaces $(2^{\ell_\infty})_F(M)$, is symmetric but the classes $(2^{C_\infty})_F(M)$ and $(2^{C_o})_F(M)$ are not symmetric.*

Theorem 3.4.5. *[80] The class Z^F ($M,\ \lambda,\ \xi$)where $Z^F = \ \ell_\infty^F$, C^F, C_o^F defined by*

$$\ell_\infty^F(M, \lambda, \xi) = \left\{(X_{nk}) \in \omega^F : sup_{n,k}\, \mathrm{M}\left(\frac{\lambda(X_{nk},\overline{0})}{\rho}\right) < \infty\ ;\ sup_{n,k}\, \mathrm{M}\left(\frac{\xi(X_{nk},\overline{0})}{\rho}\right) < \infty\right\},$$

$$\mathrm{C}^F(M, \lambda, \xi) = \left\{(X_{nk}) \in \omega^F : lim_{n,k}\, \mathrm{M}\left(\frac{\lambda\,(X_{nk}, L)}{\rho}\right) = 0\ ;\ lim_{n,k}\, \mathrm{M}\left(\frac{\xi\,(X_{nk}, L)}{\rho}\right) = 0\right\},$$

$$C_o^F\,(\,M,\ \lambda,\xi) = \left\{ (X_{nk}) \in \omega^F : lim_{n,k}\ \mathrm{M}\left(\frac{\lambda\left(X_{nk},\ \bar{0}\right)}{\rho}\right) = 0\ \ ;\ lim_{n,k}\ \mathrm{M}\left(\frac{\xi\left(X_{nk},\ \bar{0}\right)}{\rho}\right) = 0 \right\},$$

for some $\rho > 0, L \in \mathbb{R}\,(\mathrm{I})$ *, are linear spaces.*

Proof. Let $X = (X_{nk})$ and $Y = (Y_{nk})$ be two elements of $\ell_\infty^F\ (M,\ \lambda,\xi)$, then there exist $\rho_1,\ \rho_2 > 0,$ such that

$$sup_{n,k}\, M\left(\frac{\lambda\left(X_{nk},\ \bar{0}\right)}{\rho_1}\right) {<}\infty\ ;\ sup_{n,k}\, M\left(\frac{\xi\left(X_{nk},\ \bar{0}\right)}{\rho_1}\right) {<}\infty$$

and

$$sup_{n,k}\, M\left(\frac{\lambda(Y_{nk},\ \bar{0})}{\rho_2}\right) {<}\infty\ ;\ sup_{n,k}\, M\left(\frac{\xi(Y_{nk},\ \bar{0})}{\rho_2}\right) {<}\infty.$$

Then for any scalars $\alpha,\ \beta,$ and $\rho = \max\{2\alpha\rho_1,\ 2\beta\rho_2\ \}$, we have

$$\begin{aligned} sup_{n,k}\ M\left(\frac{\lambda\left(\alpha X_{nk}+\beta Y_{nk},\ \bar{0}\right)}{\rho}\right) &\leq\ sup_{n,k}\ M\left(\frac{\lambda\left(\alpha X_{nk},\ \bar{0}\right)+\lambda\left(\beta Y_{nk},\ \bar{0}\right)}{\rho}\right) \\ &= sup_{n,k}\ \left[M\ \left\{\frac{\alpha}{\rho}\lambda\left(X_{nk},\ \bar{0}\right)+\frac{\beta}{\rho}\lambda\left(Y_{nk},\ \bar{0}\right)\right\}\right] \\ &\leq sup_{n,k}\ \left[M\ \left\{\frac{1}{2}\frac{\lambda\left(X_{nk},\ \bar{0}\right)}{\rho_1}+\frac{1}{2}\frac{\lambda\left(Y_{nk},\ \bar{0}\right)}{\rho_2}\right\}\right] \\ &\leq \frac{1}{2} sup_{n,k}\ M\left(\frac{\lambda\left(X_{nk},\ \bar{0}\right)}{\rho_1}\right)+\frac{1}{2} sup_{n,k}\ M\left(\frac{\lambda\left(Y_{nk},\bar{0}\right)}{\rho_2}\right) \\ &< \infty. \end{aligned}$$

Thus, $sup_{n,k}\ M\left(\frac{\lambda\left(\alpha X_{nk}+\beta Y_{nk},\ \bar{0}\right)}{\rho}\right) \leq \frac{1}{2}\ sup_{n,k}\ \ M\left(\frac{\lambda\left(X_{nk},\ \bar{0}\right)}{\rho_1}\right)+\frac{1}{2}\ sup_{n,k}\ M\left(\frac{\lambda\left(Y_{nk},\ \bar{0}\right)}{\rho_2}\right) {<}\infty.$

Similarly , we show that

$sup_{n,k}\ M\left(\frac{\xi\left(\alpha X_{nk}+\beta Y_{nk},\ \bar{0}\right)}{\rho}\right) \leq \frac{1}{2}\ sup_{n,k}\ \ M\left(\frac{\xi\left(X_{nk},\ \bar{0}\right)}{\rho_1}\right)+\frac{1}{2}\ sup_{n,k}\ M\left(\frac{\xi\left(Y_{nk},\ \bar{0}\right)}{\rho_2}\right) {<}\infty.$

So that,

$$sup_{n,k}\ M\left(\frac{\lambda\left(\alpha X_{nk}+\beta Y_{nk},\ \bar{0}\right)}{\rho}\right) < \infty \text{ and } sup_{n,k}\ M\left(\frac{\xi\left(\alpha X_{nk}+\beta Y_{nk},\ \bar{0}\right)}{\rho}\right) {<}\infty.$$

$\Rightarrow \alpha X + \beta Y\ \in \ell_\infty^F\,(\,M,\ \lambda,\xi)$ and hence $\ell_\infty^F\,(\,M,\ \lambda,\xi)$ is linear space.

Similarly, we show that the classes $\mathrm{C}^F\,(\,M,\ \lambda,\xi)$ and $C_o^F\,(\,M,\ \lambda,\xi)$ form linear spaces. □

Theorem 3.4.6. *[80] The class $Z^F(M,\lambda,\xi)$ forms a metric space with the relation defined by*

$$\overline{d}(X,Y) = inf\left\{\rho > 0 : sup_{n,k}\, M\left(\frac{\lambda(X_{nk},Y_{nk})}{\rho}\right) < 1;\ sup_{n,k}\ M\left(\frac{\xi(X_{nk},Y_{nk})}{\rho}\right) < 1,\ \text{for some } \rho > 0\right\},$$

where $X = (X_{nk})$, $Y = (Y_{nk}) \in Z^F(M, \lambda, \xi)$ for $n, k \in \mathbb{N}$.

Proof. Let $X, Y, Z \in Z^F(M, \lambda, \xi)$, where $X = (X_{nk})$, $Y = (Y_{nk})$, $Z = (Z_{nk})$. Then,

1. $\overline{d}(X, Y) = 0$

   $\Rightarrow inf\left\{\rho > 0 :\ sup_{n,k}\ M\left(\frac{\lambda(X_{nk},\ Y_{nk})}{\rho}\right) < 1\ ;\ sup_{n,k}\ M\left(\frac{\xi(X_{nk},\ Y_{nk})}{\rho}\right) < 1\right\} = 0$

   $\Rightarrow\ M\left(\frac{\lambda(X_{nk},\ Y_{nk})}{\rho}\right) = 0$ and, $M\left(\frac{\xi(X_{nk},\ Y_{nk})}{\rho}\right) = 0$

   $\Rightarrow \lambda(X_{nk},\ Y_{nk}) = 0$ and $\xi(X_{nk},\ Y_{nk}) = 0$

   $$\Rightarrow\quad \min\left\{|X^{\alpha}_{ni} - Y^{\alpha}_{ni}|\ ,\ |X^{\alpha}_{jk} - Y^{\alpha}_{jk}|\right\} = 0 \tag{3.1}$$

   $$\max\left\{|X^{\alpha}_{ni} - Y^{\alpha}_{ni}|\ ,\ |X^{\alpha}_{jk} - Y^{\alpha}_{jk}|\right\} = 0 \tag{3.2}$$

   From these two relations, we see that $X_{nk} = Y_{nk} \implies X = Y$.

   Thus $\overline{d}(X, Y) = 0 \implies X = Y$.

   Conversely, we assume that $X = Y$. Then by the definition of λ_{α}, ξ_{α}, we have

   $\lambda_{\alpha}(X^{\alpha}_{nk}, Y^{\alpha}_{nk}) = 0$ and, $\xi_{\alpha}(X^{\alpha}_{nk}, Y^{\alpha}_{nk}) = 0$, $\forall\, n, k \in \mathbb{N}$, $\alpha \in (0, 1)$

   $sup_{n,k}\, \lambda_{\alpha}(X^{\alpha}_{nk}, Y^{\alpha}_{nk}) = 0$, and $sup_{n,k}\, \xi_{\alpha}(X^{\alpha}_{nk}, Y^{\alpha}_{nk}) = 0$

   $\Rightarrow \lambda(X_{nk}, Y_{nk}) = 0$, and $\xi(X_{nk}, Y_{nk}) = 0$

   By the definition of Orlicz function M, we have

   $M\left(\frac{\lambda(X_{nk},\ Y_{nk})}{\rho}\right) = 0$ and, $M\left(\frac{\xi(X_{nk},\ Y_{nk})}{\rho}\right) = 0$, $\forall\, n, k \in \mathbb{N}$ and $\rho > 0$

   $$\Rightarrow \inf\left\{\rho > 0 : \sup_{n,k} M\left(\frac{\lambda(X_{nk},\ Y_{nk})}{\rho}\right) < 1\ ;\ \sup_{n,k} M\left(\frac{\xi(X_{nk},\ Y_{nk})}{\rho}\right) < 1\right\} = 0$$

   Therefore, $\overline{d}(X, Y) = 0$.

2. We have,

   $\overline{d}(X, Y) = inf\left\{\rho > 0\ :\ sup_{n,k}\ \mathrm{M}\left(\frac{\lambda(X_{nk},\ Y_{nk})}{\rho}\right) \leq 1\ ;\ sup_{n,k}\ \mathrm{M}\left(\frac{\xi(X_{nk},\ Y_{nk})}{\rho}\right) \leq 1\right\}$.

   Here, $\lambda(X_{nk},\ Y_{nk}) = \sup_{\alpha\in(0,1)} \lambda_{\alpha}(X^{\alpha}_{nk}, Y^{\alpha}_{nk})$

   $$= \sup_{\alpha\in(0,1)} \left[\,\min\left\{|X^{\alpha}_{ni} - Y^{\alpha}_{ni}|\ ,\ |X^{\alpha}_{jk} - Y^{\alpha}_{jk}|\right\}\,\right]$$

   $$= \sup_{\alpha\in(0,1)} \left[\,\min\left\{|Y^{\alpha}_{ni} - X^{\alpha}_{ni}|\ ,\ |Y^{\alpha}_{jk} - X^{\alpha}_{jk}|\right\}\,\right]$$

$$= \sup_{\alpha\in(0,1)} \lambda_\alpha(Y_{nk}^{\alpha}, X_{nk}^{\alpha})$$

$$=\lambda\,(Y_{nk},\ \ X_{nk}).$$

Similarly, we can show that, $\xi\,(X_{nk},\ \ Y_{nk})$= $\xi\,(Y_{nk},\ \ X_{nk})$.

Now,

$$\overline{d}\,(\,X,\,Y) = inf\left\{\rho>0:\ sup_{n,k}\ M\left(\frac{\lambda\,(X_{nk},\ \ Y_{nk})}{\rho}\right)\leq 1\ \ ;sup_{n,k}\,M\left(\frac{\xi\,(X_{nk},\ \ Y_{nk})}{\rho}\right)\leq 1\right\}$$

$$= inf\left\{\rho>0:\ sup_{n,k}\,M\left(\frac{\lambda\,(Y_{nk},\ \ X_{nk})}{\rho}\right)\leq 1\ \ ;sup_{n,k}\,M\left(\frac{\xi\,(Y_{nk},\ \ X_{nk})}{\rho}\right)\leq 1\right\}$$

$$=\overline{d}\,(\,Y,\,X)\,.$$

3. Let $\rho_1{>}0$ and $\rho_2>0$ such that

$$sup_{n,k}\ M\left(\tfrac{\lambda(X_{nk},\ Z_{nk})}{\rho_1}\right)\leq 1 \text{ and } sup_{n,k}\,M\left(\tfrac{\lambda(Z_{nk},\ Y_{nk})}{\rho_2}\right)\leq 1$$

Let $\rho=\rho_1+\ \rho_2$, then $\rho>0$. By the definition of λ, we have

$$\lambda\,(X_{nk},\ \ Y_{nk}) = \sup_{\alpha\in(0,1)} \lambda_\alpha\,(X_{nk}^{\alpha}\ ,\ Y_{nk}^{\alpha})$$

$$= \sup_{\alpha\in(0,1)}\ [\ \min\left\{|X_{ni}^{\alpha}-Y_{ni}^{\alpha}|\ ,\ \left|X_{jk}^{\alpha}-Y_{jk}^{\alpha}\right|\right\}\ ]$$

Using the definition of λ_α, we have

$$\lambda_\alpha\,(X_{nk}^{\alpha}\ ,\ Y_{nk}^{\alpha})\leq\ \lambda_\alpha\,(X_{nk}^{\alpha}\ ,\ Z_{nk}^{\alpha})+\lambda_\alpha\,(Z_{nk}^{\alpha}\ ,\ Y_{nk}^{\alpha})\,,\ \text{for all } n,\ k\in\mathbb{N} \text{ and } \alpha\in(0,1)$$

$$\Rightarrow \sup_{\alpha\in(0,1)} \lambda_\alpha\,(X_{nk}^{\alpha}\ ,\ Y_{nk}^{\alpha})\leq \sup_{\alpha\in(0,1)} \lambda_\alpha\,(X_{nk}^{\alpha}\ ,\ Z_{nk}^{\alpha})+\sup_{\alpha\in(0,1)} \lambda_\alpha\,(Z_{nk}^{\alpha}\ ,\ Y_{nk}^{\alpha})$$

$$\therefore\ \lambda\,(X_{nk},\ \ Y_{nk})\ \ \leq\lambda\,(X_{nk},\ \ Z_{nk})+\lambda\,(Z_{nk},\ \ Y_{nk}) \tag{3.3}$$

Using the continuity of M, we get

$$sup_{n,k}\,\mathrm{M}\left\{\frac{\lambda\,(X_{nk},\ \ Y_{nk})}{\rho}\right\}\ \leq sup_{n,k}\,\mathrm{M}\left\{\frac{\lambda\,(X_{nk},\ \ Z_{nk})}{\rho_1+\rho_2}+\frac{\lambda\,(Z_{nk},\ \ Y_{nk})}{\rho_1+\rho_2}\right\}$$

$$\leq sup_{n,k}\,\mathrm{M}\left\{\frac{\rho_1}{\rho_1+\rho_2}\left(\frac{\lambda\,(X_{nk},\ \ Z_{nk})}{\rho_1}\right)+\frac{\rho_2}{\rho_1+\rho_2}\left(\frac{\lambda\,(Z_{nk},\ \ Y_{nk})}{\rho_2}\right)\right\}$$

$$\leq sup_{n,k}\frac{\rho_1}{\rho_1+\rho_2}\mathrm{M}\left(\frac{\lambda\,(X_{nk},\ \ Z_{nk})}{\rho_1}\right)+sup_{n,k}\frac{\rho_2}{\rho_1+\rho_2}\mathrm{M}\left(\frac{\lambda\,(Z_{nk},\ \ Y_{nk})}{\rho_2}\right)$$

$$\leq\ \frac{\rho_1}{\rho_1+\rho_2}.1+\ \frac{\rho_2}{\rho_1+\rho_2}.1=1$$

Therefore, $\sup_{n,k} M\left\{\frac{\lambda(X_{nk},\ Y_{nk})}{\rho}\right\} \le 1$.
This relation is true for all $\rho > 0$, so that

$$\inf\left\{\ \rho > 0\ :\ sup_{n,k}\, M\left\{\frac{\lambda\,(X_{nk},\ \ Y_{nk})}{\rho}\right\}\ \le 1\right\}.$$

Then from (3.3)

$$\inf\left\{\rho > 0 :\ sup_{n,k}\, M\left(\frac{\lambda(X_{nk},\ Y_{nk})}{\rho}\right) \le 1\right\}\ \le \inf\left\{\rho_1 > 0 :\ sup_{n,k}\, M\left(\frac{\lambda(X_{nk},\ Z_{nk})}{\rho_1}\right) \le 1\right\}$$

$$+\inf\left\{\rho_2 > 0 :\ sup_{n,k}\, M\left(\frac{\lambda(Z_{nk},\ Y_{nk})}{\rho_2}\right) \le 1\right\}.$$

Similarly, we can show that

$$\inf\left\{\rho > 0 : sup_{n,k}\, M\left(\frac{\xi\,(X_{nk},\ \ Y_{nk})}{\rho}\right) \le 1\right\}\ \le \inf\left\{\rho_1 > 0 :\ sup_{n,k}\, M\left(\frac{\xi\,(X_{nk},\ \ Z_{nk})}{\rho_1}\right) \le 1\right\}$$

$$+\inf\left\{\rho_2 > 0 :\ sup_{n,k}\, M\left(\frac{\xi(Z_{nk},\ Y_{nk})}{\rho_2}\right) \le 1\right\}.$$

Thus, we have,

$$\overline{d}\,(\,X,\ Y) = inf\left\{\rho > 0\ \ : sup_{n,k}\ \ \mathrm{M}\left(\frac{\lambda\,(X_{nk},\ \ Y_{nk})}{\rho}\right) \le 1\ ;\ sup_{n,k}\ \ \mathrm{M}\left(\frac{\xi\,(X_{nk},\ \ Y_{nk})}{\rho}\right) \le 1\right\}$$

$$\le \inf\left\{\rho_1 > 0 : sup_{n,k}\, M\left(\frac{\lambda\,(X_{nk},\ Z_{nk})}{\rho_1}\right) \le 1\right\} + \inf\left\{\rho_2 > 0 : sup_{n,k}\, M\left(\frac{\lambda\,(Z_{nk},\ \ Y_{nk})}{\rho_2}\right) \le 1\right\}$$

$$+\inf\left\{\rho_1 > 0 :\ sup_{n,k}\, M\left(\frac{\xi\,(X_{nk},\ \ Z_{nk})}{\rho_1}\right) \le 1\right\} +\inf\left\{\rho_2 > 0 :\ sup_{n,k}\, M\left(\frac{\xi\,(Z_{nk},\ \ Y_{nk})}{\rho_2}\right) \le 1\right\}$$

$$= \overline{d}\,(\,X,\ Z) + \ \overline{d}\,(\,Z,\ Y).$$

$$\therefore\ \ \overline{d}\,(\,X,\ Y) \le \overline{d}\,(\,X,\ Z) + \ \overline{d}\,(\,Z,\ Y).$$

Hence, $\overline{d}\,(\,X,\ Y) = inf\left\{\rho > 0\ :\ sup_{n,k}\ \mathrm{M}\left(\frac{\lambda(X_{nk},Y_{nk})}{\rho}\right) < 1\ ;\ sup_{n,k}\ \mathrm{M}\left(\frac{\xi(X_{nk},\ Y_{nk})}{\rho}\right) < 1\right\}$
is a matric for $X = (X_{nk})\,,\ Y = (Y_{nk}) \in Z^F\,(\,M,\ \lambda, \xi\,)$ and $\rho > 0$ for $n,\ k \in \mathbb{N}$.
So $Z^F\,(\,M,\ \lambda, \xi)$ is a metric space with the metric $\overline{d}\,(\,X,\ Y)$. □

Theorem 3.4.7. *[80] The class of sequence spaces $Z^F\,(\,M,\ \lambda,\ \xi)$ are complete metric space with the metric defined by*
$\overline{d}\,(\,X,\ Y) = \inf\left\{\rho > 0 : \sup_{n,k}\ M\left(\frac{\lambda(X_{nk},\,\overline{0})}{\rho}\right) < 1\ ;\ \sup_{n,k}\ M\left(\frac{\xi(X_{nk},\,\overline{0})}{\rho}\right) < 1\right\}$
for $X = (X_{nk})\,,\ Y = (Y_{nk}) \in Z^F\,(\,M,\ \lambda,\ \xi)$ and $\rho > 0$ for $n,\ k \in \mathbb{N}$.

Proof. To complete the proof of the theorem, we first show that $\ell_\infty^F(\,M,\ \lambda, \xi\,)$ is complete and the rest can be similarly done.
For, let $(X^i) = (X^i_{nk})$ be a Cauchy sequence in $\ell_\infty^F(\,M,\ \lambda, \xi\,)$ and let $\varepsilon > 0$ be given, then there

exists $i_o \in \mathbb{N}$ such that

$$\overline{d}\left(X_{nk}^{i},\ X_{nk}^{j}\right) < \varepsilon\ ,\ \forall\, i\ ,\ \mathrm{j} \geq i_o\ ,\ \ n,\ k\ \in \mathbb{N}$$

$$\Longrightarrow \inf\left\{\rho > 0\ :\ sup_{n,k}\, M\left(\frac{\lambda\left(X_{nk}^{i}, X_{nk}^{j}\right)}{\rho}\right) \leq 1\ ;\ \sup_{n,k} M\left(\frac{\xi\left(X_{nk}^{i}, X_{nk}^{j}\right)}{\rho}\right) \leq 1\ \text{ for } \rho > 0\right\} < \varepsilon \tag{3.4}$$

$$\Longrightarrow \sup_{n,k}\ M\left(\frac{\lambda(X_{nk}^{i}\ ,\ X_{nk}^{j})}{\rho}\right) \leq 1,\ \text{ and }\ \sup_{n,k}\ \ M\left(\frac{\xi(X_{nk}^{i}\ ,\ X_{nk}^{j})}{\rho}\right) \leq 1$$

Now,

$$\sup_{n,k}\ M\left(\frac{\lambda\left(X_{nk}^{i}\ ,\ \ X_{nk}^{j}\right)}{\rho}\right) \leq\ 1 \tag{3.5}$$

For, $\varepsilon > 0$ let us choose $\mu\ ,\eta > 0$ such that $M\ \left(\frac{\mu\eta}{2}\right) \geq 1$.

Then from 3.5, $\sup_{n,k} M\left(\frac{\lambda(X_{nk}^{i}\ ,\ X_{nk}^{j})}{\rho}\right)\ \leq 1\ \leq M\left(\frac{\mu\eta}{2}\right)$.

Since, M is non- decreasing function, we have

$$\frac{\lambda(X_{nk}^{i}\ , X_{nk}^{j})}{\rho}\ \leq \frac{\mu\eta}{2} \Longrightarrow \lambda\left(X_{nk}^{i}\ ,\ X_{nk}^{j}\right)\ \leq \frac{\mu\eta}{2}\ .\ \rho$$

Since, $\rho > 0$, so choose $\rho = \frac{\varepsilon}{\mu\eta}$ then we have

$$\lambda\left(X_{nk}^{i},\ X_{nk}^{j}\right)\ \leq \frac{\varepsilon}{2} <\ \varepsilon.$$

Thus, $\lambda\left(X_{nk}^{i}\ , X_{nk}^{j}\right) <\ \varepsilon,\ \forall\, i\ ,\ \mathrm{j} \geq i_o\ ,$

This shows that, $\left(X_{nk}^{i}\right)$ is a Cauchy sequence in $\mathbb{R}$ (I) and $\mathbb{R}$ (I) is complete, so we find $X_{nk} \in \mathbb{R}$ (I) such that $\lim_i X_{nk}^{i} = X_{nk}$, for $n,\ k \in \mathbb{N}$.

To complete the proof, we show that $X_{nk} \in \ell_\infty^F(\ M,\ \lambda, \xi)$.

We have, $\sup_{n,k} M\left(\frac{\lambda(X_{nk}^{i}\ , X_{nk}^{j})}{\rho}\right) \leq 1$ for $\rho > 0$.

Let us fix *i* and taking $j \rightarrow \infty$, then we get

$\sup_{n,k} M\left(\frac{\lambda(X_{nk}^{i}\ ,\ X_{nk})}{\rho}\right) \leq\ 1,$ for $\rho > 0$ and $i\ \geq i_o$.

Similarly, we can show that

$$\sup_{n,k}\ M\left(\frac{\xi(X_{nk}^{i}, X_{nk})}{\rho}\right) \leq 1\ ,\ \text{for } \rho > 0 \text{ and }\ i \geq i_o\ .$$

Thus, $sup_{n,k}\ M\left(\frac{\lambda(X_{nk}^{i}\ ,\ X_{nk})}{\rho}\right) \leq 1$; $\sup_{n,k}\ M\left(\frac{\xi(X_{nk}^{i}\ ,\ X_{nk})}{\rho}\right) \leq 1$, for ρ>0, and $i\ \geq i_o$.

Taking infimum over such $\rho > 0$, and using 3.4 , we get

$$\inf\left\{\rho > 0\ :\ \sup_{n,k} M\left(\frac{\lambda\left(X_{nk}^{i}\ ,\ \ X_{nk}\right)}{\rho}\right) \leq 1\ \ ;\ \sup_{n,k} M\left(\frac{\xi\left(X_{nk}^{i}\ ,\ \ X_{nk}\right)}{\rho}\right) \leq 1\ \right\} < \varepsilon.$$

$$\Longrightarrow \overline{d}\left(X_{nk}^{i},\ X_{nk}\right) < \varepsilon\ ,\ \forall\, i\ , \geq i_o\ .$$

$\Longrightarrow \lim_{i\to\infty} X_{nk}^{i} \;=\; X_{nk}$ for $n,\ k\ \in \mathbb{N}$

By the triangle inequality of the metric $\overline{d}$

$$\overline{d}\left(X_{nk},\,\overline{0}\right) \le\ \overline{d}\left(X_{nk},\,X_{nk}^{i}\right)\ +\overline{d}\left(X_{nk}^{i},\overline{0}\right) < \varepsilon +\ \overline{d}\left(X_{nk}^{i},\overline{0}\right) < \infty$$

$$i.e\ \inf\left\{\rho>0\ :\sup_{n,k}\ M\left(\frac{\lambda\left(X_{nk}\ ,\ \overline{0}\right)}{\rho}\right)\le 1\ ;\sup_{n,k}\ M\left(\frac{\xi\left(X_{nk}\ ,\ \overline{0}\right)}{\rho}\right)\le 1\ \right\}<\infty$$

$\Longrightarrow X_{nk} \in \ell_{\infty}^{F}(\ M,\ \lambda,\xi\ )$and so $\ell_{\infty}^{F}(\ M,\ \lambda,\xi\ )$ is complete. This completes the proof. ☐

Definition 21. *(**Solidity of double sequence space**)[109] A double sequence space E^F of fuzzy real numbers is said to be solid if $Y=(Y_{nk})\in E^F$ whenever $X=(X_{nk})\in E^F$ and $|Y_{nk}|\le|X_{nk}|$ for all $n,\ k\ \in\mathbb{N}$.*

Theorem 3.4.8. *[80] The spaces $\ell_{\infty}^{F}(M,\ \lambda,\xi\ )$, $C^F\ (\ M,\ \lambda,\xi\ )$ and $C_o^F(\ M,\ \lambda,\xi\ )$ are solid.*

Proof. Suppose $X=(X_{nk})\ \in\ \ell_{\infty}^{F}(\ M,\ \lambda,\xi\ )$ and let $Y=(Y_{nk})$ be double sequence of fuzzy numbers such that

$\lambda\ (Y_{nk},\ \overline{0}\ )\le\lambda\ (X_{nk},\ \overline{0}\ )$ and $\xi\ (Y_{nk},\ \overline{0}\ )\le\xi\ (X_{nk},\ \overline{0}\ )$ for $n,\ k\ \in\mathbb{N}$.

Let $\rho>0$, then we have

$\frac{\lambda(Y_{nk},\,\overline{0})}{\rho}\le\frac{\lambda(X_{nk},\,\overline{0})}{\rho}$ and $\frac{\xi(Y_{nk},\,\overline{0}\ )}{\rho}\ \le\frac{\xi(X_{nk},\,\overline{0}\ )}{\rho}$.

Since M is non-decreasing function, we have

$M\left(\frac{\lambda(Y_{nk},\,\overline{0})}{\rho}\right)\le M\left(\frac{\lambda(X_{nk},\,\overline{0})}{\rho}\right)$ and $M\left(\frac{\xi(Y_{nk},\,\overline{0}\ )}{\rho}\right)\ \le M\left(\frac{\xi(X_{nk},\,\overline{0}\ )}{\rho}\right)$ for all $n,\ k\ \in\mathbb{N}$.

Since this relation is true for all $n,\ k\ \in\mathbb{N}$, so

$\sup_{n,k} M\left(\frac{\lambda(Y_{nk},\,\overline{0})}{\rho}\right)\le\sup_{n,k}\ M\left(\frac{\lambda(X_{nk},\,\overline{0})}{\rho}\right)$, and

$\sup_{n,k} M\left(\frac{\xi(Y_{nk},\,\overline{0}\ )}{\rho}\right)\ \le\sup_{n,k}\ M\left(\frac{\xi(X_{nk},\,\overline{0}\ )}{\rho}\right)$

Since, $X=(X_{nk})\in\ell_{\infty}^{F}(M,\lambda,\xi)$ we have,

$$\sup_{n,k} M\left(\frac{\lambda(X_{nk},\overline{0})}{\rho}\right)<\infty,\ \text{and}\ \sup_{n,k} M\left(\frac{\xi(X_{nk},\overline{0})}{\rho}\right)<\infty.$$

Thus, $\sup_{n,k} M\left(\frac{\lambda(Y_{nk},\,\overline{0})}{\rho}\right)<\infty$, and $\sup_{n,k} M\left(\frac{\xi(Y_{nk},\,\overline{0}\ )}{\rho}\right)<\infty$, for $\rho>0$.

$\Rightarrow Y=(Y_{nk})\ \in\ell_{\infty}^{F}(\ M,\ \lambda,\xi\ )$. Thus $\ell_{\infty}^{F}\ (\ M,\ \lambda,\xi\ )$ is solid.
Similarly, the solidness of other spaces can be shown . ☐

Chapter 4

Difference Sequence Spaces of Fuzzy Real Numbers and its Generalized Form

4.1 Introduction

A large number of research projects have been carried out in mathematical structures built with real or complex numbers so far. In recent years, many researchers have investigated many results of replacing these mathematical structures with fuzzy numbers and interval numbers. The study of fuzzy sequence space is an appealing topic of research in mathematics and its application. As a result, various scholars made major contributions, and several new concepts and conclusions emerged in this field. Savas[92] introduced some classes of fuzzy number sequences, studied and assessed some of their key characteristics such as completeness, solidity, symmetry, and convergence-free, and also studied different inclusion relations related to these classes. Numerous studies have been conducted recently using fuzzy numbers and their mathematical structures, looking at a variety of findings. In 1981, Kizmaz [51] was the first to introduce the idea of difference sequence space $l_\infty(\Delta)$, $C(\Delta)$, $C_o(\Delta)$. Later, the idea of Kizmaz was expanded by Et. and Colka [3] in 1995 as $l_\infty(\Delta^m)$, $C(\Delta^m)$, $C_o(\Delta^m)$. Then, Burch and Tripathy[1] applied this concept to the fuzzy domain. The researchers in their works [10], [11], [32], [34],[35], [37], [51], [79], [83] and many more have studied various types of difference sequence spaces.

Definition 22. *[51] Let* ℓ_∞, C, C_o *are the sequence spaces of bounded, convergent, and null sequences respectively with* $\Delta X_k = X_k - X_{k+1}$, *then the difference sequence space corresponding to the spaces* ℓ_∞, C, C_o *defined as follows:*

$$\ell_\infty(\Delta) = \{\, X = (X_k) \in \omega \;:\; (\Delta X_k) \in \ell_\infty\}$$

$$C(\Delta) = \{X = (X_k) \in \omega \;:\; (\Delta X_k) \in C\}$$

$$C_o(\Delta) = \{\ X = (X_k) \in \omega\ :\ (\Delta X_k) \in C_o\}.$$

Definition 23. *[107] Let* $\omega(F)$ *denote the set of sequences of fuzzy numbers. For* $\Delta X = X_k - X_{k+1}, k \in \mathbb{N}$*, the difference sequence spaces* $\ell_\infty^F(\Delta X)$, $C^F(\Delta X)$, $C_o^F(\Delta X)$ *of fuzzy real numbers are defined as follow:*

$$\ell_\infty^F(\Delta X) = \{X = (X_k) \in \omega(F) :\ (\Delta X_k) \in \ell_\infty^F\}$$

$$C^F(\Delta X) = \{X = (X_k) \in \omega(F) :\ (\Delta X_k) \in C^F\}$$

$$C_o^F(\Delta X) = \{X = (X_k) \in \omega(F) :\ (\Delta X_k) \in C_o^F\}$$

where, $\ell_\infty^F(\Delta X)$ *= set of all bounded sequences of fuzzy numbers ,*

$C^F(\Delta X)$ *= set of all convergent sequences of fuzzy numbers , and*

$C_o^F(\Delta X)$ *= set of all null sequences of fuzzy numbers.*

4.2 Topological Properties of the Class $\mathcal{S}(\ X\ , M, \alpha,\ P)$ of Difference Sequences of Fuzzy Real Numbers

Lemma 4.2.1. *[72] Let* (p_k) *be a bounded sequence of strictly positive real numbers with* $0 < \mathrm{p}_k \leq sup_k\, p_k = L,\ \mathcal{H} = max\{1,\ 2^{L-1}\}$ *then*

1. $|X+Y|^{p_k} \leq \mathcal{H}\{|X|^{p_k} +\ |Y|^{p_k}\}$;
2. $|\alpha|^{p_k}\ \leq\ max\left\{1,\ [\alpha]^L\right\}$

Theorem 4.2.1. *[83] Let* $P = (p_k)$ *and* $Q = (q_k)$ *be any two sequences of strictly positive real numbers and* (α_k) *be a sequence of non-zero real numbers. Then the class*

$$\mathcal{S}(\ X\ , M, \alpha,\ P) = \left\{X = (X_k) \in \omega^F\ :\ \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\ \Delta X_k\|^{p_k}}{\rho}\right) < \infty,\ for\ some\ \rho > 0\right\}$$

is linear.

Proof. Suppose $X = \ (X_k)$ and $Y = \ (Y_k)$ are two elements of $\mathcal{S}(X\ , M, \alpha,\ P)$. Then there exist ρ_1>0 and ρ_2>0 such that

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\ \Delta X_k\|^{p_k}}{\rho_1}\right) < \infty\ \ and\ \ \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\ \Delta Y_k\|^{p_k}}{\rho_2}\right) < \infty\ .$$

Let us choose $\rho_3 > 0$, then for any scalar a and b we have

$\frac{\mathcal{H}}{\rho_3} \leq \frac{1}{2\rho_1 \max(1,\ |a|)}$ and $\frac{\mathcal{H}}{\rho_3} \leq \frac{1}{2\rho_2 \max(1,\ |b|)}$,

where, $\sup p_k = L$ and $\mathcal{H} = \max\left\{1\ ,\ 2^{L-1}\right\}$.

Then for any scalars a and b, we have

$$\begin{aligned}\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k(a\Delta X_k + b\Delta Y_k)\|^{p_k}}{\rho_3}\right) &= \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k a\Delta X_k\ +\ \alpha_k b\Delta Y_k\|^{p_k}}{\rho_3}\right)\\ &= \sum_{k=1}^{\infty} M\left(\frac{\|(a\ \alpha_k\Delta X_k) + (b\ \alpha_k\ \Delta Y_{k_k})\|^{p_k}}{\rho_3}\right)\\ &\leq \sum_{k=1}^{\infty} M\left(\frac{\mathcal{H}}{\rho_3}\|a\ \alpha_k\Delta X_k\|^{p_k} + \frac{\mathcal{H}}{\rho_3}\ \|b\ \alpha_k\Delta Y_k\|^{p_k}\right)\\ &\leq \sum_{k=1}^{\infty} M\left(\frac{1}{2\rho_1}\|\alpha_k\Delta X_k\|^{p_k} + \frac{1}{2\rho_2}\ \|\ \alpha_k\Delta Y_k\|^{p_k}\right) < \infty\end{aligned}$$

$\Longrightarrow aX + bY\ \in\ \mathcal{S}\,(\ X, M, \alpha,\ P)$ for $X,\ Y \in\ \mathcal{S}\,(\ X, M, \alpha,\ P)$ and hence the class is linear. □

Theorem 4.2.2. *[83] Let $P = (p_k)$ and $Q = (q_k)$ be any two sequences of strictly positive real numbers. If $0 < p_k\ \leq q_k < \infty$ for all but finitely many values of k, then*

$$\mathcal{S}\,(X, M, \alpha,\ P) \subseteq \mathcal{S}\,(X, M, \alpha,\ Q)\,.$$

Proof. Let $X = (X_k) \in\ \mathcal{S}\,(\ X, M, \alpha,\ P)$, then there exists $\rho > 0$ such that

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\ \Delta X_k\|^{p_k}}{\rho}\right) < \infty \tag{4.1}$$

This relation shows that there exists an integer $K \geq 1$ such that $\|\alpha_k\ \Delta X_k\| < 1$. Since, $p_k\ \leq\ q_k$, we have

$$\|\alpha_k\ \Delta X_k\|^{q_k}\ \leq \|\alpha_k\ \Delta X_k\|^{p_k}, \text{ for every } k \geq K.$$

Since *M* is a non-decreasing function, we have

$M\left(\frac{\|\alpha_k\ \Delta X_k\|^{q_k}}{\rho}\right) \leq M\left(\frac{\|\alpha_k\ \Delta X_k\|^{p_k}}{\rho}\right)$ for $\rho > 0$

$\Longrightarrow X = (X_k) \in \mathcal{S}\,(X, M, \alpha,\ Q)$ and hence $\mathcal{S}\,(\ X, M, \alpha,\ P) \subseteq \mathcal{S}\,(\ X, M, \alpha,\ Q)$. □

Theorem 4.2.3. *[83] The sequence space $\mathcal{S}(X, M, \alpha, P)$ is solid.*

Proof. Let $X = (X_k) \in \mathcal{S}\,(X, M, \alpha, P)$, then there exists $\rho > 0$ such that

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\Delta X_k\|^{p_k}}{\rho}\right) < \infty.$$

Let $a = (a_k)$ be a sequence of scalars such that $|a_k| \leq 1$ for all $k \geq 1$. Since M is non-decreasing, we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k (a_k \Delta X_k)\|^{p_k}}{\rho}\right) = \sum_{\mathrm{k}=1}^{\infty} M\left(\frac{|a_{\mathrm{k}}|^{\mathrm{p}_k}\ \|\alpha_{\mathrm{k}}\ \ \Delta X_{\mathrm{k}}\|^{p_k}}{\rho}\right)$$

$$\leq \sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\alpha_{\mathrm{k}}\ \ \Delta X_{\mathrm{k}}\|^{p_k}}{\rho}\right) < \infty.$$

Thus, $(a_k X_k) \in \mathcal{S}(\,X, M, \alpha,\ P)$ for all sequences $(a_k\ )$ of scalars with $|a_k| \leq 1$, whenever $X = (X_k) \in \mathcal{S}(\,X, M, \alpha,\ P)$. So $\mathcal{S}(\,X, M, \alpha,\ P)$ is solid. □

Theorem 4.2.4. *[83] Consider the collection $\mathcal{A}(X)$ defined by the relation*

$$\mathcal{A}(X) = \left\{\ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\ \Delta\ X_k\|^{p_k}}{\rho}\right) \leq 1\right\}$$

where, $X = (X_k) \in \mathcal{S}(\,X, M, \alpha,\ P)$, then the space $\mathcal{S}(\,X, M, \alpha,\ P)$ is a paranormed space with respect to the paranorm defined by

$$g(X) = \inf\left\{\ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\ \Delta\ X_k\|^{p_k}}{\rho}\right) \leq 1\right\} \tag{4.2}$$

Proof. 1. Clearly , we can see that $g(0) = 0$.

2. $g(-X) = \inf\left\{\ \rho > 0\ :\ \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\ (-\ \Delta X_k)\|^{p_k}}{\rho}\right) \leq 1\right\}$

$$= \inf\left\{\ \rho > 0\ :\ \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\ \Delta X_k\|^{p_k}}{\rho}\right) \leq 1\right\}$$

$$= g(X).$$

3. Let $X,\ Y\ \in\ \mathcal{S}(\,X, M, \alpha,\ P)$. Then there exist ρ_1>0 and ρ_2>0 such that

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k\ \ \Delta X_k\|^{p_k}}{\rho_1}\right) < \infty\ \ and\ \sum_{\mathrm{k}=1}^{\infty}\ M\left(\frac{\|\alpha_k\ \ \Delta Y_k\|^{p_k}}{\rho_2}\right) < \infty.$$

Let $\rho_3 =\ \rho_1 + \rho_2$ where, $\rho_1\ ,\ \rho_2 \in \mathcal{A}(X\ )$, then $\rho_3 > 0$, and

$$\sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\alpha_{\mathrm{k}}(\Delta X_{\mathrm{k}} + \Delta Y_k)\|^{\mathrm{p}_k}}{\rho_3}\right) = \sum_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\alpha_{\mathrm{k}} \Delta X_{\mathrm{k}} + \alpha_k \Delta Y_k\|^{p_k}}{\rho_1 + \rho_2}\right)$$

$$\leq \frac{\rho_1}{\rho_1 + \rho_2} \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_{\mathrm{k}} \Delta X_{\mathrm{k}}\|^{p_k}}{\rho_1}\right) + \frac{\rho_2}{\rho_1 + \rho_2} \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_{\mathrm{k}} \Delta Y_{\mathrm{k}}\|^{p_k}}{\rho_2}\right)$$

$$\leq \frac{\rho_1}{\rho_1 + \rho_2}.1 + \frac{\rho_2}{\rho_1 + \rho_2}.1 = 1$$

and therefore, $\sum\limits_{\mathrm{k}=1}^{\infty} M\left(\frac{\|\alpha_{\mathrm{k}}(\Delta X_{\mathrm{k}}+\Delta Y_k)\|^{p_k}}{\rho_3}\right) \leq 1.$

This relation shows that $\rho_1 + \rho_2 = \rho_3 \in \mathcal{A}(X)$. Thus we have

$g(X+Y) \leq \rho_1 + \rho_2$ for $\rho_1 \in \mathcal{A}(X)$ and $\rho_2 \in \mathcal{A}(Y)$.

i.e, $g(X+Y) \leq \rho_1 + \rho_2$

i.e, $g(X+Y) \leq g(X) + g(Y)$.

Hence the triangle inequality holds.

4. Suppose $X^n = (X_k^n)$ be a sequence in $\mathcal{S}(X, M, \alpha, P)$ such that $g(X^n) \to 0$ as $n \to \infty$ and (α_{n}) be a sequence of scalars such that $\alpha_{\mathrm{n}} \to \alpha$. Then

$$g(\lambda_{\mathrm{n}} X^n) = \inf\left\{\rho : \sum_{k=1}^{\infty} M\left(\left\|\frac{\lambda_n \alpha_k \Delta X_k^n}{\rho}\right\|^{p_k}\right) \leq 1\right\}$$

$$= \inf\left\{\rho : \sum_{k=1}^{\infty} M\left(|\lambda_n|^{p_k}\left\|\frac{\alpha_k \Delta X_k^n}{\rho}\right\|^{p_k}\right) \leq 1\right\}$$

$$\leq \inf\left\{\rho : \sum_{k=1}^{\infty} M\left(D^{p_k}\left\|\frac{\alpha_k \Delta X_k^n}{\rho}\right\|^{p_k}\right) \leq 1\right\}, where\ \boldsymbol{D} = \sup_n |\lambda_{\mathrm{n}}|$$

.

Then for $r = \max\{1, D\}$, we have

$$g(\lambda_{\mathrm{n}} X^n) \leq \inf\left\{\rho : \sum_{k=1}^{\infty} M\left(\frac{r\,\|\alpha_k \Delta X_k^n\|^{p_k}}{\rho}\right) \leq 1\right\}$$

.

Taking $\frac{\rho}{\mathrm{r}} = s$ then $s > 0$, and

$$g(\lambda_{\mathrm{n}} X^n) \leq \inf\left\{rs : \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta X_k^n\|^{p_k}}{s}\right) \leq 1\right\}.$$

$$= rg(X^{\mathrm{n}}) \to 0 \text{ as } n \to \infty.$$

Therefore, $g(\lambda_{\mathrm{n}} X^n) \to 0$ as $n \to \infty$.

Now, let (λ_{n}) be a sequence of scalars such that $\lambda_{\mathrm{n}} \to 0$ as $n \to \infty$ then for ε with $0 < \varepsilon < 1$, we can find a positive integer N such that $|\lambda_{\mathrm{n}}| \leq \varepsilon$ for all $n \geq N$ and let $X = (X_k) \in \mathcal{S}(X, M, \alpha, P)$. Then

$$\sum_{k=1}^{\infty} M\left(\frac{\|\lambda_n \alpha_k \Delta X_k\|^{p_k}}{\rho}\right) = \sum_{k=1}^{\infty} M\left(\frac{|\lambda_n|^{p_k}\,\|\alpha_k \Delta X_k\|^{p_k}}{\rho}\right)$$

$$\leq \sum_{k=1}^{\infty} M\left(\frac{\varepsilon^{p_k}\ \|\alpha_k \Delta X_k\|^{p_k}}{\rho}\right).$$

This relation shows that if $\rho \in \varepsilon^k \mathcal{A}(X)$ then $\rho \in \mathcal{A}(\lambda_n X)$ and so $\varepsilon^k \mathcal{A}(X) \subseteq \mathcal{A}\ (\lambda_n X)$.

$$\Longrightarrow \inf\{\rho > 0\ :\ \rho \in \mathcal{A}(\lambda_n X)\}\ \leq\ \inf\{\rho > 0\ :\ \rho \in \mathcal{A}(\varepsilon^{p_k} X)\}$$

$$= \varepsilon^{p_k} \inf\{\rho > 0 :\ \rho \in \mathcal{A}(X)\}$$

Therefore, $g\,(\lambda_n X) \leq \varepsilon^{p_k}\ g(X)$ for all $n \geq N$, which implies that $g\,(\lambda_n X) \to 0$ as $n \to \infty$.

Thus, g satisfies all the conditions of paranorm on $\mathcal{S}\ (\ X, M, \alpha,\ P)$ and hence $(\mathcal{S}\ \ (X, M, \alpha,\ P)\,,\ \ g)$ is a paranormed space. □

Theorem 4.2.5. *[83] The paranormed space $(\mathcal{S}\ \ (X, M, \alpha,\ P)\,,\ g)$ is complete with respect to the paranorm defined in (4.2).*

Proof. Let $X^n = (X_k^n)$ be a Cauchy sequence in $(\mathcal{S}\ \ (X, M, \alpha,\ P)\,,\ \ g)$. Then there exists $\delta > 0$ such that for all $m, n >$ N$\in\mathbb{Z}^+$, we have

$$g\,(X_{\mathrm{k}}^n - X_{\mathrm{k}}^m) < \delta. \tag{4.3}$$

By the definition of paranorm, we see that

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta X_{\mathrm{k}}^n - \alpha_{\mathrm{k}} \Delta X_{\mathrm{k}}^m\|^{p_k}}{g\left(X_{\mathrm{k}}^n\right) - g(X_{\mathrm{k}}^m)}\right) \leq 1,\ \text{ for all } m,\ n\ \geq N. \tag{4.4}$$

Let us choose a fixed real number $r > 0$ such that $M(r) \geq 1$. Then

$$M\left(\frac{\|\alpha_k \Delta X_{\mathrm{k}}^n - \alpha_{\mathrm{k}} \Delta X_{\mathrm{k}}^m\|^{p_k}}{g\left(X_{\mathrm{k}}^n\right) - g(X_{\mathrm{k)}}^m}\right)\ \leq M(r),\ \text{ for all } m,\ n\ \geq N\ \text{ and for all } k \geq 1.$$

Since *M* non -decreasing, we have

$$\frac{\|\alpha_k \Delta X_{\mathrm{k}}^n - \alpha_{\mathrm{k}} \Delta X_{\mathrm{k}}^m\|^{p_k}}{g\left(X_{\mathrm{k}}^n\right) - g(X_{\mathrm{k}}^m)} \leq r\ , \forall\ \ m\ ,\ n\ \geq N$$

$$\Longrightarrow \|\Delta X_{\mathrm{k}}^n - \Delta X_{\mathrm{k}}^m\|^{p_k}\ \leq \frac{r}{|\alpha_k|^{p_k}}.\ g\ (X_{\mathrm{k}}^n - X_{\mathrm{k}}^m),\ \ \forall\, m, n\ \geq N \tag{4.5}$$

and therefore, $\|\Delta X_{\mathrm{k}}^n -\ \ \Delta X_{\mathrm{k}}^m\| < \frac{(r\delta)^{\frac{1}{p_k}}}{|\alpha_k|} = \varepsilon'$ (say) $, \forall\ \ m\ ,\ n\ \geq N$

This shows that $(\Delta X_{\mathrm{k}}^n)$ is a Cauchy sequence in $\mathbb{R}(I)$. Since $\mathbb{R}(I)$ is complete, so the sequence $(\Delta X_{\mathrm{k}}^n)$ converges in $\mathbb{R}(I)$ say $\Delta X_{\mathrm{k}}^n \longrightarrow \Delta X_{\mathrm{k}}$ as $n \longrightarrow \infty$.

To complete the proof we show that $X=(X_{\mathrm{k}})\in\left(\mathcal{S}\left(X,M,\alpha,\ P\right),\ g\right).$

Let us choose $\delta>0$ such that $g\left(X_{\mathrm{k}}^{m}-X_{\mathrm{k}}^{n}\right)<\delta<\varepsilon$, for all $m\ ,\ n\ \geq N.$

$$M\left(\tfrac{\|\alpha\Delta X_{\mathrm{k}}^{n}-\alpha_{\mathrm{k}}X_{\mathrm{k}}^{m}\|^{p_k}}{\rho}\right)\leq\tfrac{\|\alpha_k\left(\Delta X_{\mathrm{k}}^{n}-\Delta X_{\mathrm{k}}^{m}\right)\|^{p_k}}{g(X_{\mathrm{k}}^{n}-X_{\mathrm{k}}^{m})}$$

$$\Longrightarrow\ \sum_{k=1}^{\infty}M\left(\frac{\|\alpha\Delta X_{\mathrm{k}}^{n}-\alpha_{\mathrm{k}}\Delta X_{\mathrm{k}}^{m}\|^{p_k}}{\rho}\right)\leq\sum_{k=1}^{\infty}M\left(\frac{\|\alpha_k\left(\Delta X_{\mathrm{k}}^{n}-\Delta X_{\mathrm{k}}^{m}\right)\|^{p_k}}{g(X_{\mathrm{k}}^{n}-X_{\mathrm{k}}^{m})}\right)\leq 1,\ \forall\ m\,,n\ \geq N.$$

Taking limit as $m\longrightarrow\infty$, we get

$$\sum_{k=1}^{\infty}M\left(\frac{\|\alpha_k\Delta X_{\mathrm{k}}^{n}-\alpha_{\mathrm{k}}\Delta X_k\|^{p_k}}{\rho}\right)\leq\ 1,\ \ \forall n\geq N.$$

This relation is true for all such $\rho>0$, so taking infimum of such ρ's, we get

$$\inf\left\{\rho>0\ :\ \sum_{k=1}^{\infty}M\left(\frac{\|\alpha_k\left(\Delta X_{\mathrm{k}}^{n}-\Delta X\right)\|^{p_k}}{\rho}\right)\leq\ 1\right\}$$

$$\Longrightarrow\ g\left(X^n-X\right)=\inf\left\{\rho:\sum_{k=1}^{\infty}M\left(\frac{\|\alpha_k\left(\Delta X_{\mathrm{k}}^{n}-\Delta X\right)\|^{p_k}}{\rho}\right)\leq 1\right\}\leq\rho<\varepsilon,\ \ \forall\ n\geq N.$$

$\Longrightarrow g\left(X^n-X\ \right)<\varepsilon,\ \forall\ \ n\geq N.$

This shows that $X^n\longrightarrow X$. Also, we have

$\|X^n-X\|\longrightarrow 0$ as $n\longrightarrow\infty$

$\Longrightarrow\ |\alpha_k|\ \|\Delta X^n-\Delta X\|\longrightarrow 0$ as $n\longrightarrow\infty$

$\Longrightarrow\ \ \frac{\|\alpha_k(\Delta X^n-\Delta X)\|^{p_k}}{\rho}\longrightarrow 0$ as $n\longrightarrow\infty$

By the continuity of *M* , we have

$M\left(\frac{\|\alpha_k(\Delta X^n-\Delta X)\|^{p_k}}{\rho}\right)\ \longrightarrow M\left(0\right)\ =0\ as\ n\longrightarrow\infty$ for every k,

$$\Longrightarrow\ \sum_{k=1}^{\infty}M\left(\frac{\|\alpha_k\left(\Delta X^n-\Delta X\right)\|^{p_k}}{\rho}\right)\leq 1$$

$\Longrightarrow X^n-X\ \in\ \mathcal{S}\left(\ X,\ M,\ \alpha,\ P\right)$

Since, $\mathcal{S}\left(\ X,\ M,\ \alpha,\ P\right)$ is linear space, so , $X^n-\left(X^n-X\right)\ \in\mathcal{S}\left(\ X,\ M,\ \alpha,\ P\right)$, that is

$X\in\mathcal{S}\left(\ X,\ M,\ \alpha,\ P\right).$

Hence $X^n \to X$ in $\mathcal{S}\,(\,X,\;M,\;\alpha,\;P)$. So the space $(\mathcal{S}\,(\,X,\;M,\;\alpha,\;P)\,,\,g)$ is complete . □

4.3 Topological Properties of the Class $bV_p^F(\Delta_m)$

The notion of fuzzy set has been successfully applied in studying the different classes of sequence spaces. In this section we study the class $bV_p{}^F$ of p- bounded variation of sequence space of fuzzy real numbers and extended it to p- bounded variation of difference sequence of fuzzy real numbers in generalized form. Further, we will discuss the topological properties of difference sequence space of fuzzy real numbers defined by the Orlicz function in its generalized form. In 2008 [100] Talo and Bassar and in 2018 Das [24] used the notation $bV_p^F(X)$ of p - bounded variation of difference sequence space to study some properties like completeness, solidness, symmetric, and convergence free. Then, in 2019, Tripathy and Das [108] modified ℓ_p^F studied by Nanda in 1989 and introduced the class of fuzzy real numbers $bV_p^F\,(\Delta)\,,\; for\ 1 \leq p < \infty$ in generalized form.

Definition 24. *[108] The class of p- bounded variation of difference sequences space of fuzzy real numbers is denoted by $bV_p^F(\Delta)$, for $1 \leq p < \infty$ and defined as follow*

$$bV_p^F(\Delta) = \{X = (X_k) \in \omega(F):\ \sum_{k=1}^{\infty}\{\bar{d}(\triangle X_k, \bar{0})\}^p < \infty\}$$

.

Theorem 4.3.1. *[108] The class of p- bounded variation of difference sequence space of fuzzy real numbers $bV_p{}^F$ for $1 \leq p < \infty$ is a complete metric space with respect to the metric*

$$\rho(\,X\,,\,Y\,)\ = \bar{d}\,(\,X_1\,,Y_1\,) + \left[\sum_{k=1}^{\infty}\{\bar{d}(\Delta X_k\,,\ \Delta Y_k)\}^p\right]^{1/\mathrm{p}}$$

for $X\ =\ (X_k)$ and $Y\ =\ (Y_k) \in bV_p{}^F$

Theorem 4.3.2. *[77] The generalized class of $p-$ bounded variation of difference sequence space $bV_p^F(\triangle_m)$ for, $1 \leq p < \infty$ is a complete metric space with respect to the metric defined by*

$$\bar{d}_p(X,Y) = \left[\sum_{k=1}^{\infty}\{\bar{d}(\Delta_m X_k, \Delta_m Y_k)\}^p\right]^{1/p} ,1 \leq p < \infty$$

where,

$$bV_p^F(\Delta_m) = \{X = (X_k) \in \omega(F):\ \sum_{k=1}^{\infty}\{\bar{d}(\Delta_m X_k, \bar{0})\}^p < \infty\},$$

$$X = (X_k), Y = (Y_k) \in bV_p^F(\Delta_m) \textit{ and } \Delta_m X_k = X_k - X_{k+m}\,.$$

Proof. Let (X^i) be a Cauchy sequence in $bV_p^F(\Delta_m)$, where

$X^i = (X^i_k) = (X^i_1, X^i_2, X^i_3, X^i_4, ..., ..., ...)$

Then for all $\varepsilon > 0$, there exists a positive integer n_o such that

$$\bar{d}_p(X^i, X^j) = \left[\sum_{k=1}^{\infty} \bar{d}(\triangle_m X^i_k, \triangle_m X^j_k)^p\right]^{\frac{1}{p}} < \varepsilon \tag{4.6}$$

which implies

$$\sum_{k=1}^{\infty} \bar{d}(\triangle_m X^i_k, \triangle_m X^j_k) < \epsilon.$$

This shows that $(\triangle_m X^i_k)$ is a Cauchy sequence in $\mathbb{R}(I)$ for all $k \in \mathbb{N}$. Since $\mathbb{R}(I)$ is complete , so for fixed k, the sequence $(\triangle_m X^i_k)$ is convergent in $\mathbb{R}(I)$ and suppose that

$$lim_{i\to\infty} \triangle_m X^i_k \qquad = \triangle_m X_k$$

i.e. $lim_{i\to\infty} (X^i_k - X^i_{k+m}) = (X_k - X_{k+m})$

i.e. $lim_{i\to\infty} X^i_k = X_k$, for all $k \in \mathbb{N}$

This implies that $lim_{i\to\infty} X^i = X$, for all $k \in \mathbb{N}$, where $X = (X_k)$

To complete the proof , we need to show that $X \in bV^F_p(\Delta_m)$. In view of 4.6 , we have

$$\bar{d}_p(X^i, X^j) = \left[\sum_{k=1}^{\infty} \{\bar{d}(\triangle_m X_k, \triangle_m Y_k)\}^p\right]^{1/p} < \varepsilon$$

Now,

$$\begin{aligned}\left[\sum_{k=1}^{\infty} \bar{d}(\triangle_m X_k, \bar{0})^p\right]^{1/p} &= \bar{d}_p(X, \bar{0}) \\ &\le \bar{d}_p(X, X^i) + \bar{d}_p(X^i, \bar{0}) \\ &< \varepsilon' + \bar{d}_p(X^i, \bar{0}) < \varepsilon\end{aligned}$$

$$\therefore \quad \left[\sum_{k=1}^{\infty} \bar{d}(\Delta_m X_k, \bar{0})^p\right]^{1/p} < \varepsilon, \ \ for \ \ 1 \le p < \infty.$$

This implies that $X = (X_k) \in bV^F_p(\Delta_m)$ and hence $bV^F_p(\Delta_m)$ is a complete metric space. □

Theorem 4.3.3. *[77] For 1 $\le p < \infty$, the relation $\ell^F_p \subseteq bV^F_p(\Delta_m)$ holds.*

Proof. Let $X = (X_k) \in \ell^F_p$, then

$$\sum_{k=1}^{\infty} \bar{d}_p(X_k, \bar{0})^p < \infty.$$

To complete the proof we show that

$$\sum_{k=1}^{\infty} \bar{d}(\triangle_m X_k, \bar{0})^p < \infty.$$

For

$$\begin{aligned}\bar{d}(\triangle_m X_k, \bar{0})^p &= \left[\bar{d}(X_k - X_{k+m}, \bar{0})\right]^p \\ &\leq \left[\bar{d}(X_k, \bar{0}) + \bar{d}(X_{k+m}, \bar{0})\right]^p \\ &\leq 2^p\, max\{\left[\bar{d}(X_k, \bar{0})\right]^p, [\bar{d}(X_{k+m}, \bar{0})]^p\} \\ &\leq 2^p\{\left[\bar{d}(X_k, \bar{0})\right]^p + \left[\bar{d}(X_{k+m}, \bar{0})\right]^p\} < \infty.\end{aligned}$$

Thus $\bar{d}(\triangle_m X_k, \bar{0})^p < \infty$. Hence $X = (X_k) \in bV_p^F(\Delta_m)$ and so $\ell_p^F \subseteq bV_p^F(\Delta_m)$. □

Theorem 4.3.4. *[108] The class of sequences of p-bounded variation* bV_p^F *solid.*

Theorem 4.3.5. *[77] For 1* $\leq q < p < \infty$*, the relation* $bV_q^F(\Delta_m) \subseteq bV_p^F(\Delta_m)$ *holds.*

Proof. Let $X = (X_k) \in bV_q^F(\Delta_m)$ then $[\bar{d}(\Delta_m X_k, \bar{0})]^q < \infty$.
Here, $\triangle_m X_k \to \bar{0}$ as $k \to \infty$, so we can choose a positive integer n_o such that $\bar{d}(\triangle_m X_k, \bar{0}) < 1$ for all $k > n_o$.

Now,

$$\sum_{k=1}^{\infty} \left[\bar{d}(\triangle_m X_k, \bar{0})\right]^p = \sum_{k=1}^{n-1} \left[\bar{d}(\triangle_m X_k, \bar{0})\right]^p + \sum_{k=n}^{\infty} \left[\bar{d}(\triangle_m X_k, \bar{0})\right]^p$$

Since ,

$$\sum_{k=n}^{\infty} \left[\bar{d}(\triangle_m X_k, \bar{0})\right]^p \leq \sum_{k=n}^{\infty} \left[\bar{d}(\triangle_m X_k, \bar{0})\right]^q.$$

Hence $X = (X_k) \in bV_p^F(\Delta_m)$. So for $1 \leq q < p < \infty$, the relation

$$bV_q^F(\Delta_m) \subseteq bV_p^F(\Delta_m)$$

holds. □

Theorem 4.3.6. *[79] Let* $\overline{p} = (p_k)$*, a sequence of strictly positive real numbers,* M *is an Orlicz function, and* $A = (a_k)$*, a sequence of real numbers, then the classes* $F_\infty(M, \overline{p}, A)$, $F(M, \overline{p}, A)$ *and* $F_o(M, \overline{p}, A)$ *defined by*

$$F_\infty(M,\overline{p}, A) = \left\{X = (X_k) : X_k \in \omega(F) : \sup_k \left[M\left(\frac{\overline{d}\left(a_k(\Delta_m X_k), \overline{0}\right)}{\rho}\right)\right]^{p_k} < \infty, for\ some\ \rho > 0\right\}$$

$$F(M, \overline{p}, A) = \left\{X = (X_k) : X_k \in \omega(F) : \lim_k \left[M\left(\frac{\overline{d}\ (a_k(\Delta_m X_k), X_o)}{\rho}\right)\right]^{p_k} = 0, for\ some\ \rho > 0\right\}$$

$$F_o\,(M,\bar{p},A) = \left\{ X = (X_k) : X_k \in \omega\,(F) : \lim_k \left[M\left(\frac{\overline{d}\ (a_k(\Delta_m X_k),\ X_o)}{\rho} \right) \right]^{p_k} = 0, for\ some\ \rho > 0 \right\}$$

are linear.

Proof. To prove the linearity of the classes, we first show that $F_\infty\,(M\ ,\bar{p}\ ,\ A\ )$ is linear and then the proof for other classes follows similarly.

Let $X = (X_k)$ and $Y = (Y_k)$ are from $F_\infty\,(M\ ,\bar{p}\ ,\ A\ )$. Then there exist $\rho_1 > 0,\ \ \rho_2 > 0$ such that

$$\sup_k \left[M\left(\frac{\overline{d}\left(a_k(\Delta_m\ X)\ ,\bar{0}\ \right)}{\rho_1} \right) \right]^{p_k} < \infty \text{ and } \sup_k \left[M\left(\frac{\overline{d}\left(a_k(\Delta_m\ Y)\ ,\bar{0}\ \right)}{\rho_2} \right) \right]^{p_k} < \infty.$$

Then for any scalars $\alpha,\ \beta$ and $\rho = \max\left\{2\alpha\rho_1\,,\ 2\beta\ \rho_2\right\}$, we have

$$\begin{aligned} \sup_k \left[M\left(\frac{\overline{d}\left(a_k(\Delta_m \alpha X_k) + a_k(\Delta_m \beta Y_k), \bar{0}\right)}{\rho} \right) \right]^{p_k} &\leq \sup_k \left[M\left(\alpha \frac{\overline{d}\left(a_k(\Delta_m X_k), \bar{0})\right)}{\rho} + \beta \frac{\overline{d}\left(a_k(\Delta_m\ Y_k), \bar{0})\right)}{\rho} \right) \right]^{p_k} \\ &\leq \sup_k \left[M\left(\frac{1}{2} \frac{\overline{d}\left(a_k(\Delta_m\ X_k), \bar{0})\right)}{\rho_1} + \frac{1}{2} \frac{\overline{d}\left(a_k(\Delta_m\ Y_k), \bar{0})\right)}{\rho_2} \right) \right]^{p_k} \\ &\leq G \sup_k \left[M\left(\frac{\overline{d}\left(a_k(\Delta_m\ X_k), \bar{0})\right)}{\rho_1} \right) \right]^{p_k} + G \sup_k \left[\frac{\overline{d}\left(a_k(\Delta_m\ Y_k), \bar{0})\right)}{\rho_2} \right]^{p_k}. \\ &< \infty, \end{aligned}$$

where $G = \max\left\{1\ , 2^{\mu - 1}\right\}$, and $\mu = \sup_k\ p_k$.

So that

$$\sup_k \left[M\left(\frac{\overline{d}\ \left(a_k\left(\Delta_m \alpha\ X_k\right) + a_k\left(\Delta_m \beta\ Y_k\right)\ , \bar{0}\right)}{\rho} \right) \right]^{p_k} < \infty$$

Therefore, $\alpha X +\ \beta Y\ \in F_\infty\,(\ M, \bar{p}\ ,\ A\ )$ and hence the sequence of class $F_\infty\,(M\ ,\bar{p}\ ,\ A\ )$ is linear space. □

Theorem 4.3.7. *[79] Suppose $\bar{p} = (p_k)$ and $\bar{q} = (q_k)$ be two strictly increasing sequences of positive terms such that $0 < p_k \leq q_k < \infty$ for all values of k, then the following relations hold:*

(i) $F(M,\ \bar{p},\ A) \subseteq F(M,\ \bar{q},\ A)$.

(ii) $F_o(M,\ \bar{p},\ A) \subseteq F_o(M,\ \bar{q},\ A)$.

Proof. Suppose $\bar{p} = (p_k)$ and $\bar{q} = (q_k)$ are two positive strictly increasing sequences such that $0 < p_k \leq q_k < \infty$ for all values of k. Let $X = (X_k) \in F(M,\ \bar{p},\ A)$. Then there exists $\rho > 0$ such that

$$lim_k \left[M\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k\right),\ X_o\right)}{\rho} \right) \right]^{p_k} = 0.$$

This relation is true only for $\left[M\left(\frac{\bar{d}(a_k(\Delta_m X_k), X_o)}{\rho} \right) \right]^{p_k} < 1$ for sufficiently large value of k. Since $p_k \leq q_k$ for all values of k, we have

$$\left[M\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k\right), X_o\right)}{\rho} \right) \right]^{q_k} \leq \left[M\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k\right), X_o\right)}{\rho} \right) \right]^{p_k}$$

Taking limit $k \to \infty$,

$$lim_k \left[M\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k\right), X_o\right)}{\rho}\right)\right]^{q_k} \leq lim_k \left[M\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k\right), X_o\right)}{\rho}\right)\right]^{p_k} = 0$$

$$\implies lim_k \left[M\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k\right), X_o\right)}{\rho}\right)\right]^{q_k} = 0$$

$$\implies X = (X_k) \in F(M, \bar{q}, A)$$

$$\implies F(M,\ \bar{p},\ A) \subseteq F(M,\ \bar{q},\ A)$$

Similarly, we can show that $F_o(M,\ \bar{p},\ A) \subseteq F_o(M,\ \bar{q},\ A)$.

$\square$

Theorem 4.3.8. *[16] For non-negative integer m, the class*

$$Z(\Delta_m) = \left\{(X_k) \in \omega^F : (\Delta_m X_k) \in Z\right\} \textit{ for } Z = C^F(\Delta_m),\ C_o^F(\Delta_m),\ \ell_\infty^F(\Delta_m),$$

where $\Delta_m X_k = X_k - X_{k+m}$ for all $k \in \mathbb{N}$ is a complete metric space with respect to the metric defined by

$$G(X,\ Y) = \sum_{k=1}^{m} \bar{d}(X_k,\ Y_k) + \sup_k \bar{d}\left(\Delta_m X_k,\ \Delta_m Y_k\right).$$

Theorem 4.3.9. *[79] The space $F_\infty\left(M, \bar{d}, A\right)$ is complete metric space with the metric defined by*

$$\bar{d_G}\left(X, Y\right) = inf\left\{\rho^{\frac{p_k}{T}} : \sup_k \left[\left\{M\left(\frac{\bar{d}(a_k\left(\Delta_m X_k\right),\ a_k\left(\Delta_m Y_k\right))}{\rho}\right)\right\}^{p_k}\right]^{\frac{1}{T}} \leq 1\right\},$$

where $T = max\{1,\ p_k\}$.

Proof. Let $\left(X^i\right)$ be a Cauchy sequence in $F_\infty\left(M, \bar{d}, A\right)$. Then for $\varepsilon > 0$, let us choose $u > 0$ and $\eta > 0$ such that

$$M\left(\frac{u\eta}{2}\right) \geq\ 1.$$

Then for $\varepsilon > 0, \exists n_o \in \mathbb{N} : \forall i, j \geq n_o$ we have

$$\bar{d_G}\left(X^i, X^j\right) = \inf\left\{\rho^{\frac{p_k}{T}} : \sup_k \left[\left\{\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k^i\right),\ a_k\left(\Delta_m X_k^j\right)\right)}{\rho}\right)\right\}^{p_k}\right]^{\frac{1}{T}} \leq 1\right\}$$

$$\implies \left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k^i\right),\ a_k\left(\Delta_m X_k^j\right)\right)}{\rho}\right)^{p_k} \leq 1$$

$$\implies \left\{M\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k^i\right),\ a_k\left(\Delta_m X_k^j\right)\right)}{\rho}\right)\right\}^{p_k} \leq 1 \leq M\left(\frac{u\eta}{2}\right)$$

Since M is non-decreasing function,

$$\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k^i\right),\ a_k\left(\Delta_m X_k^j\right)\right)}{\rho}\right) \leq \frac{u\eta}{2}$$

Since $\rho > 0$, let us choose $\rho = \frac{\varepsilon}{u\eta}$, so that

$$\bar{d}\left(a_k\left(\Delta_m X_k^i\right),\ a_k\left(\Delta_m X_k^j\right)\right) \leq \frac{u\eta}{2}.\frac{\varepsilon}{u\eta} < \varepsilon$$

Therefore, $\frac{\bar{d}\left(a_k\left(\Delta_m X_k^i\right),\, a_k\left(\Delta_m X_k^j\right)\right)}{\rho} < \varepsilon$, for all $\forall i,\ j \geq n_o$

This shows that $\left(a_k\left(\Delta_m X_k^i\right)\right)$ is a Cauchy sequence in $\mathbb{R}(I)$ for all $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, the sequence $\left(a_k\left(\Delta_m X_k^i\right)\right)$ converges in $\mathbb{R}$ and say

$$lim_i\ \left[a_k(\Delta_m X_k^i)\right]\ =\ a_k\, \Delta_m X_k$$

$$\implies lim_i\ (X_k^i - X_{k+m}^i)\ =\ X_k - X_{k+m}$$

$$\implies lim_i\ X_k^i\ =\ X_k$$

Let $(X_k) = X$. To complete the proof, we show that $X \in F_\infty\left(M,\ \bar{p},\ A\right)$. We have

$$\bar{d_G}\left(X, Y\right) = inf\left\{\rho^{\frac{p_k}{T}} : \sup_k\ \left[\left\{M\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k\right),\ a_k\left(\Delta_m Y_k\right)\right)}{\rho}\right)\right\}^{p_k}\right]^{\frac{1}{T}} \leq 1\right\}$$

Now,

$$\inf\left\{\rho^{\frac{p_k}{T}} : \sup_k\ \left[\left\{M\left(\frac{\bar{d}\left(a_k\left(\Delta_m X_k\right),\ \bar{0}\right)}{\rho}\right)\right\}^{p_k}\right]^{\frac{1}{T}}\right\} = \bar{d_G}(X,\ \bar{0})$$

$$\leq \bar{d_G}(X,\ X^i) + \bar{d_G}(X^i, \bar{0})$$

$$< \varepsilon' +\ \bar{d_G}(X^i, \bar{0}) < \infty.$$

$\implies X \in F_\infty\left(M,\ \bar{p},\ A\right)$. Hence $F_\infty\left(M,\ \bar{p},\ A\right)$ is complete.

□

Chapter 5

Applications of Fuzzy Logic and Fuzzy Set

5.1 Introduction

Over the past 50^+ years, the fields of fuzzy sets and fuzzy logic have seen significant progressions. Fuzzy logic, a mathematical concept introduced by Lotfi A. Zadeh[117] in the 1965, provides a way to handle uncertainty in decision-making process. Unlike traditional binary logic based on strict true or false values, fuzzy logic allows for degrees of truth between 0 and 1, enabling a more flexible and nuanced approach to problem-solving. Fuzzy sets and fuzzy logic have been actively explored and applied to real-world situations by a large number of academics and researchers [103]. As a result, it has been incorporated into a number of fields, including engineering, medicine, agriculture, and many more. Fuzzy logic can handle complicated systems where exact, deterministic models might not accurately reflect the environment due to their flexibility and adaptability. One of the fundamental applications of fuzzy logic is in decision-making process [12]. Making decisions is an integral part of problem-solving and involves selecting a course of action from several potential options to achieve a particular goal. This ability to make informed choices is crucial in numerous fields, such as business, finance, management, economics, social and political science, engineering, computer science, biology, and medicine. However, the decision-making process is frequently complicated by many elements encountered in real-life scenarios. Some of these factors include:

Incomplete and Inaccurate Information: Decision-makers may not have access to all the relevant information needed to make a well-informed choice. Some data may be missing or unreliable, making it challenging to arrive at a definitive solution[113] .

Subjectivity: Decision-making often needs human judgment, which can be subjective and impacted by personal biases or preferences. The same scenario may be viewed differently by many people, which might result in decisions being made differently [112].

Language and Linguistic Variables: In many cases, the criteria for making a decision may be described using linguistic terms rather than precise numerical values. For example, describing a product's quality as "very good" or "average" introduces linguistic uncertainty[119].

The presence of these traits in real-life situations gives rise to a "fuzzy environment" for decision-making. Fuzzy logic provides a powerful framework to deal with this uncertainty by allowing decision-makers to work with imprecise, vague, or uncertain information. Through fuzzy sets and fuzzy logic, decision-makers can model and process linguistic variables, subjective judgments, and incomplete data in a way that enables them to make more flexible and robust decisions [14].

Thus, the development of fuzzy sets and fuzzy logic has significantly impacted various fields by providing effective tools to handle uncertainty in decision-making. The ability to tackle complex and vague situations makes fuzzy logic a valuable asset in addressing real-world challenges across numerous disciplines, ultimately leading to improved problem-solving and more efficient, adaptive systems.

5.1.1 Application of Fuzzy Logic Through Bellmen-Zadeh Maximin Method

Making a choice between various options is a problem-solving process that results in a specific action known as decision-making. The idea of fuzzy decision-making was first put forth by Bellman and Zadeh [12] in 1970 and is based on a compromise between goals and constraints that are represented by fuzzy sets. By using membership functions to represent goals and constraints, this optimization concept seeks to identify the best possible outcome. Decision-making is the process of making an important decision. So, in order to arrive at a solution for a particular problem, decision-making entails choosing a course of action from two or more feasible alternatives A decision process in a fuzzy environment is one in which the goals and constraints are both fuzzy in character. Bellman–Zadeh [12] defined decision-making as the intersection of goals and constraints expressed by fuzzy sets. In a simple decision-making problem if a fuzzy set G denotes the goal and the fuzzy set C denotes the constraints with the membership $\mu_G(x)$ and $\mu_C(x)$ respectively. Then according to Bellman- Zadeh the decision is the fuzzy set denoted by D is defined as $D = G \cap C$ and its membership function is $\mu_D(x)$=$\mu_G(x) \wedge \ \mu_C(x)$.

More generally, suppose $G_1, G_2, , G_3, ..., ..., ..., G_n$ be n goals and $C_1, C_2, , C_3, ..., ..., ..., C_m$ be m constraints. Then the result decision is the intersection of given goals and constraints, i. e.

$$D = G_1 \cap G_2 \cap G_3 \cap \dots \dots \dots \cap G_n \cap C_1 \cap C_2 \cap C_3 \cap \dots \dots \dots \cap C_m.$$

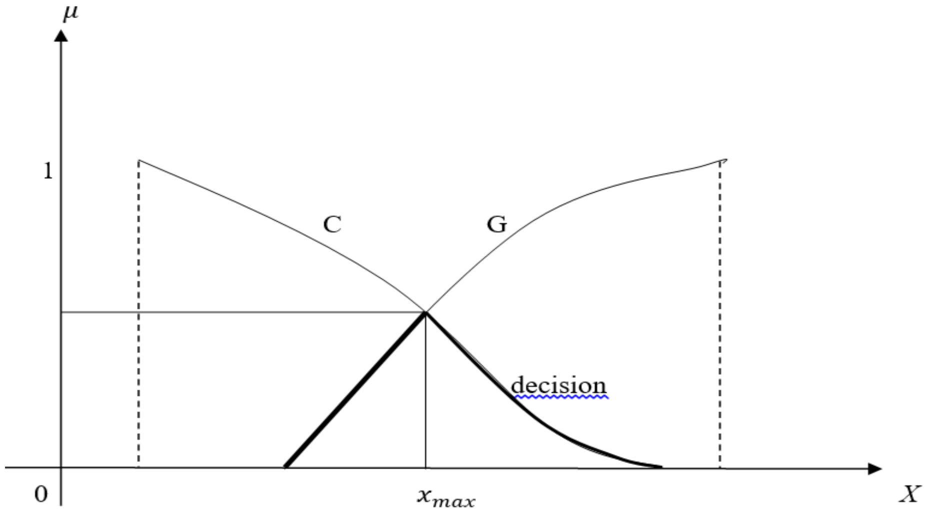


Figure 5.1: Fuzzy Decision

And the corresponding membership function is

$$\mu_D = \mu_{G_1} \wedge \mu_{G_2} \wedge \mu_{G_3} \wedge \dots \dots \dots \wedge \mu_{G_n} \wedge \mu_{C_1} \wedge \mu_{C_2} \wedge \mu_{C_3} \wedge \dots \dots \dots \wedge \mu_{C_m}.$$

Thus, the decision is the confluence of goals and constraints.

Suppose a reputed corporation posts a job posting for a vacant position. Candidates $x_1, x_2, ..., ..., ..., x_n$ apply for the available position.They form a discrete set of alternatives, say $\{x_1, x_2, x_3, ..., ..., x_n\}$. The selection committee looks for certain attributes in candidates, such as experience, youth,competence, and so on in a specific field, which is referred to as goals, $G_i, i = 1, 2, 3, ..., ..., m$. The selection committee also intended to set some limits, such as low pay, which can be considered constraints, $C_j, j = 1, 2, ..., ..., ..., p$. At the conclusion of the interviewing process, each candidate's x_k is graded on a scale of 0 to 1 with the help of a membership function based on their goals and constraints. The grade of the candidate x_k for the goals G_i are denoted by g_{ki} , and the candidate's grades for the constraints C_j are denoted by c_{kj}. Then the fuzzy sets corresponding to goals G_i and constraints C_j are

$$G_i = \{(x_1, g_{1i}), (x_2, g_{2i}), (x_3, g_{3i}), ..., ..., ..., (x_n, g_{ni})\}, i = 1, 2, 3, ..., ..., ..., m$$

$$C_j = \{(x_1, c_{1j}), (x_2, c_{2j}), (x_3, c_{3j}), ..., ..., ...(x_n, c_{nj})\}, j = 1, 2, 3, , ..., ..., ..., p.$$

According to Bellman- Zadeh,[12] the decision is fuzzy set

$$D = G_1 \cap G_2 \cap G_3 \cap \dots \quad \dots \quad \dots \cap G_m \cap C_1 \cap C_2 \cap C_3 \cap \dots \quad \dots \quad \dots \cap C_p$$

with grade,

$$\mu_D = \mu_{G_1} \wedge \ \mu_{G_2} \wedge \dots \quad \dots \quad \dots \wedge \mu_{G_n} \wedge \mu_{C_1} \wedge \mu_{C_2} \wedge \dots \quad \dots \quad \dots \wedge \mu_{C_m}$$

i.e, μ_l =min $\{g_{l1}, g_{l2}, ..., ..., ..., g_{lm}, c_{l1}, c_{l2}, ..., ..., ..., c_{lp}\}$, $1 \leq l \leq n.$

5.1.2 Case Study

Suppose an educational institute has declared to provide scholarship to support higher study. Excellent result in different subjects: S_1, S_2, S_3, S_4, and S_5 is the major factor in selecting students to get scholarships. But the term ”excellent” creates a fuzzy environment and makes it difficult to select a student in a crisp environment. So, fuzzy logic is more suitable here. The marks obtained by five students in different subjects are given below:

Students	S_1	S_2	S_3	S_4	S_5
A	90	93	94	87	88
B	92	89	93	86	96
C	91	97	88	92	84
D	88	97	90	91	90
E	95	92	86	90	92

Table 5.1: Marks of Students

The obtained marks are out of 100 and an excellent result is considered when the marks are more than 80. So the domain of the marks is [80, 100].To select a student for a scholarship, we use the Bellmen- Zadeh maximin [12] method by converting the crisp set of obtained marks into a fuzzy set. For this we need corresponding membership values of marks, which are obtained by the following membership function.

$$\mu_X(x) = \begin{cases} 0, & \text{for } 0 \leq x \leq 80 \\ \frac{x-80}{90-80}, & \text{for } 80 \leq x \leq 90 \\ 1, & \text{for } 90 \leq x \leq 100 \end{cases}$$

Using the membership function defined above, we calculate the grading value of the marks obtained by the students in different subjects

$$\mu_A(90) = 1, \mu_A(93) = 1,\ \mu_A(94) = 1,\ \mu_A(87) = 0.7,\ \mu_A(88) = 0.8$$

$$\mu_B(92) = 1, \mu_B(89) = 0.9, \mu_B(93) = 1, \mu_B(86) = 0.6, \mu_B(96) = 1$$

$$\mu_C(91) = 1, \mu_C(97) = 1, \mu_C(88) = 0.8, \mu_C(92) = 1, \mu_C(84) = 0.4$$

$\mu_D(88) = 0.8, \mu_D(97) = 1, \mu_D(90) = 1, \mu_D(91) = 1, \mu_D(92) = 1$

$\mu_E(95) = 1, \mu_E(92) = 1, \mu_E(86) = 0.6, \mu_E(90) = 1, \mu_E(92) = 1$

Students	S_1	S_2	S_3	S_4	S_5
A	1	1	1	0.7	0.8
B	1	0.9	1	0.6	1
C	1	1	0.8	1	0.4
D	0.8	1	1	1	1
E	1	1	0.6	1	1

Table 5.2: Grading Marks

The fuzzy set corresponding to the marks obtained in subjects S_1, S_1, S_3, S_4, and S_5 respectively are:

$$G_1 = \{(A, 1), (B, 1), (C, 1), (D, 0.8), (E, 1)\}$$

$$G_2 = \{(A, 1), (B, 0.9), (C, 1), (D, 1), (E, 1)\}$$

$$G_3 = \{(A, 1), (B, 1), (C, 0.8), (D, 1), (E, 0.6)\}$$

$$G_4 = \{(A, 0.7), (B, 0.6), (C, 1), (D, 1), (E, 1)\}$$

$$G_5 = \{(A, 0.8), (B, 1), (C, 0.4), (D, 1), (E, 1)\}$$

The selection committee has to give their decision for excellency by using Bellmen-Zadeh[12] method,

$$D = G_1 \cap G_2 \cap G_3 \cap G_4 \cap G_5$$

$$= \{(A, 0.7), (B, 0.6), (C, 0.4), (D, 0.8), (E, 0.6)\}$$

By the Max-min method, we conclude that D has the best preference, with a membership value of 0.8.

5.1.3 Case Study

Suppose a reputed organization has to fill a post, for which five candidates, say A, B, C, D, and E, are shortlisted. These candidates form a set of alternatives, say{ A, B, C, D, E}. The selection committee looks for certain attributes in candidates, such as computer skills, experience in years, academic qualifications, and youthfulness. These qualities can be considered the goal of the problem. The domain of experience is [5, 10]. The selection committee has taken an exam for computer skills and with domains [30, 40]. Also, the organization imposed another condition that the salary should be moderated from Rs 30,000 to Rs 40,000, and this can be taken as a constraint. The marks obtained in computer skills, experience, scored percentage at bachelor level, and age in years is given below:

Candidates	Experience(E)	Computer skill(C)	Qualification(Q)	Youngness(Y)
A	8	37	76	29
B	6	38	79	26
C	9	33	75	33
D	5	38	72	25
E	7	35	75	32

Table 5.3: Results in Different Sectors

The shortlisted candidates who are ready to work for domain [30000 40000] of salary and their response for it is:

A	B	C	D	E
37000	38000	36000	35000	39000

Table 5.4: Salary in Rs

The corresponding membership value for each component E , C, Q, and Y are respectively calculated by defining the functions.

$$\mu_E(x) = \begin{cases} 0, & \text{for} \quad 0 \leq x \leq 4 \\ \frac{x-4}{10-4}, & \text{for} \quad 4 \leq x \leq 10 \\ 1, & \text{for} \quad 10 \leq x \qquad ; \end{cases}$$

$$\mu_C(x) = \begin{cases} 0, & \text{for} \quad 0 \leq x \leq 30 \\ \frac{x-30}{40-30}, & \text{for} \quad 30 \leq x \leq 40 \\ 1, & \text{for} \quad x \geq 40 \qquad ; \end{cases}$$

$$\mu_Q(x) = \begin{cases} 0, & \text{for} \quad 0 \leq x \leq 70 \\ \frac{x-70}{80-70}, & \text{for} \quad 70 \leq x \leq 80 \\ 1, & \text{for} \quad x \geq 80 \qquad ; \end{cases}$$

$$\mu_Y(x) = \begin{cases} 0, & \text{for} \quad 0 \leq x < 20 \\ \frac{35-x}{35-20}, & \text{for} \quad 20 \leq x \leq 35 \\ 0, & \text{for} \quad x \geq 35 \end{cases}$$

The membership values in different sectors are:

Candidates	E	C	Q	Y
A	0.67	0.7	0.6	0.4
B	0.33	0.8	0.9	0.6
C	0.83	0.3	0.5	0.13
D	0.16	0.8	0.2	0.83
E	0.5	0.5	0.5	0.2

Table 5.5: Membership Values

The fuzzy set corresponding to the marks obtained in sector E , C, Q,and Y respectively are:

$G_1 = \{(A, 0.67), (B, 0.33), (C, 0.83), (D, 0.16), (E, 0.5)\}.$

$G_2 = \{(A, 0.7), (B, 0.8), (C, 0.3), (D, 0.8), (E, 0.5)\}$

$G_3 = \{(A, 0.6), (B, 0.9), (C, 0.5), (D, 0.2), (E, 0.5)\}$

$G_4 = \{(A, 0.4), (B, 0.6), (C, 0.13), (D, 0.83), (E, 0.2)\}$

Also, to calculate the grading value for the salary, let us define the membership function as :

$$\mu_S(x) = \begin{cases} 1, & \text{for} \quad 0 \leq x < 30000 \\ \frac{40000-x}{40000-30000}, & \text{for} \quad 30000 \leq x \leq 40000 \\ 0, & \text{for} \quad x \geq 40000 \end{cases}$$

And the fuzzy set corresponding to salary is:

$C = \{(A, 0.3), (B, 0.2), (C, 0.4), (D, 0.5), (E, 0.1)\}.$

The selection committee has to give their decision for the best candidate. For this we use Bellmen-Zadeh [12] method.

$$\begin{aligned}\text{Fuzzy decision } (D) &= G_1 \cap G_2 \cap G_3 \cap G_4 \cap C \\ &= \{(A, 0.3), (B, 0.2), (D, 0.13), (E, 0.1)\}\end{aligned}$$

Then by the Maximin method A with the largest grade 0.3 is selected for the vacant post.

Making a decision is a problem-solving process that leads to a certain action. It is a choice between numerous methods for achieving a goal. Here, we have discussed how can we select the best person from a set of persons of the same environment using the Bellman- Zadeh method of decision-making using maxi-min method. Also, we examined how best to select the best of the best for a specific outcome.

5.2 Fuzzy Arithmetic–Based Algorithm for Identifying Medical Conditions

5.2.1 Introduction

Making the right medical decision is challenging work because, in our daily life, decision-making problems may have the components of membership and non-membership degrees with the possibility of hesitation. But fuzzy set theory is considered to have only a membership degree. So, this theory could not be considered for solving such problems. In 1996, Atanassov [7] introduced the concept of intuitionistic fuzzy sets (IFS), which is capable of capturing information that includes membership and non-membership values with some possible degree of hesitation. Then intuitionistic fuzzy set has been applied in various fields of research in making a decision and medical diagnoses. Molodtsov [68] first introduced the soft set theory in 1999 to handle objects whose definitions used a very broad and general set of characteristics. The theory has the potential to be used to solve real-world problems in economics, engineering, the environment, social science, medicine, and business management. It is very convenient and easily applicable because there are no restrictions on the approximate description. The traditional soft set theory was fuzzified by Yang et al [115], and the fuzzy membership is used to describe parameters-approximate elements of the fuzzy soft set. In the study of decision models and their applications for simulating ambiguity and uncertainty, Deli and Çağman [26] presented a method of making decisions that were based on intuitionistic fuzzy parameterized-soft set theory. Intuitionistic fuzzy sets and intuitionistic fuzzy soft sets are more useful for the application point of view in the field of uncertainty due to vagueness. Fuzzy soft set theory is gaining importance for finding a coherent and logical solution to various real-life problems.

In this section, we apply Sanchez's[89] idea to medical diagnosis and present a case study to illustrate the method.

Definition 25. *[62] Suppose X is a universal set and $P(X)$ be the power set of X. Let us denote E as the set of parameters and A as a subset of E. Then the collection (F, A) is defined as:*

$$(F, A) = \{(x, F_A(x)) : x \in E, F_A(x) \text{ in } P(X)\}$$

where F_A is a function from E to $P(X)$. Here, $F_A(x)$ is known as the x-approximate function of A. We note that $F_A(x)$= ϕ if $x \notin A$.

Definition 26. *[61] Let X be a universal set, then the collection of pairs*

$$A = \{(x, \mu_A(x)) : \mu_A(x) : X \to [0, 1], x \in X\}$$

defines a fuzzy set A on X.
Here, μ_A is called a ***membership function*** *defined as*

$$\mu_A(x) = \begin{cases} 0, & \textit{if } x \in A \ \ and \ \ there \ \ is \ \ no \ \ ambiguity \\ 1, & \textit{if } x \in A \ \ and \ \ there \ \ is \ \ no \ \ ambiguity \\ (0, 1), & \textit{if } \ there \ \ is \ \ ambiguity \ \ whether \ \ x \in A \ \ or \ \ x \notin A. \end{cases}$$

The value of $\mu_A(x)$ represents the degree of element x belonging to the set A.

Definition 27. *[42] Let X be a non-empty set. An* ***intuitionistic fuzzy set*** *A in X is an object having the form*

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where, $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$
are the membership and non-membership function respectively, of the element $x \in X$ to the set A.

Here, $\mu_A(x)$ and $\nu_A(x)$ are respectively called the degree of membership and degree of non-membership function of the element $x \in X$ to the set A, and for every $x \in X$, we have

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Furthermore, in the fuzzy set, there is a lack of knowledge of whether $x \in X$ belongs to A or not. This lack of knowledge for $x \in X$ to the set A is called ***hesitation*** *of x in A and is denoted by $\pi_A(x)$ and is defined as:*

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$$

where, $\pi_A : X \to [0, 1]$ with $0 \leq \pi_A(x) \leq 1$.

5.2.2 Operations on Fuzzy Sets and Intuitionistic Fuzzy Sets [36]

Suppose X is a universe of discourse and let A and B be two intuitionistic fuzzy sets in X, then we have

1. $A = B$ if and only if $\mu_A(x) = \mu_B(x), \nu_A(x) = \nu_B(x), \forall x \in X$.
2. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x), \nu_A(x) \geq \nu_B(x), \forall x \in X$.
3. The complement of the intuitionistic fuzzy set A is denoted by A^c and defined by $A^c = \{(\nu_A(x), \mu_A(x)) : x \in X\}$.
4. The union $A \cup B$ is defined as

   $A \cup B = \{\langle x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x))\rangle : x \in X\}$
5. The intersection $A \cap B$ is defined as

   $A \cap B = \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x))\rangle : x \in X\}$
6. $A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle : x \in X\}$
7. $A.B$ =$\{\langle x, \mu_A(x).\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x).\nu_B(x)\rangle : x \in X\}$
8. The Cartesian product [42] of A and B is defined by:
   $A \times B$= $\{\langle\langle x, y\rangle, \mu_A(x).\mu_B(x), \nu_A(x).\nu_B(x)\rangle : x \in A, y \in B\}$

Let X and Y be two non-empty sets and R be an intuitionistic fuzzy relation from X to Y. Suppose A is an intuitionistic fuzzy set in X then max-min-max composite relation of R with A being an intuitionistic fuzzy set, B of Y denoted by $B = RoA$ such that its membership function and non-membership function are defined as

$$\mu_B(y) = max_x\{min[\mu_A(x), \mu_R(x, y)]\},$$

and

$$\nu_B(y) = max_x\{min[\nu_A(x), \nu_R(x, y)]\}$$

for all $x \in X, y \in Y$

Definition 28. ***(Fuzzy soft set)****[43] Let X be the universal set and E be the set of all parameters and $F(X)$ be the set of all fuzzy sets in X. For $A \subseteq E$, the fuzzy soft set F_A over $F(X)$ is defined by:*

$$F_A = (F, A) = \{(p, F_A(p)) : p \in E,\ and\ F_A(p) \in F(X)\},$$

where F_A is a function, $F_A : E \to F(X)$.

If F(X) is the collection of all intuitionistic fuzzy set over X, then F_A is an intuitionistic fuzzy soft set. The value $F_A(x)$ is an intuitionistic fuzzy set and is called x- element of F_A for all $x \in E$ and is defined as:

$$F_A(x) = (x, \mu_A(x), \nu_A(x) : x \in X)$$

5.2.3 Methodology and Algorithm

Here we apply Sanche's [89] idea for the medical diagnosis and present a case study to illustrate the method. In order to do this, we use intuitionistic fuzzy soft set theory, and for that we present a fuzzy–based algorithm for diagnosing a medical condition. Assume that there is a set of m patients P and a set of n symptoms, E that is caused by a set of k diseases, D. Consider an intuitionistic fuzzy soft set (F, P) over E. This intuitionistic fuzzy set gives a patient-symptom matrix R. With the help of the intuitionistic fuzzy set (F, E) over D, we construct two matrices R_1 and R_2 named symptom-disease and non-symptom-disease matrix respectively. The relation matrices M_1 and M_2 are obtained from RoR_1 and RoR_2 using max-min-max composition. Then we calculate the medical diagnosis table DS_k, where

$$DS_k = max_x SD_{M_1}(p_i, d_j) - SD_{M_2}(p_i, d_j) \text{ with } d_j = \mu_j - \nu_j \pi_j$$

From the diagnosis table DS_k, we conclude that the patient p_i is suffering from the disease d_j

5.2.4 Algorithm

1. The output matrix R is obtained via input intuitionistic fuzzy soft set (F, P)
2. The output matrices R_1 and R_2 are obtained through intuitionistic fuzzy sets (F, E) and $(F, E)^c$
3. Calculate $M_1 = RoR_1$ and $M_2 = RoR_2$ using max-min-max rule.
4. The diagnosis matrices SD_{M_1} and SD_{M_2} are de-fuzzify of M_1 and M_2 respectively.
5. Calculate, $DS_k = max_j\{SD_{M_1}(p_i, d_j) - SD_{M_2}(p_i, d_j)\}$ with $d_j = \mu_j - \nu_j \pi_j$
6. Conclude that the patient p_i is suffering from the disease d_j

5.2.5 Case Study

Let us consider a universal set $P = \{p_1, p_2, p_3, p_4\}$ of patients in a hospital with different symptoms : body temperature, headache, dizziness, and body pain. Consider a set S=$\{e_1, e_2, e_3, e_4\}$ of parameters where,

$e_1 =$ body temperature , $e_2 =$ headache, e_3= dizziness , e_4= body pain.

Let $D = \{d_1, d_2, d_3\}$ be the set of diseases, where d_1= typhoid, d_2= malaria , d_3= covid.

Here the temperature is measured with the help of a medical instrument (digital thermometer), while headache, dizziness and body pain are the rating scale in the interval, which are obtained via the questions to the patients and we prepared the following table

patient	e_1	e_2	e_3	e_4
p_1	103.6	6	6	7
p_2	102.8	7	4	8
p_3	104.4	5	8	8
p_4	102	4	4	6

Table 5.6: Patient -Symptoms Table

To fuzzify the above data, we use the membership functions $\mu_T : [98, 106] \to [0, 1]$ and $\mu : [0, 10] \to [0, 1]$. Here μ_T indicates the membership function defined for the body temperature and the second one indicates for other symptoms. We note that the body temperature 98 means there is no fever in the body and 106 means the extreme level of body temperature and we have $\mu_T(98) = 0$ and $\mu_T(106) = 1$. Also $\mu(0) = 0$ and $\mu(10) = 1$ for the rest of others symptoms. For any $x \in [98, 106]$, we define the membership function as follow

$$\mu_T(x) = \frac{x - 98}{106 - 98} = \frac{x - 98}{8}$$

and for, $x \in [0, 10]$, the membership function is defined as

$$\mu(x) = \frac{x - 0}{10 - 0} = \frac{x}{10}$$

Then the corresponding fuzzified data of above table is as:

patient	e_1	e_2	e_3	e_4
p_1	0.7, 0.1, 0.2	0.6, 0.2, 0.2	0.6, 0.1, 0.3	0.7, 0.2, 0.1
p_2	0.6, 0.1, 0.3	0.7, 0.1, 0.2	0.4, 0.4, 0.2	0.8, 0.0, 0.2
p_3	0.8, 0.1, 0.1	0.5, 0.3, 0.2	0.8, 0.1, 0.1	0.8, 0.2, 0.0
p_4	0.5, 0.2, 0.3	0.4, 0.3, 0.3	0.4, 0.5, 0. 1	0.6, 0.3, 0.1

Table 5.7: Patient -Symptoms Table

Now, we construct a matrix to show the relation between patients and symptoms:

$$R = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array}\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} (0.7,0.1,0.2) & (0.6,0.2,0.2) & (0.6,0.1,0.3) & (0.7,0.2,0.1) \\ (0.6,0.1,0.3) & (0.7,0.1,0.2) & (0.4,0.4,0.2) & 0.7,0.2,0.1) \\ (0.8,0.1,0.1) & (0.5,0.3,0.2) & (0.8,0.1,0.1) & (0.8,0.0,0.2) \\ (0.5,0.2,0.3) & (0.4,0.3,0.3) & (0.4,0.5,0.1) & (0.6,0.3,0.1) \end{bmatrix} \end{array}$$

This matrix is known as the patient- symptom matrix. Now, we consider a table to show the relationship between diseases and their corresponding parametric values.

symptoms	d_1	d_2	d_3
e_1	100.4	103.6	102
e_2	6	3	3
e_3	4	2	4
e_4	4	5	7

Table 5.8: Symptoms-Diseases Table

The corresponding fuzzified table is given below:

symptoms	d_1	d_2	d_3
e_1	(0.3, 0.3, 0.4)	(0.7, 0.2, 0.1)	(0.5, 0.3, 0.2)
e_2	(0.6, 0.3, 0.1)	(0.3, 0.4,0.3)	(0.5, 0.2, 0.3)
e_3	(0.4, 0.5, 0.2)	(0.2, 0.2, 0.3)	(0.4, 0.4, 0.2)
e_4	(0.4, 0.3, 0.3)	(0.5, 0.2, 0.3)	(0.7, 0.2, 0.1)

Table 5.9: Symptoms-Diseases Table

Now, we introduce two matrices R_1 and R_2 named as $symptom - disease\ matrix$ and $non - symptom{-}disease\ matrix$ as follow:

$$R_1 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array}\begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{bmatrix} (0.3,0.3,0.4) & (0.7,0.2,0.1) & (0.5,0.3,0.2) \\ (0.6,0.3,0.1) & (0.3,0.4,0.3) & (0.5,0.2,0.3) \\ (0.4,0.5,0.1) & (0.2,0.5,0.3) & (0.4,0.4,0.2) \\ (0.4,0.3,0.3) & (0.5,0.2,0.3) & (0.7,0.2,0.1) \end{bmatrix} \end{array}$$

$$R_2 = \begin{array}{c|ccc|} & d_1 & d_2 & d_3 \\ e_1 & (0.3, 0.3, 0.4) & (0.2, 0.7, 0.1) & (0.3, 0.5, 0.2) \\ e_2 & (0.3, 0.6, 0.1) & (0.4, 0.3, 0.3) & (0.2, 0.5, 0.3) \\ e_3 & (0.5, 0.4, 0.1) & (0.5, 0.2, 0.3) & (0.4, 0.4, 0.2) \\ e_4 & (0.3, 0.4, 0.3) & (0.2, 0.5, 0.3) & (0.2, 0.7, 0.1) \end{array}$$

To diagnose the diseases of the patients, we construct two new matrices, say M_1 and M_2 called patient-disease and patient-non-disease matrix respectively using the max-min-max method:

$\mu_{M_1}(p_i, d_j)$ = $\vee\{\mu_R(p_i, e_j) \wedge \mu_{R_1}(e_j, d_k)\}$ and $\mu_{M_1}(p_i, d_j)$=$\wedge\{\nu_R(p_i, e_j) \vee \nu_{R_1}(e_j, d_k)\}$

So,we have

$$M_1 = RoR_1 = \begin{array}{c|ccc|} & d_1 & d_2 & d_3 \\ e_1 & (0.6, 0.3, 0.1) & (0.7, 0.2, 0.1) & (0.7, 0.2, 0.1) \\ e_2 & (0.6, 0.3, 0.1) & (0.6, 0.2, 0.2) & (0.7, 0.2, 0.1) \\ e_3 & (0.5, 0.3, 0.2) & (0.7, 0.2, 0.1) & (0.7, 0.2, 0.1) \\ e_4 & (0.4, 0.3, 0.3) & (0.5, 0.2, 0.3) & (0.6, 0.3, 0.1) \end{array}$$

Similarly,

$$M_2 = RoR_2 = \begin{array}{c|ccc|} & d_1 & d_2 & d_3 \\ e_1 & (0.5, 0.3, 0.2) & (0.5, 0.2, 0.3) & (0.4, 0.4, 0.2) \\ e_2 & (0.4, 0.3, 0.3) & (0.4, 0.3, 0.3) & (0.4, 0.4, 0.2) \\ e_3 & (0.5, 0.3, 0.2) & (0.5, 0.2, 0.3) & (0.4, 0.4, 0.2) \\ e_4 & (0.4, 0.3, 0.3) & (0.4, 0.3, 0.3) & (0.4, 0.5, 0.1) \end{array}$$

Now, we calculate the diagnosis score which helps us to conclude that, the patient's p_i is suffering from the diseases d_k, and for this, we use the formula:

DS_k = $max_j\{SD_{M_1}(p_i, d_j) - SD_{M_2}(p_i, d_j)\}$ with $d_j = \mu_j - \nu_j\pi_j$

Here, d_1 = 0.6 $-$ 0.3 $\times$ 0.1 = $\dot{5}$7 , d_2= 0.7 $-$ 0.2 $\times$ 0.1 = 0.68 , d_3= 0.7 $-$ 0.2 $\times$ 0.1 = 0.68

Similarly, we can calculate the remaining and also for M_2 and we have

$$DS_{M_1} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{bmatrix} 0.57 & 0.68 & 0.68 \\ 0.57 & 0.56 & 0.68 \\ 0.44 & 0.68 & 0.68 \\ 0.31 & 0.44 & 0.57 \end{bmatrix} \end{array}$$

$$DS_{M_2} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{bmatrix} 0.44 & 0.44 & 0.32 \\ 0.31 & 0.31 & 0.32 \\ 0.44 & 0.44 & 0.32 \\ 0.31 & 0.31 & 0.35 \end{bmatrix} \end{array}$$

The following diagnosis table is obtained by using the formula $DS_{M_1} - DS_{M_2}$, shows relations of patients and their corresponding diseases.

patients	disease(d_1)	disease(d_2)	disease(d_3)
p_1	0.13	0.24	0.36
p_2	0.26	0.25	0.36
p_3	0	0.24	0.36
p_4	0	0	0.22

Table 5.10: Disease-Diagnosis Table

From this $disease - diagnosis$ table, we can conclude that the patient $p_1, p_2, p_3, p_4 \& p_5$ all are suffering from disease d_3.

Since soft theory offers a theoretical framework for dealing with ambiguous, fuzzy, and ill-defined objects, it is a key for solving multiple attribute decision-making with intuitionistic fuzzy information. The best medical decision can be challenging to implement because, in daily life, membership and non-membership degrees with the potential for hesitation can be included in decision-making issues. Here, we have looked into how Sanchez's medical theory can be used for diagnosing patients using an intuitionistic fuzzy set through a fuzzy arithmetic-based algorithm.

5.3 Fuzzy Model for the Validation and Authenticity of Avoid Fraud at the Time of Claim Process

5.3.1 Introduction

Fraud during claim settlement has become a growing concern for insurance firms in recent times. Deliberately misleading an insurance company or agent for monetary gain is referred to as insurance fraud. Fraudulent intentions towards an insurance provider are also common types of fraud that may occur in the course of business. Applicants, policyholders, third-party claimants, or professionals who help claimants with their needs may all commit fraud at various stages of the process. Inflating claims, lying on an insurance application, claiming for damage or injuries that never happened, and staging accidents are all examples of common fraud. Therefore, businesses need dependable specialist systems that can assist them in verifying or authenticating the processed statements as a result of this rising development and preventive costs on human knowledge. To address this issue, fuzzy models have been developed to assist internal auditors in identifying claims that may involve fraudulent activities. A description of fuzzy pattern recognition techniques to be used in the cluster analysis of risk and classification of claims was developed by Derrig et al.[27], while Deshmukh et al.[28] suggested a fuzzy reasoning approach based on rules to quantify the threat of management fraud.

A fuzzy-based algorithm has also been developed that enables auditors to identify fraud elements in insurance claims handled by Pathak et al.[75]. Furthermore, Kumar et al.[57] worked on the risk of policy cancellation by insurers applying the fuzzy inference method, where they developed a model that observes the effects of policy cancellation to avoid unwanted risk. Azar et al.[8] have recently worked to create and validate a measure of the perception of justice using the vague principle in the context of justice theory and have shown good results in this work. A mathematical model developed by Kumar and Tiwari [58] uses the idea of the fuzzy expert system to assess the risk of policy cancellation. Abdallah et al.[1] sought to present a methodical and in-depth summary of these problems and difficulties that hinder the effectiveness of fraud prevention systems. Majhi[61] proposed hybrid clustering based on a modified whale optimization algorithm (MWOA) and FCM, which is employed as an under-sampling technique to optimize the cluster centroids for auto insurance fraud detection.

In 2021, Yan et al.[114] applied the SAGFCM-Apriori algorithm for the identification of auto insurance fraud and mined the auto insurance fraud data to obtain fuzzy association rules that could identify fraud claims, with the aim of improving auto insurance fraud management and exploring the application of data mining technology in auto insurance fraud identification. In 2022, Panda, G., et al.'s [73] developed a mechanism that can be used as a predictive model to ascertain whether a policy claim is genuine or not. This chapter of the dissertation demonstrates the use of fuzzy sets to calculate red flags, ambiguous laws to combine several red flags, and a particular measure of administration fraud threat.

5.3.2 Key Feathers:

Fuzzy logic seems to be a very helpful method of coping with uncertainty, approximation, inaccuracy, misunderstanding, inaccuracy, or causes of inaccuracy that are not statistics in the human reasoning and decision-making process. We may calculate the relevance of a collection of information to different criteria of unclear significance by introducing fuzzy logic. If Δ_{ij} is the variable $0 - 1(\Delta_{ij} = 1,$ if there is any discrepancy/difference between the information supplied by the claimant and that obtained by the auditor, otherwise zero) and W_{ij} is the weighting or impact factor for the i^{th} section for the j^{th} information, then the inaccuracy index for the i^{th} section of the information is provided by Ross[87]

$$X_i = \frac{(\sum_{i=1}^{I} \sum_{j=1}^{J} W_{ij}\Delta_{ij})}{I} \tag{5.1}$$

It must be remembered that the details critical in the decision to solve the argument, as well as all weightings for a collection of i^{th} data $\sum_{j=1}^{J} W_{ij}$ to the unit added, are given a greater weighting/ effect factor. Likewise, it is possible to determine the values of the other entries. A triangular association function is used here for the " degree of incompleteness" input variable, while trapezoidal association functions are used for the " inaccuracy index", " rating level" and "claimant's credit report " input variables. The claim must be verified and validated when handling the claim in order to reduce the relation time of settlements.

Membership functions of inputs

(a) Vagueness index (x_1)	(b) Degree of incompleteness (x_2)
(c)Level of judgment used by claim settler (x_3)	(d) Credit report for claimants (x_4)

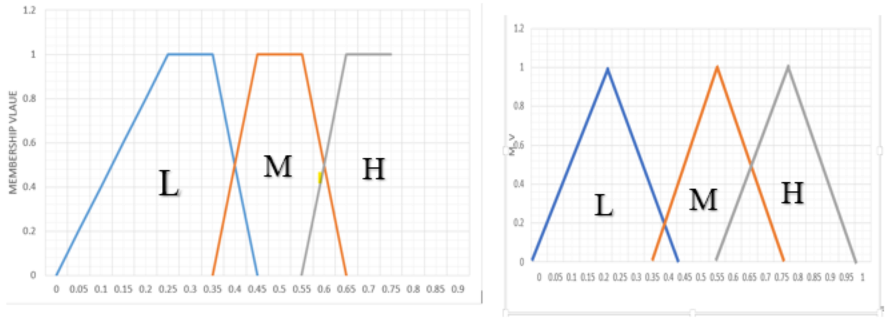
5.3.3 Algorithm

(i) **Input**: Clear value of settlement claim and other collected data.

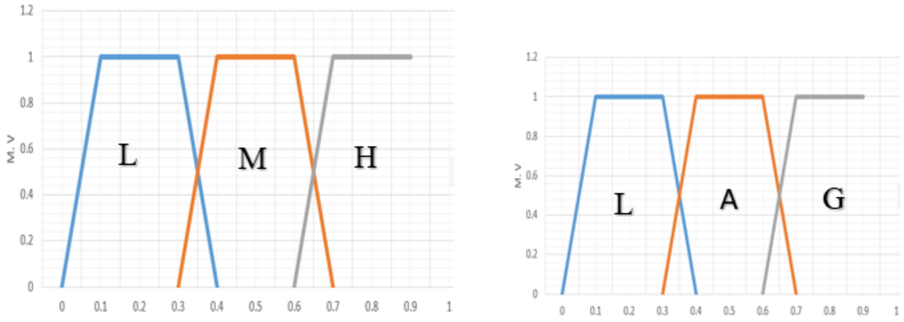
(ii) **Compare with the limit:** Whether the settlement benefit of the lawsuit, as calculated by the auditors, is less than a certain predetermined sum(say alpha = Rs. 2,000,00), so the settled claim can be deemed authentic and proceed to stage 8.

(iii) **Incoming rating:** Rating the inaccuracy index x_1, the level of the incompleteness of the claims x_2, the rating level used by claim settlers x_3 and the claimants' credit report x_4.

(iv) **Crisp input values are fuzzified:** Use the membership functions specified to every fuzzy set for each linguistic variable to decide the degree of participation of each fuzzy set with crisp



(a) **(a)** Vagueness index (x_1) **Degree of incompleteness(x_2)**



(c)Level of judgment used by claim settler (x_3) **(d) Credit report for claimants (x_4)**

Figure 5.2: Input Variables

value. The membership equation for computation is:

$$f(x,a,b,c) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \end{cases} \tag{5.2}$$

$$\mu_L\left(x_1\right) = \begin{cases} \frac{x_1-0}{0.25} & \text{if } 0 \leq x_1 \leq 0.25 \\ 1 & \text{if } 0.25 \leq x_1 \leq 0.35 \\ \frac{0.45-x_1}{0.10} & \text{if } 0.35 \leq x_1 \leq 0.45 \end{cases} \tag{5.3}$$

$$\mu_M\left(x_1\right) = \begin{cases} \frac{x_1-0.35}{0.10} & \text{if } 0.35 \leq x_1 \leq 0.45 \\ 1 & \text{if } 0.45 \leq x_1 \leq 0.55 \\ \frac{0.65-x_1}{0.10} & \text{if } 0.55 \leq x_1 \leq 0.65 \end{cases} \tag{5.4}$$

$$\mu_H\left(x_1\right) = \begin{cases} \frac{x_1-0.55}{0.25} & \text{if } 0.55 \leq x_1 \leq 0.65 \\ 1 & \text{if } x_1 \geq 0.65 \end{cases} \tag{5.5}$$

$$\mu_L\left(x_2\right) = \begin{cases} \frac{x_2}{0.23} & \text{if } x_2 \leq 0.23 \\ \frac{0.45-x_2}{0.22} & \text{if } 0.23 \leq x_2 \end{cases} \tag{5.6}$$

$$\mu_M\left(x_2\right)=\begin{cases}\frac{x_2-0.35}{0.20} & \text{if } x_2 \leq 0.55\\ \frac{0.74-x_2}{0.20} & \text{if } 0.55 \leq x_2\end{cases} \tag{5.7}$$

$$\mu_H\left(x_2\right)=\begin{cases}\frac{x_2-0.55}{0.21} & \text{if } x_2 \leq 0.76\\ \frac{1-x_2}{0.24} & \text{if } 0.76 \leq x_2\end{cases} \tag{5.8}$$

$$\mu_L\left(x_3\right)=\begin{cases}\frac{x_3-0}{0.15} & \text{if } 0 \leq x_3 \leq 0.15\\ 1 & \text{if } 0.15 \leq x_3 \leq 0.30\\ \frac{0.45}{0.15} & \text{if } 0.30 \leq x_3 \leq 0.45\end{cases} \tag{5.9}$$

$$\mu_M\left(x_3\right)=\begin{cases}\frac{x_3-0.30}{0.15} & \text{if } 0.30 \leq x_3 \leq 0.45\\ 1 & \text{if } 0.45 \leq x_3 \leq 0.60\\ \frac{0.75-x_3}{0.15} & \text{if } 0.60 \leq x_3 \leq 0.75\end{cases} \tag{5.10}$$

$$\mu_H\left(x_3\right)=\begin{cases}\frac{x_3-0.60}{0.15} & \text{if } 0.60 \leq x_3 \leq 0.75\\ 1 & \text{if } \quad x_3 \geq 0.75\end{cases} \tag{5.11}$$

$$\mu_L\left(x_4\right)=\begin{cases}\frac{x_4-0}{0.10} & \text{if } 0 \leq x_4 \leq 0.10\\ 1 & \text{if } 0.10 \leq x_4 \leq 0.30\\ \frac{0.4-x_4}{0.10} & \text{if } 0.30 \leq x_4 \leq 0.40\end{cases} \tag{5.12}$$

$$\mu_A\left(x_4\right)=\begin{cases}\frac{x_4-0.30}{0.10} & \text{if } 0.30 \leq x_4 \leq 0.40\\ 1 & \text{if } 0.40 \leq x_4 \leq 0.60\\ \frac{0.70-x_4}{0.10} & \text{if } 0.60 \leq x_4 \leq 0.70\end{cases} \tag{5.13}$$

$$\mu_G\left(x_4\right)=\begin{cases}\frac{x_4-0.60}{0.10} & \text{if } 0.60 \leq x_4 \leq 0.70\\ 1 & \text{if } \quad x_4 \geq 0.70\end{cases} \tag{5.14}$$

Here, (a, b, c, d) = vertices of the functions of the trapezoidal membership,

(a, b, c) = vertices of the functions of the triangular membership, and

L, M, H, A, and G represent the fuzzy collection for low, medium, high, average, and good, respectively.

(v) **Trigger the rule bases corresponding to these inputs:**

”If-then” rules that are based on fuzzy logic are used by all expert systems. The ”if” portion is linked to this as a precedent as well as an assumption, although the ”then” portion is known to it as the end result or conclusion.

(vi) **Perform the inference engine:**

A fuzzy inference method uses aggregation and composition to derive numeric values for intermediate and output variables. Aggregation calculates the principles of the component and assigns a level of reality based on the membership function of a linguistic variable. The component's degree of fact is determined by clipping the truth from all portions.

(vii) **Defuzzification**

In order to defuzzify linguistic descriptions into numerical values, the fuzzy inference technique has recently developed. The most often used version, center-of-maximum (CoM), determines the crisp value for both principles and associated degrees of association by first determining the meaning of each output word and component.

(viii) **The output of Decisions of the Expert System:**

The development forms represent the settlement of lawsuits and charges with defendants or scammers. Controllers' characteristics are determined by models and efficiency computations. Fuzzy logic-dependent expert methods evaluate system relationships by replicating human reasoning, establishing optimal fuzzy management rules and databases.

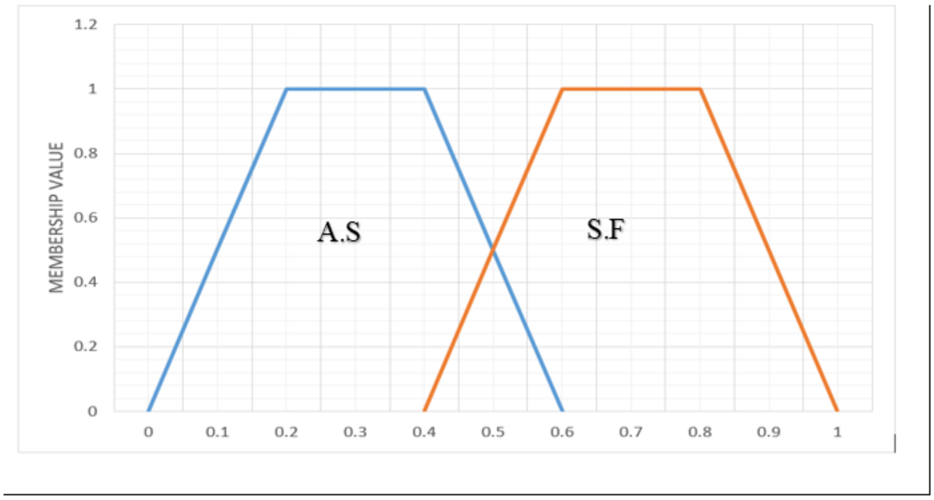


Figure 5.3: (Membership Function of Output)

5.3.4 Case Study

For instance, we assume that four inputs are used by the insurance provider, namely the vagueness indicator x_1 the extent of incomplete claims x_2, the amount of judgment used among claim invaders x_3, and the credit report of a claimant x_4. Such measures were also indicative of the provenance of an insurance payout settlement. The degree of vagueness/doubt with data as an indication of an inaccurate insurance settlement reflects these inputs and the extent of control used by the claimants to assess the settlement.

(i) **Input:** Value of the claim = Rs 80,000.00

(ii) **Compare against a threshold:** Although the value is higher than the present value (Rs.25000.00). The system of experts may review the validity of the settled claim.

(iii) **Test the validity of claim settlement:** The value of the statements' inputs must be measured, $x_1 = 0.38, x_2 = 0.70, x_3 = 0.65, x_4 = 0.62$ (say)

(iv) **Fuzzification of inputs crisp values:** Though using the membership characteristics defined for language variable for each fuzzy package. The degree of membership in the fuzzy set of crisp values is defined as follows:

$$\mu_L\left(x_1\right) = \frac{0.45 - x_1}{0.10} = 0.70 \tag{5.15}$$

$$\mu_M\left(x_1\right) = \frac{x_1 - 0.35}{0.10} = 0.30 \tag{5.16}$$

$$\mu_M\left(x_1\right) = 0 \tag{5.17}$$

$$\mu_L\left(x_2\right) = \max\left\{0, \frac{0.45 - x_2}{0.22}\right\} = 0 \tag{5.18}$$

$$\mu_M\left(x_2\right) = \max\left\{0, \frac{0.75 - x_2}{0.20}\right\} = 0.25 \tag{5.19}$$

$$\mu_H\left(x_2\right) = \max\left\{0, \frac{x_2 - 0.55}{0.21}\right\} = 0.667 \tag{5.20}$$

$$\mu_L\left(x_3\right) = 0 \tag{5.21}$$

$$\mu_M\left(x_3\right) = \frac{0.75 - x_3}{0.15} = 0.67 \tag{5.22}$$

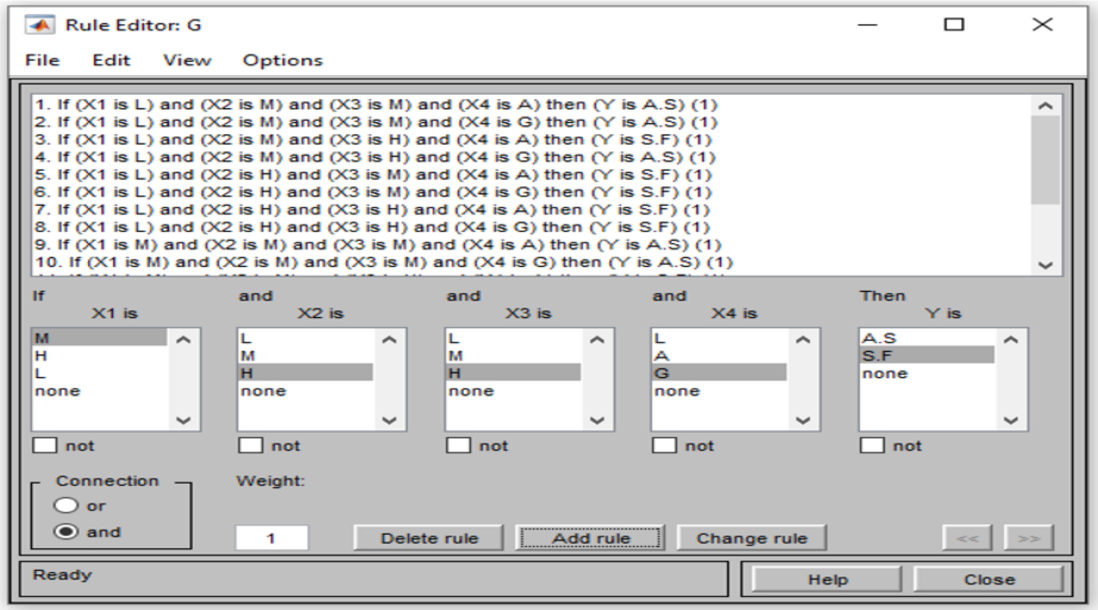
$$\mu_H(x_3) = \frac{x_3 - 0.60}{0.15} = 0.33 \tag{5.23}$$

$$\mu_L(x_4) = 0 \tag{5.24}$$

$$\mu_M\left(x_4\right) = \frac{0.70 - x_4}{0.10} = 0.80 \tag{5.25}$$

$$\mu_H(x_4) = \frac{x_4 - 0.60}{0.10} = 0.20 \tag{5.26}$$

(v) **Trigger the rule bases corresponding to these inputs :** The following rules, based on the importance of the fuzzy membership function values, refer to the example under consideration.



1. If (X1 is L) and (X2 is M) and (X3 is M) and (X4 is A) then (Y is A.S) (1)
2. If (X1 is L) and (X2 is M) and (X3 is M) and (X4 is G) then (Y is A.S) (1)
3. If (X1 is L) and (X2 is M) and (X3 is H) and (X4 is A) then (Y is S.F) (1)
4. If (X1 is L) and (X2 is M) and (X3 is H) and (X4 is G) then (Y is A.S) (1)
5. If (X1 is L) and (X2 is H) and (X3 is M) and (X4 is A) then (Y is S.F) (1)
6. If (X1 is L) and (X2 is H) and (X3 is M) and (X4 is G) then (Y is S.F) (1)
7. If (X1 is L) and (X2 is H) and (X3 is H) and (X4 is A) then (Y is S.F) (1)
8. If (X1 is L) and (X2 is H) and (X3 is H) and (X4 is G) then (Y is S.F) (1)
9. If (X1 is M) and (X2 is M) and (X3 is M) and (X4 is A) then (Y is A.S) (1)
10. If (X1 is M) and (X2 is M) and (X3 is M) and (X4 is G) then (Y is A.S) (1)

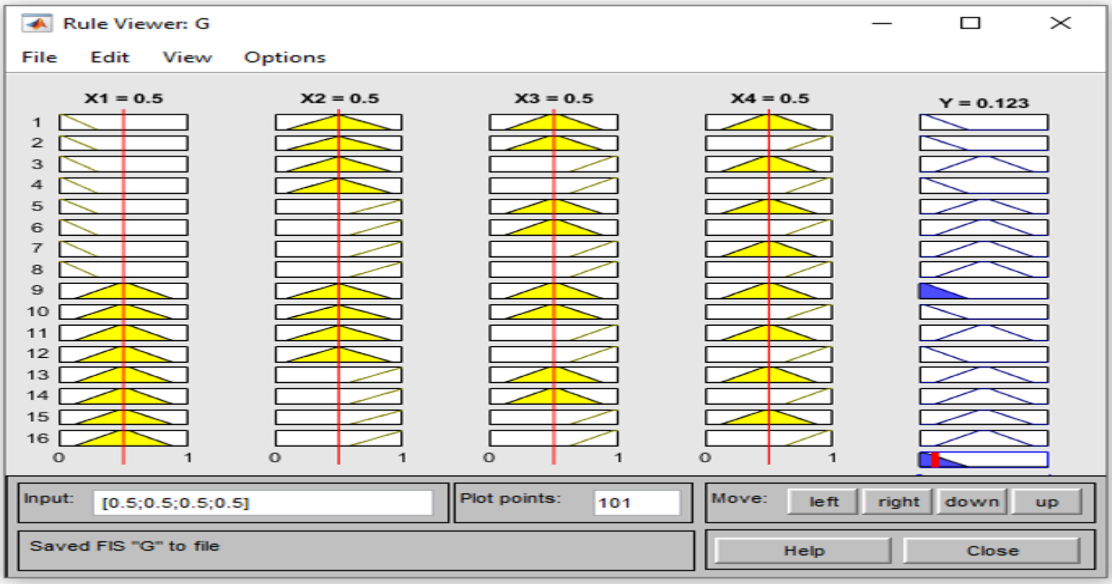
If X1 is	and X2 is	and X3 is	and X4 is	Then Y is
M	L	L	L	A.S
H	M	M	A	S.F
L	H	H	G	none
none	none	none	none	
☐ not	☐ not	☐ not	☐ not	☐ not

Connection
○ or
● and

Weight: 1

Ready

Figure 5.4: Fuzzy Rule-Base



Input: [0.5;0.5;0.5;0.5]

Plot points: 101

Saved FIS "G" to file

Figure 5.5: Fuzzy Rule Base Output

(vi) **Execute the engine of inference:** We see the “root sum squares” (RSS) types to combine the results of all relevant laws and scale the functions to their respective strengths (range [0,1]) of the output membership function from possible rules (R 1- 81) are :

$$
\begin{aligned}
A.S =&\sqrt{(0.25)^2+(0.20)^2+(0.20)^2+(0.25)^2+(0.20)^2+(0.20)^2}\\
=&\sqrt{0.2850}\\
&= 0.53
\end{aligned}
$$

$$
\begin{aligned}
S.F &= \sqrt{(0.25)^2+(0.67)^2+(0.2)^2+(0.33)^2+(0.2)^2+(0.25)^2+(0.3)^2+(0.2)^2+(0.3)^2+(0.2)^2}\\
&= \sqrt{1.0187}\\
&= 1.0(\text{ approx. })
\end{aligned}
$$

(vii) **Defuzzification:** By incorporating the outcomes of an evaluation procedure, we just use the ”weighted average approach” to achieve the defuzzification of such information into the crisp output. By measuring almost every membership function in the outcome by the corresponding maximum membership value, the weighted average technique was created and the result is called the crisp outcome. 0.48 is the crisp performance. The crisp production belongs more to the SF collection than to the true settlement set (as obvious with its basis functions). The judgment in this case, then, is SF.

(viii) **Through the expert system assessment:** The process means that misconduct has been fixed in the case of the proceedings under examination. In the above example, it is in between *A. S* and *S.F.* but towards *S.F.* So we will give only the threshold amount which is Rs. 2500.00

5.3.5 Result and Discussion

The study provides a fuzzy model to assist internal auditors in distinguishing between all resolved claims including an element of fraud. The model is intended to validate the argument process, and it offers indicative results for the credibility and validity of the statement in the process. A case study is also used to support the strategy. The analysis yielded a sharp result of 0.48, which belongs more to the SF collection than to the genuine settlement set. In this situation, the decision is *S. F*. The two outputs of the model are a suspect's authentic factor *A. S.* and ”cases settled with fraud element *S. F.*” The importance of fraud prevention in the insurance sector is highlighted in the paper's discussion, along with how the fuzzy model might help auditors spot false claims. The report also addresses the model's shortcomings and makes suggestions for future studies to enhance its efficacy.

Chapter 6

Summary and Conclusion

6.1 Summary

Fuzzy logic and the fuzzy sets are the study of uncertainty and vagueness, which have been applied in a wide range of mathematical fields. This thesis is based on our three years research work on fuzzy logic. Our research work is divided into two parts. The first part includes the study of pure mathematics using fuzzy real numbers and fuzzy set theory with their fundamental concepts. On the other hand, the second part deals with the application of fuzzy logic in our daily lives. This thesis is mainly divided into five main chapters, including a summary and conclusion.

Chapter one consists of the introduction of fuzzy logic, fuzzy sets, and difference sequence spaces of fuzzy real numbers. Fundamental definitions, basic concepts of fuzzy logic, and fuzzy set theory have been introduced in this chapter. Moreover, some of the introductions to sequence spaces of fuzzy real numbers have been studied in this chapter . In addition, the rationale for the study of this topic is also mentioned in the same chapter.

In chapter two, notions and preliminaries of fuzzy logic and fuzzy set theory are studied. Moreover, operations on fuzzy sets, fuzzy relations, the composition of fuzzy relations, fuzzy matrices, and max-min-max composition on fuzzy relations have been incorporated as well.

In chapter three and four, some well-known sequence spaces of complex numbers are studied. In this chapter, basic topological properties of the classes $l_F(X,\bar{\lambda},\bar{p})$ and $l_F(X,\bar{\lambda},\bar{p},L)$ of fuzzy real numbers are studied with the help of Orlicz function and suitable natural paranorm function. Besides of these, double sequence spaces of fuzzy real numbers are studied and investigated some linearity, completeness and solidity properties of the classes of double sequences $l_\infty^F(M,\lambda,\xi),\ C^F(M,\lambda,\xi)$ and $C_o^F(M,\lambda,\xi)$. Fundamental topological properties of the difference sequence spaces of fuzzy real numbers and their generalized forms also have been discussed in this chapter. This chapter, also covers the study of completeness property of the class bV_p^F of p -bounded variation in generalized form. In addition, some inclusion relations regarding to the class bV_p^F have been studied.

In Chapter 5, some real-life applications of fuzzy sets have been studied. By using the membership functions defined on fuzzy sets, the chapter addressed the challenges seen in making decisions for selecting the best of the best as studied by the Bellmen-Zadeh method. A fuzzy arithmetic-based technique for recognizing medical problems is offered in the same chapter using Sanchez's medical theory. Fraud throughout the claim-settlement process is becoming a major problem for insurance companies. A fuzzy model that helps internal auditors spot claims that might be the result of fraudulent activity has been created to solve this problem. A case study is presented to support the model's effectiveness at the end of this chapter.

6.2 Conclusion

Fuzzy logic is the study of uncertainty and vagueness. It is a flexible, uncertainty-based reasoning method for rational decision-making that addresses vague or incomplete information and solves specific problems. The fuzzy set theory has been successfully applied in a wide range of mathematical fields, and this thesis is divided into two parts: pure and applied mathematics. The first part of the thesis includes the study of the sequence space of fuzzy real numbers. In sequence space of fuzzy real numbers, the fundamental topological properties of the classes $l_F(X, \bar{\lambda}, \bar{p})$ and $l_F(X, \bar{\lambda}, \bar{p}, L)$ with the help of Orlicz function and paranorm have been established through the fuzzy real numbers. Similarly, some algebraic structure of the double sequence spaces of fuzzy real numbers have been developed and studied their linear topological properties. Linearity, completeness, and solidity of the difference sequence spaces of fuzzy real numbers are also studied. Basic topological properties of the generalized form of the difference sequence spaces of fuzzy real numbers are proven with the help of the Orlicz and paranorm functions. On the other hand, in the applied part, fuzzy sets have been used in certain real-world applications. This thesis addressed the problems of making-decisions in selecting the best with the help of the membership function defined on the fuzzy set using the Bellmen-Zadeh method. Moreover, Sanchez's medical theory is used to provide a fuzzy arithmetic-based method for identifying medical issues and a case study is presented to support this work. For insurance firms, fraud in the claim-settlement process is turning into a serious issue. This issue has been resolved by developing a fuzzy model that aids internal auditors in identifying claims that might be the outcome of fraudulent conduct. To demonstrate the usefulness of the methodology, a case study is provided.

6.3 Recommendation for the Future Work

Before the introduction of fuzzy logic, mathematics was limited to only two conclusions: true and false. It is a multivalued logic in which the truth value of variables is any real number between 0 and 1, as determined by the membership function. Using fuzzy logic and fuzzy set theory, some of the sequence spaces are studied and extended into difference sequence spaces and double sequence spaces of fuzzy real numbers. Fuzzy logic is a strategy for dealing with linguistic variables and decision modifiers, so it is more appropriate in applied mathematics. Fuzzy logic, which incorporates fuzzy sets and fuzzy rules, is a valuable tool for modeling and analyzing complex systems. This enables research to address issues with missing or unclear non-linear connections and complicated decision-making processes. On the basis of our research study related to fuzzy logic and fuzzy set theory, we would like to make recommendations for possible future research work as follows:

1. to study the difference sequence spaces of double sequence spaces of fuzzy real numbers with their various topological properties.
2. to study more algebraic and topological properties of sequence spaces and difference sequence spaces of fuzzy real numbers.
3. to study and generalize the sequence spaces and difference sequence spaces of fuzzy real numbers extended it in 2-normed spaces.
4. to apply the fuzzy model to diagnose diseases having similar symptoms.

REFERENCES

[1] Abdallah, A., Maarof, M. A., & Zainal, A.: Fraud detection system: A survey, *Journal of Network and Computer Applications*, 68(2016), 90-113. 1. https://doi.org/10.1016/j.jnca.2016.04.007

[2] Adams, C. R.: Transformations of double sequences with application to Cesàro summability of double series, *Bulletin of the American Mathematical Society*, 37(10)(1931), 741-748.

[3] Adamu, I. M.: Application of intuitionistic fuzzy sets to environmental management, *Notes on Intuitionistic Fuzzy Set*s, 27(3) (2021), 40-50. 2. https://doi.org/10.7546/nifs.2021.27.3.40-50

[4] Aggarwal, H., Arora, H. D., & Kumar, V. A: Decision-making problem as an applications of intuitionistic fuzzy set, *Int. J. Eng. Adv. Technol*, 9(2019), 5259-5261. 3. https://doi.org /10.35940/ijeat.A1053.129219

[5] Alexopoulos, J. : *A brief introduction to N–functions and Orlicz function spaces*, (2004).

[6] Altay, B., & Başar, F.: Some new spaces of double sequences, *Journal of Mathematical Analysis and Applications*, 309(1)(2005), 70-90.4. https:// doi.org /10.1016/j.jmaa.2004.12.020.

[7] Atanassov, K.: Review and new results on intuitionistic fuzzy sets, *Sofia*, 5(1)(1988).

[8] Azar, A., and Darvishi, Z.A. : Development and validation of a measure of justice perception in the frame of fairness theory-fuzzy approach, *The Journal of Expert System with Application: An International Journal*, 38(6) (2011), 7364-7372. https://doi.org/10.1016/j.eswa.2010.12.090

[9] Baruah A. & Tripathy B. C.: N¨orlund and Riesz mean of sequences of fuzzy real numbers, *Applied Mathematics Letters*, 23(2010), 651- 655. https://doi.org/10.1016/j.aml.2010.02.006

[10] Baruah A. & Tripathy B. C: New type of difference sequence spaces of fuzzy real numbers, *Mathematical Modeling and Analysis*, 14(3)(2009) ,391-397. https://doi.org/10.3846/1392-6292.2009.14.391-397

[11] Basu, A.: On new fuzzy metric Orlicz sequence space. *International Journal of Pure and Applied Mathematics*, 118(4) (2018), 931-947.

[12] Bellman, R. E., & Zadeh, L. A.: Decision-making in a fuzzy environment, *Management science*, 17(4) (1970), B-141. https://doi.org/10.1287/mnsc.17.4.B141

[13] Bhattacharya, P., & Mukherjee, N. P.: Fuzzy relations and fuzzy groups, *Information Sciences*, 36(3)(1985), 267-282. https://doi.org/10.1016/0020-0255(85)90057-X

[14] Bojadziev, G., & Bojadziev, M.: Fuzzy logic for business, finance, and management, *World Scientific*. 23(2007).

[15] Brockett, P. L., Xia, X., & Derrig, R. A. : Using Kohonen's self-organizing feature map to uncover automobile bodily injury claims fraud, *Journal of Risk and Insurance*, (1998),245-274. https://doi.org/10.2307/253535

[16] Burrough, P. A., & Frank, A. : *Geographic objects with indeterminate boundaries*, CRC Press (2020). https://www.taylorfrancis.com/chapters/edit/10.1201/9781003062660-3/prevalence-objects-sharp-boundaries-gis-andrew-frank?context=ubx

[17] Cantor, G. : Beiträge zur begründung der transfiniten mengenlehre, *Mathematische Annalen*, 46(4) (1895), 481-512. https://doi.org/10.1007/BF02124929.

[18] Chen, J. and Ye, J.: Some single-valued neutrosophic dombi weighted aggregation operators for multiple attribute decision-making, *Symmetry*, 9(6) 2017, 82. https://doi.org/10.3390/sym9060082.

[19] Cheng, C. B.: Group opinion aggregation based on a grading process: A method for constructing triangular fuzzy numbers, *Computers and Mathematics with Applications*, 48(10-11)(2004), 1619-1632. https://doi.org/10.1016/j.camwa.2004.03.008.

[20] Choudhary, B., and Tripathy B. C.: "On fuzzy real valued l_p^F sequences," In Proc. International Conf. 8th Joint Con. Inf. Sci.(10th International Conf. on Fuzzy Theory and Technology), Saltlake City, Utha, USA, July 21-25,(2005),184-190.

[21] Dabbas, A. F., & Battor, A. H.: Convergent double sequence of fuzzy real numbers defined by double Orilz function, *International Journal of Academic and Applied Research*, 4(6) (2020),22-31. https://doi.org/10.29350/ jops.2020.25. 2.1068

[22] Das, M.: Difference operator on double sequences with an Orlicz function, *International Journal of Science, Environment and Technology*, 1(1)(2012), 1-6.

[23] Das, N. R., & Choudhury, A. : Boundedness of fuzzy real-valued sequences, *Bull. Cal. Math. Soc*, 90 (1998), 35-44.

[24] Das, P. C.: Fuzzy normed linear sequence space bV_p^F , *Proyecciones* (*Antofagasta, On line*), 37(2)(2018), 389-403.

[25] Das, Resul, Ibrahim T., and Abdulkadir S.: Effective diagnosis of heart disease through neural networks ensembles, *Expert systems with applications* 36(4) (2009), 7675-7680. https://doi.org/10.1016/j.eswa.2008.09.013.

[26] Deli, I., & Çağman, N.: Intuitionistic fuzzy parameterized soft set theory and its decision making, *Applied Soft Computing*, 28(2015), 109-113. https://doi.org/10.1016/j.asoc.2014.11.053

[27] Derrig, R. and Ostaszewski, K.: Fuzzy techniques of pattern recognition in risk and claim classification, *The Journal of Risk and Insurance*, 62 (3) (1995), 447-482. https://doi.org/10.2307/253819.

[28] Deshmukh, A., & Talluru, L. : A rule-based fuzzy reasoning system for assessing the risk of management fraud, *Intelligent Systems in Accounting, Finance & Management*, 7(4) (1998), 223-241. https://doi.org/10.1002/(SICI)1099-1174(199812)7:4¡3C223::AID-ISAF158¿3E3.0.CO;2-I

[29] Ebadullah, K. : Sequence spaces of fuzzy numbers defined by a sequence of moduli, *J.Math. Comput. Sci.*, 8(4): 494-505. https://doi.org/10.28919/jmcs/3149.

[30] Ejegwa, P. A., & Onasanya, B. O.: Improved intuitionistic fuzzy composite relation and its application to medical diagnostic process, *Note IFS*, 25(1) (2019) , 43-58. https:// doi.org/10.7546/nifs.2019.25.1.43-58.

[31] Ejegwa, P. A., & Onyeke, I. C.: Medical diagnostic analysis on some selected patients based on modified Thao et al.'s correlation coefficient of intuitionistic fuzzy sets via an algorithmic approach, *Journal of fuzzy extension and applications*, 1(2) (2020), 130-141. https://doi.org/10.22105/jfea.2020.250108.1014.

[32] Et, M., & C¸olak, R. : On some generalized difference sequence spaces, *Soochow Journal of Mathematics*, 21(4) (1995), 377-386.

[33] Et, M., Savas, E. & Altnok, H.: On some difference sequence spaces of fuzzy numbers, *Soft Computer*, 20 (2016), 4395-4401. https://link.springer.com/article/10.1007/s00500-015-1998-5.

[34] Faried, N., & Bakery, A. : The difference sequence space defined on Orlicz-Cesaro function, *Text Journal of American Science*, 6(10) (2010).

[35] Faried, N., & Bakery, A.: The difference sequence space defined on Musielak-Orlicz function, *New York Science Journal*, 3(8) (2010).

[36] Feng, F., Xu, Z., Fujita, H., & Liang, M. : Enhancing PROMETHEE method with intuitionistic fuzzy soft sets, *International Journal of Intelligent Systems*, 35(7) (2020), 1071-1104. https://doi.org/10.1002/int.22235.

[37] Ganie, A. H., & Gupkari, S. A.: Generalized difference sequence space of fuzzy numbers, *Annals of Fuzzy Mathematics and Informatics*, 21 (2020), 161-168. https: // doi.org/ 10.37394/23206.2023.22.68.

[38] Habil, E. D. : Double sequences and double series, *IUG Journal of Natural Studies*, 14(1) (2016).

[39] Hardy, G. H. : On the convergence of certain multiple series proc, *London Math. Soc*, 1 (1904), 124-128.

[40] Hardy, G.H., Littlewood, J.E., & Pólya, G. : *Inequalities. Cambridge University Press*, (1934).

[41] Hndoosh, R., Saroa, M., & Kumar, S. : Fuzzy and adaptive neuro-fuzzy inference system of washing machine, *European Journal of Scientific Research*, 86(3) (2012), 443-459.

[42] Hooda, D. S., Kumari, R., & Sharma, D. K.: Intuitionistic fuzzy soft set theory and its application in medical diagnosis, *International Journal of Statistics in Medical Research*, 7(3) (2018), 70-76.

[43] Hu, J., Pan, L., Yang, Y., & Chen, H. : A group medical diagnosis model based on intuitionistic fuzzy soft sets, *Applied Soft Computing*, 77 (2019), 453-466. https://doi.org/10.1016/j.asoc.2019.01.041.

[44] Hussain, A., Chun, J. & Khan, M.: A novel multi-criteria decision making (MCDM) approach for precise decision making under a fuzzy environment, *Soft Computing*, 25(7) 2021, 5645-5661. https://doi.org/10.1007/s00500-020-05561-9.

[45] Jalal, T. : A note on multiordered fuzzy difference sequence spaces. *Filomat*, 32(8) (2018), 2867-2874. https://doi.org/10.2298/FIL1808867J.

[46] John, T. M. & Kuriakose, S.: Decision making under uncertainty in a fuzzy environment, *International Journal of Fuzzy Mathematics and Systems*, 1(3) 2011, 273-279.

[47] Jordan, C. : *Sur la série de fourier*, *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 105(1887). 369-371.

[48] Kandasamy, W. V., Smarandache, F., & Ilanthenral, K.: Elementary fuzzy matrix theory and fuzzy models for social scientists, *Infinite Study*, 2007.

[49] Kılınç, G., & Solak, İ. : Some double-sequence spaces are defined by a modulus function, *Gen. Math. Notes*, 25(2) (2014), 19-30.

[50] Kishore, A. H., & Jayanthi V. E.: Neuro-fuzzy based medical decision support system for coronary artery disease diagnosis and risk level prediction, *Journal of Computational and Theoretical Nanoscience* 15 (3) (2018), 1027-1037. https://doi.org/10.1166/jctn.2018.7198.

[51] Kizmaz, H. : On certain sequence spaces, *Canadian Mathematical Bulletin*, 24(2) (1981), 169– 176. https://doi.org/10.4153/CMB-1981-027-5.

[52] Klir, G.J. & Yuan, B.: Fuzzy sets and fuzzy logic theory and application, *Prentice Hall of India* (1997).

[53] Kora, P., Meenakshi, K., Swaraja, K., Rajani, A., & Islam, M. K. : Detection of cardiac arrhythmia using fuzzy logic, *Informatics in Medicine Unlocked*, 17 (2019), 100257. https://doi.org/10.1016/j.imu.2019.100257.

[54] Kripa, K. & Govindarajan, R.: Fuzzy sequencing problem using generalized triangular fuzzy numbers, *International Journal of Engineering Research and Application*, 5 (2016), 61-64.

[55] Kumar, A. S. :Diagnosis of heart disease using fuzzy resolution mechanism, *J. Artif. Intell.*, 5(1) (2012), 47–55. https://doi.org/10.3923/jai.2012.47.55.

[56] Kumar, S. & Pathak, P.: Premium allocation-fuzzy approach in insurance business, *Proceeding of the 3rd National Conference, INDIA Com.*, (2009), 703-706.

[57] Kumar, S., & Jain, H. : A fuzzy logic based model for life insurance underwriting when insurer is diabetic, *European Journal of Applied Sciences*, 4(5) (2012), 196-202. https://doi.org/10.5829/idosi.ejas.2012.4.5.2027.

[58] Kumar, S., & Tiwari, N. : Mathematical model for the risk of cancellation of life insurance policies, *SRM International Journals of Engineering and Sciences*, 19 (2015).

[59] Lindenstrauss, J., & Tzafriri, L. : On Orlicz sequence spaces, *II. Israel Journal of Mathematics*, 11 (1972), 355-379. http://dx.doi.org/10.1007/2F978-3-540-37732-0-4.

[60] Lu, J., Zhang, S., Wu, J. & Wei, Y.: COPRAS method for multiple attribute group decision-making under picture fuzzy environment and their application to green supplier selection, *Technological and Economic Development of Economy*, 27(2) (2021), 369-385. https://doi.org/10.3846/tede.2021.14211.

[61] Majhi, S. K. : Fuzzy clustering algorithm based on modified whale optimization algorithm for automobile insurance fraud detection, *Evolutionary intelligence*, 14(1), 35-46. https://doi.org/10.1007/s12065-019-00260-3.

[62] Maji, P. K., Biswas, R.,& Roy, A. R. : Intuitionistic fuzzy soft sets, *Journal of Fuzzy Mathematics*, 9(3) (2001), 677-692. https://doi.org/10.4236/ajcm.2023.131008.

[63] Maji, P. K., Biswas, R., & Roy, A. R.: Soft set theory, *Computers & Mathematics with Applications*, 45(4-5), 555-562. https://doi.org/10.1016/S0898-1221(03)00016-6.

[64] Maji, P. K., Roy, A. R., & Biswas, R. : An application of soft sets in a decision-making problem, *Computers & Mathematics with Applications*, 44(8-9)(2002), 1077-1083. https://doi.org/10.1016/S0898-1221(02)00216-X.

[65] Mansoor, M. M., & Battor, A. H. : Some new double sequence spaces of fuzzy numbers defined by double Orlicz functions using a fuzzy metric, *International Journal of Physics: Conference Series* 1818 (1) (2021), 012092. IOP Publishing. 10.1088/1742-6596/1818/1/012092.

[66] Matloka, M. : Sequence of fuzzy numbers, *Busefal*, 28(1986), 28-37. https://doi.org/10.4236/jamp.2017.511181.

[67] Méray, C. : Sur la représentation des fonctions par des séries trigonométriques, *Bulletin de la Société Mathématique de France*, 18 (1890), 31-96.

[68] Molodtsov, D. : Soft set theory—first results, *Computers & Mathematics with Applications*, 37(4-5) (1999), 19-31. https://doi.org/10.1016/S0898-1221(99)00056-5.

[69] Nanda, S. : On sequences of fuzzy numbers, *Fuzzy Sets and System*, 33 (1989), 123-126. https://doi.org/10.1016/0165-0114(89)90222-4.

[70] Nian, K., Zhang, H., Tayal, A., Coleman, T., & Li, Y. : Auto insurance fraud detection using unsupervised spectral ranking for anomaly, *The Journal of Finance and Data Science*, 2(1) (2016), 58-75. https://doi.org/10.1016/j.jfds.2016.03.001.

[71] Pahari, N. P. : On normed space valued total paranormed Orlicz space of null sequences and its topological structures, *International Journal of Mathematics Trends and Technology*, 6 (2014), 105-112.

[72] Pahari, N. & Srivastava, J. K.: On Banach space valued sequence space $\ell_M(X, \bar{\lambda}, \bar{p}, L)$ defined by Orlicz function, *South East Asian J. Math. and Math. Sc*, 10(1)(2011), 39-49.

[73] Panda, G., Dhal, S. K., Satpathy, R., & Pani, S. K. : ANFIS for fraud automobile insurance detection system, *In Advances in Data Science and Management*,(2022) 519-530. https://doi.org/10.1007/978-981-16-5685-9-50.

[74] Parashar, S. D., and Choudhary, B. : Sequence spaces defined by Orlicz functions, *Indian Journal of Pure and Applied Mathematics*, 25(4)(1994), 419-419. https://doi.org/10.12691/ajams-1-4-1.

[75] Pathak J., Vidhyarthi N. and Summers S.L.: A fuzzy-based algorithm for auditors to detect elements of fraud in settled insurance claims, *Managerial Auditing Journal*, 20(6) (2005), 632 -644. https://doi.org/10.1108/02686900510606119.

[76] Paudel, G. P. and Pahari, N. P.: 2021, On fundamental properties in fuzzy metric space, *Academic Journal of Mathematics Education*, 4(1), 20-25.

[77] Paudel, G. P., Pahari, N. P. and Kumar, S. : Generalized form of p-bounded variation of sequences of fuzzy real numbers, *Pure and Applied Mathematics Journal*, 11(3) (2022): 47-50. https://doi.org/ 10.11648/j.pamj.20221103.12

[78] Paudel, G. P., Pahari, N. P., & Kumar, S. : Application of fuzzy logic through the Bellmen-Zadeh maximin method, *Journal of Nepal Mathematical Society*, 5(1) (2022), 41-47. https://doi.org/10.3126/jnms.v5i1.47376.

[79] Paudel, G. P., Sahani, S. K., & Pahari, N. P. : Generalized form of difference sequence space of fuzzy real numbers defined by Orlicz function, *Nepal Journal of Mathematical Sciences*, 3(2)(2022). https://doi.org/10.3126/njmathsci.v3i2.49200

[80] Paudel, G. P., Pahari, N. P., & Kumar, S. : Double sequence space of fuzzy real numbers defined by Orlicz function, *The Nepali Mathematical Sciences Report*, 39(2)(2022), 85-94. https://doi.org/10.3126/nmsr.v39i2.51698.

[81] Paudel, G. P., Pahari, N. P., & Kumar, S.: Implementation of fuzzy based clustering system, *NeuroQuantology*, 20(9)(2022), 3221-3224. https://doi.org/10.14704/nq.2022.20.9.NQ44370.

[82] Paudel, G. P., Pahari, N. P., & Kumar, S. (2022): On sequence space of fuzzy real numbers defined by Orlicz function and paranorm, *Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)*, 42(2-2023)(2023), 494-509. https://doi.org/10.17605/OSF.IO/G9TFA

[83] Paudel, G. P., Pahari, N. P., & Kumar, S.: Topological properties of difference sequence space through Orlicz -paranorm function, *Advances and Applications in Mathematical Sciences*, 22(8)2023, 1689-1703.

[84] Paudel, G. P., Upadhyay, P. K., & Baral, A. : Fuzzy arithmetic–based algorithm for identifying medical conditions for better treatment, *Mikailalsys Journal of Mathematics and Statistics*, 1(1), 35-46, 2023. https://doi.org/10.58578/mjms.v1i1.2074

[85] Reddy, G. T., Reddy, M. P. K., Lakshmanna, K., Rajput, D. S., Kaluri, R., & Srivastava, G. : Hybrid genetic algorithm and a fuzzy logic classifier for heart disease diagnosis, *Evolutionary Intelligence*, 13 (2020), 185-196. https://doi.org/10.1007/s12065-019-00327-1.

[86] Rifat, C., Hsfisi, A. & Et, M. : Generalized difference sequences of fuzzy numbers, *Chaos, Solution and Fractions*, 40 (2009): 1109-1117. https://doi.org/10.1016/j.chaos.2007.08.065.

[87] Ross, T. J. : *Fuzzy logic with engineering applications*,John Wiley & Sons, (2009).

[88] SaLJr, B., Duyar, C., & OLur, O. : A new double sequence space $m_2(F, \phi, p)$ defined by a double sequence of modulus functions, *International Journal of Analysis*,2014.

[89] Sanchez, E. : Resolution of composite fuzzy relation equations, *Information and control*, 30(1) (1976), 38-48.

[90] Sarma, B. : Some sequence spaces of fuzzy numbers defined by Orlicz function, *Acta Scientiarum Technology*, 37(1) (2015), 85-87. https://doi.org/10.4025/actascitechnol.v37i1.16966

[91] Sarma, B. : Double sequence spaces of fuzzy real numbers of paranormed type under an Orlicz function, *Mathematica Sciences a Springer Journal*,8(7) (2018), 46- 50

[92] Savas, E., and Sava¸s, R. : Some sequence spaces defined by Orlicz functions, *Archivum Mathematicum*, 40(1)(2004), 33-40.

[93] Savas, E. : New double sequence spaces of fuzzy numbers, *Quaestiones Mathematicae*, 33(4) (2010), 449-456. https://doi.org/10.2989/16073606.2010.541619.

[94] Savaş, E. : A note on double sequences of fuzzy numbers, *Turkish Journal of Mathematics*, 20(2) (1996), 175-178.

[95] Savas, E. : On some double lacunary sequence spaces of fuzzy numbers, *Mathematical and Computational Applications*, 15(3) (2010), 439-448. https://doi.org/10.3390/mca15030439.

[96] Savas, E., & Patterson, R. F. : Double sequence spaces defined by Orlicz functions, *Iranian Journal of Science and Technology (Sciences)*, 31(2) (2007), 183-188.

[97] Sharma, M., & Hazarika, D. : Fuzzy bounded linear operator in fuzzy normed linear spaces and its fuzzy compactness, *New Mathematics and Natural Computation*, 16(1) (2020), 177-193. https://doi.org/10.1142/S1793005720500118.

[98] Stojic, G., : Using fuzzy logic for evaluating the level of countries economic development, *PANOECONOMICUS*, 59(3) (2011), 293-310. https://doi.org/10.2298/PAN1203293S.

[99] Subramanian, N., Krishnamoorthy, S., & Balasubramanian, S. : The generalized difference gai sequences of fuzzy numbers defined by Orlicz functions, *Department of Mathematics Northwest University*, 8(4) (2012), 51-60.

[100] Talo Ö., & Bassar, F.: On the space $bV_p^{(F)}$ of sequences of p-bounded variation of fuzzy numbers, *Acta. Math. Sin.-English Ser.*, 24 (2008), 1205-1212. https://doi.org/10.1007/s10114-007-6552-7.

[101] Talo, Ö. : Some properties of limit inferior and limit superior for sequences of fuzzy real numbers, *Information Sciences*, 279(2014), 560-568. https://doi.org/10.1016/j.ins.2014.04.011.

[102] Talo, Ö. : The core of a double sequence of fuzzy numbers, *Fuzzy Sets and Systems*, 424 (2021), 132-154. https://doi.org/10.1016/j.fss.2021.02.007.

[103] Tamir, D. E., Rishe, N. D., & Kandel, A. (Eds.): *Fifty years of fuzzy logic and its applications*, *Springer*, 326(2015).

[104] Tripathy, B. C., & Borgohain, S.: Some classes of difference sequence spaces of fuzzy real numbers defined by Orlicz function, (2011), 8-8. https://doi.org/10.1155/2011/216414.

[105] Tripathy, B. C. & Borgohain, S. : Sequence space $(M, \phi)^F$ of fuzzy real numbers defined by Orlicz functions with fuzzy metric, *Kyungpook Mathematical Journal*, 53(3) (2013), 319-332. http://dx.doi.org/10.5666/KMJ.2013.53.3.319.

[106] Tripathy, B. C., & Borgohain, S. : Sequence spaces of fuzzy real numbers using fuzzy metric, *Kyungpook Mathematical Journal*, 54(1) (2014), 11-22. http://dx.doi.org/10.5666/KMJ.2014.54.1.11.

[107] Tripathy, B. C., & Das, P. C. : Some classes of difference sequences of fuzzy real numbers, *Fasciculi Mathematici*, 40(2008), 105-117.

[108] Tripathy, B.C & Das P.C : On the sequence space of fuzzy number sequence bV_p^F , *Songklanakarin J. Sci. Technol*, 41(4) (2019), 11 – 14.

[109] Tripathy, B. C., & Sarma, B. : Some double sequence spaces of fuzzy numbers defined by Orlicz functions, *Acta Mathematica Scientia*, 31(1) (2011), 134-140. https://doi.org/10.1016/S0252-9602(11)60215-4.

[110] Tripathy, B., & Sarma, B. : Vector-valued double sequence spaces defined by Orlicz function, *Mathematica Slovaca*, 59(6) (2009), 767-776. https://doi.org/10.2478/s12175-009-0162-z.

[111] Tripathy, B.C. & Sharma B. : Sequences of fuzzy numbers defined by Orlicz function, *Mathematic Slovaca*, 58(2008), 621-628. https://doi.org/10.2478/s12175-008-0097-9.

[112] Türkşen, I. B.: Type I and type II fuzzy system modeling, *Fuzzy Sets and Systems*, 106(1)(1999), 11-34. https://doi.org/10.1016/S0165-0114(98)00354-6.

[113] Tversky, A., & Kahneman, D. : Judgment under uncertainty: Heuristics and biases, *science*, 185(4157), (1974),1124-1131. https://doi.org/10.1126/science.185.4157.1124.

[114] Yan, C., Liu, J., Liu, W., & Liu, X.: Research on automobile insurance fraud identification based on fuzzy association rules, *Journal of Intelligent & Fuzzy Systems*,(2021) 1-14. https://doi.org/10.3233/JIFS-201301

[115] Yang, X., Yu, D., Yang, J., & Wu, C. : Generalization of soft set theory: from crisp to fuzzy case, *In Fuzzy Information and Engineering*, (2007), 345-354. https://doi.org/10.1007/978-3-540-71441-5-39.

[116] Yuan, B. & George, J. K. : Fuzzy sets and fuzzy logic: theory and applications, *PHI New Delhi*, (1995), 443-455.

[117] Zadeh, L. A. : Fuzzy sets, *Information and Control*, 8(3) (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[118] Zadeh, L. A. : Is there a need for fuzzy logic?, *Information Sciences*, 178(13) (2008), 2751-2779. https://doi.org/10.1016/j.ins.2008.02.012.

[119] Zadeh, L. A.: Fuzzy logic computing with words, *IEEE Transactions on Fuzzy Systems*, 4(2) (1996), 103-111. https://doi.org/10.1109/91.493904.

APPENDIX

PUBLICATIONS

1. Paudel, G. P., & Pahari, N. P. : On fundamental properties in fuzzy metric space, *Academic Journal of Mathematics Education*, 4(1)(2021), 20-25.

2. Paudel, G. P., Pahari, N. P., & Kumar, S. : Generalized form of p-bounded variation of sequences of fuzzy real numbers, *Pure and Applied Mathematics journal*, 11(3)(2022), 47-50.

3. Sahani, S. K., Paudel, G. P., Ghimire, J. L., & Thakur, A. K.: On certain series to series transformation and analytic continuations by matrix methods, *Nepal Journal of Mathematical Sciences*, 3(1)(2022), 75-80.

4. Paudel, G. P., Pahari, N. P., & Kumar, S. : Application of fuzzy logic through Bellmen-Zadeh maximin method, *Journal of Nepal Mathematical Society*, 5(1)(2022), 41-47.

5. Paudel, G. P., Sahani, S. K., & Pahari, N. P. : Generalized Form of difference sequence space of fuzzy real numbers defined by Orlicz function, *Nepal Journal of Mathematical Sciences*, 3(2)(2022).

6. Sahani, S. K., Paudel, G. P., & Thakur, A. K. : On a New application of positive and decreasing sequences to double fourier series associated with $(N, p^(1)_m, p^(2)_n)$. *Journal of Nepal Mathematical Society*, 5(2)(2022), 58-64.

7. Paudel, G. P., Pahari, N. P., & Kumar, S. : Double sequence space of fuzzy real numbers defined by Orlicz function, *The Nepali Mathematical Sciences Report*, 39(2)(2022), 85-94.

8. Paudel, G. P., Pahari, N. P., & Kumar, S.: Implementation of fuzzy based clustering system, *Neuro Quantology*, 20(9)(2022), 3221-3224.

9. Paudel, G. P., Pahari, N. P., & Kumar, S. : On Sequence Space of Fuzzy Real Numbers Defined by Orlicz Function and Paranorm, *Jilin Daxue Xuebao (Gongxueban)/Journal of Jilin University (Engineering and Technology Edition)*, 42(2-2023)(2023), 494-509.

10. Paudel, G. P., Pahari, N. P., & Kumar, S.: Topological properties of difference sequence space through Orlicz-paranorm function, *Advances and Applications in Mathematical Sciences*, 22(8)2023, 1689-1703.

11. Paudel, G. P., Upadhyay, P. K., & Baral, A. : Fuzzy arithmetic–based algorithm for identifying medical conditions for better treatment, *Mikailalsys Journal of Mathematics and Statistics*, 1(1), 35-46, 2023.

CONFERENCES/ WORKSHOPS/ SEMINARS

1. Paper presented at International Conference on Analysis and Its Applications- 2021 April 09-11, 2021.

2. Paper presented at ”8th Graduate Conference” held at Mid-West University on April 4-5, 2022.

3. Paper presented at National Conference of Mathematics and its Application (NCMA-2022), ILAM, June 11-13, 2022.

4. Paper presented at 5th International Conference on Mathematics, Statistics and Computing 2023 held during January 07- 08, 2023, organized by the International Society for Research (ISR) Chapter Malaysia.

5. Paper presented at ”Mathematics and Computational Analysis” organized T.D. P. G Collage, Jaunpur (U. P) India and Ramanujan Society of Mathematics and Mathematical Sciences on January 28-30, 2023.

6. Paper presented at ”International Conference on Mathematical Sciences and Applications” at Bhimrao Ambedkar University, Khandari Campus Agra, March 24-26, 2023.

7. Paper presented at ”Third International Conference on Application of Mathematics to Nonlinear Sciences (AMNS-2023)” held in Pokhara, May 25-28, 2023.

CERTIFICATES OF THE CONFERENCES

NATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS

(NCMA-2022), ILAM, Jestha 28-30, 2079 (June 11-13, 2022).

Certificate of Participation / Presentation

This is to certify that

Prof./Dr./Mr./Ms. Gyan Prasad Paudel

Mid-West University, Surkhet, Nepal

has participated / presented the paper entitled

Application of Fuzzy Through Bellmen-Zadeh Maximum Method

in NCMA-2022 at ILAM held on June 11-13, 2022.

Narayan Pahari

Prof. Dr. Narayan Prasad Pahari
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~~talk~~ on ... Application of Fuzzy Logic in Making decision (Selecting Best Candidates)................... *in*

National Conference held in the Department of Mathematics, T.D.P.G. College, Jaunpur.

(Prof. Alok Kumar Singh)
Principal
T.D.P.G. College
Jaunpur

(Prof. Satya Prakash Singh)
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has ~~participated~~/presented a paper (Oral/~~Poster~~) entitled Fuzzy model for the Validation and Authenticity of Avoid fraud at The Time of The claim Process during the Conference at **Department of Mathematics, Dr. Bhimrao Ambedkar University,** Khandari Campus, Agra.

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Prof. Nanda Bahadur Singh, PhD
Patron
Vice-Chancellor
Mid-West University

Sudeep Thakuri, PhD
Convener
Dean, Faculty of Science and Technology
Mid-West University

Third International Conference on
Applications of Mathematics to Nonlinear Sciences
(AMNS-2023)
May 25-28, 2023, Pokhara, Nepal
http://anmaweb.org/AMNS-2023

This certificate is awarded to

Gyan Prasad Paudel

for presenting a talk on

Fuzzy Arithmetic–Based Algorithm for Identifying Medical Conditions

in a contributed session of the

Third International Conference on Applications of Mathematics to Nonlinear Sciences (AMNS-2023)

held in Pokhara, Nepal from May 25-28, 2023.

Dr. Gokarna Aryal

Dr. Shree Ram Khadka

Date: May 28, 2023

Conveners

Dr. Gokarna Raj Aryal
Department of Mathematics and Statistics
Purdue University Northwest, USA
aryalg@pnw.edu, http://www.anmaweb.org

Dr. Shree Ram Khadka
Central Department of Mathematics
Tribhuvan University, Nepal
shree.khadka@cdmath.tu.edu.np, https://www.cdmathtu.edu.np

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Certificate

This is to certify that **Gyan Prasad Paudel** of **Graduate of Science and Technology, Mid-West University, Surkhet, Nepal** has presented a paper entitled **"On Sequence Space of Fuzzy Real Numbers Defined by Orlicz Function and Paranorm"** in ICMSC -2023 held during January 07-08, 2023, organized by International Society for Research (ISR)-Chapter Malaysia.

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International Conference on Analysis and its Applications - 2021
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We are pleased to award this certificate to

Mr. Gyan Prasad Paudel

from Mid-Western University, Central Campus, Surkhet, Nepal for participating and presenting as a **Contributory** speaker on the topic entitled "**p- Bounded Variation Difference Sequence Space of Fuzzy Real Numbers**" on ICAA_NEPAL_2021 organized by Nepal Mathematical Society (NMS) in collaboration with the Central Department of Mathematics, Tribhuvan University (TU); Department of Mathematics, School of Science, Kathmandu University (KU); South Asian University (SAU), New Delhi, India; Association of Nepalese Mathematicians in America (ANMA) USA; Department of Mathematics Valmeeki Campus, Nepal Sanskrit University (NSU) from April 9 – 11, 2021.

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On Generalized Form of Difference Sequence Space of Fuzzy Real Numbers Defined by Orlicz Function

Gyan Prasad Paudel[1], Suresh Kumar Sahani[2] and Narayan Prasad Pahari[3]
[1]*Central Campus of Science and Technology, Mid-Western University, Surkhet, Nepal*
[2]*Department of Mathematics, MIT Campus, Janakpurdham, Nepal*
[3]*Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal.*
Email: [1]*gyan.math725114@gmail.com*, [2]*sureshkumarsahani35@gmail.com* & [3]*nppahari @gmail.com*

Abstract: *Lotfi A. Zadeh developed the concept of fuzzy logic in 1965, and it has since been applied to the study of various class spaces. This research aims to investigate the linearity and some topological properties of the generalized form of difference sequence spaces* $F_\infty(\bar{p}, \mathcal{M}, p, A)$, $F(\bar{p}, \mathcal{M}, p, A)$ *and* $F_0(\bar{p}, \mathcal{M}, p, A)$ *of fuzzy real numbers defined by the Orlicz function in its generalized form by defining new metrics.*

Keywords: *Fuzzy real numbers, Orlicz function, Difference sequence space.*

AMS Subject Classification 03B52, 46A45.

1. Introduction

A large number of research projects have been carried out in mathematical structures built with real or complex numbers so far. In recent years, many researchers have investigated many results of replacing these mathematical structures with fuzzy numbers and interval numbers. Prior to the introduction of fuzzy logic and fuzzy sets, mathematics could only reach two conclusions: true or false (denoted by 1 and 0). In 1965, Zadeh [26] introduced the concept of fuzzy logic and fuzzy sets based on the notion of the relative degree of membership, which is inspired by the process of human perception and knowledge. Then, slowly and gradually, the use of fuzzy logic and fuzzy sets is increased. Many researchers are motivated towards further investigation and application of it. The fuzzy set and fuzzy real numbers have been studied by a wide number of academics in many classes of sequence space with their different properties. Matloka [13] analyzed bounded and convergent sequences of fuzzy numbers in 1986 and proved that every convergent sequence of fuzzy numbers is bounded. In 2004, Savas and Savas [21] proposed a new idea of λ-strong convergence with regard to an Orlicz function and investigated some of its features. Rifat, and colleagues [18], in 2009 proposed the difference operator Δ^m and an Orlicz function to analyze several sequence spaces of fuzzy numbers and studied some of their properties such as completeness, solidity, symmetry, and so on and also provided some relationships connected to these spaces. In 2010, Faried and Barkey [7] proposed the Orlicz-Cesaro difference sequence space with distinct paranormed sequences. Also, in 2010, Faried and Barkey [8] presented the Musielak-Orlicz difference sequence space, paranormed the difference, and examined several inclusion relations. Sarma [19] in 2012 proposed some I-

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ON SEQUENCE SPACE OF FUZZY REAL NUMBERS DEFINED BY ORLICZ FUNCTION AND PARANORM

GYAN PRASAD PAUDEL
Central Campus of Science and Technology, Mid-Western University, Surkhet, Nepal. Corresponding Author.

NARAYAN PRASAD PAHARI
Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal.

SANJEEV KUMAR
Department of Mathematics, Dr. Bhimrao Ambedkar University, Agra, India

Abstract

In this paper, we use the fuzzy set and fuzzy real numbers to define the classes of fuzzy sequence spaces $l_M(X,\bar{\lambda},\bar{p})$ and $l_M(X,\bar{\lambda},\bar{p},L)$ by using the Orlicz function. Also, we explore some linear topological structures of the spaces. Further, we introduce a paranorm to study some properties of the spaces. Finally, we define a class $\bar{l}_M(X,\bar{\lambda},\bar{p},L)$, and using the concept of $\Delta_2 -$ condition, to show the relation between $l_M(X,\bar{\lambda},\bar{p},L)$ and $\bar{l}_M(X,\bar{\lambda},\bar{p},L)$.

Keywords: Fuzzy Real Number, Paranorm Space, Limit Supremum and Limit Infimum

1. INTRODUCTION

Before the introduction of Fuzzy Logic, mathematics was limited only to two conclusions that are true and false (1 & 0) only. But in 1965, Zadeh [1] was the first to establish the concept of fuzzy set and fuzzy set operations. After that several authors have studied various branches of its theory and applications and an enormous number of authors have employed the fuzzy set and fuzzy real numbers in various sequence spaces. Motloka [2] defined the boundedness and convergence of a fuzzy sequence and demonstrated that any convergent sequence of fuzzy numbers is bounded. Similarly, Nanda [3] defined a new metric to show that a space of a convergent and bounded sequence of Fuzzy real numbers is complete. Later on, Et. M. Savas and Altinok H. [4] proposed certain classes of fuzzy number sequences, examined them, and analyzed some of their properties such as completeness, solidity, symmetry, convergence free, and also looked at various inclusion relations that were pertinent to these classes. Kim, J.M., and et al. [10] established fuzzy norms for the novel idea of a fuzzy normed space and investigated how to express a dual space of sequences. Furthermore, the systematic investigation of fuzzy normed linear spaces with various features is discussed [11, 12]. In 2021, Paul and Pahari [7] used the concept of fuzzy to study a few topological structures in fuzzy metric space. Also, in 2022, Paudel and et al. [8] studied the topological structure of P- bounded variation of difference sequence space and introduced the generalized form of the P- bounded variation of fuzzy real numbers.

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On Certain Series to Series Transformation and Analytic Continuations by Matrix Methods

Suresh Kumar Sahani[*1], **Gyan Prasad Paudel**[2], **Jhavi Lal Ghimire**[3] **and Amarnath Kumar Thakur**[4]

[*1]*Department of Mathematics, MIT campus, Janakpurdham- Nepal*
[2]*Central Department of Mathematics, Mid-Western University, Surkhet, Nepal*
[3]*Central Department of Mathematics, Tribhuvan University, Kritipur, Nepal*
[4]*Central Department of Mathematics, Dr. C.V.Raman University, Kota, Bilaspur (C.G) 495113, India*

Email: [*1]*sureshkumarsahani35@gmail.com*, [2]*gyan.math725114@gmail.com*,
[3]jhavighimire@gmail.com *and* [4]*akthakur@cvru.ac.in*

Abstract: *In this research paper, we proved some general theorems on the absolute convergence transformation of matrix which is expressed in terms of preserving transformation under the very general conditions. This works is motivated by the works of* [3],[12] *and* [14].

Keywords: *Summability, Transformation, Convergence.*

AMS Mathematics Subject Classification: 40A05, 40C05, 40F05.

1. Introduction

Mathematical analysis is primarily concerned with the notion of limit of a sequence of real or complex numbers which forms the basis for the study of infinite series. One important branch of the field of infinite series is the study of summability of divergent sequence (series). This study is an attempt to attach (in some series) a generalized limit to those sequences which do not converge in the usual sense, realizing at the same time that when the generalized limit is applied to a convergent sequence then it must agree with the limit in the ordinary sense. This procedure of assigning a new limit in generalized sense to divergent sequences is called a summability method. A sequence space is a linear space whose elements are sequences chosen from another linear space. The summability theory deals with the study of linear transformations on sequence spaces. In the earliest stage, the idea of summability theory were perhaps contained in a letter written by Leibnitz to C. Wolf (1713). In 1880, Frobenius introduced the method of summability by arithmetic means, which was generalized by Ces`aro (1890) (see [9]) as the (C, k)-method of the summability. These types of summability can also be presented by the use of infinite matrix transformation. So, we now turn to the fact that how infinite matrix transformation can be used to define generalized limits. A very important application of matrices, namely to the theory of summability of divergent sequence and the series was initiate by Toeplitz [12] in 1911. Although, the concept of absolute summability was introduced as early as in 1911, by Fekete [4] in case of Cesàro [9]method, and the same for Reisz [9] and Abel [9] methods

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TOPOLOGICAL PROPERTIES OF DIFFERENCE SEQUENCE SPACE THROUGH ORLICZ-PARANORM FUNCTION

GYAN PRASAD PAUDEL, NARAYAN PRASAD PAHARI
and SANJEEV KUMAR

Central Campus of Science and Technology
Mid-Western University, Surkhet, Nepal
E-mail: gyan.math725114@gmail.com

Central Department of Mathematics
Tribhuvan University
Kirtipur, Kathmandu, Nepal
E-mail: nppahari@gmail.com

Department of Mathematics
Dr. Bhimrao Ambedkar University, Agra, India
E-mail: sanjeevibs@yahoo.co.in

Abstract

In this paper, we introduce the difference sequence space $\mathcal{S}(\mathcal{F}, M, \alpha, P)$ and $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ of fuzzy real numbers using the Orlicz function. We also discussed some of the linear topological properties of the space and demonstrated $\mathcal{S}(\mathcal{F}, M, \alpha, P, G)$ is complete by defining a new paranorm on it.

Introduction

In 1965, L. A. Zadeh [24] developed the fuzzy set theory to address ambiguity and uncertainty in mathematics. Since then, both pure and applied fuzzy mathematics have been the subject of extensive research. The sequence space of fuzzy real numbers is the area where the majority of research has been conducted. Sequence space refers to a linear subspace of a vector space.

2020 Mathematics Subject Classification: 03B52, 46A45.
Keywords: fuzzy real number, paranorm space, limit supremum and limit infimum.
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Application of Fuzzy Logic Through Bellmen-Zadeh Maximin Method

Gyan Prasad Paudel[1,2,*], Narayan Prasad Pahari[2], Sanjeev Kumar[3]

[1] Central Campus of Science and Technology, Mid Western University, Surkhet, Nepal
[2] Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal
[3] Department of Mathematics, Dr. Bhimrao Ambedkar University, Agra, India

*Correspondence to: Gyan Prasad Paudel, Email: gyan.math725114@gmail.com

Abstract: *Fuzzy Logic is a multiple valued logic, from which the truth values of variables might be any real numbers between* 0 *and* 1*. It has wide application in different fields of real life. We might face different challenges in our daily life to select the best of the best for the concrete result. There are various criteria in the literature for ranking alternatives in the realm of decision-making under uncertainty in a crisp environment. In this paper, we address the challenges seen in making a decision for selecting the best of the best as studied by Bellmen-Zadeh maximin method. It is used to address such challenges using fuzzy real numbers.*

Keywords: Fuzzy logic, Membership function, Fuzzy number, Decision problem, Alternatives

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1 Introduction

Fuzzy logic is a multiple-valued logic form, in which the truth values of variables might be any real number between 0 and 1 [16]. It is a strategy for dealing with linguistic variables and describing modifiers such as vary, fairly, and not, among others. It aids common sense reasoning with imprecise and ambiguous propositions in natural language and acts as a foundation for decision analysis and control action.

Lots of the applications of fuzzy logic can be found in various sectors of the real world. Fuzzy logic has been studied and investigated with many results in economic fields. Stojic [14] presented a model for evaluating the level of economic development of countries and regions using fuzzy logic. In 2016, to obtain optimal solution, Kripa and Govindarajan [10] proposed various strategies for solving fuzzy sequencing problems with trapezoidal fuzzy numbers. In 2017, Sahoo [13] proposed a solution procedure to solve the fuzzy job sequencing problem, in which processing time is represented as a trapezoidal fuzzy number, and Yager's Ranking Index method is used to convert the fuzzy processing time into crisp ones, in which the optimal job solution (order) and ideal time for each machine are determined. Leelavathy and Kowsalya [8] suggested a new fuzzy arithmetic operation and ranking algorithm in 2019 to find an ideal sequence for fuzzy sequencing problem using trapezoidal fuzzy number. Making a decision is a problem-solving process that leads to a certain action and is a choice between numerous methods for achieving a goal [1]. In business, finance, management, economics, social and political science, engineering and computer science, biology and medical science, decision-making plays a vital role. Due to elements such as insufficient and inaccurate information, subjectivity, and language, which tend to be present in real-life circumstances to varying degrees, it is a tough procedure. These characteristics suggest that a decision-making process occurs in a fuzzy environment. According to Bellman–Zadeh [1], decision making is defined as the intersection of goals and constraints expressed by fuzzy sets. A decision process in a fuzzy environment is one in which the goals and constraints are both fuzzy characters. This means that the goal and constraints define classes of alternatives whose boundaries are not clearly defined. John and Sunny [7] conducted a research in 2011 on decision making in a fuzzy environment employing preference relations and comparative uncertainty, in which the probabilities of natural states are unknown a priori. There are various criteria in the literature for ranking alternatives in the realm of decision-making under uncertainty in a crisp environment. Jianping and et al. [9], in 2021 designed a method that is connected to the green supplier selection(GSS) and conducted some comparative analysis to illustrate the designed method's superiority. The best alternative can be chosen based on the final score calculated by combining the weights of various

On a New Application of Positive and Decreasing Sequences to Double Fourier Series Associated with $(N, p_m^{(1)}, p_n^{(2)})$

Suresh Kumar Sahani[1], Gyan Prasad Paudel[2,*], A. K. Thakur [3]

[1] Department of Mathematics, MIT Campus (T.U.), Janakpurdham, Nepal
[2] Central Campus of Science and Technology, Mid-Western University, Surkhet, Nepal
[3]Department of Mathematics, Dr. C.V. Raman University, Bilaspur (C.G)

*Correspondence to: Gyan Pradad Paudel, Email: gyan.math725114@gmail.com

Abstract: *In this paper, we introduce a new application of positive and decreasing sequences to double Fourier series associated with* $(N, p_m^{(1)}, p_n^{(2)})$. *Further, by considering some suitable conditions for previously known results, we have validated the current findings. This work was motivated by the works of* [5] *and* [12].

Keywords: Positive and decreasing sequences, Summability, Double Fourier series

DOI: `https://doi.org/10.3126/jnms.v5i2.50011`

1 Introduction

A series is built on the concept of sequence. As a result, sequence and series are concepts that are related and several authors have studied the sequence. Paudel et al. [12] studied sequence and generalized sequence space. Sahani et al. [13] studied series-to-series transformations and analytical continuations using matrix methods. In this paper, we have studied about the double Fourier series. The double Fourier series associated with the function $\varphi(\alpha, \beta)$ is defined in the following way

$$\sum_{g=0}^{\infty}\sum_{h=0}^{\infty} \gamma_{gh}\{r_{gh}\cos g\alpha\,\cos h\beta + s_{gh}\sin h\alpha\,\cos h\beta + t_{gh}\cos g\alpha\,\sin h\beta + q_{gh}\sin g\alpha\,\sin h\beta\} \tag{1}$$

where $\varphi(\alpha, \beta)$ is a Lebesgue integrable mapping in the rectangle $R(-\pi,\ \pi; -\pi,\ \pi)$ and is f period 2π [5, 11] and

$$\gamma_{gh} = \begin{cases} 4^{-1} & \text{for} \quad g = 0,\ h = 0 \\ 2^{-1} & \text{for} \quad g = 0,\ h > 0 \,\text{or}\, g > 0,\ h = 0 \\ 1 \quad \text{for} & g,\, h\ > 0 \end{cases}$$

$$r_{gh} = \frac{1}{\pi^2}\iint_R \varphi(\alpha,\ \beta)\ \cos g\alpha\ \cos h\beta\ d\alpha\, d\beta \tag{2}$$

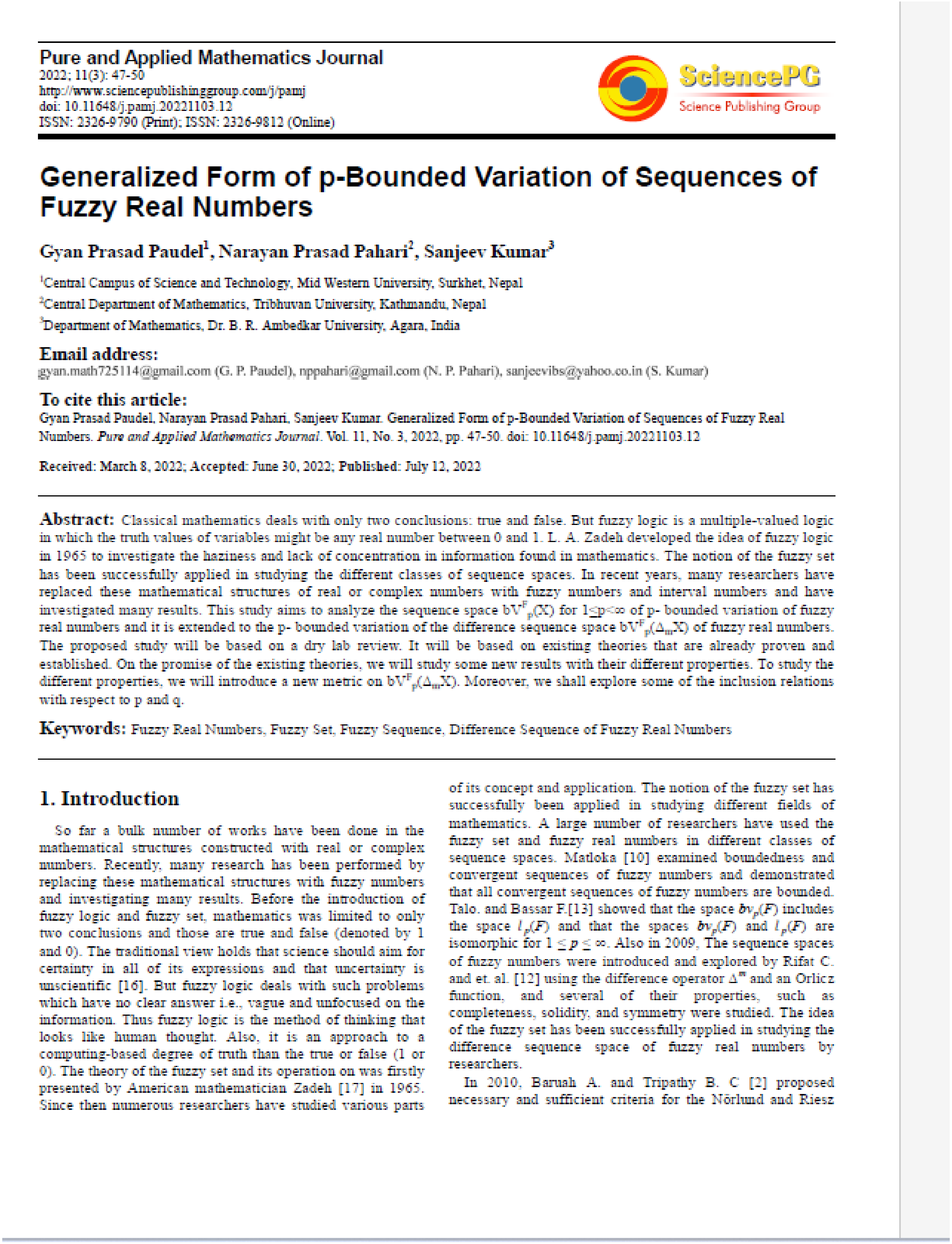
$$s_{gh} = \frac{1}{\pi^2}\iint_R \varphi(\alpha,\ \beta)\ \sin g\alpha\ \cos h\beta\ d\alpha\, d\beta \tag{3}$$

$$t_{gh} = \frac{1}{\pi^2}\iint_R \varphi(\alpha,\ \beta)\ \cos g\alpha\ \sin h\beta\, d\alpha\, d\beta \tag{4}$$

$$q_{gh} = \frac{1}{\pi^2}\iint_R \varphi(\alpha,\ \beta)\ \sin g\alpha\ \sin h\beta\, d\alpha\, d\beta \tag{5}$$

Also, let

$$\chi(\alpha, \beta) = \chi_{g,\,h}(\alpha, \beta) = 4^{-1}\{\varphi(g+\alpha,\, h+\beta) + \varphi(g-\alpha,\, h+\beta) + \varphi(g+\alpha,\, h-\beta) + \varphi\,(g-\alpha,\ h-\beta) - 4\varphi(\alpha, \beta)\} \tag{6}$$



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Generalized Form of p-Bounded Variation of Sequences of Fuzzy Real Numbers

Gyan Prasad Paudel[1], Narayan Prasad Pahari[2], Sanjeev Kumar[3]

[1]Central Campus of Science and Technology, Mid Western University, Surkhet, Nepal
[2]Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal
[3]Department of Mathematics, Dr. B. R. Ambedkar University, Agara, India

Email address:
gyan.math725114@gmail.com (G. P. Paudel), nppahari@gmail.com (N. P. Pahari), sanjeevibs@yahoo.co.in (S. Kumar)

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Abstract: Classical mathematics deals with only two conclusions: true and false. But fuzzy logic is a multiple-valued logic in which the truth values of variables might be any real number between 0 and 1. L. A. Zadeh developed the idea of fuzzy logic in 1965 to investigate the haziness and lack of concentration in information found in mathematics. The notion of the fuzzy set has been successfully applied in studying the different classes of sequence spaces. In recent years, many researchers have replaced these mathematical structures of real or complex numbers with fuzzy numbers and interval numbers and have investigated many results. This study aims to analyze the sequence space $bV^F{}_p(X)$ for $1 \leq p < \infty$ of p- bounded variation of fuzzy real numbers and it is extended to the p- bounded variation of the difference sequence space $bV^F{}_p(\Delta_m X)$ of fuzzy real numbers. The proposed study will be based on a dry lab review. It will be based on existing theories that are already proven and established. On the promise of the existing theories, we will study some new results with their different properties. To study the different properties, we will introduce a new metric on $bV^F{}_p(\Delta_m X)$. Moreover, we shall explore some of the inclusion relations with respect to p and q.

Keywords: Fuzzy Real Numbers, Fuzzy Set, Fuzzy Sequence, Difference Sequence of Fuzzy Real Numbers

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1. Introduction

So far a bulk number of works have been done in the mathematical structures constructed with real or complex numbers. Recently, many research has been performed by replacing these mathematical structures with fuzzy numbers and investigating many results. Before the introduction of fuzzy logic and fuzzy set, mathematics was limited to only two conclusions and those are true and false (denoted by 1 and 0). The traditional view holds that science should aim for certainty in all of its expressions and that uncertainty is unscientific [16]. But fuzzy logic deals with such problems which have no clear answer i.e., vague and unfocused on the information. Thus fuzzy logic is the method of thinking that looks like human thought. Also, it is an approach to a computing-based degree of truth than the true or false (1 or 0). The theory of the fuzzy set and its operation on was firstly presented by American mathematician Zadeh [17] in 1965. Since then numerous researchers have studied various parts of its concept and application. The notion of the fuzzy set has successfully been applied in studying different fields of mathematics. A large number of researchers have used the fuzzy set and fuzzy real numbers in different classes of sequence spaces. Matloka [10] examined boundedness and convergent sequences of fuzzy numbers and demonstrated that all convergent sequences of fuzzy numbers are bounded. Talo. and Bassar F.[13] showed that the space $bv_p(F)$ includes the space $l_p(F)$ and that the spaces $bv_p(F)$ and $l_p(F)$ are isomorphic for $1 \leq p \leq \infty$. Also in 2009, The sequence spaces of fuzzy numbers were introduced and explored by Rifat C. and et. al. [12] using the difference operator Δ^m and an Orlicz function, and several of their properties, such as completeness, solidity, and symmetry were studied. The idea of the fuzzy set has been successfully applied in studying the difference sequence space of fuzzy real numbers by researchers.

In 2010, Baruah A. and Tripathy B. C [2] proposed necessary and sufficient criteria for the Nörlund and Riesz

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DOUBLE SEQUENCE SPACE OF FUZZY REAL NUMBERS DEFINED BY ORLICZ FUNCTION

GYAN PRASAD PAUDEL [1] NARAYAN PRASAD PAHARI [2] SANJEEV KUMAR[3]

[1] *Graduate School of Science andTechnology, Mid-Western University, Surkhet, Nepal*
[2] *Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal*
[3] *Department of Mathematics ,Dr.B. R. Ambedkar University, Agara, India.*
Email: [1]*gyan.math725114@gmail.com,* [2]*nppahari @gmail.com,* [3]*sanjeevibs@yahoo.co.in*

Abstract: The theory of fuzzy logic and the fuzzy set has been successfully applied in various fields of research in social science, management science, and mathematics. In this paper, we use the concept of fuzzy real numbers to introduce and study the new double sequence spaces $l_\infty^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$, $C^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ and $C_o^{\mathcal{F}}(\mathcal{M},\lambda,\rho)$ of fuzzy real numbers defined by the Orlicz function and study some of their properties like linear space structure, completeness, and solidness.

Key Words: Fuzzy set and logic, Fuzzy real numbers, Orlicz function, Difference sequence space

AMS (MOS) Subject Classification. Classification 46A45, 03B50, 03B52.

1. INTRODUCTION

Mathematics has traditionally been restricted to two conclusions: true and false. According to the traditional viewpoint, science should strive for certainty in all statements, and uncertainty is seen as unscientific[1]. However, American Mathematician L. A. Zadeh[2], initially introduced the concept of fuzzy set and fuzzy logic in 1965 to deal with difficulties seen in mathematics that have no clear answer. Since then, a number of authors have researched various aspects of its theory and applications. A large number of authors have used the fuzzy set and fuzzy numbers in different classes of sequence spaces. Motloka [3] has studied the boundedness and convergent sequence of fuzzy numbers and has shown that every convergent sequence of fuzzy numbers is bounded. The notion of the fuzzy set has been successfully applied in studying the double sequence of fuzzy real numbers by researchers. Hardy [4] developed the concept of regular convergence for double sequences in the sense that the double sequence has a limit in the Pringsheim sense and has one-sided limits. In 2005, Altay and Basar[5] defined the double sequence spaces, looked at some of their properties, and demonstrated that they are fully paranormed or normed spaces under certain conditions. In order to study the double statistical convergence of a sequence of fuzzy numbers, Savas[7] introduced some new double sequence spaces of fuzzy numbers in

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GYAN PRASAD PAUDEL,NARAYAN PRASAD PAHARI and SANJEEV KUMAR/ IMPLEMENTATION OF FUZZY BASED CLUSTERING SYSTEM

IMPLEMENTATION OF FUZZY BASED CLUSTERING SYSTEM

[1]GYAN PRASAD PAUDEL, [2]NARAYAN PRASAD PAHARI and [3]SANJEEV KUMAR
[1]Central Campus of Science and Technology, Mid-Western University, Surkhet, Nepal
[2]Central Department of Mathematics,Tribhuvan University,Kirtipur, Kathmandu,Nepal.
[3]Department of Mathematics, Dr.BhimraoAmbedkar University, Agra, India
Email: [1]gyan.math725114@gmail.com,[2]nppahari @gmail.com, and [3]sanjeevibs@yahoo.co.in

Principle author: Gyan Prasad Paudel

Abstract

Since most associations conveying security devices use encryption to protect records and external network affiliations, Gatecrasher will have an advantage in places where encryption/protection of data transmission is missing or particularly irrelevant. Here the data is managed as well as given to trusted in hosts and affiliations. A plan is a relation of some sections that are spread over a small or wide area over a short or large distance, so that the interconnected parts can go with each other.

Keywords: Fuzzy, Clustering, Data

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Introduction

Each of the explicit information from the sensor system remains desensitized until it is consumed and separated by the applications present in the application layer. To achieve this undertaking a connection has indeed been made and the message based middleware strategy is motivated to promote a more censored web middleware orchestrating. The system manages the information examination and awesome structure for the installation of sensors. The plan revolves around the advancement of web affiliations and message lines. SOAP show is used between parts and webservice message passing, being the most lightweight show to use over web http connection. Information access and evaluation integrates several interconnected frameworks for various purpose applications. The methodology is seen by parts. Parts can call different parts using SOAP based approach. Corresponding client applications that show the final type of decomposed information and driven structures can be worked with moving steps, models and so forth. Middleware is enabled to oversee deals and efforts and cross-stage and deliver results to appropriated clients. The further created Sensor Web middleware will help build and organize orchestrating designed WSN applications that collect and manage never-ending information from heterogeneous steps. important and concrete way

Shocking improvement in choosing types of progress at the end of various years, drawn in fast and efficient data systems. Closer to the progress of monitoring progress, correspondence progress also proceeded on an indistinguishable track and empowered the improvement of PC systems. Choosing hand tied for running two structures connection and dispersive time. Soon a slew of applications abuse the potential growth of communicative joins.

Soon an alternative type of data district needs to be considered to make arrangements with the working conditions under the class. Pack lots of information is created continuously in our situation. Regardless, if it is brought up, corrected and put to use, it remains as cruel information worth noting. Hopefully we have an electronic part to view, store, assess, process and forward applications that information will benefit various applications. Up to this point, such information gathering was done in a standard way, for example by installing rain survey meters at various locations, recording readings and organizing reports subject to the information gathered. Despite this, the current need is to refine designs that can continuously collect, process these information including best-in-class

On Fundamental Properties in Fuzzy Metric Space

- Gyan Prasad Paudel
- Narayan Prasad Pahari

Abstract: *Fuzzy metric spaces theory is important in mathematics, statistics, computer science, and other fields. In this paper fundamental characteristics of the fuzzy metric space are examined. The notions of fuzzy convergent sequence, fuzzy Cauchy sequence, fuzzy open ball, are reviewed with theorems associated with these concepts.*

Key words : t-norm , Fuzzy logic, Fuzzy metric space, Fuzzy open ball, Fuzzy convergence,

Introduction

Prof. Zadeh introduced fuzzy sets and fuzzy logic in 1965 to deal with the vagueness and uncertainty in mathematics. Since then, there has been a lot of research in the fields of fuzzy logic and fuzzy sets since then. In comparison to classical set theory, fuzzy set theory takes a different approach. Fuzzy logic, in particular, might be understood as an attempt to combine two unique skills. First, the ability to reason and make rational decisions in the face of ambiguity, uncertainty, incompleteness of knowledge, contradictory information, the partiality of truth, and the partiality of possibility. Second, without any measurement or computation, is the ability to do a wide range of physical and mental tasks [Zadeh 2008]. In the discipline of topology, fuzzy logic and fuzzy set theory are commonly used. Fuzzy topology is a key branch of fuzzy theory with a huge research area and a diverse set of applications. Many writers have actively participated in solving difficulties in fuzzy topology to get a suitable definition of fuzzy metric, as time has demanded. Many authors have looked into such issues in fuzzy topology from various perspectives. Karmosil and Michalek [6] created fuzzy metric space in 1975 as a generalization of Menger Space, which is a statistical metric notion. The fuzzy metric was defined by O. Kaleva and S. Seikkala [7] as the distance between two points in a collection expressed in fuzzy real values. Georage and Veeramani [2] adapted the fuzzy metric approach introduced by Karmosil and Michalek in 1994. Hausdorff Space was created for that fuzzy metric space. They've also demonstrated that each metric generates a fuzzy metric. Fuzzy metrics can be used to solve challenges with uncertain and imprecise data. In a fuzzy metric space, A. George Veeramani [10] defined the Hausdorff topology. Every ordinary metric space can produce a fuzzy metric space that is complete whenever the original one does,