Numerical Approaches for Solving Equations Involving Special Functions in Fractional Calculus

Mini Research Project in Mathematics

<u>**Submitted to**</u>

Dean's Office
Institute of Science and Technology
Tribhuvan University
Kirtipur, Kathmandu, Nepal

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spariyar

DEDICATION

To

My Late Father

Gaj Bahadur Pariyar

My Mother

Budhi Maya Pariyar

My Wife

Uma Pariyar

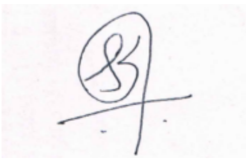
DECLARATION

This mini research project entitled "**Numerical Approaches for Solving Equations Involving Special Functions in Fractional Calculus**", which has been submitted to the Dean's Office, Institute of Science and Technology (IOST), Tribhuvan University, Nepal for the faculty mini research project in Mathematics, is a genuine work that I carried out under my supervisor Assistant Professor Dr. Jeevan Kafle and that no sources other than those listed in the Bibliography have been used in this work. Moreover, this work has not been published or submitted elsewhere for the requirement of any degree programme.

Shankar Pariyar
Assistant Professor
Tri-chandra Multiple Campus
Date: March 12, 2024

RECOMMENDATION

This is to recommend that Mr **Shankar Pariyar** has prepared this mini research project entitled “**Numerical Approaches for Solving Equations Involving Special Functions in Fractional Calculus**” in Mathematics under my supervision. To my knowledge, this work has not been submitted for any other degree. He has fulfilled all the requirements laid down by the Dean’s Office, Institute of Science and Technology (IOST), Tribhuvan University (TU), Kirtipur for the mini research project in Mathematics.

..

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Shankar Pariyar

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ABSTRACT

The study uses systemic diagrams to compare fractional equations at α values ($0 < \alpha < 1$) with non-fractional equations at $\alpha = 1$. Under Caputo Fabrizio's fractional differential equation, it looks at the identity, sine, and cosine functions. While avoiding non-zero fractional derivatives of constant functions, the Grunwald-Letinikov (G-L) numerical solution assures precision. Ensuring numerical results from fractional calculus and traditional calculus remain comparable is the goal of this research. It is necessary to use non-linear operators because deviations arise when fractional orders diverge. Ideas such as curve smoothness and value fluctuations are made clear by illustrative examples. Additionally, utilizing MLF at different values of α, β, and γ within the range $0 < \alpha < 1$, the study examines the numerical solution of one, two, and three parameters. It is shown how Magnus Gosta (M.G.) Mittag-Leffler's computations are used in scientific and practical fields. In order to familiarize with the latest research trends and MLF consequences, this study provides a thorough overview of the several types of Mittag-Leffler functions (MLF) that can be found in the literature. This research significantly enhances the knowledge of fractional derivatives and the integration of transcendental forms of functions. Using the Lagrange interpolation approach, we will demonstrate some numerical strategies, including the L_1 scheme for $0 < \alpha < 1$, the L_2 scheme for $1 < \alpha < 2$, and the L_1 method for the Caputo Fabrizio derivative.

Keywords: Caputo fractional derivatives, Grunwald-Letinikov, Numerical solution, Mittag-Leffler, Analytical Solution, Lagrange interpolation

LIST OF ACRONYMS AND ABBREVIATIONS

FC: Fractional Calculus

CFD: Caputo Fractional Calculus Derivative

RL: Riemann Liouvellie

FD: Fractional Derivative

GL: Grunwald-Letinikov

MLF: Mittag Leffler Function

MG : Magnus Gosta

FDE: Fractional Differential Equation

GW.: Gottfried Wilhelm

FOD: Fractional Order Derivatives

LT: Laplace Transform

C-F: Caputo and Fabrizio

A-B: Algahtani and Balgairi

LIST OF SYMBOLS

$\sum$	Summation.
∞	Infinity.
x, y, z	spatial coordinates.
C_β	space of functions.
$\Gamma(n)$	Second Eulerian integral.
$(\gamma)_n$	Pochhammer's symbol.
$!$	Factorial.
D^α	Fractional Derivatives of arbitrary order.
I^α	Fractional Integral of order.
$E_\alpha(Z)$	Mittag Leffler Function of single parameter.
$E_{\alpha,\beta}(Z)$	Mittag Leffler Function of double parameters.
$E^\gamma_{\alpha,\beta}(Z)$	Mittag Leffler Function of Three parameters.
${}^{GL}D^-_a$	Left Grunwald Fractional Derivatives.
${}^{GL}D_{b^-}$	Right Grunwald Fractional Derivative.
$P(x)$	Lagrange's interpolating polynomial.
M_0, M_1, M_2	Lagrange's interpolating basis functions.
$1D, 2D, 3D$	one, two, three dimensional.
$N(\alpha)$	Normalizing Factor.
$\lfloor a \rfloor$	Floor Function.
$\mathbf{D}^\alpha[f(x)]$	Liouville derivative.
$\mathbf{D}^\alpha_{0+}[f(x)]$	Liouville left-sided derivative.
$\mathbf{D}^\alpha_{0-}[f(x)]$	Liouville Right-sided derivative.
${}^{RL}\mathrm{D}^\alpha_{a+}[\,f(x)\,]$	Riemann-Liouville left-sided derivative.
${}^{RL}\mathrm{D}^\alpha_{b-}[\,f(x)\,]$	Riemann-Liouville Right-sided derivative.
$\star D^\alpha_{a+}[\,f(x)\,]$	Caputo left-sided derivative.
$\star D^\alpha_{b-}[\,f(x)\,]$	Caputo Right-sided derivative.

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Chapter 1

INTRODUCTION

Fractional calculus introduces two important concepts: fractional integral equations and fractional differential equations. When an equation involves fractional integrals, it is referred to as a fractional integral equation. On the other hand, An equation is a fractional differential equation if it has fractional derivatives. Remarkably, there exist systems that exhibit characteristics of both these equations, and they are classified as having a different order [45].

There has been a notable increase in the study of fractional calculus in this century, leading to its widespread adoption across various scientific and technological fields. Fractional calculus has emerged as a powerful tool for modeling diverse processes in physics, biology, and chemistry. Scientists have successfully applied fractional calculus to accurately describe and analyze a broad spectrum of phenomena in these disciplines. Its versatility and effectiveness have contributed to its increasing prominence in contemporary research [14, 45]. There has been an increase in the use of fractional-order differential equation models in the scientific community. These models have proven to be valuable in elucidating physical phenomena and understanding complex dynamic systems. By incorporating fractional-order derivatives into the equations, researchers have been able to capture intricate dynamics and phenomena that traditional integer-order models often struggle to explain. This growing adoption of fractional-order models signifies their effectiveness in enhancing our comprehension of diverse systems across various scientific disciplines [10]. The field of fractional calculus has undergone significant advancements and efforts to enhance its theory and applications. While the term "fractional calculus" has been defined in numerous ways by over various mathematicians, it is important to note that there is no universally applicable set of rules for fractional derivatives. Among the various definitions, the Riemann-Liouville (R-L) fractional derivative is commonly utilized and referenced [13]. However, the Riemann-Liouville definition presents a challenge as it shows that a constant term's derivative is non-zero when examining fractional derivatives using classical calculus. This divergence from traditional calculus principles adds complexity to the analysis of fractional derivatives. Researchers have been actively addressing these challenges to develop comprehensive frameworks and methodologies for effectively studying and utilizing fractional calculus [13, 35].

With this current version, Jumarie addressed this problem by updating the notion of fractional derivative of R-L type. The modified R-L derivatives of Caputo, (G-L), and Jumarie are invaluable for addressing non-zero derivatives of constant functions [45, 14, 5]. For analytical approaches, the R-L and Caputo definitions are better, but theG-Ldefinition is better for numerical applications. It is noteworthy that linear fractional differential equations do not have a universal solution. For example, one may use Jumarie's modified fractional derivative to find the derivative of the Mittag-Leffler function [25].

Swedish mathematician M.G. Mittag Leffler(ML) established the ML function for averaging divergent series. Mittag - Leffler function (MLF) is a unique transcendental function and it has attracted the attention of many researchers in the field of science and technology, due to its key role in providing solutions with integral and derivatives of fractional order [37]. The fractional order differentiation of the expanded forms of complex systems, random walks, and superdiffusive transport are naturally resolved by the MLF [29].

Within the MLF framework, the conventional kinetic equations and their fractional counterparts, which range from exponential to power-law dynamics, are essential for controlling the interpolation of common and extended phenomena.The FDof the MLF function may be obtained by applying Jumarie's revised definition of the fractional derivative [45], likewise, there is no general method for solving linear fractional differential quations. In an integer-order differential equations, e^z, is crucial. MLF [8, 35, 37] was the first to present the function that is now represented by, introducing its one-parameter generalization [7], We find a strong connection between fractional calculus and the MLF and its expanded variants, can help us to explain some unusual phenomena and solve fractional differential equations, A second complex parameter was added to this formulation immediately after it was first introduced, Goreflo et al [22] and Agarwal [24] established the two-parameter MLF β, $R(\beta) > 0$.

1.1 Fractional Calculus

The required preliminary steps for the work are listed below

Definition [34]: Space C_β, $\beta \in$ **R** is a real valued function $h(x), x > 0$ that has the property that $h(x) = x^m\ h_1(x)$ Where $h_1 \in$ C $[0, \infty)$.

Definition [22]: The function $g(x), x > 0$ is the space as a real valued C^p_β, $p \in \mathbf{N} \cup \{0\}, if g^p \in C_\beta$.

Definition [16]: Let, N is the category of analytical functions of form $P(y)$ of type $P(y) = y + \sum_{n=2}^{\infty} a_n y^n$, then the FD of which are analytical in $U = \{y \in \mathbf{C} : |y| < 1\}$, of α, for $f(y) \in \mathbf{M}$, is defined as
$\mathbf{D}^\alpha_x k(x) = \frac{1}{\Gamma(1-\alpha)}\ \frac{d}{dx} \int_0^x\ k(\tau)(x-\tau)^{-\alpha} d\tau, \quad (0 < \alpha < 1)$

Definition [12]: The integral provides the first Eulerian formula represented by $\beta(p, q)$ such that $\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}, p, q > 0$. The beta function is connected to the gamma function via the relation, and the integral function converges for $p, q > 0$ $\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Definition [14]: The function $g(y)$ of n^{th} order finite differences with regard to y can be defined, according to GL, if we have α is a positive number.

GL Left sided FD:

$$^{GL}D^\alpha_{a^+}[P(y)] = \lim_{h \to\ 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor n \rfloor} (-1)^k \frac{\Gamma(\alpha+1)P(y-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \ \ nh = (y-a).$$

GL Right-sided Fractionla Derivative:

$$^{GL}D^\alpha_{b^-}[P(y)] = \lim_{h \to\ 0} \frac{1}{h^\alpha} \sum_{t=0}^{\lfloor m \rfloor} (-1)^t \frac{\Gamma(\alpha+1)P(y+th)}{\Gamma(t+1)\Gamma(\alpha-t+1)}, \ \ mh = (b-y). \tag{1.1}$$

Definition [14]: The Riemann Liouville fractional Integral.

$${}_a\mathbf{I}^\alpha_y(m(y)) = \frac{1}{\Gamma(\alpha)} \int_a^y (y-z)^{\alpha-1} m(t)dt; y > a. \tag{1.2}$$

$${}_x\mathbf{I}^\alpha_a(m(y)) = \frac{1}{\Gamma(\alpha)} \int_y^a (y-z)^{\alpha-1} m(x)dx; a > y. \tag{1.3}$$

Properties [42]
Prop.1: ${}_aI^\alpha_x(K(T(x))) = K \cdot_a I^\alpha_x(T(x))$.
Prof.2: ${}_aI^\alpha_y(M(x) \pm N(x))) =_a I^\alpha_y(M(x) \pm_a I^\alpha_y N(x)))$.
Prop.3: ${}_aI^\alpha_y(R(x) \pm S(x))) =_a I^\alpha_y(R(x) \pm_a I^\alpha_y S(x))$.

Defintion [9]: FC are defined by the R-L formula and have non-integer orders of differentiation.

$${}_b\mathbf{D}^\alpha_t(v(x)) = \begin{cases} \frac{1}{\Gamma(\nu-\gamma)} \cdot \left(\frac{d}{dx}\right)^t \int_b^t (q-t)^{t-\gamma-1} v(t).dt; & (t-1) < \alpha < t, \alpha \in \mathbb{R}^+, t \in \mathbb{N} \\ \frac{d^t}{dx^t} v(x) \quad if \quad \alpha = t \end{cases} \tag{1.4}$$

Defintion [32]: The Caputo FD is a kind of fractional derivative that is often used.

$$ {}_a\mathbf{D}_z^{\alpha}(m(z)) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^z (z-t)^{n-\alpha-1} \cdot \left(\frac{d}{dz}\right)^n m(t).dt; \\ \frac{d^n}{dz^n} m(z); \quad if \quad \alpha = n \end{cases} \tag{1.5} $$

1.2 Mittag-Leffler Function

Definition [18], single parameter MLF,

$$ E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{1.6} $$

where α is a complex parameter and $\Gamma(z)$ is the gamma function.
Definition [22, 24], Two-parameters *MLF* β, $R(\beta) > 0$

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^k}{\Gamma(\alpha \cdot k + \beta)} \tag{1.7} $$

The gamma function, abbreviated as $\Gamma(z)$, is a real and complex number extension of the factorial function that frequently uses the complex parameter α. The expression (1.7) using a series representation was first introduced by Prabhakar ([37]).

$$ E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_n z^k}{\Gamma(\alpha \cdot k + \beta) \cdot n!}, \tag{1.8} $$

In our mini research project, we examine fractional operators utilizing three-parameter generalized ML kernels, incorporating Pochhammer's symbol $(\gamma)_n$. In their work, Caputo and Fabrizio have developed a unique fractional derivative with no singularities in its kernel [10]. The kernel of this recently introduced fractional derivative is analogous to the kernel of an exponential function.

${}_a^{CF}D_x(g(x)) = \frac{N(\alpha)}{(1-\alpha)} \int_a^x e^{-\frac{\alpha(x-t)}{(1-\alpha)}} \cdot g'(t) \cdot dt$; $N(0) = N(1) = 1$. Where $N(\alpha)$ is a normalizing factor.

Interpolation [8]**,** Interpolation refers to the method of finding function values at intermediary points given the values at the initial two points.

Linear Interpolation [7]**,** The linear polynomial that connects the specified coordinates (a_0, b_0) and (a_1, b_1) follows,

$$ P_1(x) = b_0 \frac{a_1 - x}{a_1 - a_0} + b_1 \frac{x - a_0}{a_1 - a_0} \tag{1.9} $$

Linear interpolation can be used to calculate a function's value at a specific location within a specified range, when only the values of the function at certain discrete points are known. It is widely used in numerical analysis, data analysis, and computer graphics, among other fields, to approximate the values of functions, and to generate smooth curves and surfaces from discrete data points.

Quadratic Interpolation [8]**,** Quadratic polynomial through (a_0, b_0), (a_1, b_1), and (a_2, b_2)

$$\begin{aligned}
M_2(z) &= b_0P_0(z) + b_1P_1(z) + b_2P_2(z) \\
P_0(z) &= \frac{(z-a_1)(z-a_2)}{(a_0-a_1)(a_0-a_2)} \\
P_1(z) &= \frac{(z-a_0)(z-a_2)}{(a_1-a_0)(a_1-a_2)} \\
P_2(z) &= \frac{(z-a_0)(z-a_1)}{(a_2-a_0)(a_2-a_1)}
\end{aligned} \tag{1.10}$$

Where $M(z)$ is Lagrange's interpolating polynomial, and P_0, P_1, and P_2 are the Lagrange's interpolating basis functions.

1.3 Motivation

Within the field of fractional calculus, the accuracy of solutions to different fractional differential equations depends on a number of definitions and theorems. Agar [3, 41] have described how to solve a fractional wave equation, a fractional diffusion equation, and the *(FD)* harmonic oscillator using computational fluid dynamics *(CFD)*, Riemann-Liouville *(RL)*, and Grunwald-Letnikov *(GL)* methods. Using ideas like Computational Fluid Dynamics, Riemann-Liouville, and Grunwald-Letnikov approaches, several fractional differential equations (FDEs) have been successfully solved. These include the FD harmonic oscillator, a fractional wave equation, and a fractional diffusion equation. These methods, fundamental to fractional calculus, have greatly enhanced engineering and scientific fields, and are vital resources for practitioners and scholars alike [41], Abdullah [1]. Our research aims to contribute to the creation of a mathematical model for special functions derived from experimental data, utilizing these functions to initiate a fractional order derivatives (FOD) [2]. Additionally, the study compares answers from ordinary and fractional calculus and evaluates computational results using computer software.

This study investigates novel mathematical models and approaches to problem-solving, offering insights into the wide range of scientific and engineering domains in which Fractional

Calculus (FC) is applied. In mathematical analysis, special functions such as Hypergeometric, Fox's H-, Wright, Bessel, and ML functions are essential and useful in several fields [11]. Their essential importance is highlighted by the fact that they emerge as solutions to fractional differential equations. Among these functions, interesting relationships can be found using fractional calculus operators. It might be difficult to get exact analytical answers, particularly for fractional derivatives or integrals. To approximate or compute these functions, numerical techniques such as L_1, L_2, and L_{1-2} are frequently employed [29]. While L_1 and L_2 are useful numerical approaches, there exist fractional differential equations for which there are no analytical solutions. Furthermore, analytical solutions exist for some equations, especially when the Mittag-Leffler function is involved.

1.4 Literature Review

L' Hospital asked Leibniz [23] about the n^{th} derivative of $f^n(t) = t$ and what would happen if n = $\frac{1}{2}$. Mathematics has a long history with fractional calculus.This is a very important fractional calculus question. What would happen if n were any number, such as n = $\frac{1}{2}$, in general? This appears to be a paradox, but it has positive implications for humanity," Leibniz said. This discussion between the two mathematicians revealed, a new branch of mathematics was established on september 30, 1695 [33]. Three decades later, G.W. Leibniz expanded fractional calculus with the introduction of the beta and gamma functions. A number of well-known mathematicians have made significant definitional contributions to this topic. Lagrange established the notion of exponents for differential operators that functioned with integer orders in 1772 [45]. Furthermore, Laplace [44] and Lacroix employed Legendre's symbol Γ in 1812 to streamline the n^{th} order derivative's formulation, demonstrating the significant contributions of several eminent mathematicians to its development. Fractional calculus was driven to develop by the accomplishments of many mathematicians, including Fourier, Riemann, Laplace, Euler, and many more. There are several definitions for fractional derivatives, but we generally prefer the ones from Riemann- Liouville and Caputo since they produce conclusions that are compatible with calsssical calculus [44]. A recent development in fractional calculus was made possible by numerous national and international mathematicians. Iterative and finite difference approaches have been developed by many of them to solve linear and non-linear FPDE [27]. R.P. Agrawal [3] and Erdely was first developed the MLF in the fractional calculation with parameters and Additionally, by using the Laplace transform technique, Humbert and Agarwal [17] created a number of links for their inquiry. In the context of fractional derivatives, Rudolf Gerenflo and Francesco Mainardi [21] examined the fundamental equations for fractional relaxation and vibration [10, 39].

Fourier, Riemann, Green, Holmgren, Grunwald,Letnikov, Liouville, Laplace, Caputo [45] and many other mathematicians, for example, H. Laurent was born in 1884, followed by P. A. Nekrassow in 1888, A. Krug in 1890, J. Hadamard in 1892, O. Heaviside from 1892 to 1912, S. Pincherle [27, 31, 15].

1.5 Objectives of the Study

The following are the main objectives of this mini research project:

Objective-I: to gain insights into approximate solutions of non-fractional and fractional equations, examine fractional differentiation equations for special functions, and explore the applications of the Caputo-Fabrizio identity in understanding fractional differentiation.

Objective-II: to examine the application of the Mittag-Leffler function in studying fractional differentiation, comparing it to traditional calculus methods. Additionally, it seeks to develop numerical approximations for the Caputo-Fabrizio derivative and assess their convergence rates using interpolation techniques.

1.6 Methodology

In this work, we collect the necessary articles as part of the survey, assess them, and choose an appropriate strategy, the Caputo Fractional Derivatives L1 and L2 methods and the Caputo Fabrizio sense L1-2 numerical method. The fractional differential equation of simple elementary functions to some special functions, such asthe Mittag-Leffler function, can be solved analytically using a variety of techniques. In order to interpret the result in terms of figures, we compare the exact solution and analytical solutions using computer tools.

Chapter 2

Approximation Methods for Special Functions in Fractional Calculus

Fractional calculus helps the scientific and non-scientific sectors and is widely used in modern scientific applications across several areas. It is essential for using fractional differential methods to derive Caputo fractional derivatives and integrations of different functions, including sines, cosines, polynomials, and exponentials [42]. For constant functions, non-zero fractional derivatives are avoided by using the Grunwald-Letnikov numerical solution [27]. In the context of fractional calculus, this study explores analytical and approximation solutions. It compares the findings with conventional differential equation models and describes the solutions as Riemann-Liouville functions. This approach's usefulness in real-world circumstances is demonstrated through a case study that provides a practical example of these concepts.

$$\begin{aligned} & y = x^n \\ & \frac{d^\alpha y}{dx^\alpha} = \frac{n!}{(n-\alpha)!} \cdot x^{n-\alpha} \end{aligned} \tag{2.1}$$

The study investigates solutions for fractional calculus by comparing it with classic models and applying the Grunwald-Letnikov approach. Riemann-Liouville functions are recognized as the results, and a case study is provided to illustrate this. Euler invented the gamma function in 1729, which allows factorial calculations to be performed on non-integers. In reference to equation (2.1)

$$D^\alpha(y) = \frac{n!}{\Gamma(n-\alpha+1)} \cdot x^{n-\alpha} \tag{2.2}$$

2.1 $\frac{1}{2}^{th}$ Order Fractional Derivative of Constant Function(C)

Assumed that, y = constant function, such that $y = Cx^0$
In specifically, if we replace $n = 0$ and $\alpha = \frac{1}{2}$ in equation (2.1),

$$D^{\frac{1}{2}}(C) = \frac{0!}{\Gamma(0 - \frac{1}{2} + 1)} \cdot x^{0-\frac{1}{2}} = \frac{1}{\sqrt{\pi x}}$$

The results are incongruent with conventional calculus, the function is discontinuous at $x = 0$. Inherently non-continuous are half-order derivatives of polynomial functions. To better explore these issues, a novel relation is offered [27].

$$k'(x) = \lim_{h\to 0}\left(\frac{k(x) - k(x-h)}{h}\right)$$

$$k'''(x) = \lim_{h\to 0}\frac{1}{h^3}\sum_{j=0}^{3}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix} k(x - jh)$$

$$k^{\alpha}(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)^{\alpha}\sum_{j=0}^{n}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix} k(x - jh) \tag{2.3}$$

$$k^{\alpha}(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)^{\alpha}\sum_{j=0}^{n}(-1)^j \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha - j + 1)}.k(x - jh)$$

$$k^{\alpha}(x) = \lim_{h\to 0}\left(\frac{1}{h}\right)^{\alpha}\sum_{j=0}^{n}(-1)^j \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha - j + 1)}.k(x - jh)$$

$$k^{\alpha}(x) = \lim_{n\to\infty}\left(\frac{n}{x-a}\right)^{\alpha}\sum_{j=0}^{n}(-1)^j \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha - j + 1)}.K(x - \frac{j}{n}(x-a)) \text{ [14].} \tag{2.4}$$

A well-known idea that needs us to identify and assess its limit for any given function is the Grunwald-Letnikov fractional derivative. We must, however, make some further changes to the formulation because this work can be extremely difficult. The formulation is clearly complicated, and the existence of the limit is necessary for it to be legitimate.
$\because \alpha \to$ - α, from (2.4)

$$k^{-\alpha}(x) = \lim_{n\to\infty}\left(\frac{1}{h}\right)^{-\alpha}\sum_{j=0}^{n}(-1)^j \begin{bmatrix} -\alpha \\ j \end{bmatrix} k(x - jh)$$

$$I^{\alpha}k(x) = h^{\alpha}\lim_{n\to\infty}\left(\frac{1}{h}\right)^{-\alpha}\sum_{j=0}^{n}\frac{\Gamma(\alpha+j)}{j!\Gamma(\alpha)}.k(x - j.h),\ \ h = \frac{x-a}{n},\ a < x$$

If h is small enough, fractional integration, also known as fractional differentiation, can be approximated with high levels of accuracy. Furthermore, it has been proven that this approximation is accurate to within an order of h. However, the *G-L* fractional derivative is typically not applicable to non-integer terms, and the R-L formulation is the one that is most frequently applied in these circumstances [35]. For analytical purposes, the G-L and Caputo fractional derivatives are equivalent. In a constant function, there is minimal variation. While the G-L derivative yields a fractional derivative of zero, the Caputo sense presents a different result. The Caputo fractional derivative is frequently employed in the FC context [9].

The following characteristics of this operator are crucial,

2.2 Half-Order Fractional Differentiation of x

A particular method for fractional derivatives is used in equation (2.1), which is especially useful for $\alpha = \frac{1}{2}$ when $n = 1$. This approach works with all polynomial functions and accepts any real integer or fraction for α. For different values of the fractional differentiation parameter α, this study investigates diverse behaviors by analyzing the identity function $x(t) = t$ and its fractional derivatives. The study shows fractional differentiation shifts in the function, with transitions from identity function to constant function and $\int(x(t)) = \frac{t^2}{2}$ at various α values. The geometric depiction in Figure 2.1 clarifies the evolving behaviors of $x(t)$ for orders 0^{th}, 0.5^{th}, 0.8^{th}, and 1^{st} with various x values. Surprisingly, at these orders, these fractional derivatives nearly match the classical calculus result $x'(t) = 1$.

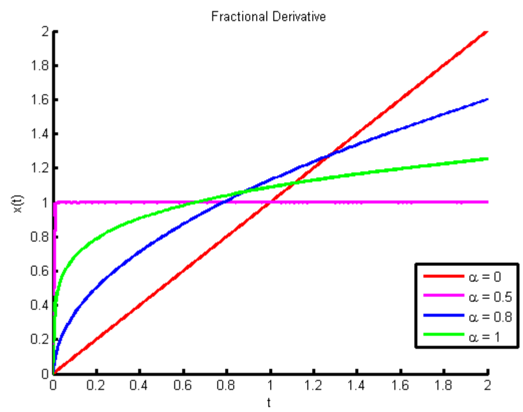

Figure 2.1: 0^{th}, 0.5^{th}, 0.8^{th}, 1^{th}derivatives of x(t)=t

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with $h = 10^{-2}$ at $0 < \alpha < 1$, the family of FD of $x(t) = t$ is depicted in the $3D$ plot.

The classical calculus line $x'(t) = 1$ is consistently approached by all curves on the left side for $\alpha = 0.1$. The identity function adheres to this classical property, as seen by the family with $\alpha = 0.5$ on the right side, which also shows a similar tendency to approach $x'(t) = 1$.

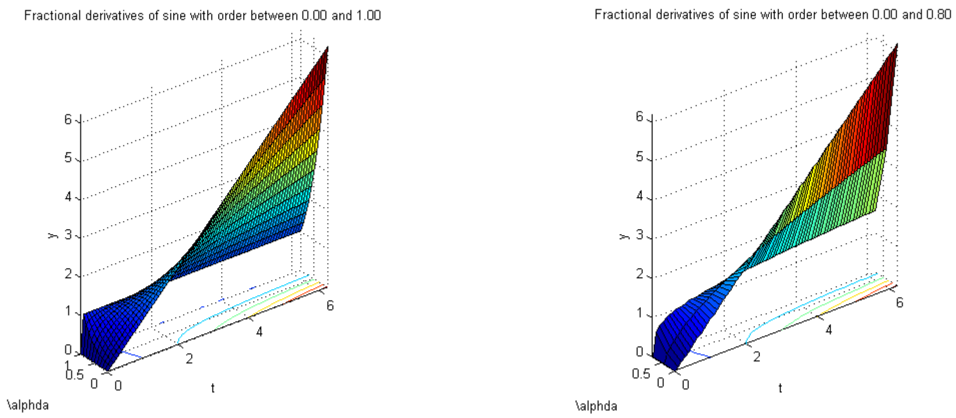


Figure 2.2: **Left:** The FD of the function $x(t) = t$ isrepresented on a $3D$ graph with step size $h = 0.01$ and order α changing from 0 to 1 in steps of 0.1. **Right:** The FD of $x(t) = t$ with order α ranging from 0 to 1 with a step of 0.5 isshown in the $3D$ graph on the right. The step size is $h = 0.01$

2.2.1 Analyzing Analytic Approaches for FI

Properties of FI [39].

i. ${}_aI_x^\alpha$ (f) $= \frac{1}{\Gamma(\alpha)} . \int_a^x (x-t)^{\alpha-1}$f(t).dt

ii. ${}_aI_x^\alpha$ (Cf) $=$ C ${}_aI_x^\alpha$(f)

iii. ${}_aI_x^\alpha\ (f \pm g) = {}_aI_x^\alpha(f) \pm {}_aI_x^\alpha(g)$

iv. ${}_aI_x^\alpha(\ {}_aI_x^\beta)) = {}_aI_x^{\alpha+\beta}(f) = {}_aI_x^\beta(\ {}_aI_x^\alpha(f))$

2.3 Non-Integer Orders of Differentiation to Integrate Constant Function

From the existing relation,
${}_aI_x^\alpha$ (Cf) $=$ C$\cdot {}_aI_x^\alpha$(f) $= \frac{1}{\Gamma(\alpha)} . \int_a^x (x-t)^{\alpha-1}$ C.dt $= C.\frac{(x-a)^\alpha}{\alpha\Gamma(\alpha)}$
For a $= 0$; ${}_0I_x^\alpha$ (C) $=$ C$\cdot \frac{x^\alpha}{\alpha\Gamma(\alpha)}$
For $\alpha = 1$; ${}_0I_x^1$ (C) $=$ C.x

For $\alpha = 2$; ${}_0I_x^2$ (C) $= \frac{Cx^2}{2}$.
The expression ${}_aI_x^\alpha$ (C) $= \frac{(x-a)^\alpha}{\alpha\Gamma(\alpha)}$ the two conclusions mentioned above were consistent with classical calculus, hence the expression is utilized to calculate the integral of the constant term.

2.4 Approximation Solution for Fractional Integral

In Figure 2.3, the integration outcomes at orders 0^{th}, 0.5^{th}, 0.7^{th}, and 1^{st} are almost identical to integrating $\int x(t) = \frac{t^2}{2}$. This results in a very similar approximate answer compared to the analytical solution. The above outcome indicates that analyzing a function's fractional integration yields its fractional derivative.

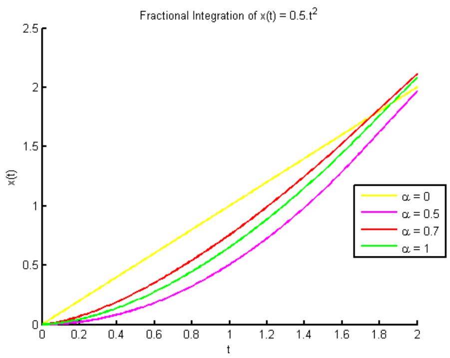


Figure 2.3: Fractional Integral of an identity function.

2.5 Computing the Half-Order Antiderivative of $\sin(x)$ through Analytical Means

To find the fractional derivative of $\sin(x)$ with respect to the $\frac{1}{2}^{th}$ order using Caputo's definition, an analytical function is defined as:

${}_a^cD_x^\alpha g(x) = \frac{N(\alpha)}{(1-\alpha)} \int_a^x e^{\frac{-\alpha(x-\alpha)}{1-\alpha}}.f'(t).dt$

when a = 0, N(α)=1, f(x) = sin(x), f'(t) = cos(t).

${}_0^cD_x^\alpha sin(x) = \frac{1}{(1-\alpha)} \int_0^x e^{\frac{-\alpha(x-\alpha)}{1-\alpha}}.cos(t).dt$

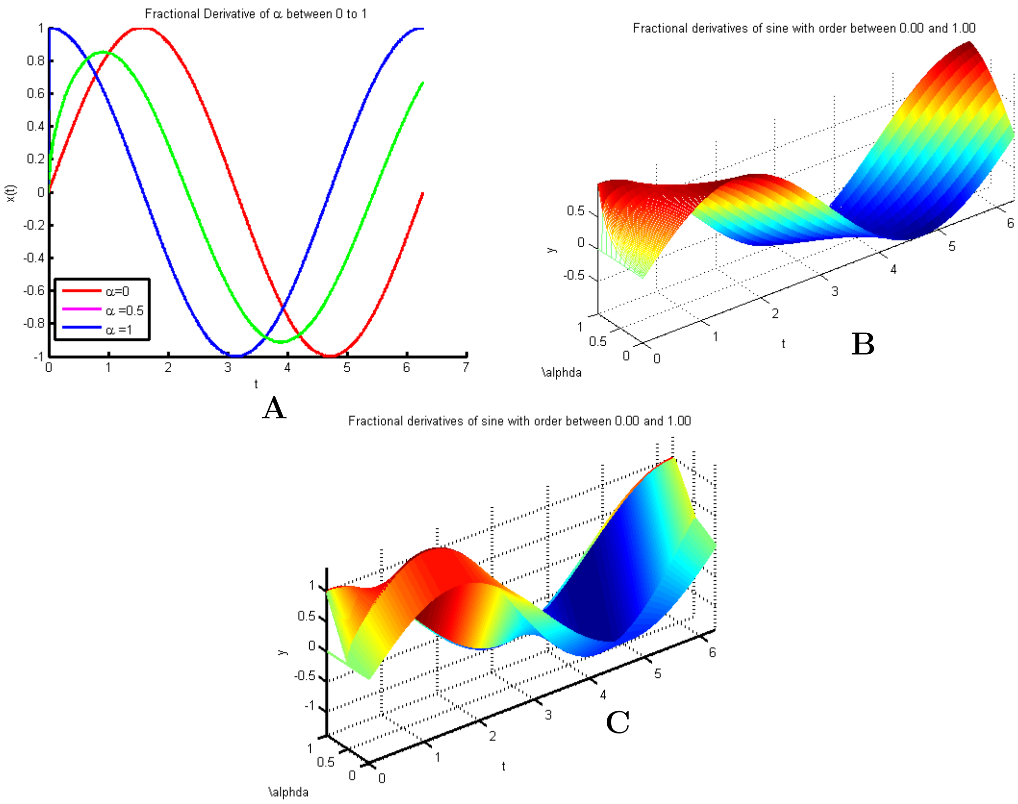
${}_0^cD_x^\alpha sin(x) = \frac{1}{(1-\alpha)}.e^{\frac{-\alpha}{(1-\alpha)x}} \int_0^x .e^{\frac{\alpha t}{1-\alpha}}\text{cost.dt}.$

${}_0^cD_x^\alpha sin(x) = \frac{1-\alpha}{(\alpha)^2+(1-\alpha)^2}$

${}_0^cD_x^\alpha sin(x) = \frac{\alpha}{1-\alpha}.cosx + sinx - \frac{\alpha}{1-\alpha}e^{\frac{-\alpha}{1-\alpha}x}$

The formula above gives, in the Caputo-Fabrizio sense, the half-order fractional sine function derivatives when $\alpha = \frac{1}{2}$. For various values of α, it is also possible to assess the fractional derivatives and integrals of additional functions such as cosine, e^x, cosine hyperbolic, and sine hyperbolic using either RL formula in the Caputo-Fabrizio sense.

Example 1 *Using computer software (Matlab), calculate the FD of the function* $y = \sin(t)$ *over* $[0, 1]$ *with the following steps:*



The figure in 2.4 displays the computation of a sine function's fractional derivatives utilizing the Grunwald-Letnikov (G-L) formulation and fractional derivatives in two dimensions. The computations are performed on a regular grid with varying step sizes and order values of α. In the initial illustration's left side, the green curve represents the fractional derivative in $2D$ at an α value of 0.5, indicating that a value halfway between 1 is utilized to construct the fractional derivative. The fractional derivative is a valuable tool for analyzing non-local behavior in functions and is defined utilizing RL fractional integral in $2D$. Yet, derivative is typically computed via a 2D grid. The Figure 2.4 describes the computation of fractional derivatives of a sine function using two different methods, Grunwald-Letnikov $(G - L)$ formulation and the fractional derivatives in 2D. The computations are performed on a regular grid with different step sizes and order values of α. In the first figure, a green curve is shown on the left side, which represents the fractional derivative in 2D at an α value of 0.5. This means that the fractional derivative is being computed using a value that is halfway between 1. The fractional derivative is defined using the Riemann-Liouville fractional integral in $2D$ and is a useful tool for analyzing non-local behavior in functions, but it is likely a method that uses a two-dimensional grid to compute the derivative. In the second figure, the fractional derivatives of the $3D$ sine function on the right-hand side are computed using the Grunwald-Letnikov formulation. This recursive formula uses the function values at various positions on a regular grid to calculate fractional derivatives. The step size is set to $h = 0.01$, and the order of the FD is changed from 0 to 1 in steps of 0.1. Furthermore, G-L derivative is also utilized to compute FD of $3D$ sine function using $h = 0.08$ and order values of α from 0.5 to 1 in steps of 0.5. The computed fractional derivatives from all sets of curves converge to $x'(t) = 1$. This indicates that as the step size and order values are altered, the calculated fractional derivatives approach the true derivative value of 1.The FD of sine in $3D$ function are also calculated using a step size of $h = 0.08$ and order values of alpha from 0.5 to 1 in steps of 0.5, in addition to the G-L derivatives. This calculation also employs the G-L formulation.

All curves converge to $x'(t) = 1$ as step size and order values vary, indicating that the computed fractional derivatives approach the real derivative value of 1.

2.5.1 Linearity

Given a linear operator and two functions, $A(x)$ and $B(x)$, let [39]

$$ {}_a^cD_x^\alpha A(x) \quad \text{and} \quad {}_a^cD_x^\alpha B(x) $$

exits. Let (n -1) < n, n $\in$ **R**, α, μ $\in$**C**. The Caputo functional derivative is

$$ {}_a^cD_x^\alpha(\mu A(x) + \gamma B(x)) = \mu \cdot_a^c D_x^\alpha A(x) + \gamma \cdot_a^c D_x^\alpha B(x) $$

The R-L operator holds,

$$ {}^RL_aD_x^\alpha(\mu A(x) + \gamma B(x)) = \mu \cdot^R L_aD_x^\alpha A(x) + \gamma \cdot^R L_aD_x^\alpha B(x) $$

Fractional Derivatives logically elaborates the one-parameter, two-parameter, and three-parameter MLF notions using mathematical expressions.

The formula for the Mittag-Leffler function of a single parameter is:

$$ E_{0,1}(z) = \frac{1}{1-z} $$

$$ \text{In general, } E_{1,m}(z) = \frac{1}{z^{m-1}}\left(e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!}\right) $$

In fact, specific instances of the MLF include hyperbolic sine and cosine functions.

$$\begin{aligned} E_{2,1}(z^2) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(2k)!} = cosh(z) \\ E_{2,2}(z^2) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(2k+1)!} = \frac{sinh(z)}{z} \end{aligned}$$

As an extension of the hyperbolic sine and cosine, the MLF can also be used to define.

$$E_i(z,p) = \sum_{k=0}^{\infty} \frac{z^{nk+i-1}}{nk+i-1} = z^{i-1} E_{n,i}(z^n),$$

as well as the n-order trigonometric function that yields the sine and cosine functions.

$$k_r(z,n) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{nj+r-1}}{(nj+r-1)!} = z^{r-1} E_{n,r}(-z^n)$$

2.5.2 Derivatives of Mittag-Leffler function

$$\frac{d}{dz} E_{\alpha,\beta}(z) = \frac{d}{dz} \sum_{\nu=0}^{\infty} \frac{z^\nu}{\Gamma(\alpha\nu+\beta)}$$

$$= \sum_{\nu=0}^{\infty} \frac{\nu z^{\nu-1}}{\Gamma(\alpha\nu+\beta-1)\Gamma(\alpha\nu+\beta-1)}$$

$$= \frac{1}{\alpha} \sum_{\nu=0}^{\infty} \frac{\Gamma(\alpha\nu+\beta-1-(\beta-1)]z^{\nu-1}}{(\alpha\nu+\beta-1)\Gamma(\alpha\nu+\beta-1)}$$

$$= \frac{1}{\alpha z} \sum_{\nu=0}^{\infty} \frac{z^{\nu-1}}{\Gamma(\alpha\nu+\beta-1)} - \frac{\beta}{\alpha z} \sum_{\nu=0}^{\infty} \frac{z^\nu}{\Gamma(\alpha\nu+\beta-1)}$$

$$\frac{d}{dz} E_{\alpha,\beta}(z) = \frac{1}{\alpha \cdot z} E_{\alpha,\beta-1}(z) - \frac{\beta-1}{\alpha z} E_{\alpha,\beta}(z)$$

Chapter 3

Physical Mathematical Model and Numerical Methods

We provide a new numerical technique to estimate Caputo-Fabrizio fractional derivatives via quadratic interpolation, which we call the L1-2 formula. In order to achieve quadratic and cubic convergence rates, the L1 and L2 methods are developed for Caputo fractional derivatives, and the L2 method for Caputo-Fabrizio derivatives. By including Lagrange interpolation, the L1-2 formula becomes more accurate and useful in a wider range of mathematical and scientific fields. We provide numerical evidence for the expected convergence rates and demonstrate how these techniques may be used to calculate Caputo-Fabrizio derivatives for standard functions. In the L1 and L1-2 approaches, Joseph-Louis Lagrange interpolation is used to provide quadratic and cubic convergence rates, guaranteeing solid and trustworthy numerical approximations.

3.0.3 L_1 Method for the CFD

The left Caputo Fractional Derivative can be defined as follows,

$$\begin{aligned}
{}_0^C\mathbf{D}_t^\alpha f(x) &\approx \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f'(s) \cdot ds, 0 < \alpha < 1 \\
{}_0^C\mathbf{D}_t^\alpha f(x_n) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{x_n} (x_n-s)^{-\alpha} f'(s) \cdot ds \\
{}_0^C\mathbf{D}_t^\alpha f(x_n) &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} (x_n-s)^{-\alpha} f'(s) ds \\
{}_0^C\mathbf{D}_t^\alpha f(x_n) &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} (x_n-s)^{-\alpha} f'(s) ds
\end{aligned} \tag{3.1}$$

proceed by substituting the forward difference quotient for the first-order derivative in the given expression

$$ {}_{0}^{C}\mathbf{D}_{t}^{\alpha} f(x_n) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} (x_n - s)^{-\alpha} \Big[\frac{(f(x_{k+1}) - f(x_k)}{h}\Big] ds $$

$$ {}_{0}^{C}\mathbf{D}_{t}^{\alpha} f(x_n) \approx \frac{-1}{h(1-\alpha)\Gamma(1-\alpha)} \sum_{k=0}^{n-1} [f(x_{k+1} - f(x_k)][(x_n - (x_{k+1})^{-\alpha+1} - (x_n - x_k)^{-\alpha+1}] \tag{3.2} $$

Under relation, x_n = nh, x_{k+1} = (k+1)h , x_k = kh

$$ {}_{0}^{C}\mathbf{D}_{t}^{\alpha} f(x_n) \approx \frac{h^{\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)][(n-k)^{(1-\alpha)} - (n-k-1)^{(1-\alpha)}] \tag{3.3} $$

The L1 method is commonly used for estimating Caputo Fractional Derivatives in functions. We've conducted a detailed computer analysis of this numerical approach, comparing it to the exact solution for validation and comprehension.

3.0.4 L_2 Method for the Caputo Fractional Derivatives

The left caputo fractional derivative

$$ {}_{0}^{C}\mathbf{D}_{x}^{\alpha} f(x) = \frac{1}{\Gamma(2-\alpha)} \int_{0}^{x} (x-s)^{1-\alpha} f''(s) \cdot ds, \quad 1 < \alpha < 2 \tag{3.4} $$

Consider a time partition $0 = x_0 < x_1 < x_2 < \cdots < x_N = T$ on the interval [0,T], where h represents step size.

$$ \int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx \tag{3.5} $$

Using this property

$$ {}_{0}^{C}\mathbf{D}_{x}^{\alpha} f(x) = \frac{1}{\Gamma(2-\alpha)} \int_{0}^{x_n} (s)^{1-\alpha} f''(x_n - s) \cdot ds, \tag{3.6} $$

From elimentary calculus,

$$ {}_{0}^{C}\mathbf{D}_{x}^{\alpha} f(x_n) = \frac{1}{h^2(2-\alpha)\Gamma(2-\alpha)} (s)^{1-\alpha} \sum_{k=0}^{n-1} [f(x_n - x_{k+1}) - 2f(x_n - x_k) + f(x_n - x_{k-1})][(x_{k+1})^{(2-\alpha)} \tag{3.7} $$

L_2 Method for the caputo fractional derivatives is obtained as;

$$ {}_{0}^{C}\mathbf{D}_{x}^{\alpha} f(x_n) = \sum_{k=0}^{n-1} w_k f(x_{n-k}) \tag{3.8} $$

where w_k are the cofficient of $f(x_{n-k})$. The L2 method for Caputo Fractional Derivative is widely recognized as a prevalent technique for approximating these derivatives in functions. We've conducted an extensive analysis of this numerical approach and compared it rigorously with the exact solution for validation.

3.0.5 L_1 approach for Caputo-Fabrizio derivative

The following defines the Caputo-Fabrizio differential operator

$$ {}_0^{CF}\mathbf{D}_x^{\alpha} f(x) = \frac{M(\alpha)}{(1-\alpha)} \int_0^x f'(s) exp[\lambda(x-s)]ds, \lambda = \frac{-\alpha}{(1-\alpha)}, \alpha \in (0,1) \tag{3.9} $$

$$ {}_0^{CF}\mathbf{D}_x^{\alpha} f(x_n) = \frac{M(\alpha)}{(1-\alpha)} \int_0^{x_n} f'(s) exp[\lambda(x_n-s)]ds, \tag{3.10} $$

$$ {}_0^{CF}\mathbf{D}_x^{\alpha} f(x_n) = \frac{M(\alpha)}{(1-\alpha)} \sum_{k=1}^{n} \int_{x_k-1}^{x_k} exp[\lambda(x_n-s)][\frac{f(x_k)-f(x_{k-1})}{h}]ds \tag{3.11} $$

$$ {}_0^{CF}\mathbf{D}_x^{\alpha} f(x_n) = \frac{M(\alpha)}{h(1-\alpha)} \sum_{k=1}^{n} [f(x_k)-f(x_{k-1})] \int_{x_{k-1}}^{x_k} exp[\lambda(x_n-s)]ds \tag{3.12} $$

Integrating by power rule;

$$ {}_0^{CF}\mathbf{D}_x^{\alpha} f(x_n) = \frac{M(\alpha)}{-\lambda h(1-\alpha)} \sum_{k=1}^{n} [f(x_k)-f(x_{k-1})][exp[\lambda(x_n-x_k)] - exp[\lambda(x_n-x_{k-1})]ds \tag{3.13} $$

Using the relations: $x_n = nh$, $x_{k-1} = (k-1)h$, $x_k = kh$.

$$ {}_0^{CF}\mathbf{D}_x^{\alpha} f(x_n) = \frac{M(\alpha)}{\alpha h} \sum_{k=1}^{n} [f(kh) - f(k-1)h][exp[\lambda h(n-k)] - exp[\lambda h(n-k+1)] \tag{3.14} $$

We introduce the L_1 Method for the Caputo Fabrizio derivative, employing linear Lagrange interpolation with three points, also known as backward approximation. This method is applicable for equations with an α range of $1 < \alpha < 2$, representing quadratic Lagrange interpolation.

3.1 Test Examples:

The relationship between the function $f(x) = e^{-x}$ and its L1, FD $D^{0.5}f(x)$ is shown in a single figure. Exponential decay is shown by the function $f(x) = e^{-x}$. Because it makes use of the whole function history and is less impacted by outliers and data noise than the first derivative,

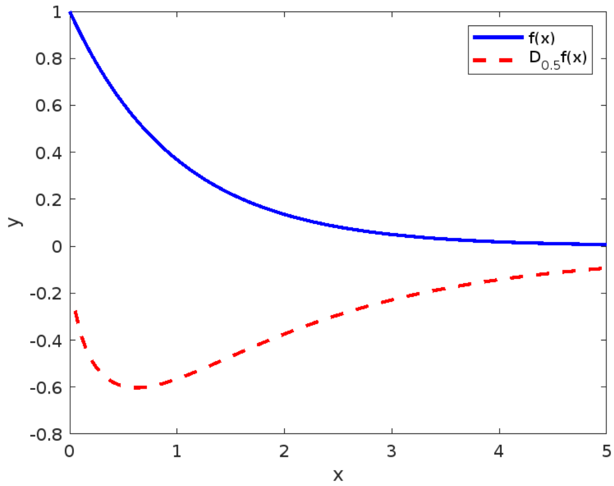


Figure 3.1: L_1 Fractional Derivative, $D^{0.5}f(x)$

Example 2 *The results are presented in the table, obtained using the L1 formula for function $f(x) = \sin(x)$ at an interval $[0, 1]$, with $\alpha = 0.5$.*

Table 3.1: Numerical result of L1 formula

Steps	Stepsize	Exact	Approximation	Error
10	1×10^{-1}	8.4606×10^{-1}	8.5592×10^{-1}	9.8624×10^{-3}
100	1×10^{-2}	8.4606×10^{-1}	8.4724×10^{-1}	1.1873×10^{-3}
1000	1×10^{-3}	8.4606×10^{-1}	8.4617×10^{-1}	1.1110×10^{-4}
10000	1×10^{-4}	8.4606×10^{-1}	8.4607×10^{-1}	1.0740×10^{-5}
100000	1×10^{-5}	8.4606×10^{-1}	8.4606×10^{-1}	1.0611×10^{-6}
1000000	1×10^{-6}	8.4606×10^{-1}	8.4606×10^{-1}	1.0569×10^{-7}

The L_1 fractional derivative provides more consistent results. Table 3.1 illustrates the numerical outcomes using the $L1$ Caputo fractional derivative scheme for $f(x) = \sin(x)$ at an interval $[0, 1]$ employing the $L1$ formula for Caputo derivative. The table consists of five columns.

From table, when the step size is reduced, the $L1$-Caputo-Der formula's numerical output becomes closer to the integral's exact value. As the step size decreases and the error decreases, the numerical approximation's accuracy rises. The error at $h = 0.0001$ should be less than 0.00001, a very little inaccuracy, to indicate a very accurate numerical approximation.

Example 3 *The table below displays the values obtained from the equation $f(x) = \sin(x)$ at an interval $[0, 1]$ with $\alpha = 1.5$ using L_2 formula.*

Table 3.2: Numerical reslut of $L2$ formula

Steps	Stepsize	Exact	Approximation	Error
10	1×10^{-1}	-6.6968$\times10^{-1}$	-6.2186$\times10^{-1}$	4.7820$\times10^{-2}$
100	1×10^{-2}	-6.6968$\times10^{-1}$	-6.6531$\times10^{-1}$	4.3702$\times10^{-3}$
1000	1×10^{-3}	-6.6968$\times10^{-1}$	-6.6926$\times10^{-1}$	4.2716$\times10^{-4}$
10000	1×10^{-4}	-6.6968$\times10^{-1}$	-6.6964$\times10^{-1}$	4.2431$\times10^{-5}$
100000	1×10^{-5}	-6.6968$\times10^{-1}$	-6.6968$\times10^{-1}$	4.2307$\times10^{-6}$

Using the scheme 3.7 throughout the range $[0, 1]$, Table 3.2 shows that $\alpha = 1.5$ is the value that was used in the approximation for the various step sizes to approximate function $f(x) = sin(x)$ applying L2 Caputo derivative. The accuracy of numbers is affected by the step size. Smaller step size frequently yields a more accurate estimate. Although it takes short span of time to compute, a bigger step size yields a less accurate approximation. Choosing a step size is a challenge since accuracy and computing efficiency must be balanced.

Example 4 *The L1-formula in the sense of Caputo Fabrizio may be used to solve $f(x) = cos(4x)$, an interval $[0, 2]$, using $\alpha = 0.1$ with various step sizes.*

Table 3.3: Numerical result of L1 formula for CFD

Steps	Steps Sizes	Exact	Approx.	Error
200	1×10^{-1}	-1.0811	-1.0807	4×10^{-4}
2000	1×10^{-2}	”	-1.0811	4×10^{-6}
20000	1×10^{-3}	”	”	4×10^{-8}
200000	1×10^{-4}	”	”	4×10^{-10}

Quadratic second-order convergence is frequently indicated by a situation where the absolute error falls by a second order of magnitude while the step size decreases by just one. Stated differently, we're not only advancing, but we're doing so really quickly.

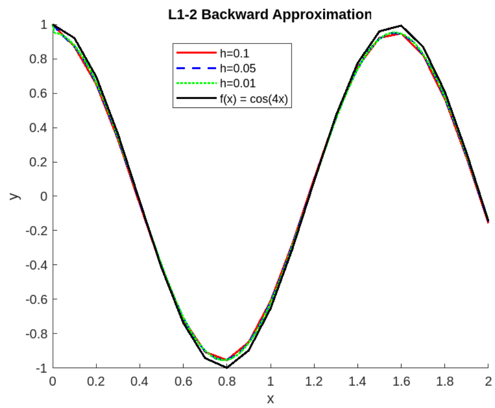
Example 5 *To tackle the task of solving $f(x) = \cos(4x)$ across the interval $[0, 2]$ with $\alpha = 0.1$ and different step sizes, the $L1-2$ (Backward Approximation) method for the Caputo-Fractional Derivative in the Caputo-Fabrizio context can be utilized.*

If we see a three order reduction in the absolute error and a one order reduction in the step size, it has a cubic rate of convergence at third order. To put it another way, relative

Table 3.4: Numerical result of L1-2 formula for CFD

Steps	Steps Sizes	Exact	Approx.	Error
20	1×10^{-1}	-1.0811	-1.0811	2.9514×10^{-7}
200	1×10^{-2}	”	”	2.9090×10^{-11}
40	0.05	”	”	1.8377×10^{-8}

to the $L1$-formula, our development is not only significant but also moving pretty quickly.



Our initial function $f(x) = \cos(4x)$ is considered, with step sizes $h_1 = 0.1$, $h_2 = 0.05$, and $h = 0.01$ across the interval $[0, 2]$. The $L1 - 2$ backward approximation technique is used to produce a curve for every step size. The image shows how, as the step size h decreases, the approximate solutions of the equation $f(x) = \cos(4x)$ tend to converge towards the real result. The goal of this approach is to increase the accuracy of predicting the derivatives of the function with minimal steps.

Chapter 4

SUMMARY

Fractional calculus is being used more and more in many fields, however it can be difficult to compute fractional integrals and derivatives numerically. Computer tools can be used in conjunction with Riemann-Liouville and Caputo-Fabrizio techniques to speed up computations. By using the G-L method, geometrical approaches demonstrate remarkable accuracy in FOD creation and function integrals. Unity of first-order fractional derivatives is obtained.

Our approach is unique across domains. We start fractional order differential equations using the MLF and check the correctness of the answers by comparing them with normal calculus. We prove numerical solutions for single, double, and triple parameters in real and complex variables, assessing MLF convergence for two parameters and concentrating on numerical solutions for FD equations.

Adomian decomposition, Homotopy perturbation, Variation iteration, and matrix methods are among the prevalent numerical techniques. Numerical approximations to the Caputo Fabrizio derivative are emphasized; they include the L_1, L_2, and $L1-2$ equations, which produce, respectively, fourth-order, quadratic, and cubic convergence rates.

Outcomes match known functions reliably in a range of order and step sizes. Every approach has benefits and cons. In contrast to the singularity-induced complexity of Riemann-Liouville, Grnwald-Letnikov is simple yet sluggish. L1-2 has promise despite accuracy issues, as seen by its quicker convergence than Caputo. An ideal approach depends on trade-offs between simplicity, precision, and efficiency that are unique to each application. Analyzing L1-2's noise and error resistance is essential for determining how well it performs versus other numerical approaches and at different disturbance levels.

Published Paper

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