QUANTUM CALCULUS: NEW DEVELOPMENT AND APPLICATIONS

A THESIS SUBMITTED TO THE

CENTRAL DEPARTMENT OF MATHEMATICS
INSTITUTE OF SCIENCE AND TECHNOLOGY
TRIBHUVAN UNIVERSITY
NEPAL

FOR THE AWARD OF
DOCTOR OF PHILOSOPHY
IN MATHEMATICS

BY

PITAMBER TIWARI

JUNE 2024

QUANTUM CALCULUS: NEW DEVELOPMENT AND APPLICATIONS

A THESIS SUBMITTED TO THE

CENTRAL DEPARTMENT OF MATHEMATICS
INSTITUTE OF SCIENCE AND TECHNOLOGY
TRIBHUVAN UNIVERSITY
NEPAL

FOR THE AWARD OF
DOCTOR OF PHILOSOPHY
IN MATHEMATICS

BY

PITAMBER TIWARI

JUNE 2024

TRIBHUVAN UNIVERSITY
Institute of Science and Technology
DEAN'S OFFICE
Kirtipur, Kathmandu, Nepal

Reference No.:

EXTERNAL EXAMINERS

The Title of Ph.D. Thesis: " Quantum Calculus: New Development and Applications

Name of Candidate: Pitamber Tiwari

Internal Examiner:

Prof. Dr. Chinta Mani Pokhrel
Institute of Engineering
Tribhuvan University, NEPAL

External Examiners:

(1) Prof. Dr. Kanhaiya Jha
Kathmandu University
Dhulikhel, NEPAL

(2) Prof. Dr. Arun Pal Singh
Dayal Singh College (University of Delhi)
Delhi, INDIA

(3) Prof. Dr. Dhruba Adhikari
Kennesaw State University
USA

13 June, 2024

(Dr. Surendra Kumar Gautam)
Asst. Dean

Telephone: 977-1-4330844, 4331755 (Administration), 4330120 (Exam)
Website: www.iost.tu.edu.np, E-mail: info@iost.tu.edu.np, exam@iost.tu.edu.np

DECLARATION

Thesis entitled “**QUANTUM CALCULUS: NEW DEVELOPMENT AND APPLICATIONS**”which is being submitted to the Central Department of Mathematics, Institute of Science and Technology (IoST), Tribhuvan University, Nepal for the award of the degree of Doctor of Philosophy (Ph.D.), is a research work carried out by me under the supervision of Prof. Dr. Chet Raj Bhatta of Central Department of Mathematics, Tribhuvan University and co-supervised by Prof. Dr. Pankaj Jain, South Asian University, New Delhi, India.

This research is original and has not been submitted earlier in part or full in this form or any other form to any University or Institute, here or elsewhere, for the award of any degree.

....
Pitamber Tiwari

RECOMMENDATION

This is to recommend that **Mr. Pitamber Tiwari** has carried out research entitled **"Quantum Calculus: New Development and Applications"** for the award of Doctor of Philosophy (Ph.D.) in **Mathematics** under our supervision. To our knowledge, this work has not been submitted for any other degree.

He has fulfilled all the requirements laid down by the Institute of Science and Technology (IoST), Tribhuvan University, Kirtipur for the submission of the thesis for the award of Ph.D. degree.

....

Prof. Dr. Chet Raj Bhatta
Supervisor
Central Department of Mathematics
Tribhuvan University, Kirtipur, Kathmandu, Nepal

Pankaj Jain 29 December 2023

....

Prof. Dr. Pankaj Jain
Co-supervisor
South Asian University
New Delhi, India

June 13, 2024

Tribhuvan University
Institute of Science and Technology
Central Department of Mathematics

LETTER OF APPROVAL

Date: June 13, 2024

On the recommendation of **Prof. Dr. Chet Raj Bhatta** and **Prof. Dr. Pankaj Jain**, this Ph.D. thesis submitted by **Mr.Pitamber Tiwari**, entitled "**Quantum Calculus: New Development and Applications**" is forwarded by Central Department Research Committee (CDRC) to the Dean, IoST, T.U.

....
Prof. Dr. Chet Raj Bhatta
Head,
Central Department of Mathematics
Tribhuvan University
Kirtipur, Kathmandu, Nepal

ACKNOWLEDGMENTS

Truly speaking, the Ph.D. study has been a great experience and an unforgettable evidence for me to energize my effort and way of thinking that how one should go for an in-depth study of the domain of study one chooses, and it would have not been possible without the support and guidance that I have received during my tenure of study.

Firstly, I owe a great deal to my humble and friendly supervisor as well as the Head of Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal, Prof. Dr. Chet Raj Bhatta, for patiently provided the track, the vision, the boost, and a frequent pushing and advice necessary to me to pursue my research domain and the reward for my dissertation. At the same time, I would like to express my heartiest thanks to my supervisor for his excellent guidance and caring as well as tremendous effort to gear up my academic achievement.

I am deeply indebted to my co-supervisor, Prof. Dr. Pankaj Jain, South Asian University, New Delhi, India, without whose proper guidance, I would have been lost in an infinite ocean, whose exemplary suggestions motivated me time and again to move ahead in my research intact.

I would also like to express my sincere gratitude to the former Head of the Central Department of Mathematics, Prof. Dr. Tank Nath Dhamala and Prof. Dr. Kedar Nath Upreti, for their valuable suggestions and ideas to carry out this endeavor. Likewise, it will be unfair if I miss to address the names of Prof. Dr. Prakash Muni Bajracharya, Prof. Dr. Narayan Prasad Pahari, Prof. Dr. Ajaya Singh, Associate Prof. Dr. Durga Jang K. C., Associate Prof. Dr. Shree Ram Khadka, and all the faculties as well as administrative staff of the Central Department of Mathematics who directly or indirectly assisted me to carry out my study ahead.

I wish to express my sincere thanks to all the academic and administrative staff of the Institute of Science and Technology (IoST), Tribhuvan University, Kirtipur, Kathmandu.

At last but not the least, I would like to express my special thanks to all my dear colleagues of Bhairahawa Multiple Campus including Campus Chief Associate Prof. Anil Kumar Jha who directly or indirectly assisted me to carry out this adventurous journey to this level. I also feel pleasant to remember and enlist my family members and relatives father Mr. Nam Dev Tiwari, mother late Dila Kumari Tiwari, my role model brother Narayan Prasad Tiwari, my caring dear wife and colleague Lec. Aruna Kumari Bhusal, son Dr. Apil Tiwari and daughter Alisha Tiwari who persistently pushed me to move for a higher study.

Likewise, my special thanks go to all my relatives, friends, students and well-wishers who wish for my success.

Pitamber Tiwari
June, 2024

शोधसार

सिमा विनाको आधारमा खोज गरिएको calculus लाई quantum calculus वा q-calculus भनिन्छ । यो calculus को माध्यमबाट discrete (खण्डित) घट्नाहरूको पनि अनुमान गर्न मद्दत मिल्छ । Theory of convex functions र Theory of Inequality मा Hermite-Hadamard प्रकारको integral inequality एक अभिन्न inequality हो । यो inequality को मद्दतबाट कुनै पनि convex function को लागि माथिल्लो र तल्लो सिमा निर्धारण गर्न सकिन्छ । यस किसिमको inequality हरू generalized means, information measures, quadratures rules इत्यादिमा प्रयोग गरिन्छ । यो Thesis मा केहि नयां प्रकारका convexities लाई generalized form मा परिभाषित गरि तिनिहरूको Hermite-Hadamard प्रकारका integral inequality प्रमाणित गरिएका छन् । यसरी प्रमाणित गरिएका inequality लाई quantum calculus को स्वरूपमा पनि प्रमाणित गरिएका छन् । B. G. Pachpatte ले प्रस्तुत गरेको product of classical convex functions नतिजाको आधारमा अन्य प्रकारका convex functions जस्तै geometrically- arithmetic, harmonically – arithmetic, geometrically- geometric convex function हरूलाई generalized form मा प्रस्तुत गरी तिनिहरूका product form को नतिजाहरू प्रमाणित गरिएको छ । यसरी प्राप्त नतिजाहरूलाई q-calculus मा extension गरिएका छन् । यो Thesis मा harmonically- convex functions को लागि Riemann-Liouville fractional integral को माध्यमबाट Hermite-Hadamard integral inequality प्रमाणित गरी तिनिहरूको product form को नतिजा पनि प्रस्तुत गरिएका छन् । भविष्यमा अन्य अनुसन्धानकर्ताहरूले Theory of convex functions र Theory of inequalities मा बिभिन्न प्रकारका convex functions को Hermite- Hadamard प्रकारका integral inequality र तिनिहरूको quantum version को domain मा काम गर्न सक्नेछन् ।

कीवर्डहरूः Hermite-Hadamard inequality, GA-convexity, HA-convexity, HG-convexity, q-calculus, fractional calculus.

ABSTRACT

The investigation of the calculus without the notion of limits is quantum calculus or q-calculus. This calculus is applied to estimate the discrete phenomena as well. The famous result on the theory of convex functions is the Hermite-Hadamard type integral inequality which provides a lower and an upper estimation for the integral average of any convex function defined on a compact interval involving the mid-point and the end points of the domain having a wider application for generalized means, information measures, quadrature rules, and many more. In this thesis, some new convexities related to harmonic convexity have been defined and for each new convexity, a corresponding Hermite-Hadamard inequality has been proved. The q-analogues of these inequalities have also been obtained. Apart from this, in this thesis we have generalized harmonically convex function to m-harmonically convex function and m-harmonically P-function and have obtained some Hermite-Hadamard type integral inequalities whose first order derivatives are m-harmonically convex functions. The integral mean of a convex function is connected with the Hermite-Hadamard inequality. From the results on the products of two classical convex functions as given by B. G. Pachpatte, in this thesis, we have been able to establish some new results on the products of generalized geometric convex functions, and these results are further extended to q-analogues which is a new paradigm in convexity theory and integral inequality. Diagnosing the idea of the classical convex functions, it is enhanced into m- convex functions which has helped to extend a convex function's equality of quantum estimate into an m-convex function. Using the resulted information, a few novel Hermite-Hadamard integral inequalities are established whose first order q-derivatives are m-convex functions, and these results are presented in q-analogues too. In this thesis, Hermite-Hadamard's inequalities via Riemann-Liouville fractional integral for the case of harmonically convex function as well as the products of two harmonically convex functions via Riemann-Liouville fractional integrals are also established. Extensions and refinements of Hermite-Hadamard type integral inequality for various types of convex functions in quantum calculus may be the interested domains for the researchers in future.

Keywords: *Hermite-Hadamard inequality, GA- convexity, HA-convexity, HG-convexity, q-calculus, fractional calculus.*

LIST OF SYMBOLS

$\mathbb{R}(-\infty,\infty)$	The set of Real Numbers.
$\mathbb{R}_+(0,\infty)$	The set of Positive Real Numbers.
$\mathbb{R}^n$	The n-dimensional space.
$[a,b]$	The closed interval.
(a,b)	The open interval.
$\int_a^b f(x)dx$	The definite integral.
$\int_a^b f(x)\,{}_ad_qx$	The q-integral of $f(x)$
D_q	q-derivative
Φ	The empty set.
$L[a,b]$	The line segment joining a and b.
Φ, Ψ, f, g	The real valued functions.
Γ	The gamma function.
$\mathbb{H}$	The harmonic convex set.
The arithmetic mean: $A(\alpha,\beta)$	$\frac{\alpha+\beta}{2}, \quad \alpha,\beta\in\mathbb{R}.$
The weighted arithmetic mean: $A_t(\alpha,\beta)$	$t\alpha+(1-t)\beta, \quad \alpha,\beta\in\mathbb{R}.$
The geometric mean: $G(\alpha,\beta)$	$\sqrt{\alpha\beta}, \quad \alpha,\beta\in\mathbb{R}_+.$
The harmonic mean: $H(\alpha,\beta)$	$\frac{2\alpha\beta}{\alpha+\beta}, \quad \alpha,\beta\in\mathbb{R}\backslash\{0\}.$
The weighted harmonic mean: $H_t(\alpha,\beta)$	$\left(\frac{t}{\alpha}+\frac{1-t}{\beta}\right)^{-1}, \quad \alpha,\beta\in\mathbb{R}\backslash\{0\}.$
The logarithmic mean: $L(\alpha,\beta)$	$\begin{cases}\alpha & \text{if}\,\alpha=\beta\\ \frac{\beta-\alpha}{\log\beta-\log\alpha} & \text{if}\,\alpha\neq\beta\end{cases}, \quad \alpha,\beta\in\mathbb{R}_+.$
The generalized logarithmic mean: $L_n(\alpha,\beta)$	$\begin{cases}\alpha & \text{if}\,\alpha=\beta\\ \left(\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right)^{\frac{1}{n}} & \text{if}\,\alpha\neq\beta\end{cases}, n\in\mathbb{R}\backslash\{-1,0\}.$
The identric mean: $I(\alpha,\beta)$	$\begin{cases}\alpha & \text{if}\,\alpha=\beta\\ \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}} & \text{if}\,\alpha\neq\beta\end{cases}, \quad \alpha,\beta\in\mathbb{R}_+.$
The power mean: $M_r(\alpha,\beta)$	$\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}}, \quad r\geq 1,\,\alpha,\,\beta\in\mathbb{R}_+.$

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

1.1 The Origin of Quantum Calculus

Calculus or infinitesimal calculus has a fascinating history. It literally means for small pebbles but stands for the study of rate of change of physical phenomena which are continuous in nature around us. It has two major wings viz. Differential Calculus and Integral Calculus which were invented by Issac Newton(1642-1727) and Gottfried Wilhelm Leibniz(1646-1716), independently in the middle of the 17th century which is based on the concept of limits although the elements of it had already been appeared in ancient Greece. The word 'quantum' appears to have been derived from the Latin word 'quantus' is termed as the smallest possible discrete unit of any physical property such as energy, matter or electromagnetic radiation, etc. The usual meaning of limits implies that the space and time are continuous and is maintained that all natural processes happen continuously on smooth curves and surfaces. However, the atomic theory in Physics and Chemistry in the 19th century paved a mark that the natural process of dividing into smaller parts will never terminate in an indivisible part or an atom, a part which lacks in proper parts ahead, i.e., continua are divisible without limits. This phenomenon necessitates the origin of developing another type of calculus, called 'Quantum Calculus' which is based on the 'Finite Difference Principle' or calculus without limits. A version of calculus where neither the smoothness of the function is required nor the limit is applied is quantum calculus, it is a sub-field of more general mathematical field of time-scale calculus which always provides a unified framework for studying dynamics equations on both discrete and continuous domains.

The study on q-calculus dates back to Leonhard Euler when he commenced in 1748 [15] by considering the infinite product

$$(q;q)_{\infty}^{-1} = \Pi_{k=0}^{\infty} \frac{1}{1-q^{k+1}}, \quad -1 < q < 1$$

as a generating function for $p(n)$, the partition function $p(n)$ is the number of ways to write n as a sum of positive integers. He, furthermore, discovered the first two q-exponential functions, prelude to binomial theorem. A hundred year later, in 1846, E. Heine [17] considered a generalization of the q-hyper-geometric series. But q-calculus became popular after the paper by Albert Einstein to its usefulness in quantum mechanics in 1905. The systematic enhancements on q-calculus were initiated by F.H. Jackson in 1908 [28] and thereafter. The q-calculus has various dialects as 'Quantum Calculus'; 'Time-Scale Calculus' or 'calculus of partitions'. Two types of quantum calculi, h and q are defined as follows: See [27] for details.

Definition 1.1.1. *Let $\Phi(x)$ be an arbitrary function. Then the q-derivative of a function Φ is defined as*

$$D_q\Phi(t) = \frac{\Phi(qt) - \Phi(t)}{(q-1)t}, \tag{1.1.1}$$

where q is a fixed number different from 1 and, the h-derivative is defined as

$$D_h\Phi(t) = \frac{\Phi(t+h) - \Phi(t)}{h}, \tag{1.1.2}$$

where h is a fixed number different from 0.

If $\Phi(t)$ is differentiable and as $q \to 1$ or $h \to 0$, then the q-derivative or h-derivative is simply the classical derivative, denoted by $\frac{d}{dt}\Phi(t)$,

$$\frac{d}{dt}\Phi(t) = \lim_{h\to 0} \frac{\Phi(t+h) - \Phi(t)}{h} \tag{1.1.3}$$

i.e.,

$$\lim_{q\to 1} D_q\Phi(t) = \lim_{h\to 0} D_h\Phi(t) = \frac{d}{dt}\Phi(t) \quad \textit{or} \quad \Phi'(t)$$

In 1908, F. H. Jackson [28] re-introduced, the Euler- Jackson q-difference operator, called the q-derivative of Φ which is defined as follows:

Definition 1.1.2. *Let $\Phi : (0,b) \to \mathbb{R}$, $\;b \in (0,\infty)$ be a continuous function. Then the q-derivative is defined as*

$$D_q\Phi(t) = \frac{\Phi(qt) - \Phi(t)}{(q-1)t}, \quad q \in (0,1), \quad t \in (0,b). \tag{1.1.4}$$

It is clear that if $\Phi(t)$ is differentiable, then

$$\lim_{q\to 1} D_q\Phi(t) = \frac{d}{dt}\Phi(t),$$

The q-derivative is a discretization of ordinary derivative and therefore it has an immediate application in numerical analysis.

In 1910, F. H. Jackson [28] introduced the concept of q-definite integrals extending the concepts of q-calculus. The q-definite integral is defined as follows:

Definition 1.1.3. *Let $\Phi : [0, \infty) \to \mathbb{R}$ be a continuous function. Then the q-integral of Φ on $(0, x)$ is*

$$\int_0^x \Phi(t) d_q t = (1-q)x \sum_{n=0}^{\infty} q^n \Phi(q^n x), \quad x \in (0, \infty), \quad 0 < q < 1 \tag{1.1.5}$$

and, the improper q-integral is defined as

$$\int_0^{\infty} \Phi(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n \Phi(q^n), \quad 0 < q < 1, \tag{1.1.6}$$

provided that the series on right hand side converges absolutely.

Tariboon and Ntouyas [55] created an entirely novel study, acquired numerous q-analogues of classical mathematical objects, and presented the notion of quantum calculus on finite intervals. They have extended some prominent integral inequalities to the q-calculus which stimulated other investigators and as a result multiple important outcomes via quantum equivalents of classical mathematical results were put forward in the literature. They extended the notion of q-derivative and q-integrals as follows:

Definition 1.1.4. *Assume that $\Phi : [\alpha, \beta] \to \mathbb{R}$ is a continuous function. Then*

$${}_{\alpha}D_q \Phi(t) = \frac{\Phi(t) - \Phi(qt + (1-q)\alpha)}{(1-q)(t-\alpha)}, \quad t \neq \alpha, \quad 0 < q < 1 \tag{1.1.7}$$

and

$${}_{\alpha}D_q \Phi(\alpha) = \lim_{t \to \alpha} \alpha D_q \Phi(t)$$

is called q-derivative of Φ at $t \in [\alpha, \beta]$.

The q-integral of a continuous function $\Phi : [\alpha, \beta] \to \mathbb{R}$ is defined by

$$\int_{\alpha}^{\beta} \Phi(t) \, {}_{\alpha}d_q t = (1-q)(t-\alpha) \sum_{n=0}^{\infty} q^n \Phi\left(q^n x + (1-q^n)\alpha\right), \quad 0 < q < 1 \tag{1.1.8}$$

and for $\gamma \in (\alpha, x)$

$$\int_{\gamma}^{x} \Phi(t) \, {}_{\alpha}d_q t = \int_{\alpha}^{x} \Phi(t) \, {}_{\alpha}d_q t - \int_{\alpha}^{\gamma} \Phi(t) \, {}_{\alpha}d_q t. \tag{1.1.9}$$

1.1.1 Application of Quantum Calculus

Quantum calculus has a numerous applications in various fields of mathematics such as orthogonal polynomials, combinatorics, number theory, hyper-geometric functions, integral inequalities, and so on. Apart from this, it has application in Physics such as quantum field theory, special relativity, knot theory, quantum hydrodynamics, string theory, elementary particle physics, chemical physics, quantum chemo-dynamics, electroweak interaction, Wess-Zumino model, q-Coulomb problem, q-hydrogen atom, molecular and nuclear spectroscopy, and so on. More specifically, it has served as the bridge between mathematics and physics in the history of the last twenty-five years. For details see [16] and the references cited there in.

1.2 Concepts of Mathematical Inequality

In Mathematics, the word 'inequality' means a disparity between the two quantities. Simply, an 'inequality' means that two quantities are not equal. With the emergence of calculus, the study of inequalities and its role has become increasingly essential.

In modern mathematics, inequalities play a significant role in almost all fields of mathematics. Several applications of inequalities are found in various areas of sciences such as physical and engineering. In numerical analysis, the approximation of a definite integral of a real function $\Phi(t)$ over an interval $[\alpha, \beta]$, i.e.,

$$\int_{\alpha}^{\beta} \Phi(t)\, dt$$

is a very interesting problem. Therefore, many methods appeared in literature to solve such problems, and one of the most famous examples of such methods is Newton-Cotes formula (e.g., midpoint, trapezoidal and Simpsons). Error bounds for these approximations involve higher order derivatives, i.e., a second order derivative for midpoint and trapezoidal formulas whereas a fourth order derivative for Simpsons' method. Therefore, these methods have many disadvantages as such requires a lot of differentiation (if we assume the derivative exists), with bounded derivatives that make this class of functions inefficient and inelastic to solve such problems. In recent years, theory of inequalities is used at large and many efforts devoted to establish several generations of the midpoint, trapezoid and Simpsons' inequalities for mappings of bounded variations.

Numerous important inequalities have been employed as powerful tools not only in mathematics but also in other areas of mathematics such as the theory of means, approximation theory, numerical analysis, and so on. For example, the famous arithmetic-geometric mean inequality was elegantly used by Erdois and Grunwald [4] to estimate integrals by rectangles and tangential triangles. The importance of inequalities is mainly highlighted by their role in analysis, but the use of inequalities can sometimes be quite unexpected, for example in graph theory. In their book, Aigner and Ziegler [4] discussed a simple proof of Turan's Theorem on the number of edges of graph without triggles whose proof is done by applying the Cauchy-Schwarz inequality. The theory of inequalities is now an important branch of mathematics. In his essay, Fink [18] traced the development of inequalities as a discipline of mathematics. Fink sketched the history of inequalities from ancient times where the inequalities were known as geometrical facts, for the awakening of inequality analysis in the early 18th century. One of the most notable books written in inequality theory is the famous 'Classic Inequalities' by Hardy, Littlewood and Polya,[21, 22] which was published in 1934. Some of the famous books in this area are Beckenbach and Bellmann's [7] "Inequalities", which was published in 1961, and Mitrinovic's "Analytic Inequalities" in 1970. Number of papers on inequalities are published after the publication of these three books. Several journals are devoted to inequalities, most notably, 'Journal of Inequalities and Applications' with the first volume in 1998, "Journal of Inequalities in Pure and Applied Mathematics" with the first volume in 2000 and 'Journal of Mathematical Inequalities' which has been launched in 2007.

One of the most important inequalities that has been attracted by many inequality experts in the last few decades is the famous Hermite-Hadamard inequality. Although it was firstly known in literature as a result of J. Hadamard in 1893, this result was actually due to C. Hermite in 1881, as pointed out by Mitrinovic and Lacovic [36] in 1985. Due to this fact, most experts refers to it as Hermite-Hadamard (or sometimes, Hadamard- Hermite) inequality.

1.2.1 The Hermite-Hadamard Inequality

For a convex function Φ, the following double inequality

$$(\beta-\alpha)\Phi\left(\frac{\alpha+\beta}{2}\right) \leq \int_{\alpha}^{\beta} \Phi(x)\,dx \leq (\beta-\alpha)\frac{\Phi(\alpha)+\Phi(\beta)}{2}, \quad \alpha, \beta \in \mathbb{R} \tag{1.2.1}$$

was known in the literature as the Hadamard's inequality. However, this inequality was actually suggested by Hermite. On 22 November 1881, Hermite sent a letter to the journal 'Mathesis'. An extract from that letter was then published in 'Mathesis', in 1883, page 82. One of the inequalities which was mentioned by Hermite in this note is the inequality (1.2.1).This note was nowhere mentioned in the mathematical literature and the important inequalities of Hermite were not widely known as Hermite's results. E. F. Bechenbach [7, 12], a leading expert on the theory of complex functions, wrote that the inequality (1.2.1) was proven by Hadamard in 1893 and apparently was not aware of Hermite's results. Fejer in 1906, while studying trigonometric polynomials obtained inequalities which generalize (1.2.1) but again Hermite's work was not acknowledged. In 1905, Jensen defined convex functions using the first and the last terms of inequality (1.2.1), that is

$$\Phi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{\Phi(\alpha)+\Phi(\beta)}{2} \tag{1.2.2}$$

for all $\alpha, \beta \in \mathbb{R}$. The inequality (1.2.2) is referred as the Jensen's inequality. It is important to note that inequality (1.2.1) provides a refinement to the Jensen's inequality. In 1974, D. S. Mitrinovic found Hermite's note in 'Mathesis'. Due to these historical facts, inequality (1.2.1) is now referred to as the Hermite-Hadamard inequality. The Hermite-Hadamard inequality plays a great role in the theory of convex functions. It provides a necessary and sufficient condition for a function to be convex in an open interval of real numbers. It also interpolates Jensen's inequality which is also an important inequality in the study of convex functions. In the monograph, Dragomir and Pierce[12] stated that the Hermite-Hadamard inequality is the first fundamental result for convex functions with a natural geometrical interpretations and many applications have attracted and continues to attract much interest in elementary mathematics. The H-H inequality has made great contributions in the fields of integral inequalities, approximation theory, special means theory, optimization theory, information theory, and numerical analysis. It has been developed for different classes of convexity such as quasi-convexity, Godunova-Levin convexity, log-convex, r-convex, P-convex functions, etc.

The Hermite-Hadamard inequality usually stated as a result valid for convex functions only, actually it holds for many other functions. The Hermite-Hadamard inequality provides a lower and an upper estimation for the integral average of any convex function defined on a compact interval involving the mid-point and the end points of the domain. This inequality has various applications for generalized means, information measures, quadrature rules, etc. and there is a growing literature providing new proofs, extensions, refinements and generalizations. It has become a corner stone in mathematical analysis and in optimization theory. It gives an estimate of mean value of a convex function which works great in analysis and partial differential equations.

Inequalities play important roles in understanding many mathematical concepts, such as probability theory, numerical integration and integral operator theory. Through the last century, H-H type inequality has been considered to be among the fastest growing fields in mathematical analysis, through which vast problems in engineering, physics and economics have been studied. Due to the enormous importance of these inequalities, many extensions, refinements, and generalizations of their related types have been equally investigated.

An interesting problem in H-H inequality attracts many researchers in the determinations of the two bounds of quantities given as follows:

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(t)\,dt-\Phi\left(\frac{\alpha+\beta}{2}\right)\right| \tag{1.2.3}$$

and

$$\left|\frac{\Phi(\alpha)+\Phi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(t)\,dt\right| \tag{1.2.4}$$

One vital problem associated with the H-H- inequality is the estimation of the mid-point and trapezoid type inequalities, when the difference between the left part of the H-H inequality and the integral of the function under study is observed, the quantity obtained is simply called the mid-point type inequality. Meanwhile, when such a difference is determined with the right hand side of the H-H inequality, here the quantity involved is called the trapezoid type inequality. Recently, different integral inequalities were obtained through differentiable convexity.

1.2.2 Physical Interpretation of Hermite-Hadamard Inequality

The core meaning of the Hermite-Hadamard inequality is translated into reality as follows:

1. The average value of a convex function Φ defined on $[\alpha,\beta]$ is between the value of Φ at the mid-point $t=\frac{\alpha+\beta}{2}$ and the average of the values of Φ at the end points α and β.
2. The mid-point formula estimates the integral from left and the trapezoidal formula estimates the integral from right.
3. The first inequality is stronger than the second inequality, i.e., if Φ is convex on $[\alpha,\beta]$, then

$$\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(t)\,dt-\Phi\left(\frac{\alpha+\beta}{2}\right)\leq\frac{\Phi(\alpha)+\Phi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(t)\,dt \tag{1.2.5}$$

1.2.3 Geometrical Interpretation of Hermite-Hadamard Inequality

The area below the graph of Φ on $[\alpha,\beta]$ lies between the areas of two trapeziums, namely the one formed by the endpoints of coordinates $(\alpha,\Phi(\alpha))$, $(\beta,\Phi(\beta))$ with the x-axis, the second one formed by the tangent to the graph of Φ at $\left(\frac{\alpha+\beta}{2},\,\Phi\left(\frac{\alpha+\beta}{2}\right)\right)$ with x-axis.

1.3 Mathematical Means

Definition 1.3.1. *A function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ is said to be a mean function if it has the following properties: (for details, see [9].)*

1. *Homogeneity:* $M(\alpha x, \alpha y) = \alpha M(x,y)$ *for all* $\alpha > 0$ *.*
2. *Symmetry:* $M(x,y) = M(y,x) \quad \forall x, y \in \mathbb{R}_+$ *.*
3. *Reflexivity:* $M(x,x) = x \quad \forall x \in \mathbb{R}_+$ *.*
4. *Monotonically: If* $x \leq x'$ *and* $y \leq y'$ *, then* $M(x,y) \leq M(x',y')$ *.*
5. *Internalized:* $min \, \{x,y\} \leq M(x,y) \leq max \, \{x,y\}$ *.*

1.3.1 Applications to Special Means

Let us consider the following means:

1. The arithmetic mean:

$$A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}.$$

2. The weighted arithmetic mean:

$$A_t(\alpha, \beta) = t\alpha + (1-t)\beta, \quad \alpha, \beta \in \mathbb{R}, \, t \in [0,1].$$

3. The geometric mean:

$$G(\alpha, \beta) = \sqrt{\alpha\beta}, \quad \alpha, \beta \in \mathbb{R}_+.$$

4. The harmonic mean:

$$H(\alpha, \beta) = \frac{2\alpha\beta}{\alpha + \beta}, \quad \alpha, \beta \in \mathbb{R} \backslash \{0\}.$$

5. The weighted harmonic mean:

$$H_t(\alpha, \beta) = \left(\frac{t}{\alpha} + \frac{1-t}{\beta} \right)^{-1}, \quad \alpha, \beta \in \mathbb{R} \backslash \{0\}.$$

6. The logarithmic mean:

$$L(\alpha,\beta)=\begin{cases}\alpha & \text{if}\,\alpha=\beta\\ \frac{\beta-\alpha}{\log\beta-\log\alpha} & \text{if}\,\alpha\neq\beta\end{cases},\quad \alpha,\beta\in\mathbb{R}_+.$$

7. The generalized logarithmic mean:

$$L_n(\alpha,\beta)=\begin{cases}\alpha & \text{if}\,\alpha=\beta\\ \left(\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right)^{\frac{1}{n}} & \text{if}\,\alpha\neq\beta\end{cases},\quad \alpha,\beta\in\mathbb{R}_+,\, n\in\mathbb{R}\backslash\{-1,0\}.$$

8. The identric mean:

$$I(\alpha,\beta)=\begin{cases}\alpha & \text{if}\,\alpha=\beta\\ \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}} & \text{if}\,\alpha\neq\beta\end{cases},\quad \alpha,\beta\in\mathbb{R}_+.$$

9. The power mean:

$$M_r(\alpha,\beta)=\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}},\quad r\geq 1,\, \alpha,\, \beta\in\mathbb{R}_+.$$

1.4 Convex Functions

The special properties of functions of real variables include continuity, convexity, monotonocity and differentiability. Of them, convexity plays a significant role in the development of several branches of mathematics since it includes the theory of convex functions that possess the following two important properties as the maximum value is attained at a boundary point and any local minimum is a global one. Although convexity is a basic notion in geometry but widely used in other areas of mathematics viz. functional analysis, complex analysis, calculus of variations, graph theory, partial differential equation, discrete mathematics, algebraic geometry, probability theory, coding theory, crystallography, and many other fields. It plays an important role outside of mathematics such as in physics, chemistry, biology, economics, finance, and more.

1.4.1 Affine and Convex Sets

Let $S\subset\mathbb{R}^n$ be a set. We say that the set

1. S is affine if $tx+(1-t)y\in S, \quad x,\, y\in S, \quad t\in\mathbb{R},$ and
2. S is convex if $tx+(1-t)y\in S, \quad x,\, y\in S, \quad t\in[0,1]$

Geometrically, when x and y are distinct points in $\mathbb{R}^n$, the set $L = \{z \in \mathbb{R}^n : z = tx + (1-t)y,\, t \in \mathbb{R}\}$ of all affine combinations of x and y is simply the line determined by x and y, and the set $C = \{z \in \mathbb{R}^n : z = tx + (1-t)y, t \in [0,1]\}$ is called the line segment between x and y.

1.4.2 Classical Convex Functions

Definition 1.4.1. *A function* $\Phi : I \subset \mathbb{R} \to \mathbb{R}, \quad I \neq \phi,$ *is said to be an Arithmetically-arithmetic convex or simply a convex function on* I *if the inequality*

$$\Phi(t\alpha + (1-t)\beta) \leq t\Phi(\alpha) + (1-t)\Phi(\beta) \tag{1.4.1}$$

holds for all $\alpha, \beta \in I$ *and* $t \in [0,1]$.

If the function Φ *is concave then the inequality reverses.*

Geometrically, the convexity of a function means that if P, Q, R are three distinct points on the graph of Φ with Q between P and R, then the point Q is on or below the chord PR. A convex function lies below its secant lines.

Remark 1.4.2. *For any real-valued function* Φ *on a closed interval* I, *we say that* Φ *is midpoint convex if,*

$$\Phi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{\Phi(\alpha)+\Phi(\beta)}{2} \quad \forall\, \alpha, \beta \in I \tag{1.4.2}$$

This can be interpreted as, 'for convex Φ, *the average of* Φ *exceeds* Φ *of the average'.*

1.4.3 Quasi-Convex Functions

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely,

Definition 1.4.3. *A function* $\Phi : [\alpha, \beta] \to \mathbb{R}$ *is said to be quasi-convex on* $[\alpha, \beta]$ *if the inequality*

$$\Phi(tx + (1-t)y) \leq max\{\Phi(x), \Phi(y)\} \tag{1.4.3}$$

holds for any $x, y \in [\alpha, \beta]$ *and* $t \in [0,1]$.

Clearly, any convex function is a quasi-convex function. However, there do exist quasi-convex functions which are not convex.

Example 1.4.1. *The function* $\Phi : [-2,2] \to \mathbb{R}$ *,*

$$\Phi(x) = \begin{cases} 1 & \text{if} \quad x \in [-2,-1] \\ x^2 & \text{if} \quad x \in (-1,2]. \end{cases}$$

is not a convex function on $[-2,2]$ *but it is a quasi-convex function on* $[-2,2]$.

So, the nature of a quasi-convexity of the function is a generalization of convexity of the function. Also, a quasi-convex function may be neither convex function nor continuous function. For details, see [50].

Example 1.4.2. *The floor function* $\Phi(x) = \lfloor x \rfloor$ *, where* x *is the largest integer not greater than* x *is an example of monotonic increasing function which is a quasi-convex but it is neither convex nor continuous.*

1.4.4 m-Convex Functions

Toader in 1985 [59] introduced the concept of m-convexity of the function Φ as follows:

Definition 1.4.4. *[59] The function* $\Phi : [0,b] \to \mathbb{R}, \quad b > 0,$ *is said to be an* m*-convex function if*

$$\Phi(t\alpha + m(1-t)\beta) \leq t\Phi(\alpha) + m(1-t)\Phi(\beta) \tag{1.4.4}$$

holds for all $\alpha, \beta \in [0,b], t \in [0,1], m \in [0,1]$ *.*

The m-convexity is intermediate concept between usual convexity and star-shaped function. We recall that a function $\Phi : [0,b] \to \mathbb{R}$ is called a star-shaped function if

$$\Phi(t\alpha) \leq t\Phi(\alpha)$$

holds for all $\alpha \in [0,b], t \in [0,1]$.

1.4.5 (α, m)-Convex Functions

Mihesan in 1993 generalized the concept of m-convexity into (α, m)- convexity as follows: For details, see [35].

Definition 1.4.5. *The function* $\Phi : [0,b] \to \mathbb{R}, \quad b > 0,$ *is said to be an* (α, m)*-convex function, where* $(\alpha, m) \in [0,1]^2$ *if we have*

$$\Phi(tx + m(1-t)y) \ \leq \ t^{\alpha}\Phi(x) + m(1-t^{\alpha})\Phi(y) \tag{1.4.5}$$

for all $x, y \in [0, b]$ *and* $t \in [0, 1]$.

1.4.6 The Harmonically Convex Functions

Definition 1.4.6. *[25] A set* $\mathbb{H} \subset \mathbb{R}^n \backslash \{0\}$ *is said to be a harmonic convex set if*

$$\frac{\alpha\beta}{t\alpha + (1-t)\beta} \in \mathbb{H}$$

holds for all $\alpha, \beta \in \mathbb{H}, t \in [0, 1]$ *.*

Definition 1.4.7. *A function* $\Phi : \mathbb{H} \subset \mathbb{R}^n \to \mathbb{R}$, *where* $\mathbb{H} \neq \phi$ *a harmonic convex set in* $\mathbb{R}^n \backslash \{0\}$, *is a harmonic convex function on* $\mathbb{H}$ *if*

$$\Phi\left(\frac{\alpha\beta}{t\alpha + (1-t)\beta}\right) \leq t\Phi(\beta) + (1-t)\Phi(\alpha)$$

holds for all $\alpha, \beta \in \mathbb{H}, \quad t \in [0, 1]$ *.*

If the inequality reverses, then Φ is harmonically concave function. See [24, 25] for details.

1.5 Evolution on q-Hermite-Hadamard Inequality

Tariboon et al. in 2014 [55] studied the concept of q-derivatives and q-integrals over the intervals of the form $[\alpha, \beta] \subset \mathbb{R}$ and settled a number of q-analogues of some well known results such as Holder's, Hermite-Hadamard's, Ostrowski's, Cauchy-Bunayakovaski-Schwarz's and Gruss- Chebysev's inequalities using classical convexity. They presented q-Hermite-Hadamard integral inequality as follows:

Theorem 1.5.1. *Let* $\Phi : [\alpha, \beta] \to \mathbb{R}$ *be a convex continuous function on* $[\alpha, \beta]$ *and* $0 < q < 1$ *be a constant. Then we have*

$$\Phi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \Phi(x)_{\alpha} d_q x \leq \frac{q\Phi(\alpha) + \Phi(\beta)}{1+q}. \tag{1.5.1}$$

Kunt and Iscan in 2016 [3] gave a counterexample and showed that the left hand side of the inequality (1.5.1) is not correct and they proved a correct form of q-Hermite- Hadamard inequality which is stated as follows:

Theorem 1.5.2. *Let* $\Phi : [\alpha, \beta] \to \mathbb{R}$ *be a convex differentiable function on* (α, β) *and*

$0 < q < 1$ *be a constant. Then we have*

$$\Phi\left(\frac{q\alpha+\beta}{1+q}\right) \le \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(x)_{\alpha}d_qx \le \frac{q\Phi(\alpha)+\Phi(\beta)}{1+q}. \tag{1.5.2}$$

In 2018, Alp et al. [3] studied the q-analogue of Hermite-Hadamard's inequality for increasing functions and provided the generalized q-Hermite-Hadamard integral inequality for differentiable convex function which is stated as follows:

Theorem 1.5.3. *Let* $\Phi : [\alpha,\beta] \to \mathbb{R}$ *be a convex differentiable function on* (α,β) *and* $0 < q < 1$ *be a constant. Then we have*

$$max\{I_1, I_2, I_3\} \le \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(x)_{\alpha}d_qx \le \frac{q\Phi(\alpha)+\Phi(\beta)}{1+q} \tag{1.5.3}$$

where

$$\begin{aligned} I_1 &= \Phi\left(\frac{q\alpha+\beta}{1+q}\right), \\ I_2 &= \Phi\left(\frac{\alpha+q\beta}{1+q}\right) + \frac{(1-q)(\beta-\alpha)}{1+q}\Phi'\left(\frac{\alpha+q\beta}{1+q}\right), \textit{ and} \\ I_3 &= \Phi\left(\frac{\alpha+\beta}{2}\right) + \frac{(1-q)(\beta-\alpha)}{2(1+q)}\Phi'\left(\frac{\alpha+\beta}{2}\right). \end{aligned}$$

Pachpatte in 2012 [46] established the two Hermite-Hadamard type inequalities for product of two classical convex functions as follows:

Theorem 1.5.4. *Let* $\Phi, \Psi : [\alpha,\beta] \subset \mathbb{R} \to [0,\infty)$ *be two convex functions on* (α,β) *and* $\alpha < \beta$. *Then*

$$\begin{aligned} &(i)\, \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(x)\Psi(x)\,dx \le \frac{1}{3}M(\alpha,\beta) + \frac{1}{6}N(\alpha,\beta), \textit{ and} \\ &(ii)\, 2\Phi\left(\frac{\alpha+\beta}{2}\right)\Psi\left(\frac{\alpha+\beta}{2}\right) \le \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(x)\Psi(x)\,dx + \frac{1}{6}M(\alpha,\beta) + \frac{1}{3}N(\alpha,\beta) \end{aligned}$$

where $M(\alpha,\beta) = \Phi(\alpha)\Psi(\alpha) + \Phi(\beta)\Psi(\beta)$; $N(\alpha,\beta) = \Phi(\alpha)\Psi(\beta) + \Phi(\alpha)\Psi(\beta)$ Sudsutad et al. in 2015 [54] established q-integral inequalities for product of two classical convex functions as follows:

Theorem 1.5.5. *Let* $\Phi, \Psi : [\alpha,\beta] \subset \mathbb{R} \to [0,\infty)$ *be two convex functions on* $[\alpha,\beta]$ *and*

$\alpha < \beta$. *Then the following inequalities hold:*

$$(i)\ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(x)\Psi(x)\,{}_{\alpha}d_qx \leq \frac{\Phi(\alpha)\Psi(\beta)}{1+q+q^2}+\frac{q(1+q^2)\Phi(\beta)\Psi(\beta)+q^2N(\alpha,\beta)}{(1+q)(1+q+q^2)}, \tag{1.5.4}$$

$$(ii)\ 2\Phi\left(\frac{\alpha+\beta}{2}\right)\Psi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(x)\Psi(x)\,{}_{\alpha}d_qx+\frac{2q^2M(\alpha,\beta)+(1+2q+q^2)N(\alpha,\beta)}{2(1+q)(1+q+q^2)}, \tag{1.5.5}$$

where $M(\alpha,\beta)=\Phi(\alpha)\Psi(\alpha)+\Phi(\beta)\Psi(\beta)$; $N(\alpha,\beta)=\Phi(\alpha)\Psi(\beta)+\Phi(\alpha)\Psi(\beta)$

1.6 Research Gap

The existing literature indicates that the refinements and enhancements of the Hermite-Hadamard type inequalities for different types of convex functions in the framework of quantum calculus are still open. Based on the literature available, this thesis has incorporated the statement of the problem as follows:

1.7 Problem Statement

In this thesis, the main problem statement is devoted to introduce and discuss several inequalities of Hermite-Hadamard on different types of convexities namely generalized harmonically-arithmetic convex function, generalized geometrically-arithmetic convex functions, differentiable m-convex functions, differentiable harmonically m-convex functions in the framework of quantum calculus. This thesis has also included H-H integral inequality for harmonically convex function through Riemann-Liouville fractional integrals. Apart from these, the refinements of Hermite-Hadamard type integral inequality is enhanced in quantum calculus, specially in q-calculus which is a paradigm- shift in the theory of convexity and theory of inequality.

1.8 Objectives of the Study

The objectives of this study are

1. To study Hermite-Hadamard (H-H) integral inequality and enhance it on various types of convex functions.
2. To develop H-H integral inequality for generalized harmonically-arithmetic and harmonically-geometric convex functions.

3. To extend H-H integral inequality for generalized geometrically-arithmetic convex functions.
4. To establish H-H inequalities for the products of generalized convex functions in q-calculus.
5. To expand H-H inequality for differentiable m-convex functions.
6. To establish H-H inequality for differentiable harmonically m-convex functions.
7. To establish H-H inequality for harmonically convex functions through Riemann-Liouville fractional integrals

1.9 Structure of the Thesis

Categorically, this thesis incorporates seven chapters. The first chapter comprises of the origin of quantum calculus, how it is different from ordinary calculus and application of quantum calculus in various domain of Mathematics and Physics. It has also included different aspects of convex functions and mathematical inequalities, Especially, Hermite-Hadamard type integral inequality for classical convex function. The mathematical means and their corresponding properties along with different types of special means have also been incorporated.

In the second chapter, the researcher diagnosed the idea of the classical convex functions and enhances it into m- convex functions which helps to extend a convex function's equality of quantum estimate to an m-convex function. Using the resulting information, a few novel Hermite-Hadamard integral inequalities are established whose first order q-derivatives are m-convex functions.Then the so obtained outcomes are presented in q-analogues too.

The third chapter introduces the extended idea of Hermite-Hadamard type integral inequality for geometrically-arithmetic (GA) convex functions and their product which is an extension of the results obtained by B. G. Pachepatte of classical convex functions. The newly obtained results are then modeled in q-calculus.

The fourth chapter incorporates the idea of harmonically-arithmetic (HA) convex function to their generalized versions as well as the products of Hermite-Hadamard inequality for harmonically-arithmetic convex functions in q-calculus. This chapter enhances some more results on generalized harmonically-convex functions.

The fifth chapter entails generalized harmonically convex function to m-harmonically convex function and applied them to develop some Hermite- Hadamard type integral inequalities whose first order derivatives are m-harmonically convex function. Apart from this, this chapter enhances harmonically P-function to m-harmonically P-function and applied them to extend Hermite-Hadamard type integral inequalities.

The sixth chapter includes the results on H-H integral inequality for harmonically convex function via Riemann-Liouville fractional integrals.

The Summary, Conclusion and the Recommendation section of this research work is presented in the seventh chapter of this thesis.

Chapter 2

q-HERMITE-HADAMARD INEQUALITY WHOSE FIRST ORDER DERIVATIVES ARE m-CONVEX FUNCTIONS

2.1 Introduction

In this section, equality of quantum estimates of a convex function is extended to an m-convex function. Using the resulted information, a few novel Hermite-Hadamard integral inequalities are established whose first order q-derivatives are m-convex functions and these outcomes are modeled in q-analogues. In this chapter, it is focused to enhance a few q-Hermite-Hadamard type integral inequalities for m-convex functions by taking into account the outcomes of quantum estimates of convex functions.

2.2 Preliminary Results

The ideas and research on q-calculus that were previously known are reviewed and stated in this section before applying in the subsequent work.

Let $K = [u, v] \subset \mathbb{R}$, $K^0 = (u, v)$ be intervals and $q \in (0, 1)$ be any constant.

Theorem 2.2.1. *[30] Let $\Phi_1, \Phi_2 : K \to \mathbb{R}$ be continuous functions, and $\alpha \in \mathbb{R}$. Then, for*

$x \in K,$ *we have*

$$(i) \int_u^x (\Phi_1(s) + \Phi_2(s))\, {}_ud_qs = \int_u^x \Phi_1(s)\, {}_ud_qs + \int_u^x \Phi_2(s)\, {}_ud_qs,$$

$$(ii) \int_u^x (\alpha\Phi_1)(s)\, {}_ud_qs = \alpha \int_u^x \Phi_1(s)\, {}_ud_qs, \text{ and}$$

$$(iii) \int_c^x \Phi_1(s) {}_uD_q\rho_2(s)\, {}_ud_qs = |\Phi_1\Phi_2|_c^x - \int_c^x \Phi_2(qs + (1-q)u) {}_uD_q\Phi_1(s)\, {}_ud_qs, \quad c \in (u, x).$$

Alp. et al. [3] in 2018 introduced the correct form of q-Hermite-Hadamard type integral inequality as follows:

Theorem 2.2.2. *[3] Let* $\Phi : K \to \mathbb{R}$ *be a continuous function on* $K,\quad q \in (0,1)$. *Then we have*

$$\Phi\left(\frac{qu+v}{1+q}\right) \leq \frac{1}{v-u}\int_u^v \Phi(s)\, {}_ud_qs \leq \frac{q\Phi(u)+\Phi(v)}{1+q}. \tag{2.2.1}$$

Lemma 2.2.3. *[54] Let* $\Phi : K \to \mathbb{R}$ *be a continuous function and* $q \in (0,1)$. *If* ${}_uD_q\Phi$ *is an integrable function on* K^0, *then the following relation holds:*

$$\frac{q\Phi(u)+\Phi(v)}{1+q} - \frac{1}{v-u}\int_u^v \Phi(s)\, {}_ud_qs = \frac{q(v-u)}{1+q}\int_0^1 (1-(1+q)s)\, {}_uD_q\Phi(sv+(1-s)u)\, {}_0d_qs.$$

Lemma 2.2.4. *[54] Let* $q \in (0,1)$. *Then the following equality holds:*

$$(i) \int_0^1 1\, {}_0d_qs = 1.$$

$$(ii) \int_0^1 s\, {}_0d_qs = \frac{1}{1+q}.$$

$$(iii) \int_0^1 s^2\, {}_0d_qs = \frac{1}{1+q+q^2}.$$

Lemma 2.2.5. *[54] Let* $q \in (0,1)$. *Then the following equality holds:*

$$\int_0^1 |1-(1+q)s|\, {}_0d_qs = \frac{q(2+q+q^2)}{(1+q)^3}.$$

Lemma 2.2.6. *[54] Let* $q \in (0,1)$. *Then the following equality holds:*

$$\int_0^1 s|1-(1+q)s|\, {}_0d_qs = \frac{q(1+4q+q^2)}{(1+q)^3(1+q+q^2)}.$$

Lemma 2.2.7. *[54] Let* $q \in (0,1)$ *. Then the following equality holds:*

$$\int_0^1 (1-s)|1-(1+q)s| \, {}_0d_q s = \frac{q(1+3q^2+2q^3)}{(1+q)^3(1+q+q^2)}.$$

Theorem 2.2.8. *[54] Let* $\Phi : K \to \mathbb{R}$ *be a continuous function. If* $|{}_uD_q\Phi|$ *is convex and integrable on* K^0 *, then the inequality below holds:*

$$\left| \frac{q\Phi(u)+\Phi(v)}{1+q} - \frac{1}{v-u}\int_u^v \Phi(s) \, {}_ud_q s \right| \leq \frac{q^2(v-u)}{(1+q)^4(1+q+q^2)} \left((1+4q+q^2)|{}_uD_q\Phi(v)| + (1+3q+2q^3)|{}_uD_q\Phi(u)| \right).$$

Theorem 2.2.9. *[54] Let* $\Phi : K \to \mathbb{R}$ *be a continuous function. If* $|{}_uD_q\Phi|^r$ *is convex and integrable on* K^0 *,and* $r \geq 1$ *, then the following result holds:*

$$\left| \frac{q\Phi(u)+\Phi(v)}{1+q} - \frac{1}{v-u}\int_u^v \Phi(s) \, {}_ud_q s \right| \leq \frac{q^2(2+q+q^2)(v-u)}{(1+q)^4} \left[\frac{(1+4q+q^2)| \, {}_uD_q\Phi(v)|^r + (1+3q^2+2q^3)| \, {}_uD_q\Phi(u)|^r}{(1+3q^2+2q^3)(2+q+q^2)} \right]^{\frac{1}{r}}.$$

Theorem 2.2.10. *[54] Let* ρ_1 *and* ρ_2 *be two real-valued, non-negative and convex functions on* K *. Then the following inequality holds:*

$$\frac{1}{v-u}\int_u^v \rho_1(x)\rho_2(x) \, {}_ud_q x \leq \frac{\rho_1(u)\rho_2(u)}{1+q+q^2} + \frac{q(1+q^2)\rho_1(v)\rho_2(v)+q^2N(u,v)}{(1+q+q^2)(1+q)}.$$

Theorem 2.2.11. *[54] Let* Φ_1 *and* Φ_2 *be two real-valued, non-negative and convex functions on* K *. Then the following results hold:*

$$(i.) \frac{(1+q)(1+q+q^2)}{(v-u)^2}\int_u^v\int_u^v\int_0^1 \Phi_1(sy+(1-s)x)\Phi_2(sy+(1-s)x) \, {}_0d_q s \, {}_ud_q x \, {}_ud_q y \leq \frac{(1+2q+q^2)}{(v-u)}\int_u^v \Phi_1(x)\Phi_2(x) \, {}_ud_q x + \frac{2q^2}{(1+q)^2}[(q^2\Phi_1(u)\Phi_2(u)+\Phi_1(v)\Phi_2(v))+qN(u,v)]$$

$$(ii)\frac{1+q+q^2}{v-u}\int_u^v\int_0^1 \Phi_1(sy+(1-s)\frac{u+v}{2})\Phi_2(sy+(1-s)\frac{u+v}{2})\,{}_0d_qs\,{}_ud_qy \leq$$

$$\frac{1}{v-u}\int_u^v \Phi_1(x)\Phi_2(x)\,{}_ud_qx+\frac{q(1+q^2)}{4(1+q)}(M(u,v)+N(u,v))$$

$$+\frac{q^2}{2(1+q)^2}(2(q\phi_1(u)\phi_2(u)+\Phi_1(v)\Phi_2(v))+(1+q)N(u,v))$$

where $M(u,v)=\Phi_1(u)\Phi_2(u)+\Phi_1(v)\Phi_2(v)$, $N(u,v)=\Phi_1(u)\Phi_2(v)+\Phi_1(v)\Phi_2(u)$.

2.3 Main Results

In this section, first of all, we extend Lemma 2.2.3 assuming that ${}_uD_q\Phi$ is q-integrable m-convex function and then present some quantum estimates of Hermite-Hadamard type integral inequalities for m-convex function on $[mu,v]$.

Lemma 2.3.1. *Let* $\Phi : K \to \mathbb{R}$ *be a continuous function and* $q \in (0,1)$. *If* ${}_uD_q\Phi$ *is* q*-integrable* m*-convex function on* K^0, *where* K^0 *is interior of* K, *then the following equality holds:*

$$\frac{1}{v-mu}\int_{mu}^v \Phi(s)\,{}_{mu}d_qs-\frac{q\phi(mu)+\Phi(v)}{1+q}=\frac{q(v-mu)}{1+q}\int_0^1(1-(1+q)s)\,{}_uD_q\Phi(sv+m(1-s)u)_0d_qs$$

Proof. Using q-derivative on a finite interval, we have

$$\int_0^1(1-(1+q)s)_uD_q\Phi(sv+m(1-s)u)\,{}_0d_qs=\int_0^1 1\,{}_uD_q\Phi(sv+m(1-s)u)\,{}_0d_qs$$

$$-(1+q)\int_0^1 s\,{}_uD_q\Phi(sv+m(1-s)u)\,{}_0d_qs$$

$$=\int_0^1\frac{\Phi(sv+m(1-s)u)-\Phi(qsv+m(1-qs)u)}{s(1-q)(v-mu)}\,{}_0d_qs$$

$$-(1+q)\int_0^1 s.\frac{\Phi(sv+m(1-s)u)-\Phi(qsv+m(1-qs)u)}{s(1-q)(v-mu)}\,{}_0d_qs$$

$$=\frac{1}{v-mu}[\sum_{n=0}^{\infty}\Phi(q^nv+m(1-q^n)u)-\sum_{n=0}^{\infty}\Phi(q^{n+1}v-m(1-q^{n+1})u)]$$

$$-\frac{1+q}{v-mu}\left[\sum_{n=0}^{\infty}q^n\rho(q^nv+m(1-q^n)u)-\sum_{n=0}^{\infty}q^n\Phi(q^{n+1}v-m(1-q^{n+1})u)\right]$$

—

$$
\begin{aligned}
&= \frac{\Phi(v)-\Phi(mu)}{v-mu} - \frac{1+q}{v-mu}\sum_{n=0}^{\infty} q^n \Phi(q^n v + m(1-q^n)u) + \\
&\qquad \frac{1+q}{v-mu}\sum_{n=0}^{\infty} q^n \Phi(q^{n+1} v + m(1-q^{n+1})u)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Phi(v)-\Phi(mu)}{v-mu} - \frac{1+q}{v-mu}\sum_{n=0}^{\infty} q^n (q^n v + m(1-q^n)u) + \\
&\qquad \frac{1+q}{q(v-mu)}\left[\Phi(v)-\Phi(u)\sum_{n=1}^{\infty} q^n \rho(q^{n+1} v + m(1-q^{n+1})u)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Phi(v)-\Phi(mu)}{v-mu} - \frac{(1+q)\Phi(v)}{q(v-mu)} \\
&\qquad + \frac{1+q}{q(v-mu)}\sum_{n=0}^{\infty} q^n \Phi(q^n v + m(1-q^n)u) \\
&\qquad - \frac{1+q}{v-mu}\sum_{n=0}^{\infty} q^n \Phi(q^n v + m(1-q^n)u).
\end{aligned}
$$

$$
= -\frac{q\Phi(mu)+\Phi(v)}{q(v-mu)} + \frac{(1+q)(1-q)(v-mu)}{q(v-mu)(v-mu)}\sum_{n=0}^{\infty} q^n \Phi(q^n v + m(1-q^n)u)
$$

$$
\begin{aligned}
&= -\frac{q\Phi(mu)+\Phi(v)}{q(v-mu)} + \frac{(1+q)}{q(v-mu)^2}\int_{mu}^{v} \Phi(s)_{mu} d_q s \\
&\qquad = \frac{1+q}{q(v-mu)}\left[\frac{1}{v-mu}\int_{mu}^{v} \Phi(s)_{mu} d_q s - \frac{q\Phi(mu)+\Phi(v)}{1+q}\right]
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\frac{1}{v-mu}\int_{mu}^{v} \Phi(s)_{mu} d_q s - \frac{q\Phi(mu)+\Phi(v)}{1+q} \\
&\qquad = \frac{q(v-mu)}{1+q}\int_{0}^{1} (1-(1+q)s)\, {}_uD_q\Phi(sv+m(1-s)u)_0 d_q s.
\end{aligned}
$$

The proof is now complete. ☐

Remark 2.3.2. *If* $m = 1$ *, and then it reduces to the result as given in Lemma 2.2.3, and if* $q \to 1$, $m = 1$ *, then it reduces to the following result:*

$$\frac{\Phi(u)+\Phi(v)}{2} - \frac{1}{v-u}\int_u^v \Phi(x)dx = \frac{v-u}{2}\int_0^1 (1 - 2s)\Phi'(sv + (1 - s)u)ds.$$

Theorem 2.3.3. *Let* $\Phi : K \to \mathbb{R}$ *be a continuous function. If* $|{}_{mu}D_q\Phi|$ *is* m *-convex and integrable on* K^0 *, then the following inequality holds:*

$$\begin{aligned}|\frac{q\Phi(mu)+\Phi(v)}{1+q} - \frac{1}{v-mu}\int_{mu}^v \Phi(s)\,{}_ud_qs| \le \\ \frac{q^2(v-mu)}{(1+q)^4(1+q+q^2)}((1+4q+q^2)|{}_{mu}D_q\Phi(v)| + (1+3q+2q^3)|{}_{mu}D_q\Phi(u)).\end{aligned}$$

Proof. Using Lemma 2.2.4 and m-convexity of ${}_{mu}D_q\Phi$ on J^0 and Lemma 2.2.6 and Lemma 2.2.7, we have

$$\begin{aligned}&\left|\frac{1}{v-mu}\int_{mu}^v \Phi(s)_{mu}d_qs - \frac{q\Phi(mu)+\rho(v)}{1+q}\right| \\ &= \left|\frac{q(v-mu)}{1+q}\int_0^1 (1-(1+q)s)_uD_q\Phi(sv+m(1-s)u)\,{}_0d_qs\right| \\ &\le \frac{q(v-mu)}{1+q}\int_0^1 |1-(1-q)s|[s|_{mu}D_q\Phi(v)+m(1-s)|_{mu}D_q\Phi(u)|]\,{}_0d_qs \\ &\le \frac{q(v-mu)}{1+q}[|_{mu}D_q\Phi(v)|\int_0^1 s|1-(1+q)s|_0d_qs + m|_{mu}D_q\Phi(u)|\int_0^1 (1-s)|1-(1+q)s|_0d_qs] \\ &= \frac{q(v-mu)}{1+q}[|_{mu}D_q\Phi(v)|\frac{q(1+4q+q^2)}{(1+q)^3(1+q+q^2)} + \frac{mq(1+3q^2+2q^3)}{(1+q)^3(1+q+q^2)}|_{mu}D_q\Phi(u)|] \\ &= \frac{q^2(v-mu)}{(1+q)^4(1+q+q^2)}[|_{mu}D_q\Phi(v)|(1+4q+q^2) + m(1+3q^2+2q^3)|_{mu}D_q\Phi(u)|].\end{aligned}$$

The proof is complete. ☐

Remark 2.3.4. *If* $m = 1$ *, then it reduces to the previous result as in Theorem 2.2.9.*

Remark 2.3.5. *If* $m = 1$ *,* $q \to 1$ *then, the above result reduces to the following already established result as*

$$\left|\frac{\Phi(u)+\Phi(v)}{2} - \frac{1}{v-u}\int_u^v \Phi(s)\,ds\right| \le \frac{(v-u)}{8}\left[|\Phi'(v)| + |\Phi'(u)|\right].$$

We prove the second result of q-integral inequality for m-convex function on $[mu, v]$.

Theorem 2.3.6. *Let* $\Phi : K \to \mathbb{R}$ *be a continuous function. If* $|{}_{mu}D_q\Phi|^r$ *is* m *-convex and* *integrable on* K^0 *and* $r \geq 1$, *then the following result holds:*

$$\left|\frac{q\Phi(mu)+\Phi(v)}{1+q} - \frac{1}{v-mu}\int_{mu}^{v}\Phi(s)\,{}_{u}d_qs\right|$$
$$\leq \frac{q^2(2+q+q^2)(v-mu)}{(1+q)^4}\left(\frac{(1+4q+q^2)|{}_{mu}D_q\Phi(v)|^r+(1+3q^2+2q^3)|{}_{mu}D_q\Phi(u)|^r}{(1+q+q^2)(2+q+q^3)}\right)^{\frac{1}{r}}$$

Proof. From Lemma 2.3.1 and using power-mean inequality and m-convexity of $|{}_{mu}D_q\Phi|^r$, and Lemma 2.2.5, Lemma 2.2.6, and Lemma 2.2.7, we have

$$\left|\frac{1}{v-mu}\int_{mu}^{v}\Phi(s)\,{}_{mu}d_qs - \frac{q\Phi(mu)+\rho(v)}{1+q}\right|$$
$$= \left|\frac{q(v-mu)}{1+q}\int_0^1 (1-(1+q)s)_uD_q\Phi(sv+m(1-s)u)_0d_qs\right|$$

$$\leq \frac{q(v-mu)}{(1+q)}\int_0^1 (1-(1+q)s)_uD_q\Phi(sv+m(1-s)u)_0d_qs|$$
$$\leq \frac{q(v-mu)}{1+q}\left(\int_0^1 |1-(1+q)s|_0d_qs)\right)^{1-\frac{1}{r}}\left(\int_0^1 |1-(1+q)s|_{ma}D_q\Phi(sb+m(1-s)a)|_0^rd_qs\right)^{\frac{1}{r}}$$

$$\leq \frac{q(v-mu)}{1+q}\left(\int_0^1 |1-(1+q)s|\,{}_0d_qs)\right)^{1-\frac{1}{r}}$$
$$\left(|{}_{mu}D_q\rho(v)|^r\int_0^1 s|1-(1+q)s_0d_qs+m|{}_{mu}D_q\Phi(u)|^r\int_0^1(1-s)|1-(1+q)s|\,{}_0d_qs\right)^{\frac{1}{r}}$$
$$\leq \frac{q(v-mu)}{1+q}\left(\frac{q(2+q+q^2)}{(1+q)^3}\right)^{1-\frac{1}{r}}$$
$$\left(\frac{q}{(1+q+q^2)(1+q)^3}\left((1+4q+q^2)|{}_{mu}D_q\Phi(v)|^r+m(1+3q^2+2q^3)|{}_{mu}D_q\Phi(u)|^r\right)\right)^{\frac{1}{r}}$$

$$= |\frac{q\Phi(mu)+\Phi(v)}{1+q} - \frac{1}{v-mu}\int_{mu}^{v}\Phi(s)\,{}_{u}d_qs|$$
$$\leq \frac{q^2(2+q+q^2)(v-mu)}{(1+q)^4}\left(\frac{(1+4q+q^2)|{}_{mu}D_q\Phi(v)|^r+(1+3q^2+2q^3)|{}_{mu}D_q\Phi(u)|^r}{(1+q+q^2)(2+q+q^3)}\right)^{\frac{1}{r}}.$$

The proof is complete. □

Remark 2.3.7. *If* $m = 1$, *then it reduces to Theorem 2.2.10.*

Remark 2.3.8. *If* $m = 1$ *and* $q \to 1$, *then we have the following previously known result as follows:*

$$\left|\frac{\Phi(u)+\Phi(v)}{2} - \frac{1}{v-u}\int_u^v \Phi(s)ds\right| \leq \frac{v-u}{4}\left[\frac{|\Phi'(u)|^r + |\Phi'(v)|^r}{2}\right]^{\frac{1}{r}}.$$

Theorem 2.3.9. *Let* Φ_1 *and* Φ_2 *be two real-valued, non-negative* m*- convex functions on* K. *Then the following inequality holds:*

$$\frac{1}{v-mu}\int_{mu}^{v} \Phi_1(x)\Phi_2(x)_{mu}d_qx \;\leq\; \frac{\Phi_1(u)\Phi_2(v)}{1+q+q^2} + \frac{m^2q(1+q^2)\Phi_1(v)\Phi_2(v) + mq^2N(u,v)}{(1+q+q^2)(1+q)},$$

where $N(u,v) = \Phi_1(u)\Phi_2(v) + \Phi_1(v)\Phi_2(u)$

Proof. Using m-convexity of Φ_1 and Φ_2 and for all $s \in [0,1]$, we have

$$\begin{aligned}\Phi_1(sv + m(1-s)u) &\leq s(\rho_1(v) + m(1-s)\Phi_1(u),\\ \Phi_2(sv + m(1-s)u) &\leq s(\Phi_2(v) + m(1-s)\Phi_2(u).\end{aligned}$$

Multiplying above inequalities, we have

$$\begin{aligned}\Phi_1(sv + m(1-s)u)\Phi_2(sv + m(1-s)u) \leq s^2\Phi_1(v)\Phi_2(v) + m^2(1-s)^2\Phi_1(u)\Phi_2(u) + \\ ms(1-s)(\Phi_1(u)\Phi_2(v) + \Phi_1(v)\Phi_2(u))\end{aligned}$$

Taking q-integral with respect to s over $[0,1]$ and using Lemma 2.2.4, we have

$$\begin{aligned}&\int_0^1 \Phi_1(sv + m(1-s)u)\Phi_2(sv + m(1-s)u)\,{}_0d_qs\\ &\leq \int_0^1 \big(s^2\Phi_1(v)\Phi_2(v) + m^2(1-s)^2\Phi_1(u)\Phi_2(u) + ms(1-s)(\Phi_1(u)\Phi_2(v) + \Phi_1(v)\Phi_2(u))\big)\,{}_0d_qs.\end{aligned}$$

Then

$$\begin{aligned}&\int_0^1 \Phi_1(sv + m(1-s)u)\Phi_2(sv + m(1-s)u)\,{}_0d_qs\\ &\qquad\leq \frac{\Phi_1(v)\Phi_2(v)}{1+q+q^2} + \frac{m^2q(1+q^2)\Phi_1(u)\Phi_2(u)}{(1+q+q^2)(1+q)} + \frac{mq^2N(u,v)}{(1+q+q^2)(1+q)}.\end{aligned}$$

Substituting $x = sv + m(1-s)u$, on left-side of above inequality, we have

$$\frac{1}{v-mu}\int_{mu}^{v}\Phi_1(x)\Phi_2(x)_{mu}d_qx \;\leq\; \frac{\Phi_1(u)\Phi_2(u)}{1+q+q^2} + \frac{m^2q(1+q^2)\Phi_1(v)\Phi_2(v) + mq^2N(u,v)}{(1+q+q^2)(1+q)},$$

where $N(u,v) = \Phi_1(u)\Phi_2(v) + \Phi_1(v)\Phi_2(u)$. □

Remark 2.3.10. *If $q \to 1$, then the above result reduces to*

$$\frac{1}{v-mu}\int_{mu}^{v}\Phi_1(x)\Phi_2(x)\,dx \leq \frac{1}{3}\left[\Phi_1(v)\Phi_2(v) + m^2\Phi_1(u)\Phi_2(u)\right] + \frac{1}{6}N(u,v).$$

Remark 2.3.11. *If $q \to 1$ and $m = 1$, then it reduces to*

$$\frac{1}{v-u}\int_{u}^{v}\Phi_1(x)\Phi_2(x)\,dx \leq \frac{1}{3}M(u,v) + \frac{1}{6}N(u,v).$$

Theorem 2.3.12. *Let Φ_1 and Φ_2 be two real-valued, non-negative, and m- convex functions on K. Then the followings hold:*

$$(i)\frac{(1+q)(1+q+q^2)}{(v-u)^2}\int_u^v\int_u^v\int_0^1 \Phi_1(sy+m(1-s)x)\Phi_2(sy+(1-s)x)\,{}_0d_qs\,{}_ud_qx_ud_qy \leq$$
$$\frac{1+q+m^2q(1+q^2)}{v-u}\int_u^v \Phi_1(x)\Phi_2(x)_ud_qx + \frac{2q^2m}{(1+q)^2}\left(q^2\Phi_1(u)\Phi_2(u) + qN(u,v) + \Phi_1(v)\Phi_2(v)\right).$$

$$(ii)\frac{1+q+q^2}{v-u}\int_u^v\int_0^1 \Phi_1(sy+m(1-s)\frac{u+v}{2})\Phi_2(sy+m(1-s)\frac{u+v}{2})\,{}_0d_qs_ud_qy \leq$$
$$\frac{1}{v-u}\int_u^v \Phi_1(y)\Phi_2(y)_ud_qy$$
$$+\frac{m^2q(1+q^2)}{4(1+q)}(M(u,v)+N(u,v)) + \frac{mq^2}{2(1+q)^2}(2(q\Phi_1(u)\Phi_2(u)+\Phi_1(v)\Phi_2(v))+(1+q)N(u,v)),$$

where $M(u,v) = \Phi_1(u)\Phi_2(u) + \Phi_1(v)\Phi_2(v)$, and, $N(u,v) = \Phi_1(u)\Phi_2(v) + \Phi_1(v)\Phi_2(u)$.

Proof. For all $x, y, \in K$ and $s \in [0,1]$, using the definition of m-convexity of Φ_1 and Φ_2, we have

$$\Phi_1(sy+m(1-s)x) \leq s(\Phi_1(y) + m(1-s)\Phi_1(x),$$
$$\Phi_2(sy+m(1-s)x) \leq s\Phi_2(y) + m(1-s)\Phi_2(x).$$

Multiplying above inequalities, we have

$$\Phi_1(sy+m(1-s)x)\Phi_2(sy+m(1-s)x) \leq s^2\Phi_1(y)\Phi_2(y) + m^2(1-s)^2\Phi_1(x)\Phi_2(x) + \\ ms(1-s)(\Phi_1(x)\Phi_2(y)+\Phi_1(y)\Phi_2(x))$$

Taking q-integral w.r.t. s over $[0,1]$ and using Lemma 2.2.4

$$\int_0^1 \Phi_1(sy+m(1-s)x)\Phi_2(sy+m(1-s)x)\,{}_0d_qs \\ \leq \int_0^1 \left(s^2\Phi_1(y)\Phi_2(x) + m^2(1-s)^2\Phi_1(x)\Phi_2(x) + ms(1-s)(\Phi_1(x)\Phi_2(y)+\Phi_1(y)\Phi_2(x))\right)\,{}_0d_qs.$$

Then

$$\int_0^1 \Phi_1(ty+m(1-t)x)\Phi_2(ty+m(1-t)x)_0d_qt \\ \leq \frac{\Phi_1(y)\Phi_2(x)}{1+q+q^2} + \frac{m^2q(1+q^2)\Phi_1(x)\Phi_2(x)}{(1+q+q^2)(1+q)} + \frac{mq^2N(u,v)}{(1+q+q^2)(1+q)}.$$

Next, taking double q-integral to both sides of the above inequality with respect to x, y on $[u,v]$, we have

$$\int_u^v\int_u^v\int_0^1 \Phi_1(sy+m(1-s)x)\Phi_2(sy+(1-s)x)_0d_qs_ud_qx_ud_qy \leq \\ \int_0^1 s^2\Phi_1(y)\Phi_2(y) + m^2(1-s)^2\Phi_1(x)\Phi_2(x) + ms(1-s)(\Phi_1(x)\Phi_2(y)+\Phi_1(y)\Phi_2(x))_0d_qs.$$

$$\int_0^1 \Phi_1(sy+m(1-s)x)\Phi_2(sy+m(1-s)x)\,{}_0d_qs \leq \\ \frac{v-u}{1+q+q^2}\int_u^v \Phi_1(y)\Phi_2(y)\,{}_ud_qy + \frac{m^2q(1+q^2)(v-u)}{(1+q+q^2)(1+q)}\int_u^v \Phi_1(x)\Phi_2(x)\,{}_ud_qx \\ + \frac{mq^2}{(1+q+q^2)(1+q)}\left(\int_u^v \Phi_2(y)\,{}_ud_qy\int_u^v \Phi_1(x)\,{}_ud_qx + \int_u^v f(y)_ud_qy\int_u^v g(x)_ud_qx\right)$$

$$\leq (v-u)\left(\frac{1}{1+q+q^2}+\frac{m^2q(1+q^2)}{(1+q)(1+q+q^2)}\right)\int_u^v \Phi_1(x)\Phi_2(x)_u d_qx+\frac{mq^2}{(1+q+q^2)(1+q)}$$
$$\left((v-u).\frac{q\Phi_2(u)+\Phi_2(v)}{1+q}.(v-u)\frac{q\Phi_1(u)+\Phi_2(v)}{1+q}+(v-u).\frac{q\Phi_1(u)+\Phi_2(v)}{1+q}.(v-u)\frac{q\Phi_2(u)+\Phi_2(v)}{1+q}\right)$$

On simplifying, we have

$$=\frac{(v-u)(1+q+m^2q(1+q^2)}{(1+q)(1+q+q^2)}\int_u^v \Phi_1(x)\Phi_2(x)_u d_qx+$$
$$\frac{2mq^2(v-u)^2}{(1+q)^3(1+q+q^2)}\left(q^2\Phi_1(u)\Phi_2(u)+qN(u,v)+\Phi_1(v)\Phi_2(v)\right).$$

Multiplying both sides by $\frac{(1+q)(1+q+q^2)}{(v-u)^2}$, we have

$$\frac{(1+q)(1+q+q^2)}{(v-u)^2}\int_u^v\int_u^v\int_0^1 \Phi_1(sy+m(1-s)x)\Phi_2(sy+(1-s)x)_0 d_qs_u d_qx_u d_qy \leq$$
$$\frac{1+q+m^2q(1+q^2)}{v-u}\int_u^v \Phi_1(x)\Phi_2(x)_u d_qx+\frac{2q^2m}{(1+q)^2}\left(q^2\Phi_1(u)\Phi_2(u)+qN(u,v)+\Phi_1(v)\Phi_2(v)\right).$$

Now, we begin the proof of (ii) part:

For all $x, y, \in K$, and $s \in [0,1]$, by the definition of m-convexity of Φ_1 and Φ_2, we have

$$\Phi_1(sy+m(1-s)\frac{u+v}{2})\leq s\Phi_1(y)+m(1-s)\Phi_1(\frac{u+v}{2})$$
$$\Phi_2(sy+m(1-s)\frac{u+v}{2})\leq s\Phi_2(y)+m(1-s)\Phi_2(\frac{u+v}{2})$$

Multiplying the above inequalities, we obtain

$$\Phi_1(sy+m(1-s)\frac{u+v}{2})\Phi_2(sy+m(1-s)\frac{u+v}{2})$$
$$\leq s^2\Phi_1(y)\Phi_2(y)+m^2(1-s)^2\Phi_1(\frac{u+v}{2})\Phi_2(\frac{u+v}{2})+$$
$$ms(1-s)\left(\Phi_1(y)\Phi_2(\frac{u+v}{2})+\Phi_1(\frac{u+v}{2})\right).$$

Obtaining q-integral with respect to s over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 \Phi_1(sy+m(1-s)\frac{u+v}{2})\Phi_2(sy+m(1-s)\frac{u+v}{2})_0d_qs\\ &\qquad\le \frac{\Phi_1(y)\Phi_2(y)}{1+q+q^2}+\frac{m^2q(1+q^2)}{(1+q+q^2)(1+q)}\Phi_1(\frac{u+v}{2})\Phi_2(\frac{u+v}{2})\\ &\qquad\qquad+\frac{mq^2}{(1+q+q^2)(1+q)}\left(\Phi_1(y)\Phi_2(\frac{u+v}{2})+\Phi_1(\frac{u+v}{2})\Phi_2(y)\right).\end{aligned}$$

Applying q-integral with respect to y over $[u,v]$ and using m-convexity of functions of Φ_1 and Φ_2, we observe

$$\begin{aligned}&\int_u^v\int_0^1 \Phi_1(sy+m(1-s)\frac{u+v}{2})\Phi_2(sy+m(1-s)\frac{u+v}{2})\,{}_0d_qs_ud_qy \le\\ &\quad\frac{1}{1+q+q^2}\int_u^v \Phi_1(y)\Phi_2(y)\,{}_ud_qy+\frac{m^2q(1+q^2)}{(1+q+q^2)(1+q)}\int_u^v \Phi_1(\frac{u+v}{2})\Phi_2(\frac{u+v}{2})\,{}_ud_qy\\ &\qquad+\frac{mq^2}{(1+q+q^2)(1+q)}\left(\int_u^v \Phi_1(y)\,{}_ud_qy\Phi_2(\frac{u+v}{2})+\Phi_1(\frac{u+v}{2})\int_u^v \Phi_2(y)\,{}_ud_qy\right)\end{aligned}$$

$$\begin{aligned}&\le \frac{1}{1+q+q^2}\int_u^v \Phi_1(y)\Phi_2(y)_ud_qy+\frac{m^2q(1+q^2)(u-v)}{(1+q+q^2)(1+q)}\left(\frac{\Phi_2(u)+\Phi_2(v)}{2}.\frac{\Phi_1(u)+\Phi_2(v)}{2}\right)+\\ &\frac{mq^2(v-u)}{(1+q+q^2)(1+q)}\left(\frac{\Phi_2(u)+\Phi_2(v)}{2}.\frac{q\Phi_1(u)+\Phi_1(v)}{1+q}+\frac{\Phi_1(u)+\Phi_1(v)}{2}.\frac{q\Phi_2(u)+\Phi_2(v)}{1+q}\right)\end{aligned}$$

$$\begin{aligned}&=\frac{1}{1+q+q^2}\int_u^v \Phi_1(y)\Phi_2(y)\,{}_ud_qy+\frac{m^2q(1+q^2)}{4(1+q+q^2)(1+q)}(M(u,v)+N(u,v))+\\ &\qquad\frac{mq^2(u-v)}{2(1+q+q^2)(1+q)^2}(2(q\Phi_1(u)\Phi_2(u)+\Phi_1(v)\Phi_2(v))+(1+q)N(u,v))\end{aligned}$$

Multiplying both sides by $\frac{(1+q)(1+q+q^2)}{v-u}$, we get

$$\begin{aligned}&\frac{(1+q)(1+q+q^2)}{(v-u)^2}\int_u^v\int_u^v\int_0^1 \Phi_1(sy+m(1-s)x)\Phi_2(sy+(1-s)x)_0d_qs_ud_qx_ud_qy\le\\ &\frac{1+q+m^2q(1+q^2)}{v-u}\int_u^v \Phi_1(x)\Phi_2(x)_ud_qx+\frac{2q^2m}{(1+q)^2}\left(q^2\Phi_1(u)\Phi_2(u)+qN(u,v)+\Phi_1(v)\Phi_2(v)\right).\end{aligned}$$

The proof is complete. □

Remark 2.3.13. *If $m=1$, then it reduces to the result as given in Theorem 2.2.11.*

Chapter 3

ON EXTENDED GEOMETRICALLY CONVEX FUNCTIONS

3.1 Introduction

In this chapter, we have extended the idea of the result of the product of classical convex functions to the products of generalized geometric convex functions, and these results are further extended to quantum calculus which is paradigm shift in convexity theory and integral inequality. The obtained results will definitely be used to estimate the integral mean of generalized geometric convex functions. The concept of geometrically convex functions was introduced by Zhang et. al in [62] as follows.

Definition 3.1.1. *[62] An operation* $f : (0, \infty) \to \mathbb{R}$ *is referred to be a geometrically-arithmetic (GA) convex function or simply a geometrically convex function on* $(0, \infty)$ *if*

$$f(a^s b^{1-s}) \leq s f(a) + (1-s) f(b) \tag{3.1.1}$$

holds for all $a, b \in (0, \infty), s \in [0, 1]$ *.*

If the inequality is reversed, then it is a GA concave function. Niculescu in [37] introduced the notion of geometrically-geometric convex function as defined below:

Definition 3.1.2. *[37] A function* $f : (0, \infty) \to \mathbb{R}$ *is said to be geometrically-geometric (GG) convex function on* $(0, \infty)$ *if*

$$f(a^s b^{(1-s)}) \leq (f(a))^s (f(b))^{1-s} \tag{3.1.2}$$

holds for all $a, b \in (0, \infty), s \in [0, 1]$ *.*

It is known as a GG-concave function if the inequality is reversed.

Xi et al. [60] generalized the geometrically-geometric convex function as follows:

Definition 3.1.3. *[60] A function* $f : (0, \infty) \to \mathbb{R}$ *is said to be* m*-geometrically-geometric (*m*-GG) convex function on* $(0, \infty)$ *if*

$$f(a^s b^{m(1-s)}) \leq (f(a))^s (f(b))^{m(1-s)} \tag{3.1.3}$$

holds for all $a, b \in (0, \infty), s \in [0, 1], m \in (0, 1]$ *.*

It is said to be an m-GG-concave function if the inequality is reversed.

Definition 3.1.4. *[60] A function* $f : (0, \infty) \to \mathbb{R}$ *is said to be* (α, m)*-geometrically-geometric* $((\alpha, m) - GG)$ *convex function on* $(0, \infty)$ *if*

$$f(a^s b^{m(1-s)}) \leq (f(a))^{s^\alpha} (f(b))^{m(1-s^\alpha)} \tag{3.1.4}$$

holds for all $a,\ b \in (0, \infty),\ s \in [0, 1],\ (\alpha, m) \in\ (0, 1]^2$ *.*

It is said to be a GH-concave function if the inequality is reversed.

Definition 3.1.5. *Let* a, b *be two positive real numbers. Then their logarithmic mean is denoted by* $L(a, b)$ *is defined as*

$$L(a, b) = \frac{b - a}{\log b - \log a}, \quad a \neq b. \tag{3.1.5}$$

Definition 3.1.6. *Let* a, b *be two positive real numbers. Then their geometric mean is denoted by* $G(a, b)$ *is defined as*

$$G(a, b) = \sqrt{ab}. \tag{3.1.6}$$

Definition 3.1.7. *Let* a, b *be two positive real numbers. Then their arithmetic mean is denoted by* $A(a, b)$ *is defined as*

$$A(a, b) = \frac{a + b}{2}. \tag{3.1.7}$$

3.2 Preliminary Results

For the class of convex functions, numerous significant inequalities are discovered. The Hermite-Hadamard inequality is among the most popular. Hermite-Hadamard is a well-known name in literature for the following double inequality.

Theorem 3.2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers $a, b \in I$ with $a < b$. Then we have*

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^a f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{3.2.1}$$

If the function f is concave, both inequalities hold in the opposite direction.

The following Hermite-Hadamard type inequality for the product of two classical convex functions was first proved by B.G. Pachpatte [44].

Theorem 3.2.2. *[44] Let f and g be real valued, non-negative and convex functions on $[a, b]$. The following inequalities are then true:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b), \tag{3.2.2}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and, $N(a, b) = f(a)g(b) + f(b)g(a)$.

3.3 Main Results

The extended definitions and the results on the products of generalized geometrically convex functions and their quantum estimates are obtained in this section, and they are as follows:

Definition 3.3.1. *A function $f : (0, \infty) \to \mathbb{R}$ is said to be m-geometrically-arithmetic $(m - GA)$ convex function on $(0, \infty)$ if*

$$f(a^s b^{m(1-s)}) \leq sf(a) + m(1-s)f(b) \tag{3.3.1}$$

holds for all $a, b \in (0, \infty), s \in [0, 1], m \in (0, 1]$.

It said to be an m-GA concave function if the inequality is reversed. It can further be generalized as (α, m)-geometrically convex function as follows:

Definition 3.3.2. *A function $f : (0, \infty) \to \mathbb{R}$ is said to be (α, m) geometrically-arithmetic $((\alpha, m) - GA)$ convex function on $(0, \infty)$ if*

$$f(a^s b^{m(1-s)}) \leq s^\alpha f(a) + m(1 - s^\alpha)f(b) \tag{3.3.2}$$

holds for all $a, b \in (0, \infty), s \in [0, 1], (\alpha, m) \in (0, 1]^2$.

It is said to be an (α, m)- GA concave function if the inequality is reversed. We further generalize geometrically-harmonic convex functions as m and (α, m)-GH convex functions as follows: We further generalize geometrically-harmonic convex functions as m and (α, m)-GH convex functions as follows:

Definition 3.3.3. *A function* $f : (0, \infty) \to \mathbb{R}$ *is said to be* m*-geometrically-harmonic (*m*-GH) convex function on* $(0, \infty)$ *if*

$$f(a^s b^{m(1-s)}) \leq \frac{1}{\frac{s}{mf(a)} + \frac{1-s}{f(b)}} = \frac{mf(a)f(b)}{sf(b) + m(1-s)f(a)} \tag{3.3.3}$$

holds for all $a, b \in (0, \infty), s \in [0, 1], m \in (0, 1]$.

It is said to be a m-GH-concave function if the inequality is reversed.

Definition 3.3.4. *A function* $f : (0, \infty) \to \mathbb{R}$ *is said to be* (α, m)*-geometrically-harmonic* (α, m)*-GH convex function on* $(0, \infty)$ *if*

$$f(a^s b^{m(1-s)}) \leq \frac{1}{\frac{s^\alpha}{mf(a)} + \frac{1-s^\alpha}{f(b)}} = \frac{mf(a)f(b)}{s^\alpha f(b) + m(1-s^\alpha)f(a)} \tag{3.3.4}$$

holds for all $a, b \in (0, \infty), s \in [0, 1], (\alpha, m) \in (0, 1]$.

It is said to be a (α, m)-GH-concave function if the inequality is reversed.

Theorem 3.3.5. *The integral mean of geometrically-arithmetic convex function is bounded above by the arithmetic mean of their functional values and the product of reciprocal of the logarithmic mean .*

Proof. Let f be a GA-convex function defined on $(0, \infty)$. Then by definition of GA-convexity, we have

$$f(a^s b^{1-s}) \leq sf(a) + (1-s)f(b).$$

Integrating with respect to s over $[0, 1]$, we have

$$\int_0^1 f(a^s b^{1-s})\, ds \leq f(a) \int_0^1 s\, dt + f(b) \int_0^1 (1-s)\, ds = \frac{f(a) + f(b)}{2}.$$

Put $x = a^s b^{1-s}, ds = \frac{dx}{x \log \frac{a}{b}}$ When $s \to 0$, then $x \to b$; when $s \to 1$, then $x \to a$. On substituting these values, we have

$$\frac{1}{b-a} \int_a^b \frac{f(x)}{x}\, dx \leq \frac{f(a) + f(b)}{2L(a, b)}.$$

This completes the proof. □

Theorem 3.3.6. *Let f and g be two (α, m)-GA convex functions. Then their product is given as follows:*

$$\frac{1}{b^m - a}\int_a^{b^m} \frac{f(x)g(x)}{x}dx \leq \frac{1}{L(a, b^m)}$$
$$\left(\frac{1}{(2\alpha+1)}K_a + \frac{2\alpha^2 m^2}{(\alpha+1)(2\alpha+1)}K_b + \frac{\alpha}{(\alpha+1)(2\alpha+1)}K_b^a\right), \tag{3.3.5}$$

where $K_a = f(a)g(a), K_b = f(b)g(b), K_b^a = f(a)g(b) + f(b)g(a)$.

Proof. Let f and g be two (α, m) GA convex functions. Then, by definition, we have

$$f(a^s b^{m(1-s)}) \leq s^\alpha f(a) + (1 - s^\alpha)f(b)$$

And,

$$g(a^s b^{m(1-s)}) \leq s^\alpha g(a) + (1 - s^\alpha)g(b).$$

Since $f, g \geq 0$, we have

$$f(a^s b^{m(1-s)})g(a^s b^{m(1-s)}) \leq f(a)g(a)s^{2\alpha} + f(b)g(b)m^2(1 - s^\alpha)^2$$
$$+ m(f(a)g(b) + f(b)g(a))s^\alpha(1 - s^\alpha).$$

Integrating with respect to s over $[0, 1]$, we have

$$\int_0^1 f(a^s b^{m(1-s)})g(a^s b^{m(1-s)})\, ds \leq f(a)g(a)\int_0^1 s^{2\alpha}\, ds + f(b)g(b)m^2\int_0^1 (1 - s^\alpha)^2\, ds$$
$$+ m(f(a)g(b) + f(b)g(a))\int_0^1 s^\alpha(1 - s^\alpha)\, ds.$$

Here,

$$\int_0^1 s^{2\alpha}\, ds = \frac{1}{2\alpha+1},$$
$$\int_0^1 (1 - s^\alpha)^2\, ds = \frac{2\alpha^2}{(\alpha+1)(2\alpha+1)},$$
$$\int_0^1 s^\alpha(1 - s^\alpha)\, ds = \frac{\alpha}{(\alpha+1)(2\alpha+1)}.$$

Put $x = a^s b^{m(1-s)}$, when $s \to 0, x \to b^m$; when $s \to 1, x \to a$. And, $ds = \frac{1}{x(\log a - \log b^m)}dx$. On substituting these values in above inequality, we have

$$\frac{1}{\log b^m - \log a}\int_a^{b^m} \frac{f(x)g(x)}{x}dx \quad \leq \quad \frac{K_a}{2\alpha+1} + \frac{2\alpha^2 m^2 K_b}{(\alpha+1)(2\alpha+1)} + \frac{m\alpha K_b^a}{(\alpha+1)(2\alpha+1)}.$$

$$\frac{b^m - a}{\log b^m - \log a}\frac{1}{b^m - a}\int_a^{b^m} \frac{f(x)g(x)}{x}\,dx \leq \frac{K_a}{2\alpha+1} + \frac{2\alpha^2 m^2 K_b}{(\alpha+1)(2\alpha+1)} + \frac{m\alpha K_b^a}{(\alpha+1)(2\alpha+1)}$$

$$L(a,b^m)\frac{1}{b^m - a}\int_a^{b^m} \frac{f(x)g(x)}{x}\,dx \leq \frac{K_a}{2\alpha+1} + \frac{2\alpha^2 m^2 K_b}{(\alpha+1)(2\alpha+1)} + \frac{m\alpha K_b^a}{(\alpha+1)(2\alpha+1)}$$

$$\frac{1}{b^m - a}\int_a^{b^m} \frac{f(x)g(x)}{x}\,dx \leq \frac{1}{L(a,b^m)}\left[\frac{K_a}{2\alpha+1} + \frac{2\alpha^2 m^2 K_b}{(\alpha+1)(2\alpha+1)} + \frac{m\alpha K_b^a}{(\alpha+1)(2\alpha+1)}\right].$$

This completes the proof. □

Remark 3.3.7. *If $\alpha = m = 1$, then it reduces to the product of two GA-convex functions.*

Theorem 3.3.8. *The integral mean of geometrically-geometric convex function is bounded above by a half of the logarithmic value of the product of their functional values and the reciprocal of the logarithmic mean.*

Proof. Let $f : (0,\infty) \to (0,\infty)$ be a geometrically- geometric convex function. Then, by definition, we have

$$f(a^s b^{1-s}) \leq (f(a))^s (f(b))^{1-s}$$

$$\log f(a^s b^{1-s}) \leq s\log(f(a)) + (1-s)\log(f(b))$$

Integrating with respect to s over $[0,1]$, we have

$$\int_0^1 \log f(a^s b^{1-s})\,ds \leq \log(f(a))\int_0^1 s\,ds + \log(f(b))\int_0^1 (1-s)\,ds = \frac{\log(f(a)) + \log(f(b))}{2}.$$
$$= \frac{\log(f(a)f(b))}{2}.$$

Using the change of variables by putting $x = a^s b^{1-s}$, we have

$$ds = \frac{L(a,b)}{a-b}\frac{dx}{x}.$$

When $s \to 0$, $x \to b$; when $s \to 1$, then $x \to a$. On substituting these values, we obtain

$$\frac{1}{b-a}\int_a^b \frac{\log f(x)}{x}\,dx \leq \frac{1}{2L(a,b)}\log(f(a)f(b)).$$

This completes the proof.

□

Theorem 3.3.9. *Let* $f, g : (0, \infty) \to (0, \infty)$ *be two* (α, m) *GG-convex functions. Then the following inequalities persist:*

$$\frac{1}{b^m - a}\int_a^{b^m} \frac{\log f(x) + \log g(x)}{x}\, dx \leq \frac{1}{L(a, b^m)(\alpha+1)}\left(\log(f(a)g(a)) + m\alpha \log(f(b)g(b))\right). \tag{3.3.6}$$

Proof. As f and g are two (α, m) GG convex functions, then by definition, we have

$$f(a^s b^{m(1-s)}) \leq (f(a))^{s^\alpha}(f(b))^{m(1-s^\alpha)},$$

and

$$g(a^s b^{m(1-s)}) \leq (g(a))^{s^\alpha}(g(b))^{m(1-s^\alpha)}.$$

Since $f, g \geq 0$, we have

$$f(a^s b^{m(1-s)})g(a^s b^{m(1-s)}) \leq (f(a)g(a))^{s^\alpha}(f(b)g(b))^{m(1-s^\alpha)}.$$

Taking $\log$ on both sides, we have

$$\log f(a^s b^{m(1-s)}) + \log g(a^s b^{m(1-s)}) \leq s^\alpha \log(f(a)g(a)) + m(1-s^\alpha)\log(f(b)g(b)).$$

Integrating with respect to s over $[0,1]$, we have

$$\int_0^1 \log f(a^s b^{m(1-s)})\, ds + \int_0^1 \log g(a^s b^{m(1-s)})\, ds \leq \log(f(a)g(a)) \int_0^1 s^\alpha\, ds$$
$$+ m\log(f(b)g(b)) \int_0^1 (1-s^\alpha)\, ds.$$

Using the change of variables by putting $x = \phi_1^s \phi_2^{m(1-s)}$, we have

$$\frac{L(a, b^m)}{b^m - a}\int_a^{b^m} \frac{\log f(x) + \log g(x)}{x}\, dx \leq \frac{\log(f(a)g(a))}{\alpha+1} + \frac{m\alpha\log(f(b)g(b))}{\alpha+1}.$$

Hence, we have

$$\frac{1}{b^m - a}\int_a^{b^m} \frac{\log f(x) + \log g(x)}{x}\, dx \leq \frac{1}{L(a, b^m)(\alpha+1)}\left(\log(f(a)g(a)) + m\alpha \log(f(b)g(b))\right).$$

This completes the proof. □

Corollary 3.3.10. *If* $\alpha = 1$, *then we get the result of the product of two* m*-GG convex functions.*

Corollary 3.3.11. *If $\alpha = m = 1$, then we get the result of the products of two GG convex functions.*

Theorem 3.3.12. *The integral mean of geometrically-harmonic (GH) convex function is bounded above by the ratio of squares of geometric mean of functional values to the product of logarithmic mean of their functional values and inputs.*

Proof. Let f be a GH convex function defined from $(0,\infty) \to \mathbb{R}$. Then, by definition, we have

$$f(a^s b^{1-s}) \leq \frac{1}{\frac{s}{f(a)} + \frac{1-s}{f(b)}} = \frac{f(a)f(b)}{sf(b) + (1-s)f(a)}.$$

Integrating with respect to s over $[0,1]$, we have

$$\int_0^1 f(a^s b^{1-s})\, ds \leq \int_0^1 \frac{f(a)f(b)}{sf(b) + (1-s)f(a)}\, ds = f(a)f(b) \int_0^1 \frac{1}{f(a) + (f(b) - f(\phi_1))s}\, ds.$$

$$\int_0^1 f(a^s b^{1-s})\, ds \leq f(a)f(b) \frac{\log f(b) - \log f(a)}{f(b) - f(a)} = \frac{G^2(f(a), f(b))}{L(f(a), f(b))}.$$

Using the change of variables by putting $x = a^s b^{1-s}$, we have

$$\frac{1}{\log b - \log a} \int_a^b \frac{\log f(x)}{x}\, dx \leq \frac{G^2(f(a), f(b))}{L(f(a), f(b))}$$

$$\frac{L(a,b)}{b-a} \int_a^b \frac{\log f(x)}{x}\, dx \leq \frac{G^2(f(a), f(b))}{L(f(a), f(b))}$$

$$\frac{1}{b-a} \int_a^b \frac{\log f(x)}{x}\, dx \leq \frac{G^2(f(a), f(b))}{L(a,b)L(f(a), f(b))}.$$

This completes the proof. ☐

Theorem 3.3.13. *Let $f, g : (0,\infty) \to \mathbb{R}$ be two geometrically-harmonic convex functions. Then the following inequality holds.*

$$\frac{1}{b-a} \int_a^b \frac{f(x)g(x)}{x}\, dx$$

$$\leq \frac{1}{L(a,b)} \frac{2K_aK_b}{\sqrt{4K_a(K_a+K_b-K_{ab}-K_{ba})}-(K_{ab}+K_{ba}-2K_a)^2}$$
$$(\tan^{-1}\frac{2(K_a+K_b-K_{ab}-K_{ba})+(K_{ab}+K_{ba}-2K_a)}{\sqrt{4K_a(K_a+K_b-K_{ab}-K_{ba})}-(K_{ab}+K_{ba}-2K_a)^2}$$
$$-\tan^{-1}\frac{K_{ab}+K_{ba}-2K_a}{\sqrt{4K_a(K_a+K_b-K_{ab}-K_{ba})}-(K_{ab}+K_{ba}-2K_a)^2}), \tag{3.3.7}$$

where $K_a = f(a)g(a), K_b = f(b)g(b), K_{ab} = f(a)g(b), K_{ba} = f(b)g(a)$.

Proof. Let f, g be two GH convex functions. Then, by definition, we have

$$f(a^sb^{1-s}) \leq \frac{f(a)f(b)}{sf(b)+(1-s)f(a)} = \frac{f(a)f(b)}{f(a)+(f(b)-f(a))s}$$

and

$$g(a^sb^{1-s}) \leq \frac{g(a)g(b)}{sg(b)+(1-s)g(a)} = \frac{g(a)g(b)}{g(a)+(g(b)-g(a))s}.$$

Since $f, g \geq 0$, we have

$$f(a^sb^{1-s})g(a^sb^{1-s}) \leq \frac{f(a)f(b)}{f(a)+(f(b)-f(a))s}\frac{g(a)g(b)}{g(a)+(g(b)-g(a))s}.$$

Then

$$f(a^sb^{1-s})g(a^sb^{1-s}) \leq \frac{K_aK_b}{K_a+(K_{ab}+K_{ba}-2K_a)s+(K_a+K_b-K_{ab}-K_{ba})s^2}.$$

Integrating with respect to s over $[0,1]$, we have

$$\int_0^1 f(a^sb^{1-s})g(a^sb^{1-s})\,ds$$

$$\leq K_aK_b\int_0^1 \frac{1}{K_a+(K_{ab}+K_{ba}-2K_a)s+(K_a+K_b-K_{ab}-K_{ba})s^2}$$

Using the change of variables by putting

$$x = a^sb^{1-s}$$

and using the formula

$$\int_0^1 \frac{1}{ax^2+bx+c}\,dx = \frac{2}{\sqrt{4ac-b^2}}\left(\tan^{-1}\frac{2ab+b}{\sqrt{4ac-b^2}} - \tan^{-1}\frac{b}{\sqrt{4ac-b^2}}\right),$$

we have

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b \frac{f(x)g(x)}{x}\,dx \\
&\qquad \leq \frac{1}{L(a,b)}\frac{2K_aK_b}{\sqrt{4K_a(K_a+K_b-K_{ab}-K_{ba})-(K_{ab}+K_{ba}-2K_a)^2}} \\
&\qquad (\tan^{-1}\frac{2(K_a+K_b-K_{ab}-K_{ba})+(K_{ab}+K_{ba}-2K_a)}{\sqrt{4K_a(K_a+K_b-K_{ab}-K_{ba})-(K_{ab}+K_{ba}-2K_a)^2}} \\
&\qquad\qquad -\tan^{-1}\frac{K_{ab}+K_{ba}-2K_a}{\sqrt{4K_a(K_a+K_b-K_{ab}-K_{ba})-(K_{ab}+K_{ba}-2K_a)^2}}),
\end{aligned}
$$

where $K_a = f(a)g(a), K_b = f(b)g(b), K_{ab} = f(a)g(b), K_{ba} = f(b)g(a)$.

This completes the proof. □

3.4 Extended Results in q-Calculus

The results on quantum estimates of the product of two $(\alpha, m) - GA$ and $(\alpha, m) - GG$ convex functions are given below:

Theorem 3.4.1. *Let $f, g : (0,\infty) \to (0,\infty)$ be two (α, m) -GA convex functions. Then the following inequality holds*

$$\frac{\log q}{q-1}\frac{1}{b^m-a}\int_a^{b^m}\frac{\log(f(x)g(x))}{x}\,{}_ad_qx \leq \frac{1}{L(a,b^m)}$$

$$\left(\frac{K_a}{[2\alpha+1]_q}+m^2K_b\left(1-\frac{2}{[\alpha+1]_q}+\frac{1}{[2\alpha+1]_q}\right)+mK_b^a\left(\frac{1}{[\alpha+1]_q}-\frac{1}{[2\alpha+1]_q}\right)\right). \quad (3.4.1)$$

Proof. As f and g are two $(\alpha, m)-$ GA convex functions, then by definition, we have

$$f(a^sb^{m(1-s)}) \leq s^\alpha f(a) + m(1-s^\alpha)f(b)$$

and

$$g(a^sb^{m(1-s)}) \leq s^\alpha g(a) + m(1-s^\alpha)g(b).$$

Since $f, g \geq 0$, we have

$$
\begin{aligned}
f(a^sb^{m(1-s)})g(a^sb^{m(1-s)}) \leq\ & s^{2\alpha}f(a)g(a) + m^2(1-s^\alpha)^2f(b)g(b) \\
& +m(f(a)g(b)+f(b)g(a))s^\alpha(1-s^\alpha).
\end{aligned}
$$

Taking q-integral with respect to s over $[0,1]$, we have

$$\int_0^1 f(a^s b^{m(1-s)}) g(a^s b^{m(1-s)}) \, {}_0d_q s \leq K_a \int_0^1 s^{2\alpha} \, {}_0d_q s + m^2 K_b \int_0^1 (1-s^\alpha)^2 \, {}_0d_q s + m K_a^b \int_0^1 s^\alpha (1-s^\alpha) \, {}_0d_q s.$$

Note that

$$\begin{aligned} \int_0^1 s^{2\alpha} \, {}_0d_q s &= (1-q) \sum_{n=0}^{\infty} q^n q^{2\alpha n} \\ &= (1-q) \sum_{n=0}^{\infty} q^{n(2\alpha+1)} \\ &= (1-q) \frac{1}{1-q^{2\alpha+1}} \\ &= \frac{1}{[2\alpha+1]_q}. \end{aligned}$$

Similarly, we obtain

$$\int_0^1 (1-s^\alpha)^2 \, {}_0d_q t = 1 - \frac{2}{[\alpha+1]_q} + \frac{1}{[2\alpha+1]_q}$$

and

$$\int_0^1 s^\alpha (1-s^\alpha) \, {}_0d_q s = \frac{1}{[\alpha+1]_q} - \frac{1}{[2\alpha+1]_q}.$$

If we take $x = a^s b^{m(1-s)}$, then $s = \frac{\log x}{\log a - \log b^m} - \frac{\log b^m}{\log a - \log b^m}$ when $s \to 0, x \to b^m$; when $s \to 1, x \to a$

And,

$${}_0d_q s = \frac{L(a, b^m) \log q}{(b^m - a)(q-1)x} \, {}_0d_q x.$$

On substituting these values, we obtain

$$\frac{1}{b^m - a} \int_a^{b^m} \frac{f(x)g(x)}{x} \, {}_0d_q x \leq \frac{q-1}{\log q L(a, b^m)} \left(\frac{K_a}{[2\alpha+1]_q} + m^2 K_b \left(1 - \frac{2}{[\alpha+1]_q} + \frac{1}{[2\alpha+1]_q} \right) + m K_a^b \left(\frac{1}{[\alpha+1]_q} - \frac{1}{[2\alpha+1]_q} \right) \right).$$

This completes the proof. □

Theorem 3.4.2. *Let $f, g : (0,\infty) \to (0,\infty)$ be two $(\alpha, m)-$ geometrically-geometric (GG)* *convex functions. Then the following inequality holds:*

$$\frac{1}{b^m - a}\int_a^{b^m} \frac{\log f(x) + \log g(x)}{x}\, {}_a d_q x \le \frac{q-1}{L(a, b^m)\log q} \left(\frac{\log K_a}{[\alpha+1]_q} + m \log K_b \left(1 - \frac{1}{[\alpha+1]_q}\right)\right). \tag{3.4.2}$$

Proof. As f and g are two (α, m)- GG convex functions, then by definitions, we have

$$f(a^s b^{m(1-s)}) \le (f(a))^{s^\alpha} (f(b))^{m(1-s^\alpha)}$$

and

$$g(a^s b^{m(1-s)}) \le (g(a))^{s^\alpha} (g(b))^{m(1-s^\alpha)}.$$

Since $f, g \ge 0$, we have

$$f(a^s b^{m(1-s)}) g(a^s b^{m(1-s)}) \le (f(a)g(a))^{s^\alpha} (f(b)g(b))^{m(1-s^\alpha)}$$

Taking $\log$ on both sides, we have

$$\log f(a^s b^{m(1-s)}) + \log g(a^s b^{m(1-s)}) \le s^\alpha \log K_a + m(1 - s^\alpha)\log K_b$$

Taking q-integral with respect to s over $[0,1]$, we have

$$\int_0^1 \log f(a^s b^{m(1-s)})_0\, d_q s + \int_0^1 \log g(a^s b^{m(1-s)})_0\, d_q s \le \log K_a \int_0^1 s^\alpha{}_0\, d_q s + m \log K_b \int_0^1 (1 - s^\alpha)_0\, d_q s.$$

Note that

$$\begin{aligned}\int_0^1 s^\alpha{}_0 d_q s &= (1-q)\sum_{n=0}^{\infty} q^n (q^n)^\alpha \\ &= (1-q)\sum_{n=0}^{\infty} q^{n(\alpha+1)} \\ &= (1-q).\frac{1}{1-q^{\alpha+1}} \\ &= \frac{1}{[\alpha+1]_q}.\end{aligned}$$

Similarly, we have

$$\int_0^1 (1 - s^\alpha)\, {}_0 d_q s = 1 - \frac{1}{[\alpha+1]_q}.$$

using the change of variables by putting $x = a^s b^{m(1-s)}$, $s = \frac{\log x}{\log a - \log b^m} - \frac{\log b^m}{\log a - \log b^m}$; when $s \to 0, x \to b^m$; when $s \to 1, x \to a$.

And,

$$_0d_q s = \frac{L(a, b^m) \log q}{(b^m - a)(q-1)x} \, _0d_q x$$

On substituting these values, we obtain

$$\frac{1}{b^m - a} \int_a^{b^m} \frac{\log f(x) + \log g(x)}{x} \, _ad_q x \le \frac{q-1}{\log q L(a, b^m)} \left(\frac{\log K_a}{[\alpha+1]_q} + m \log K_b \left(1 - \frac{1}{[\alpha+1]_q}\right)\right).$$

This completes the proof. $\square$

Chapter 4

HERMITE-HADAMARD INEQUALITY: GENERALIZED HARMONICALLY CONVEX FUNCTIONS

4.1 Introduction

The idea of convexity has undergone much developments, generalizations, and extensions in recent years. Novel classes of convex functions have been introduced, and a new version of the Hermite-Hadamard integral inequality has been found. The harmonic convex function has been defined on the concept of harmonic mean as follows: See [26] for details.

Definition 4.1.1. *An interval is defined as* $I = [\alpha, \beta]$ *. Iscan defined the notion of harmonically convex function as follows: A function* Φ *is said to be harmonically convex if* $\Phi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *if*

$$\Phi\left(\frac{\alpha\beta}{s\alpha + (1-s)\beta}\right) \leq s\Phi(\beta) + (1-s)\Phi(\alpha), \quad \alpha, \beta \in I, \forall s \in [0,1].$$

The following Hermite-Hadamard inequality was proved in [26] for harmonically convex functions.

Theorem 4.1.2. *Let* $\Phi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *be a harmonically convex function. The following inequalities are then true:*

$$\Phi\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \leq \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\Phi(x)}{x^2}dx \leq \frac{\Phi(\alpha)+\Phi(\beta)}{2}. \tag{4.1.1}$$

Inspired by the work of Iscan [6, 26] who defined harmonic convexity, in this chapter, we have generalized this convexity to m-harmonic convexity and further to (α, m)-harmonic convexity. Moreover, we also have defined new notions of harmonic-geometric (HG)-convexity, $m-HG$-convexity and $(\alpha, m)-HG$-convexity. For each of these convexities, we have proved Hermite-Hadamard inequalities and we have obtained inequalities involving the product of each of the those type of convexities.

4.2 MAIN RESULTS

4.2.1 Harmonically Convex Function

We generalize harmonic convexity as follows:

Definition 4.2.1. *A function $\Phi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be m-harmonically convex if for any $m, s \in [0,1]$*

$$\Phi\left(\frac{mxy}{sx+m(1-s)y}\right) \leq s\Phi(y)+m(1-s)\Phi(x), \quad x, y \in I.$$

The notion of m-harmonic convexity can further be generalized as follows:

Definition 4.2.2. *A function $\Phi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be (α, m)-harmonically convex if for any $m, t, \alpha \in [0,1]$*

$$\Phi\left(\frac{mxy}{sx+m(1-s)y}\right) \leq s^{\alpha}\Phi(y)+m(1-s^{\alpha})\Phi(x), \quad x, y \in I.$$

For (α, m)-harmonically convex functions, we prove the following:

Theorem 4.2.3. *Let $\Phi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be an (α, m)-harmonically convex function. Then the following inequalities hold:*

$$(i)\ \frac{m\eta_1\eta_2}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\frac{\Phi(x)}{x^2}\,dx \leq \frac{\Phi(\eta_2)+m\alpha\Phi(\eta_1)}{\alpha+1} \tag{4.2.1}$$

$$(ii)\, \Phi\left(\frac{2m\eta_1\eta_2}{m\eta_1+\eta_2}\right) \le \frac{2m\eta_1\eta_2(1-A)A^\alpha}{(\eta_2-m\eta_1)}\int_{2m\eta_1(1-A)}^{2\eta_2(1-A)}\frac{\Phi(\omega)}{\omega^2}d\omega + \frac{2m\eta_1\eta_2 A(1-A^\alpha)}{(\eta_2-m\eta_1)}\int_{2\eta_1 A}^{2\eta_2 A/m}\frac{\Phi(\omega)}{\omega^2}d\omega, \tag{4.2.2}$$

where $A=\left(\frac{m}{1+m}\right)^{\frac{1}{\alpha}}$.

Proof. (i) From the definition of (α, m)-harmonic convexity, we have

$$\Phi\left(\frac{m\eta_1\eta_2}{s\eta_2+m(1-s)\eta_1}\right) \le s^\alpha\Phi(\eta_2)+m(1-s^\alpha)\Phi(\eta_1) \tag{4.2.3}$$

which on integrating over $[0,1]$ with respect to s gives

$$\int_0^1\Phi\left(\frac{m\eta_1\eta_2}{s\eta_2+m(1-s)\eta_1}\right)ds \le \Phi(\eta_2)\int_0^1 s^\alpha\, ds + m\Phi(\eta_1)\int_0^1(1-s^\alpha)\, ds.$$

Now, by variable substitution $x=\frac{m\eta_1\eta_2}{s\eta_2+m(1-s)\eta_1}$ on the left side and solving the right side of the above inequality immediately gives (4.2.1)

(ii) Since (4.2.3) holds for all $s\in[0,1]$, in particular, it holds for $s=\left(\frac{m}{1+m}\right)^{\frac{1}{\alpha}} := A$, i.e.,

$$\Phi\left(\frac{m\eta_1\eta_2}{A\eta_2+(1-A)m\eta_1}\right) \le A^\alpha\Phi(\eta_2)+m(1-A^\alpha)\Phi(\eta_1) \tag{4.2.4}$$

Take

$$\eta_1=\frac{2xyA}{sy+mx(1-s)}, \qquad \eta_2=\frac{2mxy(1-A)}{mxs+(1-s)y} \tag{4.2.5}$$

Then

$$\frac{m\eta_1\eta_2}{A\eta_2+(1-A)m\eta_1} = \frac{\left(\frac{2mxyA}{sy+mx(1-s)}\right)\left(\frac{2mxy(1-A)}{mxs+(1-s)y}\right)}{\left(\frac{2mxy(1-A)A}{mxs+(1-s)y}+\frac{2mxyA(1-A)}{sy+mx(1-s)}\right)} = \frac{2mxy}{mx+y},$$

that is,

$$\Phi\left(\frac{m\eta_1\eta_2}{A\eta_2+(1-A)m\eta_1}\right)=\Phi\left(\frac{2mxy}{mx+y}\right)$$

which on integrating with respect to s on $[0,1]$ gives

$$\int_0^1\Phi\left(\frac{m\eta_1\eta_2}{A\eta_2+(1-A)m\eta_1}\right)ds=\Phi\left(\frac{2mxy}{mx+y}\right). \tag{4.2.6}$$

Now, with η_1 and η_2 given by (4.2.5), the right side of (4.2.4) becomes

$$A^{\alpha}\Phi\left(\frac{2mxy(1-A)}{mxs+(1-s)y}\right)+m(1-A^{\alpha})\Phi\left(\frac{2xyA}{sy+mx(1-s)}\right) \tag{4.2.7}$$

which on integrating with respect to s on $[0,1]$ becomes

$$A^{\alpha}I_1+m(1-A^{\alpha})I_2,$$

where

$$I_1=\int_0^1\Phi\left(\frac{2mxy(1-A)}{mxs+(1-s)y}\right)ds$$

and

$$I_2=\int_0^1\Phi\left(\frac{2mxyA}{sy+mx(1-s)}\right)ds.$$

For solving I_1, put $\frac{mxy(1-A)}{mxs+(1-s)y}=z$ so that $ds=\frac{mxy(1-A)}{(y-mx)z^2}dz$ and as $s\to 0, z\to mx(1-A)$ and as $s\to 1, z\to y(1-A)$. Thus

$$I_1=\frac{mxy(1-A)}{y-mx}\int_{mx(1-A)}^{y(1-A)}\frac{\Phi(2z)}{z^2}dz$$

$$=\frac{2mxy(1-A)}{y-mx}\int_{2mx(1-A)}^{2y(1-A)}\frac{\Phi(\omega)}{\omega^2}d\omega. \tag{4.2.8}$$

Similarly, by taking $\frac{xyA}{sy+mx(1-s)}=z$, it can be shown that

$$I_2=\frac{2xyA}{y-mx}\int_{2xA}^{2yA/m}\frac{\Phi(\omega)}{\omega^2}d\omega. \tag{4.2.9}$$

Thus, integrating (4.2.4) with respect to s over $[0,1]$ and using (4.2.6), (4.2.7), (4.2.8), and (4.2.9), we obtain

$$\Phi\left(\frac{2mxy}{mx+y}\right)\le\frac{2mxy(1-A)A^{\alpha}}{(y-mx)}\int_{2mx(1-A)}^{2y(1-A)}\frac{\Phi(\omega)}{\omega^2}d\omega+\frac{2mxyA(1-A^{\alpha})}{y-mx}\int_{2xA}^{2yA/m}\frac{\Phi(\omega)}{\omega^2}d\omega$$

and rewriting x and y as respectively η_1 and η_2, we get

$$\Phi\left(\frac{2m\eta_1\eta_2}{m\eta_1+\eta_2}\right)\le\frac{2m\eta_1\eta_2(1-A)A^{\alpha}}{(\eta_2-m\eta_1)}\int_{2m\eta_1(1-A)}^{2\eta_2(1-A)}\frac{\Phi(\omega)}{\omega^2}d\omega+\frac{2m\eta_1\eta_2A(1-A^{\alpha})}{\eta_2-m\eta_1}\int_{2\eta_1A}^{2\eta_2A/m}\frac{\Phi(\omega)}{\omega^2}d\omega.$$

This completes the proof. □

Remark 4.2.4. *If* $\alpha = m = 1$ *, then* $A = \frac{1}{2}$ *and therefore, in this case, both the integrals on the right side of (4.2.3) becomes equal to* $\frac{\eta_1\eta_2}{2(\eta_2-\eta_1)}\int_{\eta_1}^{\eta_2}\frac{\Phi(\omega)}{\omega^2}d\omega$ *. Consequently, (4.2.3) and (4.2.4) together become (4.1.1).*

Our next consideration is about the product of (α, m)-HA convex functions. We prove the following:

Theorem 4.2.5. *The product of two* (α, m) *-HA convex functions is also an* (α, m) *-HA convex function.*

Proof. By definition, for any $x, y \in I$ we have for two (α, m)-HA convex functions f and g , and for any $s \in [0,1]$

$$\Phi\left(\frac{mxy}{sy+m(1-s)x}\right) \le s^{\alpha}\Phi(x) + m(1-t^{\alpha})\Phi(y)$$

and

$$\Psi\left(\frac{mxy}{sy+m(1-s)x}\right) \le s^{\alpha}\Psi(x) + m(1-s^{\alpha})\Psi(y)$$

which together give that

$$\begin{aligned}\Phi\left(\frac{mxy}{sy+m(1-s)x}\right)\Psi\left(\frac{mxy}{sy+m(1-s)x}\right) &\le (s^{\alpha}\Phi(x) + m(1-s^{\alpha})\Phi(y))(s^{\alpha}\Psi(x) + m(1-s^{\alpha})\Psi(y)) \\ &\le s^{2\alpha}\Phi(x)\Psi(x) + m^2(1-s^{\alpha})^2\Phi(y)\Psi(y) \\ &\le s^{\alpha}\Phi(x)\Psi(x) + m(1-s^{\alpha})\Phi(y)\Psi(y) \\ &= s^{\alpha}(\Phi\Psi)(x) + m(1-s^{\alpha})(\Phi\Psi)(y).\end{aligned}$$

and the assertion follows. □

Remark 4.2.6. *From Theorem 4.2.5, it follows that the product of two m-HA convex functions is again an* m *-HA convex function which implies that the product of two HA convex functions is again so.*

We shall be using the following notations:

$K_{\eta_1} := \Phi(\eta_1)\Psi(\eta_1)$, $K_{\eta_2} := \Phi(\eta_2)\Psi(\eta_2)$, $K_{\eta_2}^{\eta_1} := \Phi(\eta_1)\Psi(\eta_2) + \Phi(\eta_2)\Psi(\eta_1)$.

Below, we prove a Hermite-Hadamard type inequality for the product of (α, m)-HA convex functions.

Theorem 4.2.7. *Let* $\Phi, \Psi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *be two* (α, m) *-HA convex functions. Then the following inequality holds:*

$$\frac{m\eta_1\eta_2}{m\eta_2-\eta_1}\int_{\eta_1}^{m\eta_2}\frac{\Phi(x)\Psi(x)}{x^2}dx \le \frac{K_{\eta_2}}{2\alpha+1} + \frac{2\alpha^2m^2K_{\eta_1}}{(\alpha+1)(2\alpha+1)} + \frac{m\alpha K_{\eta_2}^{\eta_1}}{(\alpha+1)(2\alpha+1)}. \tag{4.2.10}$$

Proof. Since Φ, Ψ are two (α, m)-HA convex functions, then by definition, we have

$$\Phi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right) \leq s^{\alpha}\Phi(\eta_1) + m(1-s^{\alpha})\Phi(\eta_2)$$

and

$$\Psi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right) \leq s^{\alpha}\Psi(\eta_1) + m(1-s^{\alpha})\Psi(\eta_2).$$

Since Φ, $\Psi \geq 0$, we have

$$\begin{aligned}\Phi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right)\Psi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right) \leq\; & s^{2\alpha}\Phi(\eta_1)\Psi(\eta_1) + m^2(1-s^{\alpha})^2\Phi(\eta_2)\Psi(\eta_2)\\ & + ms^{\alpha}(1-s^{\alpha})(\Phi(\eta_1)\Psi(\eta_2) + \Phi(\eta_2)\Psi(\eta_1))\end{aligned}$$

which on integrating with respect to s over $[0,1]$ gives

$$\begin{aligned}\int_0^1 \Phi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right)\Psi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-ts)\eta_1}\right) ds \leq\; & K_{\eta_2}\int_0^1 s^{2\alpha}\, ds + m^2K_{\eta_1}\int_0^1 (1-s^{\alpha})^2\, ds\\ & + mK_{\eta_2}^{\eta_1}\int_0^1 s^{\alpha}(1-s^{\alpha})ds\,.\end{aligned}$$

The assertion now follows by making variable substitution $x = \frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}$. □

Next, we provide a variant of Theorem 4.2.7, where the left side of (4.2.10) is independent of m.

Theorem 4.2.8. *Let* Φ, $\Psi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}^+$ *be two* (α, m)*-HA convex functions. Then the following inequality holds:*

$$\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\Phi(x)\Psi(x)}{x^2}\, dx \leq \frac{1}{2\alpha+1}\left(K_{\eta_2/m} + \frac{m^2\alpha^2K_{\eta_1}}{\alpha+1} + \frac{m\alpha K_{\eta_2/m}^{\eta_1}}{\alpha+1}\right). \tag{4.2.11}$$

Proof. As Φ and Ψ are (α, m)-HA convex functions, then by definition, we have

$$\Phi\left(\frac{\eta_1\eta_2}{s\eta_1 + (1-s)\eta_2}\right) = \Phi\left(\frac{m\eta_1\frac{\eta_2}{m}}{s\eta_1 + m(1-s)\frac{\eta_2}{m}}\right) \leq s^{\alpha}\Phi\left(\frac{\eta_2}{m}\right) + m(1-s^{\alpha})\Phi(\eta_1)$$

and

$$\Psi\left(\frac{\eta_1\eta_2}{t\eta_1 + (1-t)\eta_2}\right) = \Psi\left(\frac{m\eta_1\frac{\eta_2}{m}}{t\eta_1 + m(1-t)\frac{\eta_2}{m}}\right) \leq t^{\alpha}\Psi\left(\frac{\eta_2}{m}\right) + m(1-t^{\alpha})\Psi(\eta_1).$$

The assertion now follows in the same way as in Theorem 4.2.7. □

4.2.2 Harmonically-Geometric Convex Function

Here, we define certain new type of convexities as follows.

Definition 4.2.9. *A function* $\Phi : I \to \mathbb{R}$ *is said to be*

1. *harmonically-geometric (HG) convex or harmonically log convex if*

$$\Phi\left(\frac{1}{\frac{s}{x}+\frac{1-s}{y}}\right) \le (\Phi(y))^s(\Phi(x))^{1-s}, s \in [0,1], x, y \in I.$$

2. *m-harmonically-geometric (m-HG) convex or* m *-harmonically log convex if*

$$\Phi\left(\frac{1}{\frac{s}{mx}+\frac{1-s}{y}}\right) \le (\Phi(y))^s(\Phi(x))^{m(1-s)}, s \in [0,1],\ m \in [0,1],\ x, y \in I.$$

3. (α, m) *-harmonically-geometric* $((\alpha, m)$ *-HG) convex or* (α, m) *-harmonically log convex if*

$$\Phi\left(\frac{1}{\frac{s}{mx}+\frac{1-s}{y}}\right) \le (\Phi(y))^{s^\alpha}(\Phi(x))^{m(1-s^\alpha)}, s \in [0,1],\ m \in [0,1],\ \alpha \in [0,1].$$

holds.

We prove the following:

Theorem 4.2.10. *Let* Φ *be an HG convex function. Then the following Hermite-Hadamard inequality holds:*

$$\log\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right) \le \frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\log\Phi(x)}{x^2}dx \le \log\sqrt{\Phi(\eta_1)\Phi(\eta_2)}. \tag{4.2.12}$$

Proof. By the definition of HG convex function, we have

$$\Phi\left(\frac{\eta_1\eta_2}{s\eta_2+(1-s)\eta_1}\right) \le (\Phi(\eta_2))^s(\Phi(\eta_1))^{1-s}$$

Taking logarithm on both sides and integrating with respect to s over $[0,1]$, we get

$$\int_0^1 \log\Phi\left(\frac{\eta_1\eta_2}{s\eta_2+(1-s)\eta_1}\right)ds \le \log\Phi(\eta_2)\int_0^1 s\,ds + \log\Phi(\eta_1)\int_0^1(1-s)ds = \frac{\log\Phi(\eta_1)+\log\Phi(\eta_2)}{2}$$

or

$$\int_0^1 \log\Phi\left(\frac{\eta_1\eta_2}{s\eta_2+(1-s)\eta_1}\right)ds \le \log\sqrt{\Phi(\eta_1)\Phi(\eta_2)}.$$

By making the variable substitution $x = \frac{\eta_1\eta_2}{s\eta_2+(1-s)\eta_1}$ in the last inequality, we get

$$\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\log\Phi(x)}{x^2}\,dx \le \log\sqrt{\Phi(\eta_1)\Phi(\eta_2)}. \tag{4.2.13}$$

Next, taking $s=\frac{1}{2}$ in the definition of HG convex function gives

$$\Phi\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right) \le \sqrt{\Phi(\eta_1)\Phi(\eta_2)}.$$

In the last inequality, take $\eta_1 = \frac{xy}{sx+(1-s)y}$, $\eta_2 = \frac{xy}{sy+(1-s)x}$ so that we obtain

$$\Phi\left(\frac{\frac{2xy}{sx+(1-s)y}\frac{xy}{sy+(1-s)x}}{\frac{xy}{sx+(1-s)y}+\frac{xy}{sy+(1-s)x}}\right) \le \sqrt{\Phi\left(\frac{xy}{sx+(1-s)y}\right)\Phi\left(\frac{xy}{sy+(1-s)x}\right)}$$

or

$$\Phi\left(\frac{2xy}{x+y}\right) \le \sqrt{\Phi\left(\frac{xy}{sx+(1-s)y}\right)\Phi\left(\frac{xy}{sy+(1-s)x}\right)}$$

which on taking logarithm on both sides gives

$$\log\Phi\left(\frac{2xy}{x+y}\right) \le \frac{1}{2}\left[\log\Phi\left(\frac{xy}{sx+(1-s)y}\right)+\log\Phi\left(\frac{xy}{sy+(1-s)x}\right)\right].$$

Integrating with respect to s over $[0,1]$, we get

$$\log\int_0^1\Phi\left(\frac{2xy}{x+y}\right)ds \le \frac{1}{2}\left[\int_0^1\log\Phi\left(\frac{xy}{sx+(1-s)y}\right)ds+\int_0^1\log\Phi\left(\frac{xy}{sy+(1-s)x}\right)ds\right].$$

Since each integral on right side equals to $\frac{xy}{y-x}\int_x^y\frac{\log\Phi(u)}{u^2}du$, we have

$$\log\Phi\left(\frac{2xy}{x+y}\right) \le \frac{1}{2}\frac{2xy}{x+y}\int_x^y\frac{\log\Phi(u)}{u^2}\,du = \frac{xy}{y-x}\int_x^y\frac{\log\Phi(u)}{u^2}\,du$$

Replacing x and y by η_1 and η_2, respectively, we have

$$\log\Phi\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right) \le \frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\log\Phi(x)}{x^2}\,dx. \tag{4.2.14}$$

On combining the inequalities (4.2.13) and (4.2.14), we get (4.2.12) and the proof is complete. □

The next result is regarding (α,m)-HG convex functions.

Theorem 4.2.11. *Let $\Phi : I \to \mathbb{R}^+$ be an (α,m) - HG convex function. Then the following* *inequalities hold:*

$$(i)\frac{m\eta_1\eta_2}{\eta_2 - m\eta_1}\int_{m\eta_1}^{\eta_2}\frac{\log\Phi(x)}{x^2}\,dx \le \left(\frac{1}{\alpha+1}\right)[\log\Phi(\eta_2) + m\alpha\log\Phi(\eta_1)] \tag{4.2.15}$$

$$\begin{aligned}(ii)\ \log\Phi\left(\frac{2mxy}{mx+y}\right) \le\ & \frac{2mxy(1-A)A^{\alpha}}{(y-mx)}\int_{2mx(1-A)}^{2y(1-A)}\frac{\log\Phi(\omega)}{\omega^2}\,d\omega \\ & + \frac{2mxyA(1-A^{\alpha})}{y-mx}\int_{2xA}^{2yA/m}\frac{\log\Phi(\omega)}{\omega^2}\,d\omega.\end{aligned} \tag{4.2.16}$$

Proof. (i) In view of the definitions of (α, m) - HG convex functions, for any η_1 , $\eta_2 \in I$, we have

$$\Phi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right) \le (\Phi(\eta_2))^{s^{\alpha}}(\Phi(\eta_1))^{m(1-s^{\alpha})} \tag{4.2.17}$$

which on taking log on both sides and integrating with respect to s over $[0,1]$ gives

$$\int_0^1 \log\Phi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right) ds \le \log\Phi(\eta_2)\int_0^1 s^{\alpha}\,ds + m\log\Phi(\eta_1)\int_0^1 (1-s^{\alpha})\,ds.$$

The inequality (4.2.15) now follows by making variable substitution $x = \frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}$ and solving the integrals on the right side.

(ii) We proceed as in Theorem 4.2.7 (ii). We apply for (4.2.17) as $s = A$ and get

$$\Phi\left(\frac{m\eta_1\eta_2}{A\eta_2 + m(1-A)\eta_1}\right) \le (\Phi(\eta_2))^{A^{\alpha}}(\Phi(\eta_1))^{m(1-A^{\alpha})}. \tag{4.2.18}$$

Again, on taking η_1 and η_2 as in (4.2.5), the left side of (4.2.17) after taking log and integrating with respect to s over $[0,1]$ becomes

$$\log\Phi\left(\frac{2mxy}{mx+y}\right) \tag{4.2.19}$$

while the right side becomes

$$A^{\alpha}\int_0^1 \log\Phi\left(\frac{2mxy(1-A)}{mxs + (1-s)y}\right) ds + m(1-A^{\alpha})\int_0^1 \log\Phi\left(\frac{2mxyA}{sy + mx(1-s)}\right) ds. \tag{4.2.20}$$

The integrals in (4.2.20) can be simplified as done for calculating I_1 and I_2 in Theorem 4.2.3(ii) combining which and (4.2.19), we get

$$\log\Phi\left(\frac{2mxy}{mx+y}\right) \le \frac{2mxy(1-A)A^{\alpha}}{(y-mx)}\int_{2mx(1-A)}^{2y(1-A)}\frac{\log\Phi(\omega)}{\omega^2}\,d\omega + \frac{2mxyA(1-A^{\alpha})}{y-mx}\int_{2xA}^{2yA/m}\frac{\log\Phi(\omega)}{\omega^2}\,d\omega.$$

Now, rewriting x, y as respectively, a, b gives 4.2.16. This completes the proof. $\square$

Remark 4.2.12. *Similar to Remark 4.2.4, if* $\alpha = m = 1$, *then* $A = \frac{1}{2}$ *and therefore both integrals on the right side of (4.2.18) become equal to*

$$\frac{\eta_1\eta_2}{2(\eta_2-\eta_1)}\int_{\eta_1}^{\eta_2}\frac{\log\Phi(\omega)}{\omega^2}d\omega.$$

Consequently, in this case both (4.2.17) and (4.2.18) combined together give (4.1.1).

Our next consideration is about the product of (α, m)-HG convex functions. As in Theorem 4.2.5, it can be proved that the products of two (α, m)-HG convex functions is again (α, m)-HG convex function.

Now, we prove the following:

Theorem 4.2.13. *Let* $\Phi, \Psi : I \to \mathbb{R}^+$ *be* (α, m)*-HG convex functions. The following inequality thus persists:*

$$\frac{m\eta_1\eta_2}{\eta_2-m\eta_1}\int_{m\eta_1}^{\eta_2}\frac{\log\Phi(x)+\log\Psi(x)}{x^2}dx \le \frac{\log K_{\eta_2}+m\alpha\log K_{\eta_1}}{\alpha+1}. \tag{4.2.21}$$

Proof. This can be obtained similar to Theorem 4.2.7 by using the definitions of (α, m)-HG convex functions. $\square$

Below, we provide a variant of Theorem 4.2.13 in which the left side of (4.2.21) is independent of m.

Theorem 4.2.14. *Let* $\Phi, \Psi : I \to \mathbb{R}^+$ *be* (α, m)*-HG convex functions. The following inequality is then true:*

$$\frac{\eta_1\eta_2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\frac{\log\Phi(x)+\log\Psi(x)}{x^2}dx \le \frac{\log K_{\eta_2/m}+m\alpha\log K_{\eta_1}}{\alpha+1}. \tag{4.2.22}$$

Proof. As Φ and Ψ are (α, m)-HG convex functions, then by definition, we have

$$\Phi\left(\frac{\eta_1\eta_2}{s\eta_1+(1-s)\eta_2}\right)=\Phi\left(\frac{m\eta_1\frac{\eta_2}{m}}{s\eta_1+m(1-s)\frac{\eta_2}{m}}\right)\le\left(\Phi\left(\frac{\eta_2}{m}\right)\right)^{s^\alpha}(\Phi(\eta_1))^{m(1-s^\alpha)}$$

and

$$\Psi\left(\frac{\eta_1\eta_2}{s\eta_1+(1-s)\eta_2}\right)=\Psi\left(\frac{m\eta_1\frac{\eta_2}{m}}{s\eta_1+m(1-s)\frac{\eta_2}{m}}\right)\le\left(\Psi\left(\frac{\eta_2}{m}\right)\right)^{s^\alpha}(\Psi(\eta_1))^{m(1-s^\alpha)}$$

On multiplying the above inequalities and taking $\log$ on both sides, we have

$$\log\Phi\left(\frac{\eta_1\eta_2}{s\eta_1+(1-s)\eta_2}\right)+\log\Psi\left(\frac{\eta_1\eta_2}{s\eta_1+(1-s)\eta_2}\right)\leq s^\alpha\log K_{\eta_2/m}+m(1-s^\alpha)\log K_{\eta_1}$$

which on integrating with respect to s over $[0,1]$ and making variable substitution $x=\frac{\eta_1\eta_2}{s\eta_1+(1-s)\eta_2}$ on left sides, we obtain (4.2.22). $\square$

4.2.3 The Extended Results in q-Calculus

This subsection is devoted to the inequalities proved in the previous subsections for q-calculus.

The q-derivative of a continuous function $\Phi: I\to\mathbb{R}$ at any point $x\in I$ is defined by

$$_{\eta_1}D_q\Phi(x)=\frac{\Phi(x)-\Phi(qx+(1-q)\eta_1)}{(1-q)(x-\eta_1)},\ x\neq\eta_1$$

and

$$_{\eta_1}D_q\Phi(\eta_1)=\lim_{x\to\eta_1}\eta_1 D_q\Phi(x).$$

The q-integral of a continuous function $\Phi: I\to\mathbb{R}$ is defined by

$$\int_{\eta_1}^{\eta_2}\Phi(s)_{\eta_1}d_qs=(1-q)(x-\eta_1)\sum_{n=0}^{\infty}q^n\Phi\left(q^nx+(1-q^n)\eta_1\right)$$

and for $c\in(\eta_1,x)$

$$\int_c^x\Phi(s)_{\eta_1}d_qs=\int_{\eta_1}^x\Phi(s)_{\eta_1}d_qs-\int_{\eta_1}^c\Phi(s)_{\eta_1}\,d_qs.$$

The q-analogue of any real number $x\in\mathbb{R}$ is defined by

$$[x]_q:=\frac{q^x-1}{q-1}.$$

For any integer $n\geq 1$, the q-analogue of $(x-c)^n$ is the polynomial

$$(x-c)_q^n:=(x-c)(x-qc)...(x-q^{n-1}c).$$

To deal with the negative integer exponents, it is known that

$$(x-c)_q^{-n}=\frac{1}{(x-q^{-n}c)_q^n}.$$

The following derivatives are known for any integer $n \in \mathbb{Z}$:

$$ {}_{\eta_1}D_q (x-c)_q^n = [n]_q (x-c)_q^{n-1}, $$

$$ {}_{\eta_1}D_q \frac{1}{(x-c)_q^n} = [-n]_q (x-q^n c)_q^{-n-1}. $$

For more on the theory of q-calculus, we refer to the books [1, 30].

We prove the following theorem which provides a q-analogue of Hermite-Hadamard inequality for product of (α, m)-HA convex functions.

Theorem 4.2.15. *Let* $\Phi, \Psi : I \to \mathbb{R}^+$ *be two* (α, m)*-HA convex functions. The following inequalities are then true:*

$$ \begin{aligned} \frac{m\eta_1\eta_2}{q(\eta_2 - m\eta_1)} \int_{m\eta_1}^{\eta_2} \frac{\Phi(x)\Psi(x)}{x^2} \, {}_{\eta_1}d_q x \; \le \; & \frac{1}{[2\alpha+1]_q} K_{\eta_2} + m^2 K_{\eta_1} \left(1 - \frac{2}{[\alpha+1]_q} + \frac{1}{[2\alpha+1]_q}\right) \\ & + m K_{\eta_2}^{\eta_1} \left(\frac{1}{[\alpha+1]_q} - \frac{1}{[2\alpha+1]_q}\right). \end{aligned} \tag{4.2.23} $$

Proof. By the definition of (α, m)-HA convex functions applied on Φ and Ψ, we get

$$ \Phi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right) \le s^\alpha \Phi(\eta_1) + m(1-s^\alpha)\Phi(\eta_2) $$

and

$$ \Psi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right) \le s^\alpha \Psi(\eta_1) + m(1-s^\alpha)\Psi(\eta_2). $$

Since $\Phi, \Psi \ge 0$, multiplying the above inequalities and taking q-integral from 0 to 1 on both sides, we obtain

$$ \begin{aligned} \int_0^1 \Phi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right) \Psi\left(\frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1}\right) {}_0d_q s \; \le \; & K_{\eta_1} \int_0^1 s^{2\alpha} \, {}_0d_q s \\ & + m^2 K_{\eta_2} \int_0^1 (1-s^\alpha)^2 \, {}_0d_q s \\ & + m K_{\eta_2}^{\eta_1} \int_0^1 s^\alpha (1-s^\alpha) \, {}_0d_q s. \end{aligned} \tag{4.2.24} $$

Note that

$$\int_0^1 s^{2\alpha}\,{}_0d_qs = (1-q)\sum_{n=0}^{\infty} q^{n(2\alpha+1)} = (1-q).\frac{1}{1-q^{2\alpha+1}} = \frac{1}{[2\alpha+1]_q}$$

Similarly, it can be calculated that

$$\int_0^1 (1-s^\alpha)^2\,{}_0d_qs = 1 - \frac{2}{[\alpha+1]_q} + \frac{1}{[2\alpha+1]_q}$$

and

$$\int_0^1 s^\alpha(1-s^\alpha)\,{}_0d_qs = \frac{1}{[\alpha+1]_q} - \frac{1}{[2\alpha+1]_q}.$$

Further, if we take

$$x = \frac{m\eta_1\eta_2}{s\eta_2 + m(1-s)\eta_1},$$
$$s = \left(\frac{m\eta_1\eta_2}{\eta_2 - m\eta_1}\right)\left(\frac{1}{x}\right) - \frac{m\eta_1}{\eta_2 - m\eta_1}$$

and it can be concluded that

$${}_0D_qs = \left(\frac{m\eta_1\eta_2}{\eta_2 - m\eta_1}\right)[-1]_q(x)_q^{-2} = \left(\frac{m\eta_1\eta_2}{\eta_2 - m\eta_1}\right)\frac{1}{q}\left(\frac{1}{x^2}\right).$$

Moreover, when $s \to 0,\, x \to \eta_2$ and when $s \to 1,\, x \to \eta_1$. Thus by applying the above considerations, the inequality (4.2.24) becomes

$$\begin{aligned}\frac{m\eta_1\eta_2}{q(\eta_2 - m\eta_1)}\int_{m\eta_1}^{\eta_2} \frac{f(x)g(x)}{x^2}\,{}_{\eta_1}d_qx \le \frac{1}{[2\alpha+1]_q}K_{\eta_1} + m^2K_{\eta_2}\left(1 - \frac{2}{[\alpha+1]_q} + \frac{1}{[2\alpha+1]_q}\right) \\ + mK_{\eta_2}^{\eta_1}\left(\frac{1}{[\alpha+1]_q} - \frac{1}{[2\alpha+1]_q}\right)\end{aligned}$$

and the theorem is proved.

□

Remark 4.2.16. *For $\alpha = 1$, Theorem 3.3 gives the m-HA Hermite-Hadamard inequalities for product of functions. Further if $m = 1$ as well, then the corresponding inequality is obtained for HA-convex functions. Next, we prove a variant of Theorem 4.2.15 where the left side of (4.2.23) is independent of m.*

Theorem 4.2.17. *Let $\Phi, \Psi : I \to \mathbb{R}^+$ be two (α, m)-HA convex functions. Then the* *following inequality holds:*

$$\frac{\eta_1\eta_2}{q(\eta_2-\eta_1)}\int_{\eta_1}^{\eta_2}\frac{\Phi(x)\Psi(x)}{x^2}\,{}_{\eta_1}d_qx \le \frac{1}{[2\alpha+1]_q}K_{\eta_2/m}+m^2K_{\eta_1}\left(1-\frac{2}{[\alpha+1]_q}+\frac{1}{[2\alpha+1]_q}\right)$$
$$+mK_{\eta_2/m}^{\eta_1}\left(\frac{1}{[\alpha+1]_q}-\frac{1}{[2\alpha+1]_q}\right).$$

Proof. As Φ and Ψ are (α,m)-HA convex functions, then by definition, we have

$$\Phi\left(\frac{\eta_1\eta_2}{s\eta_1+(1-s)\eta_2}\right) = \Phi\left(\frac{m\eta_1\frac{\eta_2}{m}}{s\eta_1+m(1-s)\frac{\eta_2}{m}}\right) \le s^\alpha\Phi\left(\frac{\eta_2}{m}\right)+m(1-s^\alpha)\Phi(\eta_1)$$

and

$$\Psi\left(\frac{\eta_1\eta_2}{s\eta_1+(1-s)\eta_2}\right) = \Psi\left(\frac{m\eta_1\frac{\eta_2}{m}}{s\eta_1+m(1-s)\frac{\eta_2}{m}}\right) \le s^\alpha\Psi\left(\frac{\eta_2}{m}\right)+m(1-s^\alpha)\Psi(\eta_1)$$

. The assertion now follows proceeding in the same as in Theorem 4.2.15. □

The next theorem is the q-analogue of Theorem 4.2.13.

Theorem 4.2.18. *Let Φ, $\Psi : I \subset \mathbb{R}^+ \to \mathbb{R}^+$ be two (α,m)-HG convex functions. Then the following inequality persists:*

$$\frac{m\eta_1\eta_2}{q(\eta_2-m\eta_1)}\int_{m\eta_1}^{\eta_2}\frac{\log\Phi(x)+\log\Psi(x)}{x^2}\,{}_{m\eta_1}d_qx \le \frac{1}{[\alpha+1]_q}\log K_{\eta_1}+m\left(1-\frac{1}{[\alpha+1]_q}\right)\log K_{\eta_2}. \tag{4.2.25}$$

Proof. By definition, we have

$$\Phi\left(\frac{1}{\frac{s}{m\eta_1}+\frac{1-s}{\eta_2}}\right) \le (\Phi(\eta_2))^{s^\alpha}(\Phi(\eta_1))^{m(1-s^\alpha)}$$

and

$$\Psi\left(\frac{1}{\frac{s}{m\eta_1}+\frac{1-s}{\eta_2}}\right) \le (\Psi(\eta_2))^{s^\alpha}(\Psi(\eta_1))^{m(1-s^\alpha)}$$

which give that

$$\Phi\left(\frac{1}{\frac{s}{m\eta_1}+\frac{1-s}{\eta_2}}\right)\Psi\left(\frac{1}{\frac{s}{m\eta_1}+\frac{1-s}{\eta_2}}\right) \le (\Phi(\eta_1)\Psi(\eta_1))^{s^\alpha}(\Phi(\eta_2)\Psi(\eta_2))^{m(1-s^\alpha)}$$

or

$$\log\Phi\left(\frac{m\eta_1\eta_2}{s\eta_2+m(1-s)\eta_1}\right)+\log\Psi\left(\frac{m\eta_1\eta_2}{s\eta_2+m(1-s)\eta_1}\right) \le s^\alpha\log K_{\eta_1}+m(1-s^\alpha)\log K_{\eta_2}.$$

Taking q-integral with respect to s over $[0,1]$, we obtain

$$\int_0^1 \log\Phi\left(\frac{m\eta_1\eta_2}{s\eta_2+m(1-s)\eta_1}\right)\,{}_0d_qs + \int_0^1 \log\Psi\left(\frac{m\eta_1\eta_2}{s\eta_2+m(1-s)\eta_1}\right)\,{}_0d_qs$$
$$\leq \log K_{\eta_1}\int_0^1 s^\alpha\,{}_0d_qt + m\log K_{\eta_2}\int_0^1 (1-s^\alpha)\,{}_0d_qs$$

The assertion now follows proceeding in the similar way as in Theorem 4.2.15. □

Next, we give a variant of Theorem 4.2.18 so as to make the left side of (4.2.25) is independent of m.

Theorem 4.2.19. *Let* Φ, $\Psi : I \subset \mathbb{R}^+ \to \mathbb{R}^+$ *be two* (α, m)*-HG convex functions. Then the following inequality holds:*

$$\frac{\eta_1\eta_2}{q(\eta_2-\eta_1)}\int_{\eta_1}^{\eta_2}\frac{\log\Phi(x)+\log\Psi(x)}{x^2}\,{}_{\eta_1}d_qx \leq \frac{1}{[\alpha+1]_q}\log K_{\eta_1/m} + m\left(1-\frac{1}{[\alpha+1]_q}\right)\log K_{\eta_2}. \tag{4.2.26}$$

Proof. Since Φ and Ψ are (α, m)-HG convex, we have

$$\Phi\left(\frac{1}{\frac{s}{\eta_1}+\frac{1-s}{\eta_2}}\right) = \Phi\left(\frac{1}{\frac{s}{\frac{m\eta_1}{m}}+\frac{1-s}{\eta_2}}\right) \leq \left(\Phi\left(\frac{\eta_1}{m}\right)\right)^{s^\alpha}(\Phi(\eta_2))^{m(1-s^\alpha)}$$

and similarly,

$$\Psi\left(\frac{1}{\frac{s}{\eta_1}+\frac{1-s}{\eta_2}}\right) \leq \left(\Psi\left(\frac{\eta_1}{m}\right)\right)^{s^\alpha}(\Psi(\eta_2))^{m(1-s^\alpha)}$$

which give that

$$\Phi\left(\frac{\eta_1\eta_2}{s\eta_2+(1-s)\eta_1}\right)\Psi\left(\frac{\eta_1\eta_2}{s\eta_2+(1-s)\eta)1}\right) \leq \left(\Phi\left(\frac{\eta_1}{m}\right)\Psi\left(\frac{\eta_1}{m}\right)\right)^{s^\alpha}(\Phi(\eta_2)\Psi(\eta_2))^{m(1-s^\alpha)}.$$

or

$$\log\Phi\left(\frac{\eta_1\eta_2}{s\eta_2+(1-s)\eta_1}\right) + \log\Psi\left(\frac{\eta_1\eta_2}{s\eta_2+(1-s)\eta_1}\right) \leq s^\alpha\log\left(\Phi\left(\frac{\eta_1}{m}\right)\Psi\left(\frac{\eta_1}{m}\right)\right)$$
$$+m(1-s^\alpha)\log(\Phi(\eta_2)\Psi(\eta_2)).$$

Taking q-integral with respect to s over $[0,1]$ and proceeding as in Theorem 4.2.15, the assertion follows. □

Chapter 5

HERMITE-HADAMARD INTEGRAL INEQUALITIES FOR m-HARMONICALLY CONVEX FUNCTIONS

5.1 Introduction

In this chapter, we have generalized harmonically convex function to m-harmonically convex function and applied them to develop some Hermite- Hadamard type integral inequalities whose first order derivatives are m-harmonically convex function. Apart from this, we have further generalized harmonically P-function to m-harmonically P-function and applied them to extend Hermite-Hadamard type integral inequalities.

5.2 Preliminary Results

In this section, we state some fundamental ideas and findings regarding harmonically convex functions in order to enhance new inequalities.

Definition 5.2.1. *[25] A function* $\varphi : \mathbb{H} \subset \mathbb{R}^n \to \mathbb{R},$ *where* $\mathbb{H} \neq \phi$, *a harmonic convex set in* $\mathbb{R}^n \backslash \{0\},$ *is a harmonic convex function on* $\mathbb{H}$ *if*

$$\varphi\left(\frac{\alpha\beta}{\nu\alpha + (1-\nu)\beta}\right) \leq \nu\varphi(\beta) + (1-\nu)\varphi(\alpha)$$

holds for all $\alpha, \beta \in \mathbb{H}, \nu \in [0,1]$.

If the inequality reverses, then φ is harmonically concave function.

Definition 5.2.2. *[43] A function $\varphi : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically P-function if*

$$\varphi\left(\frac{\alpha\beta}{\nu\alpha + (1-\nu)\beta}\right) \leq \varphi(\alpha) + \varphi(\beta)$$

holds for all $\alpha, \beta \in I$, and, $\nu \in [0,1]$.

Authors in [6, 25] introduced the Hermite-Hadamard type integral inequality for harmonically convex function as follows:

Theorem 5.2.3. *[25] Let $\varphi : I \subset (0,\infty) \to \mathbb{R}$ be harmonically convex function and $\alpha, \beta \in I$ with $\alpha < \beta$. If $\varphi \in L[\alpha, \beta]$, then the following inequalities hold.*

$$\varphi\left(\frac{2\alpha\beta}{\alpha+\beta}\right) \leq \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\varphi(x)}{x^2}dx \leq \frac{\varphi(\alpha)+\varphi(\beta)}{2}.$$

Theorem 5.2.4. *[25] Let $\varphi : I \subset (0,\infty) \to \mathbb{R}$ be differentiable function on $I^0, \alpha, \beta \in I$, with $\alpha < \beta$ and, $\varphi' \in L[\alpha,\beta]$. If $|\varphi'|^q$ is harmonically convex on $[\alpha,\beta]$ for $q \geq 1$, then*

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\varphi(x)}{x^2}dx\right| \leq \frac{\alpha\beta(\beta-\alpha)}{2}\rho_1^{1-\frac{1}{q}}\left(\rho_2|\varphi'(\alpha)|^q + \rho_3|\varphi'(\beta)|^q\right)^{\frac{1}{q}}$$

where

$$\begin{aligned}
\rho_1 &= \int_0^1 \frac{|1-2\nu|}{(\nu\beta+(1-\nu)\alpha)^2}\,d\nu = \frac{1}{\alpha\beta} - \frac{2}{(\beta-\alpha)^2}\log\left(\frac{(\alpha+\beta)^2}{4\alpha\beta}\right).\\
\rho_2 &= \int_0^1 \frac{|1-2\nu|(1-\nu)}{(\nu\beta+(1-\nu)\alpha)^2}\,d\nu = \frac{1}{\alpha(\beta-\alpha)} - \frac{3\beta+\alpha}{(\beta-\alpha)^3}\log\left(\frac{(\alpha+\beta)^2}{4\alpha\beta}\right)\\
\rho_3 &= \int_0^1 \frac{|1-2\nu|\nu}{(\nu\beta+(1-\nu)\alpha)^2}\,d\nu = \frac{1}{\beta(\beta-\alpha)} + \frac{3\alpha+\beta}{(\beta-\alpha)^3}\log\left(\frac{(\alpha+\beta)^2}{4\alpha\beta}\right)
\end{aligned}$$

Theorem 5.2.5. *[25] Let $\varphi : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^0, \alpha, \beta \in I$ with $\alpha < \beta$ and $\varphi' \in L[\alpha,\beta]$. If $|\varphi'|^q$ if harmonically convex on $[\alpha,\beta]$ for $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{\varphi(x)}{x^2}dx\right| \leq \frac{\alpha\beta(\beta-\alpha)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\omega_1|\varphi'(\alpha)|^q + \omega_2|\varphi'(\beta)|^q\right)^{\frac{1}{q}}.$$

where,

$$\omega_1 = \int_0^1 \frac{\nu}{(\nu\beta + (1-\nu)\alpha)^{2q}}\, d\nu.$$
$$\omega_2 = \int_0^1 \frac{1-\nu}{(\nu\beta + (1-\nu)\alpha)^{2q}}\, d\nu.$$

Theorem 5.2.6. *[43]Let $\varphi : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^0, \alpha, \beta \in I$ with $a < b$ and $\varphi' \in L[\alpha,\beta]$. If $|\varphi'|^q$ if harmonically convex on $[\alpha,\beta]$, then*

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{\varphi(x)}{x^2}dx\right| \leq \frac{\alpha\beta(\beta-\alpha)}{2}(\rho_2|\varphi'(\alpha)| + \rho_3|\varphi'(\beta)|),$$

where

$$\rho_2 = \int_0^1 \frac{|1-2\nu|(1-\nu)}{\nu\beta + (1-\nu)\alpha)^2}\, d\nu = \frac{1}{\alpha(\beta-\alpha)} - \frac{3\beta+\alpha}{(\beta-\alpha)^3}\log\left(\frac{(\alpha+\beta)^2}{4\alpha\beta}\right),$$
$$\rho_3 = \int_0^1 \frac{|1-2\nu|\nu}{\nu\beta + (1-\nu)\alpha)^2}\, d\nu = \frac{1}{\beta(\beta-\alpha)} + \frac{3\alpha+\beta}{(\beta-\alpha)^3}\log\left(\frac{(\alpha+\beta)^2}{4\alpha\beta}\right).$$

Theorem 5.2.7. *[43] Let, $\varphi : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^0, \alpha, \beta \in I$ with $\alpha < \beta$ and $\varphi' \in L[\alpha,\beta]$. If $|\varphi'|^q$ if harmonically P-function on $[\alpha,\beta]$ for $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{\varphi(x)}{x^2}\, dx\right| \leq \frac{\alpha\beta(\beta-\alpha)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \psi^{\frac{1}{q}}\left(|\varphi'(\alpha)|^q + |\varphi'(\beta)|^q\right)^{\frac{1}{q}},$$

where

$$\psi = \int_0^1 \frac{1}{(\nu\beta + (1-\nu)\alpha)^{2q}}\, d\nu = \frac{\beta^{1-2q} - \alpha^{1-2q}}{(1-2q)(\beta-\alpha)}.$$

Theorem 5.2.8. *[43] Let $\varphi : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^0, \alpha, \beta \in I$ with $\alpha < \beta$ and $\varphi' \in L[\alpha,\beta]$. If $|\varphi'|^q$ if harmonically P-function on $[\alpha,\beta]$ for $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{\varphi(x)}{x^2}\, dx\right| \leq \frac{\alpha\beta(\beta-\alpha)}{2}\rho_1\left(|\varphi'(\alpha)|^q + |\varphi'(\beta)|^q\right)^{\frac{1}{q}}.$$

where,

$$\rho_1 = \int_0^1 \frac{|1-2\nu|}{\nu\beta+(1-\nu)\alpha)^2}\, d\nu = \frac{1}{\alpha\beta} - \frac{2}{(\beta-\alpha)^2}\log\left(\frac{(\alpha+\beta)^2}{4\alpha\beta}\right).$$

Theorem 5.2.9. *[43] Let* $\varphi_1, \varphi_2 : I \subset (0,\infty) \to \mathbb{R}$ *be two harmonically convex functions on* I *where* $\alpha, \beta \in I$ *with* $\alpha < \beta$ *. If* $\varphi_1\varphi_2 \in L[\alpha,\beta]$*, then*

$$\frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \left(\frac{\varphi_1(x)\varphi_2(x)}{x^2}\right) dx \leq \frac{1}{3}M(\alpha,\beta) + \frac{1}{6}N(\alpha,\beta),$$

where

$$M(\alpha,\beta) = \varphi_1(\alpha)\varphi_2(\alpha) + \varphi_1(\beta)\varphi_2(\beta), \quad N(\alpha,\beta) = \varphi_1(\alpha)\varphi_2(\beta) + \varphi_1(\beta)\varphi_2(\alpha)$$

I. Iscan [24] established the following lemma for harmonically convex function.

Lemma 5.2.10. *[24] Let* $\varphi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$*, be a differentiable function on* I^0 *and,* $\alpha, \beta \in I$ *with* $\alpha < \beta$ *. If* $f' \in L[\alpha,\beta]$ *, then*

$$\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta-\alpha}\int_\alpha^\beta \frac{\varphi(x)}{x^2}\, dx = \frac{\alpha\beta(\beta-\alpha)}{2}\int_0^1 \frac{1-2\nu}{(\nu\beta+(1-\nu)\alpha)^2}\varphi'\left(\frac{\alpha\beta}{\nu\beta+(1-\nu)\alpha}\right) d\nu.$$

5.3 Main Results

In this section, we introduce the concept of m-harmonically convex functions and m-harmonically P-function on $(0,\infty)$ as follows:

Definition 5.3.1. *A function* $\varphi : I \subset (0,\infty) \to \mathbb{R}$ *is said to be* m*-harmonically convex function on* I *if*

$$\varphi\left(\frac{m\alpha\beta}{\nu\alpha+m(1-\nu)\beta}\right) = \varphi\left(\left(\frac{\nu}{m\beta}+\frac{1-\nu}{\alpha}\right)^{-1}\right) \leq \nu\varphi(\beta) + m(1-\nu)\varphi(\alpha)$$

holds for all $\alpha, \beta \in I, \nu \in [0,1]$ *and* $m \in (0,1]$.

If the above inequality is reversed then the function φ will be m-harmonically concave.

Remark 5.3.2. *If* $m = 1$ *, then it reduces to harmonically convex function.*

Definition 5.3.3. *A function* $\varphi : I \subset (0,\infty) \to \mathbb{R}$ *is said to be* m*-harmonically* P*-function on* I *if*

$$\varphi\left(\frac{m\alpha\beta}{\nu\alpha+m(1-\nu)\beta}\right) \leq \varphi(\beta) + m\varphi(\alpha)$$

holds for all $\alpha, \beta \in I, \nu \in [0,1]$ *and* $m \in (0,1]$.

We, now, establish the following lemma for m-harmonically convex function.

Lemma 5.3.4. *Let* $\varphi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R},$ *be a differentiable function on* I^0 *and* $\alpha, \beta \in I$ *with* $\alpha < \beta$ *. If* $\varphi' \in L[\alpha, \beta]$ *, then*

$$\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_{\alpha}^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}dx = \frac{\alpha\beta(\beta-m\alpha)}{2}$$
$$\int_0^1 \frac{(1-2\nu)}{(\nu\beta+m(1-\nu)\alpha)^2}\varphi'\left(\frac{\alpha\beta}{\nu\beta+m(1-\nu)\alpha}\right)d\nu.$$

Proof. Let

$$I_1 = \frac{\alpha\beta(\beta-m\alpha)}{2}\int_0^1 \frac{(1-2\nu)}{(\nu\beta+m(1-\nu)\alpha)^2}\varphi'\left(\frac{\alpha\beta}{\nu\beta+m(1-\nu)\alpha}\right)d\nu$$

$$= \frac{\alpha\beta(\beta-m\alpha)}{2}\frac{(1-2\nu)}{(\nu\beta+m(1-\nu)\alpha)^2}$$

$$\frac{\varphi\left(\frac{\alpha\beta}{\nu\beta+m(1-\nu)\alpha}\right)}{\alpha\beta(-1)(\nu\beta+m(1-\nu)\alpha)^{-2}(\beta-m\alpha)}|_0^1 - \int_0^1 \varphi\left(\frac{\alpha\beta}{\nu\beta+m(1-\nu)\alpha}\right)d\nu$$
$$= \frac{2\nu-1}{2}\varphi\left(\frac{\alpha\beta}{\nu\beta+m(1-\nu)\alpha}\right)|_0^1 - \int_0^1 \varphi\left(\frac{\alpha\beta}{\nu\beta+m(1-\nu)\alpha}\right)d\nu$$
$$= \frac{\varphi(\alpha)+\varphi(\beta)}{2} - \int_0^1 \varphi\left(\frac{\alpha\beta}{\nu\beta+m(1-\nu)\alpha}\right)d\nu.$$

Put $x = \frac{\alpha\beta}{\nu\beta+m(1-\nu)\alpha}$ when $\nu \to 0, x \to \frac{\beta}{m}$ when $\nu \to 1, x \to \alpha$
Also, we have

$$d\nu = -\frac{\alpha\beta}{(\beta-m\alpha)}\frac{dx}{x^2}.$$

On substituting these values, we have

$$= \frac{\varphi(\alpha)+\varphi(\beta)}{2} - \int_{\frac{\beta}{m}}^{\alpha}\varphi(x)\left(\frac{-\alpha\beta}{\beta-m\alpha}\right)\frac{dx}{x^2} \quad = \frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta-m\alpha}\int_{\alpha}^{\frac{\beta}{m}}\frac{\varphi(x)}{x^2}dx.$$

Thus, we have

$$\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_{\alpha}^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}dx = \frac{\alpha\beta(\beta-m\alpha)}{2}$$
$$\int_0^1 \frac{(1-2\nu)}{(\nu\beta+m(1-\nu)\alpha)^2}\varphi'\left(\frac{\alpha\beta}{\nu\beta+m(1-\nu)\alpha}\right)d\nu.$$

This completes the proof. $\square$

Now, we apply the Lemma 5.3.4 in order to extend Hermite- Hadamard type integral inequality in terms of m-harmonically convex functions.

Theorem 5.3.5. *Let $\varphi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on $I^0, \alpha, \beta \in I$ with $\alpha < \beta$, $\varphi' \in L[\alpha, \beta]$. If $|\varphi'|$ is m-harmonically convex function on $[\alpha, \beta]$,then*

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_{\alpha}^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}dx\right| \leq \frac{\alpha\beta(\beta - m\alpha)}{2}\left(\rho_2|\varphi'(\alpha)| + m\rho_3|\varphi'(\beta)|\right),$$

where

$$\begin{aligned}\rho_2 &= \int_0^1 \frac{|1-2\nu|(1-\nu)}{(\nu\beta + m(1-\nu)\alpha)^2}\,d\nu,\\ \rho_3 &= \int_0^1 \frac{|1-2\nu|\nu}{(\nu\beta + m(1-\nu)\alpha)^2}\,d\nu.\end{aligned}$$

Proof. Using Lemma 5.3.4 and taking absolute value on both sides, we have

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_{\alpha}^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}\,dx\right|$$

$$= \left|\frac{\alpha\beta(\beta - m\alpha)}{2}\int_0^1 \frac{(1-2\nu)}{(\nu\beta + m(1-\nu)\alpha)^2}\varphi'\left(\frac{\alpha\beta}{\nu\beta + m(1-\nu)\alpha}\right)d\nu\right|$$
$$\leq \frac{\alpha\beta(\beta - m\alpha)}{2}$$
$$\int_0^1 \frac{|1-2\nu|}{(\nu\beta + m(1-\nu)\alpha)^2}\left|\varphi'\left(\frac{\alpha\beta}{\nu\beta + m(1-\nu)\alpha}\right)\right|d\nu$$

Since, $|\varphi'|$ is m-harmonically convex function, then we have

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_{\alpha}^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}\,dx\right|$$

$$\leq \frac{\alpha\beta(\beta - m\alpha)}{2}\int_0^1 \frac{|1-2\nu|}{(\nu\beta + m(1-\nu)\alpha)^2}(\nu|\varphi'(\alpha)| + m(1-\nu)|\varphi'(\beta)|)\,d\nu$$
$$= \frac{\alpha\beta(\beta - m\alpha)}{2}\left(\int_0^1 \frac{|1-2\nu|\nu}{(\nu\beta + m(1-\nu)\alpha)^2}\,d\nu|\varphi'(\alpha)| + m\int_0^1 \frac{|1-2\nu|(1-\nu)}{(\nu\beta + m(1-\nu)\alpha)^2}\,d\nu|\varphi'(\alpha)|\right)$$

$$= \frac{\alpha\beta(\beta - m\alpha)}{2}\left(\rho_3|\varphi'(\alpha)| + m\rho_2|\varphi'(\beta)|\right)$$

This completes the proof. $\square$

Remark 5.3.6. *If $m = 1$, then it reduces to Theorem 5.2.6.*

Theorem 5.3.7. *Let $\varphi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be differentiable function on $I^0, \alpha, \beta \in I$, with $\alpha < \beta$ and, $\varphi' \in L[\alpha, \beta]$. If $|\varphi'|^q$ is m- harmonically convex on $[\alpha, \beta]$ for $q \geq 1$, then*

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_{\beta}^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}\,dx\right| \leq \frac{\alpha\beta(\beta - m\alpha)}{2}\rho_1^{1-\frac{1}{q}}\left(\rho_2|\varphi'(\alpha)|^q + \rho_3|\varphi'(\beta)|^q\right)^{\frac{1}{q}},$$

where

$$\begin{aligned}
\rho_1 &= \int_0^1 \frac{|1-2\nu|}{(\nu\beta + m(1-\nu)\alpha)^2}\,d\nu,\\
\rho_2 &= \int_0^1 \frac{|1-2\nu|(1-\nu)}{(\nu\beta + m(1-\nu)\alpha)^2}\,d\nu,\\
\rho_3 &= \int_0^1 \frac{|1-2\nu|\nu}{(\nu\beta + m(1-\nu)\alpha)^2}\,d\nu.
\end{aligned}$$

Proof. Using the Lemma 5.3.4, power mean inequality and m-harmonic convexity of $|\varphi'|^q$, we have

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_{\alpha}^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}\,dx\right| \leq \frac{\alpha\beta(\beta - m\alpha)}{2}$$

$$\int_0^1 \frac{|1-2\nu|}{(\nu\beta + m(1-\nu)\alpha)^2}\left|\varphi'\left(\frac{\alpha\beta}{\nu\beta + m(1-\nu)\alpha}\right)\right|\,d\nu$$

$$\leq \frac{\alpha\beta(\beta - m\alpha)}{2}\left(\int_0^1 \left|\frac{1-2\nu}{(\nu\beta + m(1-\nu)\alpha)^2}\right|d\nu\right)^{1-\frac{1}{q}}$$

$$\left(\int_0^1 \frac{|1-2\nu|}{(\nu\beta + m(1-\nu)\alpha)^2}\left|\varphi'\left(\frac{\alpha\beta}{(\nu\beta + m(1-\nu)\alpha)}\right)\right|^q d\nu\right)^{\frac{1}{q}}$$

$$\leq \frac{\alpha\beta(\beta - m\alpha)}{2}\left(\int_0^1 \left|\frac{1-2\nu}{(\nu\beta + m(1-\nu)\alpha)^2}\right|d\nu\right)^{1-\frac{1}{q}}$$

$$\left(\int_0^1 \frac{|1-2\nu|}{(\nu\beta + m(1-\nu)\alpha)^2}(\nu|f'(\alpha)|^q + m(1-\nu)|f'(\beta)|^q)\right)^{\frac{1}{q}}$$

$$= \frac{\alpha\beta(\beta - m\alpha)}{2}\left(\int_0^1 \left|\frac{1-2\nu}{(\nu\beta + m(1-\nu)\alpha)^2}\right| d\nu\right)^{1-\frac{1}{q}}$$

$$\left(\int_0^1 \frac{|1-2\nu|\nu}{(\nu\beta + m(1-\nu)\alpha)^2}\, d\nu|\varphi'(\alpha)|^q + m\int_0^1 \frac{|1-2\nu|(1-\nu)}{(\nu\beta + (1-\nu)\alpha)^2}|f'(\beta)|^q)\right)^{\frac{1}{q}}$$

$$= \frac{\alpha\beta(\beta - m\alpha)}{2}\rho_1^{1-\frac{1}{q}}\left(\rho_3|\varphi'(\alpha)|^q + m\rho_2|\varphi'(\beta)|^q\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 5.3.8. *If $m = 1$, then the above inequality reduces to Theorem 5.2.4.*

Theorem 5.3.9. *Let $\varphi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be differentiable function on I^0, $\alpha, \beta \in I$, with $\alpha < \beta$ and, $\varphi' \in L[\alpha, \beta]$. If $|\varphi'|^q$ is m-harmonically convex on $[\alpha, \beta]$ for $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{\varphi(\alpha) + \varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_\alpha^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}\, dx\right| \leq \frac{\alpha\beta(\beta - m\alpha)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\omega_1|\varphi'(\alpha)|^q + m\omega_2|\varphi'(\beta)|^q\right)^{\frac{1}{q}}$$

where,

$$\omega_1 = \int_0^1 \frac{\nu}{(\nu\beta + (1-\nu)\alpha)^{2q}}\, d\nu, \quad \omega_2 = \int_0^1 \frac{1-\nu}{(\nu\beta + (1-\nu)\alpha)^{2q}}\, d\nu.$$

Proof. From the lemma (5.3.4), Holder's inequality and the fact that $|\varphi'|^q$ is m-harmonically convex on $[\alpha, \beta]$, then we have

$$\left|\frac{\varphi(\alpha) + \varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_\alpha^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}\, dx\right| \leq \frac{\alpha\beta(\beta - m\alpha)}{2}\int_0^1 \left|\frac{1-2\nu}{(\nu\beta + (1-\nu)\alpha)^2}\right|$$

$$\left|\varphi'\left(\frac{\alpha\beta}{(\nu\beta + m(1-\nu)\alpha)}\right)\right| d\nu$$

$$\leq \frac{ab(\beta - m\alpha)}{2}\left(\int_0^1 |1-2\nu|^p\, d\nu\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(\nu\beta + m(1-\nu)\alpha)^{2q}}\left|\varphi'(\frac{\alpha\beta}{(\nu\beta + (1-\nu)\alpha)})\right|^q d\nu\right)^{\frac{1}{q}}$$

$$\leq \frac{\alpha\beta(\beta - m\alpha)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\left(\int_0^1 \frac{\nu}{(\nu\beta + (1-\nu)\alpha)^{2q}}\, d\nu|\varphi'(\alpha)|^q + m\int_0^1 \frac{1-\nu}{(\nu\beta + (1-\nu)\alpha)^{2q}}\, d\nu|\varphi'(\beta)|^q\right)^{\frac{1}{q}}$$

$$= \frac{ab(b - ma)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\omega_1|\varphi'(\alpha)|^q + m\omega_2|\varphi'(\beta)|^q\right)^{\frac{1}{q}},$$

where

$$\int_0^1 |1-2\nu|^p \, d\nu = \frac{1}{p+1},$$
$$\omega_1 = \int_0^1 \frac{\nu}{(\nu\beta+(1-\nu)\alpha)^{2q}} \, d\nu,$$
$$\omega_2 = \int_0^1 \frac{1-\nu}{(\nu\beta+(1-\nu)\alpha)^{2q}} \, d\nu.$$

This completes the proof. $\square$

Remark 5.3.10. *If $m=1$, then the above inequality reduces to the Theorem 5.2.5.*

Theorem 5.3.11. *Let $\varphi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be differentiable function on $I^0, \alpha, \beta \in I$, with $a < b$ and, $\varphi' \in L[\alpha,\beta]$. If $|\varphi'|^q$ is m- harmonically P-function on $[\alpha,\beta]$ for $p, q \geq 1, \frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta-m\alpha}\int_\alpha^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}\,dx\right| \leq \frac{\alpha\beta(\beta-m\alpha)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \psi^{\frac{1}{q}} \left(|\varphi'(\alpha)|^q + m_3|\varphi'(\beta)|^q\right)^{\frac{1}{q}},$$

where

$$\psi = \int_0^1 \frac{1}{(\nu\beta+m(1-\nu)\alpha)^{2q}}\,d\nu = \frac{\beta^{1-2q}-(m\alpha)^{1-2q}}{(1-2q)(\beta-m\alpha)}.$$

Proof. Using the Lemma 5.3.4, Holder's inequality and the fact that $|\varphi'|^q$ is m-harmonically P-function, then we have

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{\alpha\beta}{\beta-m\alpha}\int_\alpha^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}\,dx\right| \leq \frac{\alpha\beta(\beta-m\alpha)}{2}$$

$$\int_0^1 \left|\frac{1-2\nu}{(\nu\beta+(1-\nu)\alpha)^2}\right| \left|\varphi'\left(\frac{\alpha\beta}{(\nu\beta+m(1-\nu)\alpha)}\right)\right| d\nu$$

$$\leq \frac{\alpha\beta(\beta-m\alpha)}{2}\left(\int_0^1 |1-2\nu|^p\,d\nu\right)^{\frac{1}{p}} \left(\int_0^1 \frac{1}{(\nu\beta+m(1-\nu)\alpha)^{2q}}\,d\nu(|\varphi'(\alpha)|^q + m|\varphi'(\beta)|^q)\right)^{\frac{1}{q}}$$
$$= \frac{\alpha\beta(\beta-m\alpha)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \psi^{\frac{1}{q}} \left(|f'(\alpha)|^q + m_3|f'(\beta)|^q\right)^{\frac{1}{q}},$$

where

$$\psi = \int_0^1 \frac{1}{((\nu\beta+m(1-\nu)\alpha))^{2q}}\,d\nu = \frac{\beta^{1-2q}-(m\alpha)^{1-2q}}{(1-2q)(\beta-m\alpha)}.$$

This completes the proof. $\square$

Remark 5.3.12. *If $m = 1$, then the above inequality reduces to Theorem 5.2.7.*

Theorem 5.3.13. *Let $\varphi : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be differentiable function on $I^0, \alpha, \beta \in I$, with $\alpha < \beta$ and, $f' \in L[\alpha, \beta]$. If $|\varphi'|^q$ is m- harmonically P-function on $[\alpha, \beta]$ for $p, q \geq 1, \frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left|\frac{\varphi(\alpha) + \varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_{\alpha}^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}\,dx\right| \leq \frac{\alpha\beta(\beta - m\alpha)}{2}\rho\left(|\varphi'(\alpha)|^q + m|\varphi'(\beta)|^q\right)^{\frac{1}{q}},$$

where

$$\rho = \int_0^1 \frac{|1 - 2\nu|}{(\nu\beta + m(1 - \nu)\alpha)^2}\,d\nu.$$

Proof. Using the Lemma 5.3.4, power mean inequality and the fact that $|\varphi'|^q$ be m-harmonically P-function, then we have

$$\left|\frac{\varphi(\alpha) + \varphi(\beta)}{2} - \frac{\alpha\beta}{\beta - m\alpha}\int_{\alpha}^{\frac{\beta}{m}} \frac{\varphi(x)}{x^2}\,dx\right|$$

$$= \left|\frac{\alpha\beta(\beta - m\alpha)}{2}\int_0^1 \frac{(1 - 2\nu)}{(\nu\beta + m(1 - \nu)\alpha)^2}\varphi'\left(\frac{\alpha\beta}{\nu\beta + m(1 - \nu)\alpha}\right)\,d\nu\right|$$

$$\leq \frac{\alpha\beta(\beta - m\alpha)}{2}\int_0^1 \frac{|1 - 2\nu|}{(\nu\beta + m(1 - \nu)\alpha)^2}\left|\varphi'\left(\frac{\alpha\beta}{\nu\beta + m(1 - \nu)\alpha}\right)\right|\,d\nu$$

$$\leq \frac{\alpha\beta(\beta - m\alpha)}{2}\left(\int_0^1 \left|\frac{1 - 2\nu}{(\nu\beta + m(1 - \nu)\alpha)^2}\right|\,d\nu\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1 \left|\frac{1 - 2\nu}{(\nu\beta + m(1 - \nu)\alpha)^{2q}}\right|\left|\varphi'\left(\frac{\alpha\beta}{\nu\beta + (1 - \nu)\alpha}\right)\right|^q\right)^{\frac{1}{q}}$$

$$\leq \frac{\alpha\beta(\beta - m\alpha)}{2}\left(\int_0^1 \left|\frac{1 - 2\nu}{(\nu\beta + m(1 - \nu)\alpha)^2}\right|\,d\nu\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1 \frac{|1 - 2\nu|}{(\nu\beta + m(1 - \nu)\alpha)^{2q}}(|\varphi'(\alpha)|^q + m|\varphi'(\beta)|^q)\right)^{\frac{1}{q}}$$

$$= \frac{\alpha\beta(\beta - m\alpha)}{2}\rho^{1-\frac{1}{q}}(\rho)^{\frac{1}{q}}\left(|\varphi'(\alpha)|^q + m|\varphi'(\beta)|^q\right)^{\frac{1}{q}}$$

$$= \frac{\alpha\beta(\beta - m\alpha)}{2}\rho\left(|\varphi'(\alpha)|^q + m|\varphi'(\beta)|^q\right)^{\frac{1}{q}},$$

where

$$\rho = \int_0^1 \frac{|1-2\nu|}{(\nu\beta + m(1-\nu)\alpha)^2}\, d\nu.$$

This completes the proof. $\square$

Remark 5.3.14. *If $m = 1$, then it reduces to Theorem 5.2.8.*

Theorem 5.3.15. *Let $\varphi_1, \varphi_2 : I \subset (0,\infty) \to \mathbb{R}$ be two m-harmonically convex functions where $\alpha, \beta \in I$ with $\alpha < \beta$, $m \in [0,1]$. If $\varphi_1\varphi_2 \in L[\alpha,\beta]$, then*

$$\frac{\alpha\beta}{\beta - m\alpha}\int_\alpha^{\frac{\beta}{m}}\left(\frac{\varphi_1(x)\varphi_2(x)}{x^2}\right) dx \le \frac{1}{3}\eta_1 + \frac{m^2}{3}\eta_2 + \frac{m}{6}\eta_3,$$

where

$$\eta_1 = \varphi_1(\alpha)\varphi_2(\alpha),$$

$$\eta_2 = \varphi_1(\beta)\varphi_2(\beta),$$

$$\eta_3 = \varphi_1(\alpha)\varphi_2(\beta) + \varphi_1(\beta)\varphi_2(\alpha).$$

Proof. Since φ_1, φ_2 are m-harmonically convex functions, then by definition, we have

$$\begin{aligned}\varphi_1\left(\frac{\alpha\beta}{\nu\beta + m(1-\nu)\alpha}\right) &\le \nu\varphi_1(\alpha) + m(1-\nu)\varphi_1(\beta)\\ \varphi_2\left(\frac{\alpha\beta}{\nu\beta + m(1-\nu)\alpha}\right) &\le \nu\varphi_2(\alpha) + m(1-\nu)\varphi_2(\beta)\end{aligned}$$

On multiplying the above two inequalities, we have

$$\begin{aligned}\varphi_1\left(\frac{\alpha\beta}{\nu\beta + m(1-\nu)\alpha}\right)\varphi_2\left(\frac{\alpha\beta}{\nu\beta + m(1-\nu)\alpha}\right) &\le \nu^2\varphi_1(\alpha)\varphi_2(\alpha) + m^2\varphi_1(\beta)\varphi_2(\beta)\int_0^1 (1-\nu)^2\, d\nu\\ &+ m(\varphi_1(\alpha)\varphi_2(\beta) + \varphi_1(\beta)\varphi_2(\alpha))\int_0^1 \nu(1-\nu)\, d\nu\\ &= \frac{1}{3}\varphi_1(\alpha)\varphi_2(\alpha) + \frac{m^2}{3}\varphi_1(\beta)\varphi_2(\beta) + \frac{m}{6}(\varphi_1(\alpha)\varphi_2(\beta) + \varphi_1(\beta)\varphi_2(\alpha)).\end{aligned}$$

Put $x=\frac{\alpha\beta}{\nu\beta+m(1-\nu)\alpha}$ when $\nu=0, x=\frac{\beta}{m}$, when $\nu=1, x=\alpha$

$$d\nu=-\frac{\alpha\beta}{(\beta-m\alpha)}\frac{dx}{x^2}.$$

On substituting these values, we have

$$\begin{aligned}\frac{\alpha\beta}{\beta-m\alpha}\int_{\alpha}^{\frac{\beta}{m}}\left(\frac{\varphi_1(x)\varphi_2(x)}{x^2}\right)dx \leq \frac{1}{3}(\varphi_1(\alpha)\varphi_2(\alpha)+m^2\varphi_1(\beta)\varphi_2(\beta))+\frac{m}{6}(\varphi_1(\alpha)\varphi_2(\beta)+\varphi_1(\beta)\varphi_2(\alpha)). \\ =\frac{1}{3}(\eta_1+m^2\eta_2)+\frac{m}{6}\eta_3.\end{aligned}$$

This completes the proof. □

Remark 5.3.16. *If* $m=1$*, then it reduces to Theorem 5.2.9.*

Chapter 6

HERMITE-HADAMARD INEQUALITY FOR HARMONICALLY CONVEX FUNCTIONS VIA RIEMANN-LIOUVILLE FRACTIONALS

6.1 Introduction

The notion of non-integer order calculus, the generalization of traditional integer order calculus, called fractional calculus, was introduced by Leibnitz and L'Hopital in 1695 but it was popularized in the end of nineteenth century by Riemann and Liouville. The rapid growth of the fractional calculus is because of its applications in diverse fields ranging from physical sciences to engineering to biological sciences and economics. Due to the wide application of fractional integrals and importance of Hermite-Hadamard type inequalities, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes. Some new integral inequalities involving two non-negative and integrable functions that are related to Hermite-Hadamard type are also obtained by many authors. B.G. Pachpatte proposed some Hermite-Hadamard type inequalities involving two log-convex functions. Similar results for s-convex functions are established by Kirmaci et al. M.Z. Sarikaya presented some integral inequalities for two h-convex functions. It is remarkable that M. Z. Sarikaya proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals. For more detail see [10, 25, 52] and the references therein.

The purpose of this chapter is to establish Hermite-Hadamard's inequalities via Riemann-Liouville fractional integral for the case of harmonically convex function as well as to present the results on the products of two harmonically convex functions via Riemann-Liouville fractional integrals.

6.2 Preliminary Results

In the following, we will give some necessary definitions and preliminary results of fractional calculus which are used further in this chapter.

Theorem 6.2.1. *[10] Let* $\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be a positive function with* $a < b$ *,* $\alpha \in \mathbb{R}_+$ *and* $\Phi \in L[a,b]$ *. If* f *is convex function on* $[a,b]$ *then the following inequalities for fractional integrals hold:*

$$\Phi\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathbb{J}^{\alpha}_{a^+}\Phi(b) + \mathbb{J}^{\alpha}_{b^-}\Phi(a)\right] \leq \frac{\Phi(a)+\Phi(b)}{\alpha}$$

Definition 6.2.2. *[10] Let* $\Phi \in L[a,b]$ *. The Riemann- Liouville integrals* $\mathbb{J}^{\alpha}_{a^+}\Phi$ *and* $\mathbb{J}^{\alpha}_{b^-}\Phi$ *of order* $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$\mathbb{J}^{\alpha}_{a^+}\Phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\Phi(t)\,dt, \quad x > a,$$

$$\mathbb{J}^{\alpha}_{b^-}\Phi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\Phi(t)\,dt, \quad x < b,$$

respectively.

The symbols $\mathbb{J}^{\alpha}_{a^+}\Phi(x)$ and $\mathbb{J}^{\alpha}_{b^-}\Phi(x)$ are left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$, with $a \geq 0$. Here, $\Gamma(\alpha)$ is gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\,dt$$

6.3 Main Results

Hermite-Hadamard's inequalities for harmonically convex functions via Riemann-Liouville fractional integral can be represented as follows:

Theorem 6.3.1. *Let* $\Phi : [a,b] \to \mathbb{R}$ *be a positive function with* $0 \leq a \leq b$ *and* $\Phi \in L[a,b]$ *. If* Φ *is harmonically convex function on* $[a,b]$ *, then the following inequalities for fractional inequalities hold:*

$$\Phi\left(\frac{2ab}{a+b}\right) \leq \frac{(a^\alpha - b^\alpha)\Gamma(\alpha)}{2(b-a)^\alpha}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+}\Phi(b) + \frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-}\Phi(a)\right] \leq \frac{\Phi(a)+\Phi(b)}{2\alpha}.$$

Proof. Since $\Phi : \mathbb{I} \subset \mathbb{R}\backslash\{0\}$ is a harmonically convex, for all $x, y \in \mathbb{I}$ with $t = \frac{1}{2}$, we have

$$\Phi\left(\frac{2xy}{x+y}\right) \leq \frac{\Phi(x)+\Phi(y)}{2}$$

Choosing $x = \frac{ab}{ta+(1-t)b}$, $y = \frac{ab}{tb+(1-t)a}$, we get

$$\Phi\left(\frac{2\frac{ab}{ta+(1-t)b}\frac{ab}{tb+(1-t)a}}{\frac{ab}{ta+(1-t)b}+\frac{ab}{tb+(1-t)a}}\right) \leq \frac{\Phi\left(\frac{ab}{ta+(1-t)b}\right)+\Phi\left(\frac{ab}{tb+(1-t)a}\right)}{2}$$

$$2\Phi\left(\frac{2ab}{a+b}\right) \leq \Phi\left(\frac{ab}{ta+(1-t)b}\right) + \Phi\left(\frac{ab}{tb+(1-t)a}\right).$$

Multiplying both sides by $t^{\alpha-1}$, then integrating with respect to t over $[0,1]$, we obtain

$$2\int_0^1 t^{\alpha-1}\Phi\left(\frac{2ab}{a+b}\right)dt \leq \int_0^1 t^{\alpha-1}\Phi\left(\frac{ab}{ta+(1-t)b}\right)dt + \int_0^1 t^{\alpha-1}\Phi\left(\frac{ab}{tb+(1-t)a}\right)dt,$$

i.e.,

$$\frac{2}{\alpha}\Phi\left(\frac{2ab}{a+b}\right) \leq I_1 + I_2, \tag{6.3.1}$$

where $I_1 = \int_0^1 t^{\alpha-1}\Phi\left(\frac{ab}{ta+(1-t)b}\right)dt$ Put $u = \frac{ab}{ta+(1-t)b}$ then $dt = \frac{ab}{b-a}\frac{du}{u^2}$. When $t \to 0, u \to a; t \to 1, u \to b$ And, $t = \frac{b(u-a)}{(a-b)u}$.
On substituting these values, we obtain

$$\begin{aligned}
I_1 &= \int_a^b \left(\frac{b(u-a)}{(b-a)u}\right)^{\alpha-1}\Phi(u)\frac{ab}{b-a}\frac{du}{u^2} \\
&= \frac{ab}{b-a}\int_a^b \frac{b^{\alpha-1}(u-a)^{\alpha-1}}{(b-a)^{\alpha-1}u^{\alpha-1}}\frac{\Phi(u)du}{u^2} \\
&= \frac{ab^\alpha}{(b-a)^\alpha}\int_a^b \frac{(u-a)^{\alpha-1}\Phi(u)du}{u^{\alpha+1}} \\
&= \frac{b^\alpha - a^\alpha}{\alpha(b-a)^\alpha a^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^\alpha_{a^+}\Phi(b).
\end{aligned}$$

And, $I_2 = \int_0^1 t^{\alpha-1}\Phi\left(\frac{ab}{tb+(1-t)a}\right)dt$. Put $v = \frac{ab}{tb+(1-t)a}$. Then $dt = \frac{ab}{b-a}\frac{dv}{v^2}$.

When $t \to 0, v \to b; t \to 1, v \to a$ and, $t = \frac{a(b-v)}{(b-a)v}$.
On substituting these values, we obtain $I_2 = \frac{b^\alpha - a^\alpha}{\alpha(b-a)^\alpha b^{\alpha-1}} \Gamma(\alpha) \mathbb{J}^\alpha_{b^-} \Phi(a)$
Now, we substitute the values of I_1 and I_2 in $(6.3.1)$, we obtain

$$\frac{2}{\alpha}\Phi\left(\frac{2ab}{a+b}\right) \leq \frac{b^\alpha - a^\alpha}{\alpha(b-a)^\alpha a^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^\alpha_{a^+}\Phi(b) + \frac{b^\alpha - a^\alpha}{\alpha(b-a)^\alpha b^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^\alpha_{b^-}\Phi(a),$$

i.e.,

$$2\Phi\left(\frac{2ab}{a+b}\right) \leq \frac{(b^\alpha - a^\alpha)\Gamma(\alpha)}{(b-a)^\alpha}\left(\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+}\Phi(b) + \frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-}\Phi(a)\right) \tag{6.3.2}$$

Also, as Φ is harmonically convex function, then for $t \in [0,1]$, it yields

$$\Phi\left(\frac{ab}{ta + (1-t)b}\right) \leq t\Phi(a) + (1-t)\Phi(b).$$

And,

$$\Phi\left(\frac{ab}{tb + (1-t)a}\right) \leq (1-t)\Phi(a) + t\Phi(b)$$

Adding these two inequalities,

$$\Phi\left(\frac{ab}{ta + (1-t)b}\right) + \Phi\left(\frac{ab}{tb + (1-t)a}\right) \leq \Phi(a)(t + 1 - t) + \Phi(b)(1 - t + t) = \Phi(a) + \Phi(b).$$

Multiplying both sides by $t^{\alpha-1}$ and integrating with respect to t over $[0,1]$, we have

$$\int_0^1 t^{\alpha-1}\Phi\left(\frac{ab}{ta + (1-t)b}\right) dt + \int_0^1 t^{\alpha-1}\Phi\left(\frac{ab}{tb + (1-t)a}\right) dt \leq \Phi(a)\int_0^1 t^{\alpha-1}\, dt + \Phi(b)\int_0^1 t^{\alpha-1}\, dt,$$

i.e.,

$$\frac{(b^\alpha - a^\alpha)\Gamma(\alpha)}{(b-a)^\alpha}\left(\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+}\Phi(b) + \frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-}\Phi(a)\right) \leq \frac{\Phi(a) + \Phi(b)}{\alpha}. \tag{6.3.3}$$

From $(6.3.1)$, $(6.3.2)$, *and* $(6.3.3)$, we have

$$\Phi\left(\frac{2ab}{a+b}\right) \leq \frac{(a^\alpha - b^\alpha)\Gamma(\alpha)}{2(b-a)^\alpha}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+}\Phi(b) + \frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-}\Phi(a)\right] \leq \frac{\Phi(a) + \Phi(b)}{2\alpha}.$$

This completes the proof. □

Theorem 6.3.2. *Let* Φ *and* Ψ *be two real-valued, non-negative and harmonically convex* *functions on* $[a,b]$ *.Then the following inequalities hold:*

$$\frac{(b^{\alpha}-a^{\alpha})\Gamma(\alpha)}{\alpha(b-a)^{\alpha}}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^{\alpha}_{a^{+}}\Phi(b)\Psi(b)+\frac{1}{b^{\alpha-1}}\mathbb{J}^{\alpha}_{b^{-}}\Phi(a)\Psi(a)\right]\leq\left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b),$$

where $M(a,b)=\Phi(a)\Psi(a)+\Phi(b)\Psi(b)$; $N(a,b)=\Phi(a)\Psi(b)+\Phi(b)\Psi(a)$.

Proof. Let Φ and Ψ be harmonically convex functions on $[a,b]$. Then, for $t\in[0,1]$, we have

$$\Phi\left(\frac{ab}{ta+(1-t)b}\right)\leq t\Phi(b)+(1-t)\Phi(a),$$

and

$$\Psi\left(\frac{ab}{ta+(1-t)b}\right)\leq t\Psi(b)+(1-t)\Psi(a).$$

Then their product is given by

$$\Phi\left(\frac{ab}{ta+(1-t)b}\right)\Psi\left(\frac{ab}{ta+(1-t)b}\right)\leq t^{2}\Phi(b)\Psi(b)+(1-t)^{2}\Phi(a)\Psi(a)+t(1-t)(\Phi(a)\Psi(b)+\Phi(b)\Psi(a)).$$

Similarly,

$$\Phi\left(\frac{ab}{(1-t)a+tb}\right)\Psi\left(\frac{ab}{(1-t)a+tb}\right)\leq t^{2}\Phi(a)\Psi(a)+(1-t)^{2}\Phi(b)\Psi(b)+t(1-t)(\Phi(a)\Psi(b)+\Phi(b)\Psi(a)).$$

On adding the above two inequalities, we have

$$\Phi\left(\frac{ab}{ta+(1-t)b}\right)\Psi\left(\frac{ab}{ta+(1-t)b}\right)+\Phi\left(\frac{ab}{(1-t)a+tb}\right)\Psi\left(\frac{ab}{(1-t)a+tb}\right)\leq(2t^{2}-2t+1)(\Phi(a)\Psi(a)+\Phi(b)\Psi(b))+2t(1-t)(\Phi(a)\Psi(b)+\Phi(b)\Psi(a)).$$

Multiplying the above inequality by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_{0}^{1}t^{\alpha-1}\Phi\left(\frac{ab}{ta+(1-t)b}\right)\Psi\left(\frac{ab}{ta+(1-t)b}\right)dt+\int_{0}^{1}t^{\alpha-1}\Phi\left(\frac{ab}{(1-t)a+tb}\right)\Psi\left(\frac{ab}{(1-t)a+tb}\right)dt\leq(\Phi(a)\Psi(a)+\Phi(b)\Psi(b))\int_{0}^{1}t^{\alpha-1}(2t^{2}-2t+1)\,dt+2(\Phi(a)\Psi(b)+\Phi(b)\Psi(a))\int_{0}^{1}t^{\alpha-1}t(1-t)\,dt.$$

Here,

$$\int_0^1 t^{\alpha-1}(2t^2-2t+1)\,dt = \frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}$$
$$\int_0^1 t^{\alpha-1}t(1-t)\,dt = \frac{1}{(\alpha+1)(\alpha+2)}.$$

Put $\frac{ab}{ta+(1-t)b} = u$. Then $dt = \frac{ab}{b-a}\frac{du}{u^2}$. And, when $t \to 0, u \to a$ and $t \to 1, u \to b$, and $t = \frac{b(a-u)}{u(a-b)}$. Again, put $\frac{ab}{(1-t)a+tb} = v$. Then $dt = \frac{ab}{b-a}\frac{dv}{v^2}$. Also, when $t \to 0, v \to b$, and when $t \to 1, v \to a$ On substituting these values, we obtain

$$\int_a^b \left(\frac{b(a-u)}{u(a-b)}\right)^{\alpha-1} \Phi(u)\Psi(u)\frac{ab}{(b-a)u^2}\,du + \int_b^a \left(\frac{a(v-a)}{v(b-a)}\right)^{\alpha-1} \Phi(v)\Psi(v)\frac{ab}{(b-a)v^2}\,du$$
$$\leq \left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b) + \frac{2}{(\alpha+1)(\alpha+2)}N(a,b),$$

i.e.,

$$\frac{ab^\alpha}{(a-b)^\alpha}\int_b^a \frac{(a-u)^{\alpha-1}\Phi(u)\Psi(u)}{u^{\alpha+1}}\,du + \frac{a^\alpha b}{(b-a)^\alpha}\int_b^a \frac{(v-a)^{\alpha-1}\Phi(v)\Psi(v)}{u^{\alpha+1}}\,dv$$
$$\leq \left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b) + \frac{2}{(\alpha+1)(\alpha+2)}N(a,b)$$
$$\frac{(b^\alpha-a^\alpha)\Gamma(\alpha)}{\alpha(b-a)^\alpha}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+}\Phi(b)\Psi(b) + \frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-}\Phi(a)\Psi(a)\right] \leq \left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b)$$
$$+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b).$$

This completes the proof. □

Chapter 7

SUMMARY, CONCLUSION, AND RECOMMENDATION

7.1 Summary

The origin of quantum calculus, how it is different from ordinary calculus, and application of quantum calculus in various domain of Mathematics and Physics have been explained in short in the first chapter of this thesis. It has also included different aspects of convex functions and mathematical inequalities, especially, Hermite-Hadamard type integral inequality for classical convex function. The mathematical means and their corresponding properties along with different types of special means have also been incorporated.

The second chapter of this thesis has extended the equality of convex function of quantum estimates to an m-convex function. Using the resulted information, a few novel Hermite-Hadamard integral inequalities are established whose first order q-derivatives are m-convex functions.

The third chapter of this thesis, has included some new results on the products of generalized geometric convex functions, and these results are further extended to q-analogues which is a new paradigm in convexity theory and integral inequality.

In the fourth chapter of this thesis, some new convexities related to harmonic convexity have been defined and for each new convexity, a corresponding Hermite-Hadamard inequality has been proved. The product of functions having a particular convexity has also been considered, and the q-analogues of these inequalities have also been obtained.

In the fifth chapter of this thesis, we have generalized harmonically convex function to m-harmonically convex function and applied them to develop some Hermite- Hadamard type integral inequalities whose first order derivatives are m-harmonically convex function.

Apart from this, we have further generalized harmonically P-function to m-harmonically P-function and applied them to extend Hermite-Hadamard type integral inequalities.

In sixth chapter of this thesis, we have established Hermite-Hadamard's inequalities via Riemann-Liouville fractional integral for the case of harmonically convex function, and the products of two harmonically convex functions via Riemann-Liouville fractional integrals as well.

7.2 Conclusion

The Hermite-Hadamard inequality usually stated as a result valid for convex functions only, although it holds for many other functions as well. The Hermite-Hadamard inequality provides a lower and an upper estimation for the integral average of any convex function defined on a compact interval involving the mid-point and the end points of the domain having a wider application for generalized means, information measures, quadrature rules, and many more. That's why, it has become a corner stone in mathematical analysis and in optimization theory. There is a growing literature providing new proofs, extensions, refinements and generalizations on it.

The main results of this thesis starts from chapter 2 and ends with chapter 6 which include extensions, refinements and new proofs on Hermite-Hadamard integral inequality for different convexity of functions which are concluded as follows:

- Inspired by the work of Iscan [6, 26] who defined harmonic convexity, in this thesis, we have generalized this convexity to m-harmonic convexity and further to (α, m)-harmonic convexity. Moreover, we also have defined new notions of harmonic-geometric (HG)-convexity, $m - HG$-convexity and $(\alpha, m) - HG$-convexity, and for each of these convexities, we have proved Hermite-Hadamard inequalities and we have obtained inequalities involving the product of each of the those type of convexities in q-calculus.
- Motivated by the concept of geometrically convex functions by Zhang et. al in [62], and Niculescu in [37], we have extended the definitions on geometrically-convex functions and the results on the products of generalized geometrically convex functions and those results are also established in q-calculus in this thesis.
- Inspired by the results of Sudsutad et al. [54], this thesis has extended Lemma (2.2.3) assuming that ${}_uD_q\Phi$ is q-integrable m-convex function and then presented some quantum estimates of Hermite-Hadamard type integral inequalities for m-convex function on $[mu, v]$.
- Motivated by the works of I. Iscan, [6, 26], we have generalized harmonically convex function to m-harmonically convex function and m-harmonically P-function

and obtained some Hermite-Hadamard type integral inequalities whose first order derivatives are m-harmonically convex functions.

- Inspired by the works of M. Z. Sarikaya [52], we have established the interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals for harmonically convex function.

7.3 Recommendation

Convexity is one of the four special properties viz. continuity, convexity, monotocity and differentiability of the functions of real variables having two important attributes i.e. the maximum value is attained at a boundary point and any local minimum is a global one. Hermite-Hadamard integral inequality is one of the most important integral inequalities that satisfies the necessary and sufficient condition of the convexity of the function. Inequalities play important roles in understanding many mathematical concepts, such as probability theory, numerical integration and integral operator theory. Through the last century, H-H type integral inequality has been considered to be one of the fastest growing inequality in mathematical analysis, through which vast problems of engineering, physics and economics have been addressed. Due to the enormous importance of H-H inequalities, many extensions, refinements, and generalization for various types of convex functions are still open to be dealt with. More specifically, extensions and refinements of Hermite -Hadamard type integral inequality for various types of convex functions in quantum calculus may be the interested domains for the researchers in future.

REFERENCES

[1] Ahmad, B., Ntouyas, S. K., & Tariboon, J. (2016). Quantum Calculus: New Concepts, Impulsive Ivps, Bvps and Inequalities, *World Scientific Publishing Company.*

[2] Al-Azemi, F. & Calin, O. (2015). Asian options with harmonic average, *Appl. Math. Inf. Sci.*, 9(6): 1-9.
http://dx.doi.org/10.12785/amis/090606

[3] Alp, N., Sarikaya, M. Z., Kunt, M., & Iscan, I. (2018). q-Hermite-Hadamard inequalities and quantum estimates for mid-point type inequalities via convex and quasi-convex functions, *J. King Saud University-Science*, 30: 193-203.
https://doi.org/10.1016/j.jksus.2016.09.007

[4] Aigner, M. & Ziegler, G. M. (2004). Proofs from The Book, Third ed., *Springer-Verlag, Berlin.*

[5] Bakula, M. K., Ozdemir, M. E., & Pecaric, J. (2008). Hadamard Type Inequalities For m-Convex And (α, m)-Convex Functions, *J. Inequal. Pure and Appl. Math.*,96(4): 1-12.

[6] Baloch, I. A. & Iscan, I. (2017). Some Hermite-Hadamard type Integral Inequalities for Harmonically $(p,(s,m))$-Convex Functions, *J. Inequal. and Special Functions*, 8(4): 65-84.
url:www.ILIRIAS.COM/JIASF

[7] Beckenbach, E. F. (1948). Convex Functions, *Bull. Amer. Math. Soc.*, 54: 439-460.

[8] Bullen, P. S. (2003). Handbook of Means and Their Inequalities, *Dordrecht: Kluwer Academics.*

[9] Bullen, P. S., Mitrinovic, D. S., & Vasic, M. (1998). Means and Their Inequalities, *Dordrecht: Kluwer Academic.*

[10] Chen, F. (2014). A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals, *Ital. J. Pure Appl. Math.*, 33: 299-306.
https://doi.org/10.1155/2014/173293

[11] Chen, F. & Wu, S. (2014). Some Hermite-Hadamard Type Inequalities For Harmonically s-Convex Functions, *The Scientific World Journal, Hindawi*, 2014: 1-7.
https://doi.org/10.1155/2014/279158

[12] Dragomir, S. S. (2002). On Some New Inequalities of Hermite-Hadamard Type for m-Convex functions, *J. of Mathematics*, 33(1): 45-56.
https://doi.org/10.5556/j.ckm.33.2002.304

[13] Dragomir, S. S. & Agrawal, R. P. (1948). Two Inequalities for differentiable mappings and Applications to special means of real numbers and to trapezoidal formula, *. Math. Lett.*, 11: 91-95.
https://doi.org/10.1016/S0893-9659(98)0006-x

[14] Dragomir, S. S. & Toader, G. (1993). Some inequalities for m-convex functions, *University Babes Bolyai, Mathematica*, 38: 21-28.

[15] Euler, L.(1748). *Introduction in Analysin Infitorium*, Chapter VII.

[16] Ernst, T. (2008). The Different Tongues of q-Calculus, *Proceedings of the Estonian Academy of Sciences*, 57(2): 81-99.
https://doi.org/10.3176/Proc.2008.2.03.

[17] Ernst, T. (2000). The history of q-calculus and a new method, *Department of Mathematics, Uppsala University.*

[18] Fink, A. M. (2000). An essay on the history of inequalities, *J. Math. Anal. Appl.*, 249(1): 118-134.
https://doi.org/10.0006/jmaa.2000.6934

[19] Gauchman, H. (2004). Integral Inequalities in q-calculus, *Math. Appl.*, 47: 281-300.
https://doi.org/10.1016/S0898-1221(04)90025-9

[20] Hadamard, J. (1893). Etude sur les proprietes des fonctions entieres et en particulier d'une fonction consideree par Riemann, *J. Math. Pures Appl.*, 9: 171-216.

[21] Hardy, G. H., Littlewood, J. E., & Polya, G. (1934). Inequalities, *Cambridge University Press, Cambridge, UK.*

[22] Hardy, G. H., Littlewood, J. E., & Polya, G. (1952). Inequalities, 2nd edition, *Cambridge Mathematical Library: Cambridge, UK.*

[23] Hermite, C. (1883). Sur deux limites d'une intehrale de'finite, *Mathesis.*

[24] Iscan, I. (2013). Hermite-Hadamard Type Inequalities for Harmonically Convex Functions, *:1303.6089 v1[Math.CA].* https://doi.org/10.48550/arXiv.1303.6089.

[25] Iscan, I. (2014). Hermite-Hadamard type inequalities for harmonically convex functions, *Hacetteppe, J. Math. Stats.*, 43(6): 935-942. https://doi.org/10.15672/HJMS.2014437519

[26] Iscan, I. (2014). Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions, *Journal of Mathematics*, 2014: 1-10. https://doi.org/10.1155/2014/346305

[27] Jackson, F. H. (1908). On q-Functions and a certain Difference Operators, *Trans. Roy. Soc. Edin.*, 46: 253-281. http://dx.doi.org/10.1017/S008045600002751

[28] Jackson, F. H. (1910). On q-Definite Integrals, *Pure applied Mathematics*, 41(15): 193-203.

[29] Jensen, J. L. W. V. (1906). Sur les Fonctions Convexes et les inegalites entre les valeurs moyennes, *Acta Mathematica*, 30(1): 175-193.

[30] Kac, V. & Cheung, P. (2002). Quantum Calculus, *Springer, New York.*

[31] Kikianty, E. (2010). Hermite-Hadamard Inequality in the Geometry of Banach Spaces, *University of Pretoria, Ph. D. Thesis.*

[32] Kirmaci, U. S., Bakula, M. K., Ozdemir, M. E., & Pecaric, J. (2007). Hadamard Type Inequalities For s-Convex Functions, *Appl. Math. Comput.*, 19(3): 26-35. https://doi.org/10.1016/j.amc.2007.03.030

[33] Kirmaci, U. S. (2004). Inequalities for differentiable mappings and applications to special means of real numbers and to mid point formula, *Applied Mathematics and Computation*, 147(1): 137-146. https://doi.org/10.1016/S0096-3003(02)00657-4

[34] Kirmaci, U. S. & Ozdemir, M. E. (2004). On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comput.*, 153: 361-368. https://doi.org/10.1016/S0096-3003(03)00657-4

[35] Mihesan, V. G. (1993). A generalization of Convexity, *Seminar of Functional Equations, Approx. and Convex, Cluj-Napoca, Romania.*

[36] Mitrinovic, D. S. & Lakovic, I. B. (1985). Hermite and Convexity, *Aequationes, Math.*, 28(3): 229-232.

[37] Niculescu, C. P. (2000). Convexity according to the geometric mean, *Math. Inequal. Appl.*, 3(2): 155-167.
https://doi.org/10.7153/mia-03-19

[38] Niculescu, C. P. & Perrson, L. E. (2018). Convex Functions and Their Applications: A Contemporary Approach, 2nd edition, *Books in Mathematics, Springer: New York.*

[39] Noor, M. A., Noor, K. I., Awan, M. U., & Castache, S. (2015). Some Integral Inequalities for Harmonically h-Convex Functions, *UPB Sci. Bull. Series A: Appl. Math.* 77(1): 5-16.
https://doi.org/10.1155/2019/2394021

[40] Noor, M. A. (2018). Advanced Convex and Numerical Analysis, Lecture Notes, *COMASTS, University Islamabad, Park Road, Pakistan.*

[41] Noor, M. A., Noor, K. I., Awan, M. U., & Costache, S. (2015). Some integral inequalities for harmonically h-convex functions, *U.P.B. Sci. Bull., Series A*, 77(1): 5-16.

[42] Ozdemir, M. E., Latif, M. A., & Akdemir, A. O. (2012). On Some Hadamard Type Inequalities For Product Of Two s-Convex Functions On The Co-ordinates, *J. Inequal. and Appl.*, 1-13.
https://doi.org/10.1186/1029-242x-2012-21

[43] Ozcan, S. (2019). Some Integral inequalities for Harmonically (α, s)-convex functions, *J. of Func. Spaces, Hindawi*, 2019: 1-8.
https://doi.org/10,1155/2019/2394021

[44] Pachpatte, B. G. (2003). On Some Inequalities For Convex Functions, *RGMIA Research Report Collection*, 6: 1-9.

[45] Pachpatte, B. G. (2004). A Note On Integral Inequalities Involving Two log-Convex Functions, *Math. Inequal Appl.*, 7(4): 511-515.
https://doi.org/10.7153/mia-07-51

[46] Pachpatte, B. G. (2012). Analytic Inequalities: Recent Advances, *Atlantic Press, Amsterdam-Paris.*

[47] Pearce, C. E. M., & Pecaric, J. (2000). Inequalities of differentiable mappings with application to special means and quadrature formula, *Appl. Math. Lett.*, 13: 51-55.
https://doi.org/10.1016/S0893-9659(99)00164-0

[48] Pecaric, J. E., Proschan, F., & Tong, Y. L. (1992). Convex Functions, Partial Orderings and Statistical Applications, *Math. Sci. Eng., Academic Press, Boston, MA.*

[49] Pecaric, J. E., Mitronovic, D. S., & Fink, A. M. (1993). Classical and New Inequalities in Analysis, *Springer: Cham, Switzerland*, 1-2.

[50] Robert, A. W. & Varberg, D. E. (1973). Convex Functions, *London Academic Press.*

[51] Sarikaya, M. Z., Saglam, A., & Yildirim, H. (2008). On Some Hadamard Type Inequalities For h-convex Functions, *J. Math. Inequal.*, 2(3): 335-341.
https://doi.org/10.7153/jmi-02-30

[52] Sarikaya, M. Z., Set, E., Yaldiz, H., & Basak, N. (2013). Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.*, 57: 2403-2407.
https://doi.org/10.1016/j.mcm.2011.12.048

[53] Set, E., Ozdemir, M. F., & Dragomir, S. S. (2010). On the Hermite- Hadamard inequality and other integral inequalities involving two functions, *J. Inequal. Appl.*, 2010: 1-9.
https://doi.org/10.1155/2010/148102

[54] Sudsutad, W., Ntouyas, S. K., & Tariboon, J. (2015). Quantum Integral Inequalities For Convex Functions, *J. Math. Sci.*, 9(3): 781-793.
https://doi.org/10.7153/jmi-09-64

[55] Tariboon, J. & Ntouyas, S. K. (2014). Quantum Integral Inequalities on Finite Intervals, *J. Inequal. Appl.*, 2014: 1-13.
https://doi.org/10.1186/1029-242x-2014-121

[56] Tariboon, J. & Ntouyas, S. K. (2013). Quantum calculus on finite intervals and application to impulsive difference equations, *Adv. Differ., Eqn.*, 2013: 1-19.
https://doi.org/10.1186/1687-1847-2013-282

[57] Tiwari, P. & Bhatta, C. R. (2023). Hermite-Hadamard inequality whose first order q-derivatives are m-convex functions, *Adv. Appl. Math. Sci.*, 22(11): 2171-2188.

[58] Tiwari, P. & Bhatta, C. R. (2023). *H-H inequality for harmonically convex functions via Riemann-Liouville fractional integrals*, Fractional Differential Calculus, 13(2): 163-169.

[59] Toader, G. (1985). Some generalizations of the convexity, proceedings of the Colloquium on Approximation and Optimization, *Univ. Cluj-Napoca*, 329-338.

[60] Xi, B. Y., Bai, R.F., & Qi, F. (2012). Hermite-Hadamard type inequalities for the m and (α, m)-geometrically convex functions, *Aequ. Math.* https://doi.org/10.1007/500010-011-0114-x.

[61] Yin, H. P. & Qi, F. (2015). Hermite -Hadamard Type Inequalities For Product of (α, m)-Convex Functions, *J. Non-linear Sci. Appl.*, 27(1): 71-79. https://doi.org/10.35834/mjms/1449161369

[62] Zhang, T. Y., Ji, A. P., & Qi, F. (2012). On Integral Inequalities of Hermite-Hadamard type for s-geometrically convex functions, *Abstr. Appl. Anal* https://doi.org/10.1155/2012/560586.

APPENDIX

PUBLICATIONS/COMMUNICATED

1. Tiwari, P. & Bhatta, C. R. (2023). *Hermite-Hadamard inequality whose first order q-derivatives are m-convex functions*, Adv. Appl. Math. Sci., 22(11): 2171-2188.
2. Tiwari, P. & Bhatta, C. R. (2023). *H-H inequality for harmonically convex functions via Riemann-Liouville fractional integrals*, Fractional Differential Calculus, 13(2): 163-169.
3. Tiwari, P. & Bhatta, C. R. (2023). *Quantum estimates of Hermite-Hadamard integral inequality on products of extended geometric convex functions*, Bulletin of Allahabad Mathematical Society (accepted).
4. Tiwari, P. & Bhatta, C. R. (2020). *A Review on Basics of Quantum Calculus in Mathematical Sciences*, Gandaki Journal of Mathematics.
5. Tiwari, P. & Bhatta, C. R. (2023). *q-Hermite-Hadamard integral inequality for the co-ordinated convex functions*, Bhairahawa Campus Journal, **6**, October.
6. A manuscript entitled ‘Generalized Convexities: Hermite-Hadamard Inequalities and their Quantum Estimates’ is communicated to the journal ‘Arabian Journal of Mathematics’ , and it is under review.
7. A manuscript entitled ‘Hermite-Hadamard Integral Inequalities for m-convex functions’ is communicated to the journal named, ‘Poincare Journal of Analysis and Applications’ and is under review.
8. A manuscript entitled ‘Hermite-Hadamard Integral Inequality for Differentiable m-convex function is communicated to ‘The Nepali Mathematical Sciences Report ’, (accepted).

ATTENDED CONFERENCES/SEMINARS

1. A paper was presented in an international conference held on April 9-11, 2021 (ICAA-2021), entitled with ‘Extension of Hermite-Hadamard type integral inequality for coordinated convex function in a rectangle from plane in quantum framework.’
2. A concept was presented paper on Two Days’ Symposium and Research in Pure Mathematics held on 3-4 June, 2022 in Butwal organized by Nepal mathematical Society Lumbini Chapter entitled with ‘Some Quantum Estimates of Hermite-Hadamard type integral inequalities whose first order q -derivatives are m-convex Functions.’
3. A paper was presented on National conference on Mathematics and Its Applications(NCMA-2022) which was held on June 11-13, 2022 in Ilam entitled with ‘Hermite-Hadamard type integral inequalities on generalized geometric convex function and their quantum estimates.’
4. A paper was presented on a Third International Conference on Application of Mathematics to Nonlinear Sciences (AMNS-2023) held in Pokhara, Nepal, from May 25-28, 2023 entitled with ‘Quantum estimates of Hermite-Hadamard integral inequality on products of extended geometric convex functions. ’

DEDICATION

I am grateful to dedicate this thesis of Ph. D. researh work to my mother late **DILA KUMARI TIWARI** (2003-2067 BS).

Advances and Applications in Mathematical Sciences
Volume 22, Issue 11, September 2023, Pages 2171-2188
© 2023 Mili Publications, India

HERMITE-HADAMARD INEQUALITY WHOSE FIRST ORDER *q*-DERIVATIVES ARE *m*-CONVEX FUNCTIONS

PITAMBER TIWARI* and CHET RAJ BHATTA

Research Scholar
Central Department of Mathematics
IOST, Tribhuvan University, Nepal

Professor of Mathematics
Central Department of Mathematics
Tribhuvan University, Kathmandu, Nepal
E-mail: chetbhatta0@gmail.com

Abstract

Convexity is a fundamental and significant function in the theory of geometric functions with extensive applications in both pure and applied mathematics. In this article, an equality of quantum estimate of convex function is extended to an m-convex function. Taking so obtained result a few novel Hermite-Hadamard type integral inequalities whose first order q-derivatives are m-convex functions are established. The results are further presented in q-analogues.

1. Introduction

Quantum calculus, or q-calculus, is the study of calculus without the concept of limits, first appeared in the eighteenth century and reached its pinnacle in the twentieth due to its extensive use in physics and several mathematical disciplines. By bringing the number q into Newton's work on infinite series, Euler launched his research in the eighteenth century. The theory of q-hyper-geometric functions and Jacobi's triple product identity were discovered in the nineteenth century. Jackson (1910) established definitive q-integrals and began a systematic study of q-calculus. Multiple areas of mathematics and physics, including number theory, fundamental hyper-geometric functions, combinatorics, orthogonal polynomials,

---

2020 Mathematics Subject Classification: 05A30, 26A51, 26D15.

Keywords: convexity, Hermite-Hadamard inequality, m-convexity, q-calculus.

*Corresponding author; E-mail: pitambertiwari40@gmail.com

Received November 3, 2022; Accepted February 9, 2023

mathematical inequalities, quantum theory, mechanics, and the theory of relativity, all have numerous uses for the subject of quantum calculus, and, hence the q-calculus is appeared as an inter-disciplinary subject between mathematics and physics, a blend of the subjects. We refer to go through the books by Ernst [9], Kac and Cheung [10], and the paper by Gauchman [4] for some recent developments on quantum calculus and the theory of inequalities.

Convexity is a fundamental and significant function in the theory of geometric functions with extensive applications in both pure and applied mathematics. It has been noted that the theories of inequalities and convex functions are very interdependent. According to the literature, a classical or normal convex function is defined as follows:

Definition 1.1. Let $\phi : I \subset \mathbb{R} \to \mathbb{R}$, be a real valued function. Then, the function ϕ is said to be a convex function, if the inequality

$$\phi(su + (1 - s)v) \leq s\phi(u) + (1 - s)\phi(v) \tag{1.1}$$

holds, for all $u, v \in I$, and, $s \in [0, 1]$.

If the inequality in (1.1) is reversed, then the concavity of ϕ holds. Toader [3] introduced the concept of m-convexity of the function ϕ as follows.

Definition 1.2[3]. The function $\phi : [0, b] \to \mathbb{R}, b > 0$, is said to be m-convex function if

$$\phi(su + m(1 - s)v) \leq s\phi(u) + m(1 - s)\phi(v)) \tag{1.2}$$

holds for all $u, v \in [0, b], s \in [0, 1], m \in [0, 1]$.

The m-convexity is intermediate concept between usual convexity and star-shaped function. We recall that a function $\phi : [0, b] \to \mathbb{R}$ is called a star-shaped function, if

$$\phi(su) \leq s\phi(u)$$

holds for all $u \in [0, b], s \in [0, 1]$.

Example 1.1. The function $\phi : [0, \infty) \to \mathbb{R}$ given by

$$\phi(u) = au + b$$

is an m-convex function $(m \in [0, 1])$ if $b \leq 0$.

There are numerous significant inequalities for the category of convex functions, but one of the most well-known is the so-called Hermite Hadamard integral inequality, which was independently found by Ch. Hermite and J. Hadamard in 1881 and 1893 and goes as follows.

Definition 1.3. Let $\phi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function where, $u, v \in I$ with $u < v$. Then, the following inequality holds:

$$\phi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \phi(x)dx \leq \frac{\phi(u)+\phi(v)}{2}. \tag{1.3}$$

This famous result is considered as a necessary and a sufficient condition for a function to be convex. It provides a lower and an upper estimation for the integral mean of any convex function defined on a compact interval involving the mid-point and the end points. Hermite-Hadamard's inequality has raised many scholars' attention, and thus, a variety of refinements, extensions, variants and generalizations have been found in literatures.

In this paper, we aim in developing some quantum estimates of Hermite-Hadamard type of integral inequalities for m-convex functions by considering the results of quantum estimates of convex functions.

2. Preliminary Results

In this section, we first review and state some previously understood ideas and findings on q-calculus that will be applied in the following paper.

Let $J = [u, v] \subset \mathbb{R}, J^0 = (u, v)$ be intervals and $0 < q < 1$ be a constant.

We define q-derivative of a function $\phi : J \to \mathbb{R}$ at a point $x \in J$ on $[u, v]$ as follows:

Definition 2.1[5]**.** Assume that $\phi : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression

$${}_uD_q\phi(x) = \frac{\phi(x) - \phi(qx + (1-q)u)}{(1-q)(x-u)},\ x \neq u,\ {}_uD_q\phi(u) = \lim_{x \to u} {}_uD_q\phi(x) \tag{2.1}$$

is called the q-derivative of the function ϕ at x on J. We say that ϕ is q-differentiable on J provided ${}_uD_q\phi(x)$ exists for all $x \in J$.

Note that if $u = 0$ in (2.1), then ${}_0D_q\phi = D_q\phi$, where D_q is the well known q-derivative of the function $\phi(x)$ defined by

$$D_q\phi(x) = \frac{\phi(x) - \phi(qx)}{(1-q)x}$$

For more details, see [5].

Definition 2.2[5]**.** Assume that $\phi : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the q-integral on J is defined by

$$\int_u^x \phi(s)_u d_q s = (1-q)(x-u)\sum_{n=0}^{\infty} q^n \phi(q^n x + (1-q^n)u) \tag{2.2}$$

for $x \in J$.

Moreover, if $c \in (u, x)$, then the definite q-integral on J is defined by

$$\int_c^x \phi(s)_c d_q s$$

$$= \int_u^x \phi(s)_u d_q s - \int_u^c \phi(s)_u d_q s = (1-q)(x-u)\sum_{n=0}^{\infty} q^n \phi(q^n x + (1-q^n)u)$$

$$- (1-q)(c-u)\sum_{n=0}^{\infty} q^n \phi(q^n c + (1-q^n)u).$$

Note that if $u = 0$, then equation (2.2) reduces to the classical q-integral of a function $\phi(x)$, defined by

$$\int_0^x \phi(s)_0 d_q s = (1-q)x\sum_{n=0}^{\infty} q^n \phi(q^n x),\ x \in [0, \infty).$$

Theorem 2.3[10]**.** *Assume* $\phi_1, \phi_2 : J \to \mathbb{R}$ *are continuous functions,* $\alpha \in \mathbb{R}$. *Then, for* $x \in J$

(i) $\int_u^x (\phi_1(s) + \phi_2(s))_u d_q s = \int_u^x \phi_1(s)_u d_q s + \int_u^x \phi_2(s)_u d_q s.$

(ii) $\int_u^x (\alpha\phi_1)(s)_u d_q s = \alpha \int_u^x \phi_1(s)_u d_q s.$

(iii) $\int_c^x \phi_1(s)_u D_q \phi_2(s)_u d_q s = | \phi_1\phi_2 |_c^x$

$$- \int_c^x \phi_2(qs + (1-q)u)_u D_q \phi_1(s)_u d_q s,\ c \in (u, x).$$

Alp et al. [7] in 2018 introduced the correct form of Hermite-Hadamard type integral inequality in quantum framework as follows:

Theorem 2.4[7]. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function on* J *and* $0 < q < 1$. *Then, we have*

$$\phi\left(\frac{qu+v}{1+q}\right) \le \frac{1}{v-u}\int_u^v \phi(s)_u d_q s \le \frac{q\phi(u)+\phi(v)}{1+q}. \tag{2.3}$$

Lemma 2.5[11]. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function and* $0 < q < 1$. *If* ${}_uD_q\phi$ *is an integrable function on* J^0, *then the following equality holds*:

$$\frac{q\phi(u)+\phi(v)}{1+q} - \frac{1}{v-u}\int_u^v \phi(s)_u d_q s$$

$$= \frac{q(v-u)}{1+q}\int_0^1 (1-(1+q)s)_u D_q \phi(sv+(1-s)u)_0 d_q s$$

Lemma 2.6[11]. *Let* $0 < q < 1$ *be a constant. Then, the following equalities hold*:

(i) $\int_0^1 1_0 d_q s = 1.$

(ii) $\int_0^1 s_0 d_q s = \frac{1}{1+q}.$

(iii) $\int_0^1 s^2{}_0 d_q s = \frac{1}{1+q+q^2}.$

Lemma 2.7[11]. *Let* $0 < q < 1$ *be a constant. Then, the following equality holds*:

$$\int_0^1 |1-(1+q)s|_0 d_q s = \frac{q(2+q+q^2)}{(1+q)^3}.$$

Lemma 2.8[11]. *Let* $0 < q < 1$ *be a constant. Then, the following equality holds*:

$$\int_0^1 s|1-(1+q)s|_0 d_q s = \frac{q(1+4q+q^2)}{(1+q)^3(1+q+q^2)}.$$

Lemma 2.9[11]. *Let* $0 < q < 1$ *be a constant. Then, the following equality holds*:

$$\int_0^1 (1-s)|1-(1+q)s|_0 d_q s = \frac{q(1+3q^2+2q^3)}{(1+q)^3(1+q+q^2)}.$$

Theorem 2.10[11]. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function. If* $|{}_uD_q\phi|$ *is convex and integrable on* J^0, *then the following inequality holds*:

$$\left|\frac{q\phi(u)+\phi(v)}{1+q} - \frac{1}{v-u}\int_u^v \phi(s)_u d_q s\right|$$

$$\leq \frac{q^2(v-u)}{(1+q)^4(1+q+q^2)}((1+4q+q^2)|{}_uD_q\phi(v)| + (1+3q+2q^3)|{}_uD_q\phi(u)).$$

Theorem 2.11[11]. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function. If* $|{}_uD_q\phi|^r$ *is convex and integrable on* J^0, *and* $r \geq 1$, *then the following inequality holds*:

$$\left|\frac{q\phi(u)+\phi(v)}{1+q} - \frac{1}{v-u}\int_u^v \phi(s)_u d_q s\right| \leq \frac{q^2(2+q+q^2)(v-u)}{(1+q)^4}$$

$$\left[\frac{(1+4q+q^2)|{}_uD_q\phi(v)|^r + (1+3q^2+2q^3)|{}_uD_q\phi(u)|^r}{(1+3q^2+2q^3)(2+q+q^2)}\right]^{\frac{1}{r}}$$

Theorem 2.12[11]. *Let* ϕ_1 *and* ϕ_2 *be two real-valued, non-negative and convex functions on J. Then, the following inequality holds*:

$$\frac{1}{v-u}\int_u^v \phi_1(x)\phi_2(x)_u d_q x \leq \frac{\phi_1(u)\phi_2(u)}{1+q+q^2} + \frac{q(1+q^2)\phi_1(v)\phi_2(v)+q^2N(u, v)}{(1+q+q^2)(1+q)}.$$

Theorem 2.13[11]. *Let* ϕ_1 *and* ϕ_2 *be two real-valued, non-negative and convex functions on* J. *Then, the following inequalities hold*:

(i) $$\frac{(1+q)(1+q+q^2)}{(v-u)^2}$$

$$\int_u^v\int_u^v\int_0^1 \phi_1(sy+(1-s)x)\phi_2(sy+(1-s)x)_0 d_q s_u d_q x_u d_q y \leq \frac{(1+2q+q^2)}{(v-u)}$$

$$\int_u^v \phi_1(x)\phi_2(x)_u d_q x + \frac{2q^2}{(1+q)^2}[(q^2\phi_1(u)\phi_2(u)+\phi_1(v)\phi_2(v))+qN(u, v)]$$

(ii) $$\frac{1+q+q^2}{v-u}\int_u^v\int_0^1 \phi_1\left(sy+(1-s)\frac{u+v}{2}\right)\phi_2\left(sy+(1-s)\frac{u+v}{2}\right)_0 d_q s_u d_q y$$

$$\leq \frac{1}{v-u}\int_u^v \phi_1(x)g\phi_2(x)_u d_q x + \frac{q(1+q^2)}{4(1+q)}(M(u, v)+N(u, v))$$

$$+\frac{q^2}{2(1+q)^2}(2(q\phi_1(u)\phi_2(u)+\phi_1(v)\phi_2(v))+(1+q)N(u, v))$$

where $M(u, v) = \phi_1(u)\phi_2(u)+\phi_1(v)\phi_2(v)$, $N(u, v) = \phi_1(u)\phi_2(v)+\phi_1(v)\phi_2(u)$.

3. Main Results

In this section, first of all, we extend Lemma 2.5 assuming that ${}_uD_q\phi$ is q-integrable m-convex function and then present some quantum estimates of Hermite-Hadamard type integral inequalities for m-convex function on $[mu, v]$.

Lemma 3.1. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function and* $0 < q < 1$. *If* ${}_uD_q\phi$ *is* q*-integrable* m*-convex function on* J^0, *where* J^0 *is interior of* J, *then the following equality holds*:

$$\frac{1}{v-mu}\int_{mu}^{v}\phi(s)_{mu}d_q s-\frac{q\phi(mu)+\phi(v)}{1+q}$$

$$=\frac{q(v-mu)}{1+q}\int_0^1(1-(1+q)s)_u D_q\phi(sv+m(1-s)u)_0 d_q s$$

Proof. Using q-derivative on a finite interval, we have

$$\int_0^1(1-(1+q)s)_u D_q\phi(sv+m(1-s)u)_0 d_q s=\int_0^1 1_u D_q\phi(sv+m(1-s)u)_0 d_q s$$

$$-(1+q)\int_0^1 s_u D_q\phi(sv+m(1-s)u)_0 d_q s$$

$$=\int_0^1\frac{\phi(sv+m(1-s)u)-\phi(qsv+m(1-qs)u)}{s(1-q)(v-mu)}\,{}_0d_q s$$

$$-(1+q)\int_0^1 s\cdot\frac{\phi(sv+m(1-s)u)-\phi(qsv+m(1-qs)u)}{s(1-q)(v-mu)}\,{}_0d_q s$$

$$=\frac{1}{v-mu}\left[\sum_{n=0}^{\infty}\phi(q^n v+m(1-q^n)u)-\sum_{n=0}^{\infty}\phi(q^{n+1}v-m(1-q^{n+1})u)\right]$$

$$-\frac{1+q}{v-mu}\left[\sum_{n=0}^{\infty}q^n\phi(q^n v+m(1-q^n)u)-\sum_{n=0}^{\infty}q^n\phi(q^{n+1}v-m(1-q^{n+1})u)\right]$$

$$=\frac{\phi(v)-\phi(mu)}{v-mu}-\frac{1+q}{v-mu}\sum_{n=0}^{\infty}q^n\phi(q^n v+m(1-q^n)u)$$

$$+\frac{1+q}{v-mu}\sum_{n=0}^{\infty}q^n\phi(q^{n+1}v+m(1-q^{n+1})u)$$

$$=\frac{\phi(v)-\phi(mu)}{v-mu}-\frac{1+q}{v-mu}\sum_{n=0}^{\infty}q^n(q^n v+m(1-q^n)u)$$

$$+\frac{1+q}{q(v-mu)}\left[\phi(v)-\phi(u)\sum_{n=0}^{\infty}q^n\phi(q^{n+1}v+m(1-q^{n+1})u)\right]$$

$$= \frac{\varphi(v)-\varphi(mu)}{v-mu} - \frac{(1+q)\varphi(v)}{q(v-mu)} + \frac{1+q}{q(v-mu)}\sum_{n=0}^{\infty} q^n \varphi(q^n v + m(1-q^n)u)$$

$$- \frac{1+q}{v-mu}\sum_{n=0}^{\infty} q^n \phi(q^n v + m(1-q^n)u)$$

$$= -\frac{q\phi(mu)+\phi(v)}{q(v-mu)} + \frac{(1+q)(1-q)(v-mu)}{q(v-mu)(v-mu)}\sum_{n=0}^{\infty} q^n \phi(q^n v + m(1-q^n)u)$$

$$= -\frac{q\phi(mu)+\phi(v)}{q(v-mu)} + \frac{(1+q)}{q(v-mu)^2}\int_{mu}^{v} \phi(s)_{mu} d_q s$$

$$= \frac{1+q}{q(v-mu)}\left[\frac{1}{v-mu}\int_{mu}^{v} \phi(s)_{mu} d_q s - \frac{q\phi(mu)+\phi(v)}{1+q}\right].$$

Thus, we have

$$\frac{1}{v-mu}\int_{mu}^{v} \phi(s)_{mu} d_q s - \frac{q\phi(mu)+\phi(v)}{1+q}$$

$$= \frac{q(v-mu)}{1+q}\int_{0}^{1} (1-(1+q)s)_{u} D_q \phi(sv + m(1-s)u)_0 d_q s.$$

The proof is now complete. □

Remark 3.2. If $m = 1$, and then it reduces to the result as given in Lemma 2.5, and if $q \to 1$, $m = 1$, then it reduces to the following result:

$$\frac{\phi(u)+\phi(v)}{2} - \frac{1}{v-u}\int_{u}^{v} \phi(x)dx = \frac{v-u}{2}\int_{0}^{1} (1-2s)\phi'(sv + (1-s)u)ds.$$

Theorem 3.3. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function. If* $|\,{}_{mu}D_q \phi\,|$ *is m-convex and integrable on* J^0, *then the following inequality holds*:

$$\left|\frac{q\phi(mu)+\phi(v)}{1+q} - \frac{1}{v-mu}\int_{mu}^{v} \phi(s)_u d_q s\right|$$

$$\leq \frac{q^2(v-mu)}{(1+q)^4(1+q+q^2)}\left((1+4q+q^2)|\,{}_{mu}D_q\phi(v)\,| + (1+3q+2q^3)|\,{}_{mu}D_q\phi(u)\,|\right).$$

Proof. Using lemma 3.1 and m-convexity of ${}_{mu}D_q\phi$ on J^0 and lemmas 2.8 and 2.9, we have

$$\left|\frac{1}{v-mu}\int_{mu}^{v}\phi(s)_{mu}d_qs-\frac{q\phi(mu)+\phi(v)}{1+q}\right|$$

$$=\left|\frac{q(v-mu)}{1+q}\int_0^1(1-(1+q)s)_uD_q\phi(sv+m(1-s)u)_0d_qs\right|$$

$$\leq\frac{q(v-mu)}{1+q}$$

$$\int_0^1|1-(1-q)s|[s|_{mu}D_q\phi(v)+m(1-s)|_{mu}D_q\phi(u)|]_0d_qs$$

$$\leq\frac{q(v-mu)}{1+q}$$

$$[|_{mu}D_q\varphi(v)|\int_0^1 s|1-(1+q)s|_0d_qs+m|_{mu}D_q\varphi(u)|\int_0^1(1-s)|1-(1+q)s|_0d_qs]$$

$$=\frac{q(v-mu)}{1+q}\left[|_{mu}D_q\phi(v)|\frac{q(1+4q+q^2)}{(1+q)^3(1+q+q^2)}+\frac{mq(1+3q^2+2q^3)}{(1+q)^3(1+q+q^2)}|_{mu}D_q\phi(u)|\right]$$

$$=\frac{q^2(v-mu)}{(1+q)^4(1+q+q^2)}[|_{mu}D_q\phi(v)|(1+4q+q^2)+m(1+3q^2+2q^3)|_{mu}D_q\phi(u)|].$$

The proof is complete. □

Remark 3.4. If $m=1$, then it reduces to the previous result as in Theorem 2.10.

Remark 3.5. If $m=1, q\to 1$ then, the above result reduces to the following already established result

$$\left|\frac{\phi(u)+\phi(v)}{2}-\frac{1}{v-u}\int_u^v\phi(s)ds\right|\leq\frac{(v-u)}{8}[|\phi'(v)|+|\phi'(u)|].$$

We, now present the second result of q-integral inequality for m-convex function on $[mu, v]$.

Theorem 3.6. *Let* $\phi : J \to \mathbb{R}$ *be a continuous function. If* $|{}_{mu}D_q\phi|^r$ *is* m*-convex and integrable on* J^0 *and* $r \geq 1$, *then the following inequality holds*:

$$\left|\frac{q\phi(mu)+\phi(v)}{1+q} - \frac{1}{v-mu}\int_{mu}^{v}\phi(s)_u d_q s\right| \leq \frac{q^2(2+q+q^2)(v-mu)}{(1+q)^4}$$

$$\left(\frac{(1+4q+q^2)|{}_{mu}D_q\phi(v)|^r + (1+3q^2+2q^3)|{}_{mu}D_q\phi(u)|^r}{(1+q+q^2)(2+q+q^3)}\right)^{\frac{1}{r}}$$

Proof. From lemma 3.1 and using power-mean inequality and m-convexity of $|{}_{mu}D_q\phi|^r$, and lemmas 2.7, 2.8 and 2.9, we have

$$\left|\frac{1}{v-mu}\int_{mu}^{v}\phi(s)_{mu}d_q s - \frac{q\phi(mu)+\phi(v)}{1+q}\right|$$

$$= \left|\frac{q(v-mu)}{1+q}\int_0^1 (1-(1+q)s)_u D_q\phi(sv+m(1-s)u)_0 d_q s\right|$$

$$\leq \frac{q(v-mu)}{(1+q)}\int_0^1 (1-(1+q)s)_u D_q\phi(sv+m(1-s)u)_0 d_q s$$

$$\leq \frac{q(v-mu)}{1+q}\left(\int_0^1 |1-(1+q)s|_0 d_q s\right)^{1-\frac{1}{r}}$$

$$\cdot\left(\int_0^1 |1-(1+q)s|_{ma}D_q\phi(sb+m(1-s)a)|_0^r \, d_q s\right)^{\frac{1}{r}}$$

$$\leq \frac{q(v-mu)}{1+q}\left(\int_0^1 |1-(1+q)s|_0 d_q s\right)^{1-\frac{1}{r}}$$

$$\left(|{}_{mu}D_q\phi(v)|^r\int_0^1 s|1-(1+q)s_0 d_q s| + m|{}_{mu}D_q\phi(u)|^r\int_0^1 (1-s)|1-(1+q)s|_0 d_q s\right)^{\frac{1}{r}}$$

$$\leq \frac{q(v-mu)}{1+q}\left(\frac{q(2+q+q^2)}{(1+q)^3}\right)^{1-\frac{1}{r}}.$$

$$\left(\frac{q}{(1+q+q^2)(1+q)^3}((1+4q+q^2)|\,{}_{mu}D_q\varphi(v)\,|^r+m(1+3q^2+2q^3)|\,{}_{mu}D_q\varphi(u)\,|^r)\right)^{\frac{1}{r}}$$

$$=\left|\frac{q\phi(mu)+\phi(v)}{1+q}-\frac{1}{v-mu}\int_{mu}^{v}\phi(s)_u d_q s\right|\le\frac{q^2(2+q+q^2)(v-mu)}{(1+q)^4}$$

$$\left(\frac{(1+4q+q^2)|\,{}_{mu}D_q\phi(v)\,|^r+(1+3q^2+2q^3)|\,{}_{mu}D_q\phi(u)\,|^r}{(1+q+q^2)(2+q+q^3)}\right)^{\frac{1}{r}}$$

This completes the proof. □

Remark 3.7. If $m=1$, then it reduces to Theorem 2.11.

Remark 3.8. If $m=1$ and $q\to 1$, then we have the following previously known result

$$\left|\frac{\phi(u)+\phi(v)}{2}-\frac{1}{v-u}\int_{u}^{v}\phi(s)ds\right|\le\frac{v-u}{4}\left[\frac{|\,\phi'(u)\,|^r+|\,\phi'(v)\,|^r}{2}\right]^{\frac{1}{r}}.$$

Theorem 3.9. *Let* ϕ_1 *and* ϕ_2 *be two real-valued, non-negative m-convex functions on J. Then, the following inequality holds*:

$$\frac{1}{v-mu}\int_{mu}^{v}\phi_1(x)\phi_2(x)_{mu}d_qx\le\frac{\phi_1(u)\phi_2(v)}{1+q+q^2}$$

$$+\frac{m^2q(1+q^2)\phi_1(v)\phi_2(v)+mq^2N(u,v)}{(1+q+q^2)(1+q)}.$$

where, $N(u,v)=\phi_1(u)\phi_2(v)+\phi_1(v)\phi_2(u)$.

Proof. Using m-convexity of ϕ_1 and ϕ_2 and for all $s\in[0,1]$, we have

$$\phi_1(sv+m(1-s)u)\le s(\phi_1(v)+m(1-s)\phi_1(u)).$$

$$\phi_2(sv+m(1-s)u)\le s(\phi_2(v)+m(1-s)\phi_2(u)).$$

Multiplying above inequalities, we have

$$\phi_1(sv+m(1-s)u)\phi_2(sv+m(1-s)u)\le s^2\phi_1(v)\phi_2(v)+m^2(1-s)^2\phi_1(u)\phi_2(u)$$

$$+\,ms(1-s)(\phi_1(u)\phi_2(v)+\phi_1(v)\phi_2(u)).$$

Taking q-integral with respect to s over $[0, 1]$ and using lemma 2.6, we have

$$\int_0^1 \phi_1(sv + m(1-s)u)\phi_2(sv + m(1-s)u)_0 d_q s$$

$$\leq \int_0^1 (s^2\phi_1(v)\phi_2(v) + m^2(1-s)^2\phi_1(u)\phi_2(u)$$

$$+ ms(1-s)(\phi_1(u)\phi_2(v) + \phi_1(v)\phi_2(u)))_0 d_q s.$$

$$\int_0^1 \phi_1(sv + m(1-s)u)\phi_2(sv + m(1-s)u)_0 d_q s$$

$$\leq \frac{\phi_1(v)\phi_2(v)}{1+q+q^2} + \frac{m^2 q(1+q^2)\phi_1(u)\phi_2(u)}{(1+q+q^2)(1+q)} + \frac{mq^2 N(u, v)}{(1+q+q)(1+q)}.$$

Substituting $x = sv + m(1-s)u$, on left-side of above inequality, we have

$$\frac{1}{v - mu}\int_{mu}^{v} \phi_1(x)\phi_2(x)_{mu} d_q x \leq \frac{\phi_1(u)\phi_2(u)}{1+q+q^2}$$

$$+ \frac{m^2 q(1+q^2)\phi_1(v)\phi_2(v) + mq^2 N(u, v)}{(1+q+q^2)(1+q)}$$

where, $N(u, v) = \phi_1(u)\phi_2(v) + \phi_1(v)\phi_2(u)$.

This completes the proof. □

Remark 3.10. If $q \to 1$, then the above inequality reduces to

$$\frac{1}{v - mu}\int_{mu}^{v} \phi_1(x)\phi_2(x)dx \leq \frac{1}{3}[\phi_1(v)\phi_2(v) + m^2\phi_1(u)\phi_2(u)] + \frac{1}{6}N(u, v).$$

Remark 3.11. If $q \to 1$ and $m = 1$, then the inequality reduces to

$$\frac{1}{v-u}\int_{u}^{v} \phi_1(x)\phi_2(x)dx \leq \frac{1}{3}M(u, v) + \frac{1}{6}N(u, v).$$

Theorem 3.12. *Let* ϕ_1 *and* ϕ_2 *be two real-valued, non-negative m-convex functions on J. Then, the following inequalities hold*:

(i) $$\frac{(1+q)(1+q+q^2)}{(v-u)^2}$$

$$\int_u^v\int_u^v\int_0^1 \phi_1(sy+m(1-s)x)\phi_2(sy+(1-s)x)\,{}_0d_qs\,{}_ud_qx\,{}_ud_qy \le \frac{1+q+m^2q(1+q^2)}{v-u}$$

$$\int_u^v \phi_1(x)\phi_2(x)\,{}_ud_qx + \frac{2q^2m}{(1+q)^2}(q^2\phi_1(u)\phi_2(u)+qN(u, v)+\phi_1(v)\phi_2(v)).$$

(ii)

$$\frac{1+q+q^2}{v-u}\int_u^v\int_0^1 \phi_1\left(sy+m(1-s)\frac{u+v}{2}\right)\phi_2\left(sy+m(1-s)\frac{u+v}{2}\right){}_0d_qs\,{}_ud_qy$$

$$\le \frac{1}{v-u}\int_u^v \phi_1(y)\phi_2(y)\,{}_ud_qy + \frac{m^2q(1+q^2)}{4(1+q)}(M(u, v)+N(u, v))$$

$$+\frac{mq^2}{2(1+q)^2}(2(q\phi_1(u)\phi_2(u)+\phi_1(v)\phi_2(v))+(1+q)N(u, v))$$

where $M(u, v)=\phi_1(u)\phi_2(u)+\phi_1(v)\phi_2(v),$ *and* $N(u, v)=\phi_1(u)\phi_2(v)+\phi_1(v)\phi_2(u).$

Proof. From the definition of m-convexity of ϕ_1 and ϕ_2, for all $s\in[0, 1]$, $x, y\in J$, we have

$$\phi_1(sy+m(1-s)x)\le s\phi_1(y)+m(1-s)\phi_1(x).$$

$$\phi_2(sy+m(1-s)x)\le s\phi_2(y)+m(1-s)\phi_2(x).$$

Multiplying above inequalities, we have

$$\phi_1(sy+m(1-s)x)\phi_2(sy+m(1-s)x)\le s^2\phi_1(y)\phi_2(y)+m^2(1-s)^2\phi_1(x)\phi_2(x)$$

$$+ms(1-s)(\phi_1(x)\phi_2(y)+\phi_1(y)\phi_2(x)).$$

Taking q-integral with respect to s over $[0, 1]$ and using lemma 2.6

$$\int_0^1 \phi_1(sy+m(1-s)x)\phi_2(sy+m(1-s)x)\,{}_0d_qs$$

$$\le \int_0^1 (s^2\phi_1(y)\phi_2(x)+m^2(1-s)^2\phi_1(x)\phi_2(x)$$

$$+ ms(1-s)(\varphi_1(x)\varphi_2(y)+\varphi_1(y)\varphi_2(x)))_0\, d_q s.$$

$$\int_0^1 \phi_1(ty + m(1-t)x)\phi_2(ty + m(1-t)x)_0\, d_q t$$

$$\leq \frac{\varphi_1(y)\varphi_2(x)}{1+q+q^2} + \frac{m^2 q(1+q^2)\varphi_1(x)\varphi_2(x)}{(1+q+q^2)(1+q)} + \frac{mq^2 N(u, v)}{(1+q+q^2)(1+q)}.$$

Next, taking double q-integral to both sides of the above inequality with respect to x, y on $[u, v]$, we have

$$\int_u^v \int_u^v \int_0^1 \phi_1(sy + m(1-s)x)\phi_2(sy + (1-s)x)_0\, d_q s_u\, d_q x_u\, d_q y$$

$$\leq \int_0^1 s^2 \varphi_1(y)\varphi_2(y) + m^2(1-s)^2 \varphi_1(x)\varphi_2(x) + ms(1-s)(\varphi_1(x)\varphi_2(y) + \varphi_1(y)\varphi_2(x))_0\, d_q s.$$

$$\int_0^1 \phi_1(sy + m(1-s)x)\phi_2(sy + m(1-s)x)_0\, d_q s$$

$$\leq \frac{v-u}{1+q+q^2}\int_u^v \phi_1(y)\phi_2(y)_u\, d_q y + \frac{m^2 q(1+q^2)(v-u)}{(1+q+q^2)(1+q)}\int_u^v \phi_1(x)\phi_2(x)_u\, d_q x$$

$$+ \frac{mq^2}{(1+q+q^2)(1+q)}\left(\int_u^v \phi_2(y)_u\, d_q y \int_u^v \phi_1(x)_u\, d_q x + \int_u^v f(y)_u\, d_q y \int_u^v g(x)_u\, d_q x\right)$$

$$\leq (v-u)\left(\frac{1}{1+q+q^2} + \frac{m^2 q(1+q^2)}{(1+q)(1+q+q^2)}\right)$$

$$\int_u^v f(x)g(x)_u\, d_q x + \frac{mq^2}{(1+q+q^2)(1+q)}$$

$$\left((v-u)\cdot\frac{qg(u)+g(v)}{1+q}\cdot(v-u)\frac{qf(u)+f(v)}{1+q}\right.$$

$$\left. + (v-u)\cdot\frac{qf(u)+f(v)}{1+q}\cdot(v-u)\frac{qg(u)+g(v)}{1+q}\right).$$

On simplifying, we have

$$= \frac{(v-u)(1+q+m^2q(1+q^2))}{(1+q)(1+q+q^2)} \int_u^v \phi_1(x)\phi_2(x)_u d_q x$$

$$+ \frac{2mq^2(v-u)^2}{(1+q)^3(1+q+q^2)} (q^2\phi_1(u)\phi_2(u) + qN(u, v) + \phi_1(v)\phi_2(v)).$$

Multiplying both sides by $\frac{(1+q)(1+q+q^2)}{(v-u)^2}$, we have

$$\frac{(1+q)(1+q+q^2)}{(v-u)^2} \int_u^v \int_u^v \int_0^1 \phi_1(sy + m(1-s)x)\phi_2(sy + (1-s)x)_0 d_q s_u d_q x_u d_q y$$

$$\leq \frac{1+q+m^2q(1+q^2)}{v-u}$$

$$\int_u^v \varphi_1(x)\varphi_2(x)_u d_q x + \frac{2q^2m}{(1+q)^2}(q^2\varphi_1(u)\varphi_2(u) + qN(u, v) + \varphi_1(v)\varphi_2(v)))$$

Now, we begin the proof of (ii) part:

From the definition of m-convexity of ϕ_1 and ϕ_2, for all $s \in [0, 1]$, $x, y \in J$, we have

$$\phi_1\left(sy + m(1-s)\frac{u+v}{2}\right) \leq s\phi_1(y) + m(1-s)\phi_1\left(\frac{u+v}{2}\right)$$

$$\phi_2\left(sy + m(1-s)\frac{u+v}{2}\right) \leq s\phi_2(y) + m(1-s)\phi_2\left(\frac{u+v}{2}\right).$$

Multiplying the above inequalities, we obtain

$$\phi_1\left(sy + m(1-s)\frac{u+v}{2}\right)\phi_2\left(sy + m(1-s)\frac{u+v}{2}\right)$$

$$\leq s^2\phi_1(y)\phi_2(y) + m^2(1-s)^2\phi_1\left(\frac{u+v}{2}\right)\phi_2\left(\frac{u+v}{2}\right)$$

$$+ ms(1-s)\left(\phi_1(y)\phi_2\left(\frac{u+v}{2}\right) + \phi_1\left(\frac{u+v}{2}\right)\right).$$

Taking q-integral with respect to s over $[0, 1]$, we have

$$\int_0^1 \phi_1\left(sy + m(1-s)\frac{u+v}{2}\right)\phi_2\left(sy + m(1-s)\frac{u+v}{2}\right)\,{}_0d_qs$$

$$\leq \frac{\phi_1(y)\phi_2(y)}{1+q+q^2} + \frac{m^2q(1+q^2)}{(1+q+q^2)(1+q)}\phi_1\left(\frac{u+v}{2}\right)\phi_2\left(\frac{u+v}{2}\right)$$

$$+ \frac{mq^2}{(1+q+q^2)(1+q)}\left(\phi_1(y)\phi_2\left(\frac{u+v}{2}\right) + \phi_1\left(\frac{u+v}{2}\right)\phi_2(y)\right).$$

Applying q-integral with respect to y over $[u, v]$ and using m-convexity of functions of ϕ_1 and ϕ_2, we observe

$$\int_u^v\int_0^1 \phi_1\left(sy + m(1-s)\frac{u+v}{2}\right)\phi_2\left(sy + m(1-s)\frac{u+v}{2}\right)\,{}_0d_qs\,{}_ud_qy$$

$$\leq \frac{1}{1+q+q^2}\int_u^v \phi_1(y)\phi_2(y)\,{}_ud_qy + \frac{m^2q(1+q^2)}{(1+q+q^2)(1+q)}\int_u^v \phi_1\left(\frac{u+v}{2}\right)\phi_2\left(\frac{u+v}{2}\right){}_ud_qy$$

$$+ \frac{mq^2}{(1+q+q^2)(1+q)}\left(\int_u^v \phi_1(y)\,{}_ud_qy\,\phi_2\left(\frac{u+v}{2}\right) + \phi_1\left(\frac{u+v}{2}\right)\int_u^v \phi_2(y)\,{}_ud_qy\right)$$

$$\leq \frac{1}{1+q+q^2}\int_u^v \phi_1(y)\phi_2(y)\,{}_ud_qy + \frac{m^2q(1+q^2)(u-v)}{(1+q+q^2)(1+q)}$$

$$\left(\frac{\phi_2(u)+\phi_2(v)}{2}\cdot\frac{\phi_1(u)+\phi_2(v)}{2}\right) + \frac{mq^2}{(1+q+q^2)(1+q)}$$

$$\left(\frac{\phi_2(u)+\phi_2(v)}{2}\cdot\frac{q\phi_1(u)+\phi_1(v)}{1+q} + \frac{\phi_1(u)+\phi_1(v)}{2}\cdot\frac{q\phi_2(u)+\phi_2(v)}{1+q}\right)$$

$$= \frac{1}{1+q+q^2}\int_u^v \phi_1(y)\phi_2(y)\,{}_ud_qy + \frac{m^2q(1+q^2)}{4(1+q+q^2)(1+q)}(M(u, v) + N(u, v))$$

$$+ \frac{mq^2(v-u)}{2(1+q+q^2)(1+q)^2}(2(q\phi_1(u)\phi_2(u) + \phi_1(v)\phi_2(v)) + (1+q)N(u, v)).$$

Multiplying both sides by $\frac{(1+q)(1+q+q^2)}{v-u}$, we get

$$\frac{(1+q)(1+q+q^2)}{(v-u)^2}\int_u^v\int_u^v\int_0^1 \phi_1(sy+m(1-s)x)\phi_2(sy+(1-s)x)_0 d_q s_u d_q x_u d_q y$$

$$\leq \frac{1+q+m^2q(1+q^2)}{v-u}$$

$$\int_u^v \phi_1(x)\phi_2(x)_u d_q x + \frac{2q^2m}{(1+q)^2}(q^2\phi_1(u)\phi_2(u)+qN(u,\, v)+f(v)g(v)).$$

This completes the proof. □

Remark 3.13. If $m = 1$, then it reduces to the result as given in Theorem 2.6.

References

[1] B. G. Pachpatte, Analytic Inequalities: Recent Advances, Atlantic Press, Amsterdam-Paris, 2012.

[2] C. E. M. Pearce and J. Pecaric, Inequalities of differentiable mappings with application to special means and quadrature formula, Appl. Math. Lett. 13 (2000), 51-55.

[3] G. Toader, Some generalizations of the convexity, Proceedings of the Colloquium on Approximation and Optimization, Univ. Cluj-Napoca, 329-338, 1985.

[4] H. Gauchman, Integral inequalities in q-calculus, Comput. Math. Appl. 47(2-3) (2004), 281-300.

[5] J. Tariboon and S. K. Ntouyas, Quantum calculus on finite intervals and application to impulsive difference equations, Adv. Differ. Eqn. 3013:283, 2013.

[6] J. Tariboon and S. K. Ntouyas, Quantum integral inequalities on finite intervals, J. Inequal. Appl. 2014:121, 2014.

[7] N. Alp, M. Z. Sarikaya, M. Kunt and I. Iscan, q-Hermite-Hadamard inequalities and quantum estimates for mid-point type inequalities via convex and quasi-convex functions, J. King Saud University-Science 30 (2018), 193-203.

[8] S. S. Dragomir and R. P. Agrawal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11 (1988), 91-95.

[9] T. Ernst, The history of q-calculus and a new method, Department of Mathematics, Uppsala University, 2000.

[10] V. Kac and P. Cheung, Quantum Calculus, Springer, New York, 2002.

[11] W. Sudsutad, S. K. Ntouyas and J. Tariboon, Quantum integral inequalities for convex functions, J. Math. Inequal. 9(3) (2015), 781-793.

Fractional
Differential
Calculus

With Compliments
of the Author

Zagreb, Croatia Volume 13, Number 2, December 2023

Pitamber Tiwari and Chet Raj Bhatta*

Hermite-Hadamard integral inequality for harmonically convex functions via Riemann-Liouville fractional integrals

FDC-13-10 163–169

FRACTIONAL DIFFERENTIAL CALCULUS

AIMS AND SCOPE

'Fractional Differential Calculus' ('FDC') aims to publish original research papers on fractional differential and integral calculus, fractional differential equations and related topics. Specifically, contributions on both the mathematical and the numerical analysis of fractional differential calculus in engineering and sciences are welcome. Only papers of the highest quality will be accepted for publication.

'FDC' is published semianually, in June and December.

SUBMISSION

Manuscripts should be submitted electronically in Adobe PDF format to the 'FDC' Editorial Office through the provided web interface.

The submitter of the paper to the website is referred to as the corresponding author and is the one expected to manage all communication and correspondence with 'FDC' regarding the paper. Authors may indicate their preference with regard to which member of the Editorial Board they consider appropriate for the article. However, assignment to that particular member is not assured.

Manuscripts are accepted for refereeing on the understanding that the same work has not been published (except in the form of an abstract), and is not under consideration for publication elsewhere.

In order to facilitate refereeing, copies of those papers (whether by the author or someone else) which are essential and referred to in the manuscript but are not conveniently accessible, should be enclosed.

The publisher strongly encourages submission of manuscripts written in TeX or one of its variants LaTeX, AMSTeX or AMSLaTeX. On acceptance of the paper, authors will be asked to send the file by electronic mail. The file must be the one from which the accompanying manuscript (finalized version) was printed out. A PDF file of this final version must also be enclosed.

PREPARATION OF MANUSCRIPT

The following data should be included on title page: the title of the article, author's name (no degrees) author's affiliation, e-mail addresses, mailing address of the corresponding author, and running head less than 60 characters.

Abstract, Keywords, Subject classification

The manuscript must be accompanied by a brief abstract, no longer than 100-150 words. It should make minimal use of mathematical symbols and displayed formulas. A list of 4-5 key words must be given. Mathematics Subject Classification (2010) with primary (and secondary) subject classification codes must be included.

Figures

Figures should be prepared in a digital form suitable for direct reproduction, grey scale only, preferrably at resolution of 300 dpi or higher, sized to fit a journal page, and in EPS, TIFF or JPEG format. Figures should have a short descriptive title and should be numbered consecutively in order of appearence in the text. The position of each figure in the text should be indicated.

References

Bibliographic references should be listed alphabetically at the end of the article. The authors should consult Mathematical Reviews for the standard abbreviations of journal names.

PROOFS

Galley proofs will be sent in PDF format to corresponding author. It should be returned in a few days.

PAGE CHARGES, OFFPRINTS

There are no page charges. Authors will receive PDF file of the printed article free of charge. Additional offprints may be ordered from FDC prior to publication.

FDC is published by Publishing House **ELEMENT**, Zagreb, Croatia.
All correspondence and subscription orders should be addressed to the Editorial Office:

www.ele-math.com

e-mail: fdc@ele-math.com

Fractional Differential Calculus
Editorial Office
Menceticeva 2, 10000 Zagreb, Croatia
Fax: +385 1 6008799

The content of this publication is indexed in Mathematical Reviews, Zentralblatt für Mathematik,
Реферативный журнал – Математика.

Fractional
Differential
Calculus
Volume 13, Number 2 (2023), 163–169
doi:10.7153/fdc-2023-13-10

HERMITE–HADAMARD INTEGRAL INEQUALITY FOR HARMONICALLY CONVEX FUNCTIONS VIA RIEMANN–LIOUVILLE FRACTIONAL INTEGRALS

PITAMBER TIWARI* AND CHET RAJ BHATTA

(*Communicated by S. S. Dragomir*)

Abstract. The concept of convexity of functions is a useful instrument that is used to solve a wide range of pure and applied scientific issues. The Hermite-Hadamard inequality which is also used frequently in many other parts of practical mathematics notably in optimization and probability is one of the most important mathematical inequalities relevant to convex maps. The fractional calculus, a calculus of non-integer order has applications in diverse fields of physical sciences. In this paper, we have established Hermite-Hadamard's inequalities via Riemann-Liouville fractional integral for the case of harmonically convex function as well as the products of two harmonically convex functions via Riemann-Liouville fractional integrals.

1. Introduction

The notion of non-integer order calculus, the generalization of traditional integer order calculus, called fractional calculus, was introduced by Leibnitz and L'Hopital in 1695 but it was popularized in the end of nineteenth century by Riemann and Liouville. The rapid growth of the fractional calculus is because of its applications in diverse fields ranging from physical sciences to engineering to biological sciences and economics. Due to the wide application of fractional integrals and importance of Hermite-Hadamard type inequalities, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes.

The concept of convexity of functions is a useful instrument that is used to solve a wide range of pure and applied scientific issues. Many researchers have recently comitted themselves to investigate the attributes and inequalities of convexity in various directions. The Hermite-Hadamard inequality which is also used frequently in many other parts of practical mathematics notably in optimization and in probability is one of the most important mathematical inequalities relevant to convex maps. This famous inequality gives error bounds for the mean value of a continuous convex mapping. The inequalities discovered independently by Ch. Hermite and J. Hadamard for convex functions are very essential in the literature of mathematical analysis. These

---

Mathematics subject classification (2020): 26A33, 26A51, 26D15.

Keywords and phrases: Convexity, harmonic-convexity, Hermite-Hadamard inequality, fractional-calculus.

[*] Corresponding author.

Paper FDC-13-10

inequalities state that if $f:\mathbb{I}\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval $\mathbb{I}$ of real numbers and $a,b\in\mathbb{I}$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2}.$$

Both the inequalities hold in reversed direction if f is concave. We note that Hermite-Hadamard inequality (hereafter it is called as H-H inequality in this paper) may be regarded as a refinement of the concept of convexity and it easily follows from Jension's inequality. H-H inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been obtained. The H-H inequality provides estimates of the mean value of a continuous convex functions. B. G. Pachpatte established new Hermite-Hadamard type inequalities for products of classical convex functions as follows:

THEOREM 1. [3] *Let f and g be real-valued, non-negative and convex functions on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx\leqslant\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)$$

and,

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)g(x)\,dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b).$$

Some new integral inequalities involving two non-negative and integrable functions that are related to Hermite-Hadamard type are also obtained by many authors. B. G. Pachpatte proposed some Hermite-Hadamard type inequalities involving two log-convex functions. Similar results for s-convex functions are established by Kirmaci et al. M. Z. Sarikaya presented some integral inequalities for two h-convex functions. It is remarkable that M. Z. Sarikaya proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals. For more detail see [1, 2, 4] and the references therein.

THEOREM 2. [1] *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is convex function on $[a,b]$ then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathbb{J}^{\alpha}_{a^+}f(b)+\mathbb{J}^{\alpha}_{b^-}f(a)\right]\leqslant\frac{f(a)+f(b)}{\alpha}.$$

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus which are used further in this paper.

DEFINITION 1. [1] Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $\mathbb{J}^{\alpha}_{a^+}f$ and $\mathbb{J}^{\alpha}_{b^-}f$ of order $\alpha>0$ with $a\geqslant 0$ are defined by

$$\mathbb{J}^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\quad x>a$$

$$\mathbb{J}^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt, \quad x<b$$

respectively.

The symbols $\mathbb{J}^{\alpha}_{a^+}f(x)$ and $\mathbb{J}^{\alpha}_{b^-}f(x)$ are left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha>0$, with $a\geqslant 0$. Here $\Gamma(\alpha)$ is gamma function defined by

$$\Gamma(\alpha)=\int_0^{\infty} e^{-t}t^{\alpha-1}\,dt.$$

DEFINITION 2. [2] A function $f:\mathbf{I}\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is said to be harmonically convex if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leqslant tf(y)+(1-t)f(x)$$

holds for all $x,y\in\mathbb{I}$ and $t\in[0,1]$.

The aim of this paper is to establish Hermite-Hadamard's inequalities via Riemann-Liouville fractional integral for the case of harmonically convex function as well as to present the results on the products of two harmonically convex functions via Riemann-Liouville fractional integrals.

2. Main results

Hermite-Hadamard's inequalities for harmonically convex functions via Riemann-Liouville fractional integral can be represented as follows:

THEOREM 3. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leqslant a\leqslant b$ and $f\in L[a,b]$. If f is harmonically convex function on $[a,b]$, then the following inequalities for fractional inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right)\leqslant\frac{(a^{\alpha}-b^{\alpha})\Gamma(\alpha)}{2(b-a)^{\alpha}}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^{\alpha}_{a^+}f(b)+\frac{1}{b^{\alpha-1}}\mathbb{J}^{\alpha}_{b^-}f(a)\right]\leqslant\frac{f(a)+f(b)}{2\alpha}.$$

Proof. Since $f:\mathbb{I}\subset\mathbb{R}\backslash\{0\}$ is a harmonically convex, for all $x,y\in\mathbb{I}$ with $t=\frac{1}{2}$, we have

$$f\left(\frac{2xy}{x+y}\right)\leqslant\frac{f(x)+f(y)}{2}.$$

Choosing $x=\frac{ab}{ta+(1-t)b}$, $y=\frac{ab}{tb+(1-t)a}$, we get

$$f\left(\frac{2\frac{ab}{ta+(1-t)b}\frac{ab}{tb+(1-t)a}}{\frac{ab}{ta+(1-t)b}+\frac{ab}{tb+(1-t)a}}\right)\leqslant\frac{f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)}{2}$$

$$2f\left(\frac{2ab}{a+b}\right)\leqslant f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right).$$

Multiplying both sides by $t^{\alpha-1}$, then integrating with respect to t over $[0,1]$, we obtain

$$2\int_0^1 t^{\alpha-1} f\left(\frac{2ab}{a+b}\right) dt \leqslant \int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right) dt + \int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt$$

$$\frac{2}{\alpha} f\left(\frac{2ab}{a+b}\right) \leqslant I_1 + I_2 \tag{1}$$

where $I_1 = \int_0^1 t^{\alpha-1}\left(\frac{ab}{ta+(1-t)b}\right) dt$. Put $u = \frac{ab}{ta+(1-t)b}$. Then $dt = \frac{ab}{b-a}\frac{du}{u^2}$ when $t \to 0$, $u \to a$; $t \to 1$, $u \to b$ and, $t = \frac{b(u-a)}{(a-b)u}$.

On substituting these values, we obtain

$$\begin{aligned}
I_1 &= \int_a^b \left(\frac{b(u-a)}{(b-a)u}\right)^{\alpha_1 - 1} f(u) \frac{ab}{b-a}\frac{du}{u^2} \\
&= \frac{ab}{b-a}\int_a^b \frac{b^{\alpha-1}(u-a)^{\alpha-1}}{(b-a)^{\alpha-1}u^{\alpha-1}} \frac{f(u)du}{u^2} \\
&= \frac{ab^{\alpha}}{(b-a)^{\alpha}}\int_a^b \frac{(u-a)^{\alpha-1}f(u)du}{u^{\alpha+1}} \\
&= \frac{b^{\alpha}-a^{\alpha}}{\alpha(b-a)^{\alpha}a^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^{\alpha}_{a^+}f(b).
\end{aligned}$$

And, $I_2 = \int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt$. Put $v = \frac{ab}{tb+(1-t)a}$, then $dt = \frac{ab}{b-a}\frac{dv}{v^2}$.

When $t \to 0$, $v \to b$; $t \to 1$, $v \to a$ and, $t = \frac{a(b-v)}{(b-a)v}$.

On substituting these values, we obtain

$$I_2 = \frac{b^{\alpha}-a^{\alpha}}{\alpha(b-a)^{\alpha}b^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^{\alpha}_{b^-}f(a).$$

Now, we substitute the values of I_1 and I_2 in (1), we obtain

$$\frac{2}{\alpha} f\left(\frac{2ab}{a+b}\right) \leqslant \frac{b^{\alpha}-a^{\alpha}}{\alpha(b-a)^{\alpha}a^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^{\alpha}_{a^+}f(b) + \frac{b^{\alpha}-a^{\alpha}}{\alpha(b-a)^{\alpha}b^{\alpha-1}}\Gamma(\alpha)\mathbb{J}^{\alpha}_{b^-}f(a)$$

$$2f\left(\frac{2ab}{a+b}\right) \leqslant \frac{(b^{\alpha}-a^{\alpha})\Gamma(\alpha)}{(b-a)^{\alpha}}\left(\frac{1}{a^{\alpha-1}}\mathbb{J}^{\alpha}_{a^+}f(b) + \frac{1}{b^{\alpha-1}}\mathbb{J}^{\alpha}_{b^-}f(a)\right). \tag{2}$$

Also, as f is harmonically convex function, then for $t \in [0,1]$, it yields

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leqslant tf(a) + (1-t)f(b)$$

and

$$f\left(\frac{ab}{tb+(1-t)a}\right) \leqslant (1-t)f(a) + tf(b).$$

Adding these two inequalities,

$$\begin{aligned} f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right) &\leqslant f(a)(t+1-t)+f(b)(1-t+t) \\ &= f(a)+f(b). \end{aligned}$$

Multiplying both sides by $t^{\alpha-1}$ and integrating with respect to t over $[0,1]$, we have

$$\begin{aligned} &\int_0^1 t^{\alpha-1} f\left(\frac{ab}{ta+(1-t)b}\right) dt + \int_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt \\ &\leqslant f(a)\int_0^1 t^{\alpha-1}\,dt + f(b)\int_0^1 t^{\alpha-1}\,dt \end{aligned}$$

$$\frac{(b^\alpha-a^\alpha)\Gamma(\alpha)}{(b-a)^\alpha}\left(\frac{1}{a^{\alpha-1}}\mathbb{J}^{\alpha}_{a^+}f(b)+\frac{1}{b^{\alpha-1}}\mathbb{J}^{\alpha}_{b^-}f(a)\right) \leqslant \frac{f(a)+f(b)}{\alpha}. \tag{3}$$

From (1), (2), and (3), we have

$$f\left(\frac{2ab}{a+b}\right) \leqslant \frac{(a^\alpha-b^\alpha)\Gamma(\alpha)}{2(b-a)^\alpha}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^{\alpha}_{a^+}f(b)+\frac{1}{b^{\alpha-1}}\mathbb{J}^{\alpha}_{b^-}f(a)\right] \leqslant \frac{f(a)+f(b)}{2\alpha}.$$

This completes the proof. □

THEOREM 4. *Let f and g be two real-valued, non-negative and harmonically convex functions on $[a,b]$. Then the following inequalities hold:*

$$\begin{aligned} &\frac{(b^\alpha-a^\alpha)\Gamma(\alpha)}{\alpha(b-a)^\alpha}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^{\alpha}_{a^+}f(b)g(b)+\frac{1}{b^{\alpha-1}}\mathbb{J}^{\alpha}_{b^-}f(a)g(a)\right] \\ &\leqslant \left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b) \end{aligned}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$; $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Let f and g be harmonically convex functions on $[a,b]$. Then, for $t \in [0,1]$, we have

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leqslant tf(b)+(1-t)f(a)$$

and

$$g\left(\frac{ab}{ta+(1-t)b}\right) \leqslant tg(b)+(1-t)g(a).$$

Then their product is given by

$$\begin{aligned} &f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{ta+(1-t)b}\right) \\ &\leqslant t^2 f(b)g(b)+(1-t)^2 f(a)g(a)+t(1-t)(f(a)b(b)+f(b)g(a)). \end{aligned}$$

Similarly,

$$f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)$$
$$\leqslant t^2f(a)g(a)+(1-t)^2f(b)g(b)+t(1-t)(f(a)b(b)+f(b)g(a)).$$

On adding the above two inequalities, we have

$$f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)$$
$$\leqslant (2t^2-2t+1)(f(a)g(a)+f(b)g(b))+2t(1-t)(f(a)g(b)+f(b)g(a)).$$

Multiplying the above inequality by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha-1}f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)dt\\&+\int_0^1 t^{\alpha-1}f\left(\frac{ab}{(1-t)a+tb}\right)g\left(\frac{ab}{(1-t)a+tb}\right)dt\\&\leqslant (f(a)g(a)+f(b)g(b))\int_0^1 t^{\alpha-1}(2t^2-2t+1)\,dt\\&+2(f(a)g(b)+f(b)g(a))\int_0^1 t^{\alpha-1}t(1-t)\,dt.\end{aligned}$$

Here,

$$\int_0^1 t^{\alpha-1}(2t^2-2t+1)\,dt=\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}$$
$$\int_0^1 t^{\alpha-1}t(1-t)\,dt=\frac{1}{(\alpha+1)(\alpha+2)}.$$

Put $\frac{ab}{ta+(1-t)b}=u$. Then $dt=\frac{ab}{b-a}\frac{du}{u^2}$. And, when $t\to 0$, $u\to a$ and $t\to 1$, $u\to b$ and, $t=\frac{b(a-u)}{u(a-b)}$. Again, put $\frac{ab}{(1-t)a+tb}=v$. Then $dt=\frac{ab}{b-a}\frac{dv}{v^2}$. Also, when $t\to 0$, $v\to b$, and when $t\to 1$, $v\to a$. On substituting these values, we obtain

$$\int_a^b\left(\frac{b(a-u)}{u(a-b)}\right)^{\alpha-1}f(u)g(u)\frac{ab}{(b-a)u^2}\,du+\int_b^a\left(\frac{a(v-a)}{v(b-a)}\right)^{\alpha-1}f(v)g(v)\frac{ab}{(b-a)v^2}\,du$$
$$\leqslant\left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b)$$

$$\frac{ab^\alpha}{(a-b)^\alpha}\int_b^a\frac{(a-u)^{\alpha-1}f(u)g(u)}{u^{\alpha+1}}\,du+\frac{a^\alpha b}{(b-a)^\alpha}\int_b^a\frac{(v-a)^{\alpha-1}f(v)g(v)}{u^{\alpha+1}}\,dv$$
$$\leqslant\left(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\right)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b)$$

$$\frac{(b^\alpha - a^\alpha)\Gamma(\alpha)}{\alpha(b-a)^\alpha}\left[\frac{1}{a^{\alpha-1}}\mathbb{J}^\alpha_{a^+} f(b)g(b) + \frac{1}{b^{\alpha-1}}\mathbb{J}^\alpha_{b^-} f(a)g(a)\right]$$
$$\leqslant \left(\frac{2}{\alpha+2} - \frac{2}{\alpha+1} + \frac{1}{\alpha}\right) M(a,b) + \frac{2}{(\alpha+1)(\alpha+2)} N(a,b).$$

This completes the proof. □

3. Conclusion

In this paper, we have established some new Hermite-Hadamard type inequalities for harmonically convex function and the products of two harmonically convex functions via Riemann-Liouville fractional integrals. An interesting concern is that whether we can further use it to establish Hermite-Hadamard inequality for other kinds of convex functions?

REFERENCES

[1] F. CHEN, *A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals*, Ital. J. Pure Appl. Math., **33**, (2014), 299–306.

[2] I. ISCAN, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacetteppe, J. Math. Stats., **43** (6), (2014), 935–942.

[3] B. G. PACHPATTE, *A note on integral inequalities involving two log convex functions*, Math. Inequal. Appl., **7** (4) (2004), 511–515.

[4] M. Z. SARIKAYA, E. SET, H. YALDIZ, AND N. BASAK, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model., **57** (2013), 2403–2407.

[5] E. SET, M. F. OZDEMIR, AND S. S. DRAGOMIR, *On the Hermite-Hadamard inequality and other integral inequalities involving two functions*, J. Inequal. Appl., (2010), 2010:148102.

(Received June 27, 2023)

Pitamber Tiwari
Central Department of Mathematics
Institute of Science and Technology, Tribhuvan University
Kathmandu, Nepal
e-mail: `pitambertiwari40@gmail.com`

Chet Raj Bhatta
Central Department of Mathematics
Tribhuvan University
Kathmandu, Nepal
e-mail: `chetbhatta0@gmail.com`

Quantum estimates of Hermite-Hadamard inequalities for the products on extended geometric convex functions

PITAMBER TIWARI[1*] and CHET RAJ BHATTA[2]

[1] *Research Scholar, Central Department of Mathematics, IoST, Tribhuvan University, Nepal*
[2] *Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal*
[*] *Corresponding author: pitambertiwari40@gmail.com*

Abstract. Convexity is one of the characteristics of functions of real variables which plays an efficient role in various branches of mathematics that possesses the two significant properties viz. the maximum value is attained at a boundary point and any local minimum is a global one. The integral mean of a convex function is connected to the Hermite-Hadamard inequality. B.G. Pachpatte established the results on the products of two classical convex functions which helps to estimate the integral mean of the product of two classical convex functions. In this paper, we have extended the idea of the result of the product of classical convex functions to the products of generalized geometric convex functions, and these results are further extended to quantum calculus which is paradigm shift in convexity theory and integral inequality. The obtained results will definitely be used to estimate the integral mean of generalized geometric convex functions. The ideas expressed in this paper might revolutionize the readers of this field to other types of convex functions.

Keywords: *Convexity, geometric-convexity, Hermite-Hadamard inequality, q-calculus.*
2020 Mathematics Subject Classification. *05A30; 26A51; 26D15.*

1. Introduction

A shape of an object if it is curved outward is convex. The special properties of functions of real variables include continuity, convexity, monotocity and differentiability. Of them, convexity plays a significant role in the development of several branches of mathematics since it includes the theory of convex functions that possess the following two important properties viz. the maximum value is attained at a boundary point and any local minimum is a global one. Although, convexity is a basic notion in geometry but widely used in other areas of Mathematics viz. functional analysis, complex analysis, calculus of variations, graph theory, partial differential equation, discrete mathematics, algebraic geometry, probability theory, coding theory, crystallography, and many other fields. It plays an important role outside of mathematics such as in physics, chemistry, biology, economics, finance, and more. Convexity is a fundamental concept in mathematics which has a combinatorial, an analytic, a geometric and a probabilistic flavor.

Inequalities play important roles in understanding many mathematical concepts, such as probability theory, numerical integration and integral operator theory. Through the last century, H-H (Hermite-Hadamard) type inequality has been considered to be among the fastest growing fields in mathematical analysis, through which vast problems in engineering, physics and economics have been studied. Due to the enormous importance of these inequalities, many extensions, refinements, and generalization of their related types have been equally investigated.

The Hermite-Hadamard inequality plays a great role in the theory of convex functions. It provides a necessary and sufficient condition for a function to be convex in an open interval of real numbers. It also interpolates Jensen's inequality which is also an important inequality in the study of convex functions. In the monograph, Dragomir and Pierce[4] stated that the Hermite-Hadamard inequality is the first fundamental result for convex functions with a natural geometrical interpretations and many applications have attracted and continues to attract much interest in elementary mathematics. The H-H inequality has made great contributions in the fields of integral inequalities, approximation theory, special means theory, optimization theory, information theory, and numerical analysis. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be arithmetically-arithmetic (AA) convex or simply a convex function in a classical sense on I, if

$$f(s\phi_1 + (1-s)\phi_2) \leq sf(\phi_1) + (1-s)f(\phi_2) \tag{1.1}$$

holds for all $\phi_1, \phi_2 \in I$ and, $s \in [0,1]$. It is said to be concave if the inequality is reversed.

The generalization of the classical convex function is (α, m)-convex function which was introduced by Mihesan [7] in the following way: The function $f : [0,b] \to \mathbb{R}$ is said to be (α, m)-convex function, where $\alpha, m \in [0,1]^2$, if for every $\phi_1, \phi_2 \in [0,b]$ and $s \in [0,1]$, we have

$$f(s\phi_1 + m(1-s)\phi_2) \ \leq \ s^{\alpha} f(\phi_1) + m(1-s^{\alpha})f(\phi_2). \tag{1.2}$$

The literature about the products on different types of convex functions are available in the following papers. A number of Hermite-Hadamard type inequalities for the products of two convex and s-convex functions were proven by Kirmaci et al.[6]. Hadamard type inequalities for product of s-convex functions on co-ordinates were established by Ozdemir et al.[10]. For products of two h-convex functions, Sarikaya et al. [13] demonstrated various Hermite-Hadamard type inequalities. Hermite-Hadamard type inequalities for products of two m-convex and (α, m)-convex functions were established by Bakula et al.[2]. For products of two convex functions and harmonically s-convex functions, Chen and Wu [3] found several Hermite-Hadamard type inequalities. For product of relative convex functions, relative h-convex functions, and harmonically convex functions, Noor et al.[9] established several Hardmard's type inequalities. For product of two convex functions, Yin and Qi [15] discovered certain Hermite- Hadamard type inequalities. For products of two convex functions, Chen[3] discovered several novel Hermite-Hadamard type inequalities and enlarged the scope of the issue to include m and (α, m)- convex functions.

Quantum calculus is the branch of Mathematics where there are no limits. By using a difference operator in place of the conventional derivative, it is possible to work with sets of functions that are non-derivable. Our next considerations, in this paper, is the quantum calculus or more commonly known as q-calculus. Although, the notion of q-calculus is very old, developed by Euler in $1740s$ and it is popularized during the last two decades by reinvesting this topic and applied it in almost all the branches of mathematical as well as in other sciences.

In this paper, by reviewing the existing literature, we find that the extension of Hermite-Hadamard integral inequality for generalized geometric convex function is the domain that we can work with. So, we have extended the idea of geometric convex functions to the generalized geometric-convex function and obtained some results on the products of generalized geometric convex functions such as (α, m)-GA(geometrically-arithmetic), (α, m) GG(geometrically-geometric) and GH(geometrically-harmonic) convex functions and we recapture these new results in terms of q-calculus which is a new paradigm for future research.

The paper has incorporated four main sections. The first section commences with introduction of the concepts of convexity of function with its importance along with one of the significant inequalities, the Hermite-Hadamard integral inequality and the classical definition of convex functions as well as the idea of quantum calculus. The preliminary section, the second section, provides the literature available regarding the products of classical convex functions as well as literature available for the product of different convex functions. The preliminary results which help to advance the Hermite-Hadamard integral inequality are also incorporated in the second section of this paper. The third chapter entails the main results obtained as the products of generalized geometric- convex functions and their results in q-analogs. The final section of this paper includes the conclusion part of the research work .

2. Preliminary Results

This section states the earlier defined definitions and obtained results which will be used to enhance our further endeavors. The concept of geometrically convex functions was introduced by Zhang et. al in [16] as follows.

Definition 2.1. [16] An operation $f : (0, \infty) \to \mathbb{R}$ is referred to be a geometrically-arithmetic (GA) convex function or simply a geometrically convex function on $(0, \infty)$, if

$$f(\phi_1^s \phi_2^{1-s}) \leq s f(\phi_1) + (1-s) f(\phi_2) \tag{2.1}$$

holds for all $\phi_1, \phi_2 \in (0, \infty), s \in [0, 1]$.

If the inequality is reversed, then it is a GA concave function. Niculescu in [8] introduced the notion of geometrically-geometric convex function as defined below:

Definition 2.2. [8] A function $f:(0,\infty)\rightarrow\mathbb{R}$ is said to be geometrically-geometric (GG) convex function on $(0,\infty)$, if

$$f(\phi_1^s\phi_2^{(1-s)})\leq(f(\phi_1))^s(f(\phi_2))^{1-s} \tag{2.2}$$

holds for all $\phi_1,\phi_2\in(0,\infty), s\in[0,1]$.

It is known as a GG-concave function if the inequality is reversed.

Xi et al. [14] generalized the geometrically-geometric convex function as follows:

Definition 2.3. [14] A function $f:(0,\infty)\rightarrow\mathbb{R}$ is said to be m-geometrically-geometric (m-GG) convex function on $(0,\infty)$, if

$$f(\phi_1^s\phi_2^{m(1-s)})\leq(f(\phi_1))^s(f(\phi_2))^{m(1-s)} \tag{2.3}$$

holds for all $\phi_1,\phi_2\in(0,\infty), s\in[0,1], m\in(0,1]$.

It is said to be an m-GG-concave function if the inequality is reversed.

Definition 2.4. [14] A function $f:(0,\infty)\rightarrow\mathbb{R}$ is said to be (α,m)-geometrically-geometric $((\alpha,m)-GG)$ convex function on $(0,\infty)$, if

$$f(\phi_1^s\phi_2^{m(1-s)})\leq(f(\phi_1))^{s^\alpha}(f(\phi_2))^{m(1-s^\alpha)} \tag{2.4}$$

holds for all $\phi_1,\phi_2\in(0,\infty), s\in[0,1], (\alpha,m)\in(0,1]^2$.

It is said to be an (α,m)-GG-concave function if the inequality is reversed.

Definition 2.5. [14] A function $f:(0,\infty)\rightarrow\mathbb{R}$ is said to be geometrically-harmonic (GH) convex function on $(0,\infty)$, if

$$f(\phi_1^s\phi_2^{(1-s)})\leq\frac{1}{\frac{s}{f(\phi_1)}+\frac{1-s}{f(\phi_2)}}=\frac{f(\phi_1)f(\phi_2)}{sf(\phi_2)+(1-s)f(\phi_1)} \tag{2.5}$$

holds for all $\phi_1,\phi_2\in(0,\infty), s\in[0,1]$.

It is said to be a GH-concave function if the inequality is reversed.

Definition 2.6. Let ϕ_1,ϕ_2 be two positive real numbers. Then, their logarithmic mean is denoted by $L(a,b)$ is defined as

$$L(\phi_1,\phi_2)=\frac{\phi_2-\phi_1}{\log\phi_2-\log\phi_1},\quad \phi_1\neq\phi_2. \tag{2.6}$$

Definition 2.7. Let ϕ_1,ϕ_2 be two positive real numbers. Then, their geometric mean is denoted by $G(\phi_1,\phi_2)$ is defined as

$$G(\phi_1,\phi_2)=\sqrt{\phi_1\phi_2}. \tag{2.7}$$

Definition 2.8. Let ϕ_1, ϕ_2 be two positive real numbers. Then, their arithmetic mean is denoted by $A(\phi_1, \phi_2)$ is defined as

$$A(\phi_1, \phi_2) = \frac{\phi_1 + \phi_2}{2}. \tag{2.8}$$

For the class of convex functions, numerous significant inequalities are discovered. The Hermite-Hadamard inequality is one of the most popular inequalities. The following double inequality in literature is a well-known Hermite-Hadamard integral inequality.

Theorem 2.9. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers $\phi_1, \phi_2 \in I$ with $\phi_1 < \phi_2$. Then, we have*

$$f(\frac{\phi_1 + \phi_2}{2}) \leq \frac{1}{\phi_2 - \phi_1} \int_{\phi_1}^{\phi_2} f(x)\, dx \leq \frac{f(\phi_1) + f(\phi_2)}{2} \tag{2.9}$$

If the function f is concave, both inequalities hold in the opposite direction.

The following Hermite-Hadamard type inequality for the product of two classical convex functions was first proved by B.G. Pachpatte [11].

Theorem 2.10. *[11] Let f and g be real valued, non-negative and convex functions on $[\phi_1, \phi_2]$. The following inequalities are then true:*

$$\frac{1}{\phi_2 - \phi_1} \int_{\phi_1}^{\phi_2} f(x)g(x)\, dx \leq \frac{1}{3} M(\phi_1, \phi_2) + \frac{1}{6} N(\phi_1, \phi_2) \tag{2.10}$$

where $M(\phi_1, \phi_2) = f(\phi_1)g(\phi_1) + f(\phi_2)g(\phi_2)$ and, $N(\phi_1, \phi_2) = f(\phi_1)g(\phi_2) + f(\phi_2)g(\phi_1)$.

We also give some preliminaries of q-calculus. We mention to the books [1, 5] for various notations and theory of q-calculus.

The q-derivative of a continuous function $f : I \to \mathbb{R}$ at any point $x \in I$ is defined by

$${}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a \tag{2.11}$$

and

$${}_aD_q f(a) = \lim_{x \to a} aD_q f(x)$$

The q-integral of a continuous function $f : I \to \mathbb{R}$ is defined by

$$\int_a^b f(s)\, {}_ad_q s = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f\left(q^n x + (1-q^n)a\right) \tag{2.12}$$

and for $c \in (a, x)$

$$\int_c^x f(s)\, {}_ad_q s = \int_a^x f(s)\, {}_ad_q s - \int_a^c f(s)\, {}_ad_q s.$$

The q-analogue of any real number $x \in \mathbb{R}$ is defined by

$$[x]_q := \frac{q^x - 1}{q - 1}. \tag{2.13}$$

For any integer $n \geq 1$, the q-analogue of $(x-c)^n$ is the polynomial

$$(x-c)_q^n := (x-c)(x-qc)...(x-q^{n-1}c). \tag{2.14}$$

To deal with the negative integer exponents, it is known that

$$(x-c)_q^{-n} = \frac{1}{(x-q^{-n}c)_q^n}. \tag{2.15}$$

The following derivatives are known for any integer $n \in \mathbb{Z}$

$${}_aD_q(x-c)_q^n = [n]_q(x-c)_q^{n-1}.$$

$${}_aD_q\frac{1}{(x-c)_q^n} = [-n]_q(x-q^nc)_q^{-n-1}.$$

3. Main Results

The extended definitions and the results on the products of generalized geometrically convex functions and their quantum estimates are obtained in this section, and they are as follows:

Definition 3.1. A function $f : (0,\infty) \to \mathbb{R}$ is said to be m-geometrically-arithmetic $(m - GA)$ convex function on $(0,\infty)$, if

$$f(\phi_1^s\phi_2^{m(1-s)}) \leq sf(\phi_1) + m(1-s)f(\phi_2) \tag{3.1}$$

holds for all $\phi_1, \phi_2 \in (0,\infty), s \in [0,1], m \in (0,1]$.

It is said to be an m-GA concave function if the inequality is reversed. It can further be generalized as (α, m)-geometrically convex function as follows:

Definition 3.2. A function $f : (0,\infty) \to \mathbb{R}$ is said to be (α, m) geometrically-arithmetic $((\alpha, m) - GA)$ convex function on $(0,\infty)$, if

$$f(\phi_1^s\phi_2^{m(1-s)}) \leq s^\alpha f(\phi_1) + m(1-s^\alpha)f(\phi_2) \tag{3.2}$$

holds for all $\phi_1, \phi_2 \in (0,\infty), s \in [0,1], (\alpha, m) \in (0,1]^2$.

It said to be an (α, m)- GA concave function if the inequality is reversed. We further generalize geometrically-harmonic convex functions as m and (α, m)-GH convex functions as follows:

Definition 3.3. A function $f:(0,\infty)\to\mathbb{R}$ is said to be m-geometrically-harmonic (m-GH) convex function on $(0,\infty)$, if

$$f(\phi_1^s\phi_2^{m(1-s)}) \leq \frac{1}{\frac{s}{mf(\phi_1)}+\frac{1-s}{f(\phi_2)}} = \frac{mf(\phi_1)f(\phi_2)}{sf(\phi_2)+m(1-s)f(\phi_1)} \tag{3.3}$$

holds for all $\phi_1,\phi_2\in(0,\infty), s\in[0,1], m\in(0,1]$.

It is said to be a m-GH-concave function if the inequality is reversed.

Definition 3.4. A function $f:(0,\infty)\to\mathbb{R}$ is said to be (α,m)-geometrically-harmonic (α,m)-GH convex function on $(0,\infty)$, if

$$f(\phi_1^s\phi_2^{m(1-s)}) \leq \frac{1}{\frac{s^\alpha}{mf(\phi_1)}+\frac{1-s^\alpha}{f(\phi_2)}} = \frac{mf(\phi_1)f(\phi_2)}{s^\alpha f(\phi_2)+m(1-s^\alpha)f(\phi_1)} \tag{3.4}$$

holds for all $\phi_1,\phi_2\in(0,\infty), s\in[0,1], (\alpha,m)\in(0,1]$.

It is said to be a (α,m)-GH-concave function if the inequality is reversed.

Theorem 3.5. *The integral mean of geometrically-arithmetic convex function is bounded above by the arithmetic mean of their functional values and the product of reciprocal of the logarithmic mean .*

Proof. Let f be a GA-convex function defined on $(0,\infty)$. Then by definition of GA-convexity, we have

$$f(\phi_1^s\phi_2^{1-s}) \leq sf(\phi_1)+(1-s)f(\phi_2).$$

Integrating with respect to s over $[0,1]$, we have

$$\int_0^1 f(\phi_1^s\phi_2^{1-s})\,ds \leq f(\phi_1)\int_0^1 s\,ds + f(\phi_2)\int_0^1(1-s)\,ds = \frac{f(\phi_1)+f(\phi_2)}{2}.$$

Put $x=\phi_1^s\phi_2^{1-s}, ds=\frac{dx}{x\log\frac{\phi_1}{\phi_2}}$ When $s\to 0$, then $x\to\phi_2$; when $s\to 1$, then $x\to\phi_1$. On substituting these values, we have

$$\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\frac{f(x)}{x}\,dx \leq \frac{f(\phi_1)+f(\phi_2)}{2L(\phi_1,\phi_2)}.$$

This completes the proof. □

Theorem 3.6. *Let f and g be two (α,m)-GA convex functions. Then, their product is given as follows:*

$$\frac{1}{\phi_2^m-\phi_1}\int_{\phi_1}^{\phi_2^m}\frac{f(x)g(x)}{x}dx \leq \frac{1}{L(\phi_1,\phi_2^m)}\left(\frac{1}{(2\alpha+1)}K_{\phi_1}+\frac{2\alpha^2m^2}{(\alpha+1)(2\alpha+1)}K_{\phi_2}+\frac{\alpha}{(\alpha+1)(2\alpha+1)}K_{\phi_2}^{\phi_1}\right) \tag{3.5}$$

where, $K_{\phi_1}=f(\phi_1)g(\phi_1), K_{\phi_2}=f(\phi_2)g(\phi_2), K_{\phi_2}^{\phi_1}=f(\phi_1)g(\phi_2)+f(\phi_2)g(\phi_1)$.

Proof. Let f and g be two (α, m) GA convex functions. Then, by definition, we have

$$f(\phi_1^s \phi_2^{m(1-s)}) \leq s^\alpha f(\phi_1) + (1 - s^\alpha) f(\phi_2)$$

And,

$$g(\phi_1^s \phi_2^{m(1-s)}) \leq s^\alpha g(\phi_1) + (1 - s^\alpha) g(\phi_2)$$

Since $f, g \geq 0$, so, we have

$$\begin{aligned} f(\phi_1^s \phi_2^{m(1-s)}) g(\phi_1^s \phi_2^{m(1-s)}) \leq f(\phi_1) g(\phi_1) s^{2\alpha} &+ f(\phi_2) g(\phi_2) m^2 (1 - s^\alpha)^2 \\ &+ m(f(\phi_1) g(\phi_2) + f(\phi_2) g(\phi_1)) s^\alpha (1 - s^\alpha). \end{aligned}$$

Integrating with respect to 's' over $[0, 1]$, we have

$$\begin{aligned} \int_0^1 f(\phi_1^s \phi_2^{m(1-s)}) g(\phi_1^s \phi_2^{m(1-s)})\, ds \leq f(\phi_1) g(\phi_1) \int_0^1 s^{2\alpha}\, ds &+ f(\phi_2) g(\phi_2) m^2 \int_0^1 (1 - s^\alpha)^2\, ds \\ &+ m(f(\phi_1) g(\phi_2) + f(\phi_2) g(\phi_1)) \int_0^1 s^\alpha (1 - s^\alpha)\, ds. \end{aligned}$$

Here,

$$\int_0^1 s^{2\alpha}\, ds = \frac{1}{2\alpha + 1}$$

$$\int_0^1 (1 - s^\alpha)^2\, ds = \frac{2\alpha^2}{(\alpha + 1)(2\alpha + 1)}$$

$$\int_0^1 s^\alpha (1 - s^\alpha)\, ds = \frac{\alpha}{(\alpha + 1)(2\alpha + 1)}$$

Put $x = \phi_1^s \phi_2^{m(1-s)}$, when $s \to 0, x \to \phi_2^m$; when $s \to 1, x \to \phi_1$. And, $ds = \frac{1}{x(\log a - \log b^m)} dx$
On substituting these values in above inequality, we have

$$\frac{1}{\log \phi_2^m - \log \phi_1} \int_{\phi_1}^{\phi_2^m} \frac{f(x)g(x)}{x}\, dx \quad \leq \quad \frac{K_{\phi_1}}{2\alpha + 1} + \frac{2\alpha^2 m^2 K_{\phi_2}}{(\alpha + 1)(2\alpha + 1)} + \frac{m\alpha K_{\phi_2}^{\phi_1}}{(\alpha + 1)(2\alpha + 1)}.$$

$$\frac{\phi_2^m - \phi_1}{\log \phi_2^m - \log \phi_1} \frac{1}{\phi_2^m - \phi_1} \int_{\phi_1}^{\phi_2^m} \frac{f(x)g(x)}{x}\, dx \leq \frac{K_{\phi_1}}{2\alpha + 1} + \frac{2\alpha^2 m^2 K_{\phi_2}}{(\alpha + 1)(2\alpha + 1)} + \frac{m\alpha K_{\phi_2}^{\phi_1}}{(\alpha + 1)(2\alpha + 1)}$$

$$L(\phi_1, \phi_2^m) \frac{1}{\phi_2^m - \phi_1} \int_{\phi_1}^{\phi_2^m} \frac{f(x)g(x)}{x}\, dx \leq \frac{K_{\phi_1}}{2\alpha + 1} + \frac{2\alpha^2 m^2 K_{\phi_2}}{(\alpha + 1)(2\alpha + 1)} + \frac{m\alpha K_{\phi_2}^{\phi_1}}{(\alpha + 1)(2\alpha + 1)}$$

$$\frac{1}{\phi_2^m - \phi_1} \int_{\phi_1}^{\phi_2^m} \frac{f(x)g(x)}{x}\, dx \leq \frac{1}{L(\phi_1, \phi_2^m)} \left[\frac{K_{\phi_1}}{2\alpha + 1} + \frac{2\alpha^2 m^2 K_{\phi_2}}{(\alpha + 1)(2\alpha + 1)} + \frac{m\alpha K_{\phi_2}^{\phi_1}}{(\alpha + 1)(2\alpha + 1)} \right].$$

This completes the proof. □

Remark 3.7. If $\alpha = m = 1$, then it reduces to the product of two GA-convex functions.

Theorem 3.8. *The integral mean of geometrically-geometric convex function is bounded above by a half of the logarithmic value of the product of their functional values and the reciprocal of the logarithmic mean.*

Proof. Let $f : (0, \infty) \to (0, \infty)$ be a geometrically- geometric convex function. Then by definition, we have

$$f(\phi_1^s \phi_2^{1-s}) \leq (f(\phi_1))^s (f(\phi_2))^{1-s}$$

$$\log f(\phi_1^s \phi_2^{1-s}) \leq s \log(f(\phi_1)) + (1-s)\log(f(\phi_2))$$

Integrating with respect to s over $[0,1]$ and using the properties of logarithms, we get

$$\int_0^1 \log f(\phi_1^s \phi_2^{1-s})\, ds \leq \log(f(\phi_1)) \int_0^1 s\, ds + \log(f(\phi_2)) \int_0^1 (1-s)\, ds = \frac{\log(f(\phi_1)f(\phi_2))}{2}.$$

Using the change of variables by putting $x = \phi_1^s \phi_2^{1-s}$, we have

$$ds = \frac{L(\phi_1, \phi_2)}{\phi_1 - \phi_2} \frac{dx}{x}.$$

When $s \to 0,\ x \to \phi_2$; when $s \to 1$, then $x \to \phi_1$. On substituting these values, we obtain

$$\frac{1}{\phi_2 - \phi_1} \int_{\phi_1}^{\phi_2} \frac{\log f(x)}{x}\, dx \leq \frac{1}{2L(\phi_1, \phi_2)} \log(f(\phi_1)f(\phi_2)).$$

This completes the proof.

□

Theorem 3.9. *Let $f, g : (0, \infty) \to (0, \infty)$ be two (α, m) GG-convex functions. Then the following inequalities persist.*

$$\frac{1}{\phi_2^m - \phi_1} \int_{\phi_1}^{\phi_2^m} \frac{\log f(x) + \log g(x)}{x}\, dx \leq \frac{1}{L(\phi_1, \phi_2^m)(\alpha + 1)} \left(\log(f(\phi_1)g(\phi_1)) + m\alpha \log(f(\phi_2)g(\phi_2))\right) \tag{3.6}$$

Proof. As f and g are two (α, m) GG convex functions, then by definition, we have

$$f(\phi_1^s \phi_2^{m(1-s)}) \leq (f(\phi_1))^{s^\alpha} (f(\phi_2))^{m(1-s^\alpha)}$$

and,

$$g(\phi_1^s \phi_2^{m(1-s)}) \leq (g(\phi_1))^{s^\alpha} (g(\phi_2))^{m(1-s^\alpha)}$$

Since $f, g \geq 0$, So, we have

$$f(\phi_1^s\phi_2^{m(1-s)})g(\phi_1^s\phi_2^{m(1-s)}) \leq \, (f(\phi_1)g(\phi_1))^{s^\alpha}(f(\phi_2)g(\phi_2))^{m(1-s^\alpha)}$$

Taking $\log$ on both sides, we have

$$\log f(\phi_1^s\phi_2^{m(1-s)}) + \log g(\phi_1^s\phi_2^{m(1-s)}) \leq \, s^\alpha \log(f(\phi_1)g(\phi_1)) + m(1-s^\alpha)\log(f(\phi_2)g(\phi_2))$$

Integrating with respect to s over $[0,1]$, we have

$$\begin{aligned} &\int_0^1 \log f(\phi_1^s\phi_2^{m(1-s)})\, ds + \int_0^1 \log g(\phi_1^s b\phi_2^{m(1-s)})\, ds \\ \leq \, &\log(f(\phi_1)g(\phi_1))\int_0^1 s^\alpha\, ds + m\log(f(\phi_2)g(\phi_2))\int_0^1 (1-s^\alpha)\, ds \end{aligned}$$

Using the change of variables by putting $x = \phi_1^s\phi_2^{m(1-s)}$, we have

$$\frac{L(\phi_1,\phi_2^m)}{\phi_2^m-\phi_1}\int_{\phi_1}^{\phi_2^m}\frac{\log f(x)+\log g(x)}{x}\, dx \, \leq \, \frac{\log(f(\phi_1)g(\phi_1))}{\alpha+1} + \frac{m\alpha\log(f(\phi_2)g(\phi_2))}{\alpha+1}$$

Hence, we have

$$\frac{1}{\phi_2^m-\phi_1}\int_{\phi_1}^{\phi_2^m}\frac{\log(f(x)g(x))}{x}\, dx \leq \, \frac{1}{L(\phi_1,\phi_2^m)(\alpha+1)}\left(\log(f(\phi_1)g(\phi_1)) + m\alpha\log(f(\phi_2)g(\phi_2))\right)$$

This completes the proof. ☐

Corollary 3.10. *If* $\alpha = 1$ *, then we get the result of the product of two* m *-GG convex functions.*

Corollary 3.11. *If* $\alpha = m = 1$ *, then we get the result of the products of two GG convex functions.*

Theorem 3.12. *The integral mean of geometrically-harmonic (GH) convex function is bounded above by the ratio of squares of geometric mean of functional values to the product of logarithmic mean of their functional values and inputs.*

Proof. Let f be a GH convex function defined from $(0,\infty) \to \mathbb{R}$. Then, by definition, we have

$$f(\phi_1^s\phi_2^{1-s}) \leq \, \frac{1}{\frac{s}{f(\phi_1)} + \frac{1-s}{f(\phi_2)}} = \frac{f(\phi_1)f(\phi_2)}{sf(\phi_2)+(1-s)f(\phi_1)}$$

Integrating with respect to s over $[0,1]$, we have

$$\int_0^1 f(\phi_1^s\phi_2^{1-s})\, ds \leq \, \int_0^1 \frac{f(\phi_1)f(\phi_2)}{sf(\phi_2)+(1-s)f(\phi_1)}\, ds = \, f(\phi_1)f(\phi_2)\int_0^1 \frac{1}{f(\phi_1)+(f(\phi_2)-f(\phi_1))s}\, ds.$$

$$\int_0^1 f(\phi_1^s\phi_2^{1-s})\,ds \le\ f(\phi_1)f(\phi_2)\frac{\log f(\phi_2)-\log f(\phi_1)}{f(\phi_2)-f(\phi_1)} = \frac{G^2(f(\phi_1),f(\phi_2))}{L(f(\phi_1),f(\phi_2)}.$$

Using the change of variables by putting $x=\phi_1^s\phi_2^{1-s}$, we have

$$\frac{1}{\log\phi_2-\log\phi_1}\int_{\phi_1}^{\phi_2}\frac{\log f(x)}{x}\,dx \le \frac{G^2(f(\phi_1),f(\phi_2))}{L(f(\phi_1),f(\phi_2)}.$$

$$\frac{L(\phi_1,\phi_2)}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\frac{\log f(x)}{x}\,dx \le \frac{G^2(f(\phi_1),f(\phi_2))}{L(f(\phi_1),f(\phi_2)}.$$

$$\frac{1}{\phi_1-\phi_1}\int_{\phi_1}^{\phi_2}\frac{\log f(x)}{x}\,dx \le \frac{G^2(f(\phi_1),f(\phi_2))}{L(\phi_1,\phi_2)L(f(\phi_1),f(\phi_2)}.$$

This completes the proof. □

Theorem 3.13. *Let $f,g:(0,\infty)\to\mathbb{R}$ be two geometrically-harmonic convex functions. Then the following inequality holds.*

$$\begin{aligned}\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\frac{f(x)g(x)}{x}\,dx \le \frac{1}{L(\phi_1,\phi_2)}&\frac{2K_{\phi_1}K_{\phi_2}}{\sqrt{4K_{\phi_1}(K_{\phi_1}+K_{\phi_2}-K_{\phi_1\phi_2}-K_{\phi_2\phi_1})}-(K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1})^2}\\ &(\tan^{-1}\frac{2(K_{\phi_1}+K_{\phi_2}-K_{\phi_1\phi_2}-K_{\phi_2\phi_1})+(K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1})}{\sqrt{4K_{\phi_1}(K_{\phi_1}+K_{\phi_2}-K_{\phi_1\phi_2}-K_{\phi_2\phi_1})}-(K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1})^2}\\ &-\tan^{-1}\frac{K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1}}{\sqrt{4K_{\phi_1}(K_{\phi_1}+K_{\phi_2}-K_{\phi_1\phi_2}-K_{\phi_2\phi_1})}-(K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1})^2})\end{aligned}\tag{3.7}$$

where, $K_{\phi_1}=f(\phi_1)g(\phi_1), K_{\phi_2}=f(\phi_2)g(\phi_2), K_{\phi_1\phi_2}=f(\phi_1)g(\phi_2), K_{\phi_2\phi_1}=f(\phi_2)g(\phi_1)$.

Proof. Let f,g be two GH convex functions. Then, by definition, we have

$$f(\phi_1^s\phi_2^{1-s})\ \le\ \frac{f(\phi_1)f(\phi_2)}{sf(\phi_2)+(1-s)f(\phi_1)} = \frac{f(\phi_1)f(\phi_2)}{f(\phi_1)+(f(\phi_2)-f(\phi_1))s}.$$

And,

$$g(\phi_1^s\phi_2^{1-s})\ \le\ \frac{g(\phi_1)g(\phi_2)}{sg(\phi_2)+(1-s)g(\phi_1)} = \frac{g(\phi_1)g(\phi_2)}{g(\phi_1)+(g(\phi_2)-g(\phi_1))s}.$$

Since, $f,g\ge 0$, so, we have

$$f(\phi_1^s\phi_2^{1-s})g(\phi_1^s\phi_2^{1-s})\ \le\ \left(\frac{f(\phi_1)f(\phi_2)}{f(\phi_1)+(f(\phi_2)-f(\phi_1))s}\right)\left(\frac{g(\phi_1)g(\phi_2)}{g(\phi_1)+(g(\phi_2)-g(\phi_1))s}\right).$$

$$f(\phi_1^s\phi_2^{1-s})g(\phi_1^s\phi_2^{1-s}) \leq \frac{K_{\phi_1}K_{\phi_2}}{K_{\phi_1}+(K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1})s+(K_{\phi_1}+K_{\phi_2}-K_{\phi_1\phi_2}-K_{\phi_2\phi_1})s^2}.$$

Integrating with respect to s over $[0,1]$, we have

$$\int_0^1 f(\phi_1^s\phi_2^{1-s})g(\phi_1^s\phi_2^{1-s})\,ds$$
$$\leq K_{\phi_1}K_{\phi_2}\int_0^1 \frac{1}{K_{\phi_1}+(K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1})s+(K_{\phi_1}+K_{\phi_2}-K_{\phi_1\phi_2}-K_{\phi_2\phi_1})s^2}.$$

Using the change of variables by putting

$$x=\phi_1^s\phi_2^{1-s}$$

and using the formula

$$\int_0^1 \frac{1}{ax^2+bx+c}\,dx=\frac{2}{\sqrt{4ac-b^2}}\left(\tan^{-1}\frac{2ab+b}{\sqrt{4ac-b^2}}-\tan^{-1}\frac{b}{\sqrt{4ac-b^2}}\right)$$

, we have

$$\begin{aligned}&\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\frac{f(x)g(x)}{x}\,dx\\ &\quad\leq \frac{1}{L(\phi_1,\phi_2)}\frac{2K_{\phi_1}K_{\phi_2}}{\sqrt{4K_{\phi_1}(K_{\phi_1}+K_{\phi_2}-K_{\phi_1\phi_2}-K_{\phi_2\phi_1})}-(K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1})^2}\\ &\qquad(\tan^{-1}\frac{2(K_{\phi_1}+K_{\phi_2}-K_{\phi_1\phi_2}-K_{\phi_2\phi_1})+(K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1})}{\sqrt{4K_{\phi_1}(K_{\phi_1}+K_{\phi_2}-K_{\phi_1\phi_2}-K_{\phi_2\phi_1})}-(K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1})^2}\\ &\qquad\quad-\tan^{-1}\frac{K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1}}{\sqrt{4K_{\phi_1}(K_{\phi_1}+K_{\phi_2}-K_{\phi_1\phi_2}-K_{\phi_2\phi_1})}-(K_{\phi_1\phi_2}+K_{\phi_2\phi_1}-2K_{\phi_1})^2})\end{aligned}$$

where, $K_{\phi_1}=f(\phi_1)g(\phi_1), K_{\phi_2}=f(\phi_2)g(\phi_2), K_{\phi_1\phi_2}=f(\phi_1)g(\phi_2), K_{\phi_2\phi_1}=f(\phi_2)g(\phi_1)$.
This completes the proof. □

The results on quantum estimates of the product of two $(\alpha,m)-GA$ and $(\alpha,m)-GG$ convex functions are given below:

Theorem 3.14. *Let $f,g:(0,\infty)\to(0,\infty)$ be two (α,m) -GA convex functions. Then the following inequality holds*

$$\frac{\log q}{q-1}\frac{1}{\phi_2^m-\phi_1}\int_{\phi_1}^{\phi_2^m}\frac{(f(x)g(x))}{x}\,{}_{\phi_1}d_qx\leq\frac{1}{L(\phi_1,\phi_2^m)}$$
$$\left(\frac{K_{\phi_1}}{[2\alpha+1]_q}+m^2K_{\phi_2}\left(1-\frac{2}{[\alpha+1]_q}+\frac{1}{[2\alpha+1]}\right)+mK_{\phi_2}^{\phi_1}\left(\frac{1}{[\alpha+1]_q}-\frac{1}{[2\alpha+1]_q}\right)\right)\tag{3.8}$$

Proof. As f and g are two $(\alpha, m)-$ GA convex functions, then by definition, we have

$$f(\phi_1^s \phi_2^{m(1-s)}) \;\leq\; s^\alpha f(\phi_1) + m(1-s^\alpha) f(\phi_2)$$

and

$$g(\phi_1^s \phi_2^{m(1-s)}) \;\leq\; s^\alpha g(\phi_1) + m(1-s^\alpha) g(\phi_2)$$

Since $f, g \geq 0$, so, we have

$$\begin{aligned} f(\phi_1^s \phi_2^{m(1-s)}) g(\phi_1^s \phi_2^{m(1-s)}) \;\leq\; & s^{2\alpha} f(\phi_1) g(\phi_1) + m^2 (1-s^\alpha)^2 f(\phi_2) g(\phi_2) \\ & + m(f(\phi_1)g(\phi_2) + f(\phi_2)g(\phi_1)) s^\alpha (1-s^\alpha). \end{aligned}$$

Taking q-integral with respect to s over $[0,1]$, we have

$$\begin{aligned} \int_0^1 f(\phi_1^s \phi_2^{m(1-s)}) g(\phi_1^s \phi_2^{m(1-s)})\, {}_0d_q s \;\leq\; & K_{\phi_1} \int_0^1 s^{2\alpha}\, {}_0d_q s + m^2 K_{\phi_2} \int_0^1 (1-s^\alpha)^2\, {}_0d_q s \\ & + m K_{\phi_1}^{\phi_2} \int_0^1 s^\alpha (1-s^\alpha)\, {}_0d_q s. \end{aligned}$$

Note that

$$\begin{aligned} \int_0^1 s^{2\alpha}\, {}_0d_q s &= (1-q) \sum_{n=0}^{\infty} q^n q^{2\alpha n} \\ &= (1-q) \sum_{n=0}^{\infty} q^{n(2\alpha+1)}. \\ &= (1-q) \frac{1}{1-q^{2\alpha+1}} \\ &= \frac{1}{[2\alpha+1]_q} \end{aligned}$$

Similarly, we obtain

$$\int_0^1 (1-s^\alpha)^2\, {}_0d_q t = 1 - \frac{2}{[\alpha+1]_q} + \frac{1}{[2\alpha+1]_q}$$

and

$$\int_0^1 s^\alpha (1-s^\alpha)\, {}_0d_q s = \frac{1}{[\alpha+1]_q} - \frac{1}{[2\alpha+1]_q}$$

If we take $x = \phi_1^s \phi_2^{m(1-s)}$, then $s = \frac{\log x}{\log \phi_1 - \log \phi_2^m} - \frac{\log \phi_2^m}{\log \phi_1 - \log \phi_2^m}$ when $s \to 0, x \to \phi_2^m$; when $s \to 1, x \to \phi_1$

And,

$${}_0d_q s = \frac{L(\phi_1, \phi_2^m) \log q}{(\phi_2^m - \phi_1)(q-1)x}\, {}_0d_q x$$

On substituting these values, we obtain

$$\frac{1}{\phi_2^m-\phi_1}\int_{\phi_1}^{\phi_2^m}\frac{f(x)g(x)}{x}\,{}_0d_qx \le \frac{q-1}{\log qL(\phi_1,\phi_2^m)}$$
$$\left(\frac{K_{\phi_1}}{[2\alpha+1]_q}+m^2K_{\phi_2}\left(1-\frac{2}{[\alpha+1]_q}+\frac{1}{[2\alpha+1]_q}\right)+mK_{\phi_1}^{\phi_2}\left(\frac{1}{[\alpha+1]_q}-\frac{1}{[2\alpha+1]_q}\right)\right)$$

This completes the proof. □

Theorem 3.15. *Let $f,g:(0,\infty)\to(0,\infty)$ be two $(\alpha,m)-$ geometrically-geometric (GG) convex functions. Then the following inequality holds:*

$$\frac{1}{\phi_2^m-\phi_1}\int_{\phi_1}^{\phi_2^m}\frac{\log(f(x)g(x))}{x}\,{}_{\phi_1}d_qx \;\le\; \frac{q-1}{L(\phi_1,\phi_2^m)\log q}$$
$$\left(\frac{\log K_{\phi_1}}{[\alpha+1]_q}+m\log K_{\phi_2}\left(1-\frac{1}{[\alpha+1]_q}\right)\right) \tag{3.9}$$

Proof. As f and g are two (α,m)- GG convex functions, then by definitions, we have

$$f(\phi_1^s\phi_2^{m(1-s)}) \;\le\; (f(\phi_1))^{s^\alpha}(f(\phi_2))^{m(1-s^\alpha)}$$

and

$$g(\phi_1^s\phi_2^{m(1-s)}) \;\le\; (g(\phi_1))^{s^\alpha}(g(\phi_2))^{m(1-s^\alpha)}$$

Since $f,g\ge 0$, so, we have

$$f(\phi_1^s\phi_2^{m(1-s)})g(\phi_1^s\phi_2^{m(1-s)}) \;\le\; (f(\phi_1)g(\phi_1))^{s^\alpha}(f(\phi_2)g(\phi_2))^{m(1-^\alpha)}$$

Taking $\log$ on both sides, we have

$$\log f(\phi_1^s\phi_2^{m(1-s)})+\log g(\phi_1^s\phi_2^{m(1-s)}) \;\le\; s^\alpha\log K_{\phi_1}+m(1-s^\alpha)\log K_{\phi_2}$$

Taking q-integral with respect to s over $[0,1]$, we have

$$\int_0^1\log f(\phi_1^s\phi_2^{m(1-s)})_0\,d_qs+\int_0^1\log g(\phi_1^s\phi_2^{m(1-s)})_0\,d_qs \;\le\; \log K_{\phi_1}\int_0^1 s^\alpha{}_0\,d_qs$$
$$+m\log K_{\phi_2}\int_0^1(1-s^\alpha)_0\,d_qs$$

Note that

$$\begin{aligned}\int_0^1 s^\alpha\,{}_0d_qs &= (1-q)\sum_{n=0}^\infty q^n(q^n)^\alpha\\ &= (1-q)\sum_{n=0}^\infty q^{n(\alpha+1)}\\ &= (1-q).\frac{1}{1-q^{\alpha+1}}\\ &= \frac{1}{[\alpha+1]_q}\end{aligned}$$

Similarly, we have

$$\int_0^1 (1 - s^\alpha)\, {}_0d_q s = 1 - \frac{1}{[\alpha+1]_q}.$$

using the change of variables by putting $x = \phi_1^s \phi_2^{m(1-s)}$, $s = \frac{\log x}{\log \phi_1 - \log \phi_2^m} - \frac{\log \phi_2^m}{\log \phi_1 - \log \phi_2^m}$ when $s \to 0, x \to \phi_2^m$; when $s \to 1, x \to \phi_1$

And,

$${}_0d_q s = \frac{L(\phi_1, \phi_2^m) \log q}{(\phi_2^m - \phi_1)(q-1)x}\, {}_0d_q x$$

On substituting these values, we obtain

$$\frac{1}{\phi_2^m - \phi_1} \int_{\phi_1}^{\phi_2^m} \frac{\log(f(x)g(x))}{x}\, {}_{\phi_1}d_q x \le \frac{q-1}{\log q L(\phi_1, \phi_2^m)} \left(\frac{\log K_{\phi_1}}{[\alpha+1]_q} + m \log K_{\phi_2} \left(1 - \frac{1}{[\alpha+1]_q} \right) \right)$$

This completes the proof. □

4. Conclusion

It is concluded that we have obtained the extension of Hermite-Hadamard integral inequality for geometric convex function as generalized geometric convex functions such as (α, m)-GA(geometrically-arithmetic), (α, m) GG(geometrically-geometric) and GH(geometrically-harmonic) convex functions and we have further recaptured these new results in terms of q-calculus which is a paradigm shift in the realm of convexity theory and integral inequality. The obtained results will definitely help to estimate the integral mean of the generalized geometric convex functions. The ideas might be employed to obtain the results for other type of convex functions.

Acknowledgment

The completion of this research work would not have been possible without the contributions and support of the researchers. We are deeply grateful to all of them who have assisted us in carrying out this research work.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] Ahmad, B., Ntouyas, S.K. and Tariboon, J. *Quantum Calculus: New Concepts, Impulsive Ivps and Bvps, Inequalities*, World Scientific Publishing Company, 2016.

[2] Bakula, M. K., Ozdemir, M. E. and Pecaric, J., *Hadamard type inequalities for m-convex and* (α, m) *-convex functions*, J. Inequal. Pure Appl. Math., **9** (2008), no.4, 1-12.

[3] Chen, F. and Wu, S., *Some Hermite-Hadamard type inequalities for harmonically s-convex functions*, Sci-World Journal, (2014), 1-7.

[4] Dragomir, S. S., *On Some New Inequalities of Hermite-Hadamard Type for m-Convex functions*, Tamakang J. of Mathematics, (2002), 33(1), 45-56.

[5] Kac, V. and Cheung, P. *Quantum Calculus* , Springer, New York, 2002.

[6] Kirmaci, U. S., Bakula, M. E. and Pecaric, J. *Hadamard type inequalities for s-convex functions*, Appl. Math. Comput.,**193**(2007), no. 1, 26-35.

[7] Mihesan, V. G. *A generalization of convexity*, In the proceedings of the seminar on Function Equations, Approximation and Convexity, (1993), Cluj-Napoca, Romania.

[8] Niculescu, C. P., *Convexity according to the geometric mean*, Math. Inequal. Appl., **3** (2000), no. 2, 155-167.

[9] Noor, M. A., Noor, K. I., Awan, M.U. and Castache, S. *Some Integral Inequalities for harmonically h-convex functions*, U.P.B. Science Bulletin Series A, **77**(2015), no. 1, 5-16.

[10] Ozdemir, M. E., Latif, M. A. and Akdemir, A. O. *On Some Hadamard type inequalities for product of two s-convex functions on the co-ordinates*, J. Inequal Appl., (2012), 1-13.

[11] Pachpatte, B. G., *On Some Inequalities for convex functions*, RGMIA Research, (2003), Rep.Call E.6, 1-9.

[12] Pachpatte, B. G., *A note on Integral Inequalities involving two log-convex functions*, Math. Inequal. Appl., **7**(2004), no. 4, 511-515.

[13] Sarikaya, M. Z., Saglam, A. and Yildirim, H. *On Some Hadamard type Inequalities for h-convex functions*, J. Math. Inequal, **2**(2008), no. 3, 335-341.

[14] Xi, B. Y., Bai, R.F. and Qi, F., *Hermite-Hadamard type inequalities for the* m *and* (α, m) *-geometrically convex functions*, Aequ. Math., (2012), 10.1007/500010-011-0114-x.

[15] Yin, H. P. and Qi, F., *Hermite -Hadamard type Inequalities for product of* (α, m) *-convex functions*, J. Non-linear Sci. Appl., **8**(2015), 231-236.

[16] Zhang, T. Y., Ji, A. P. and Qi, F. *On Integral Inequalities of Hermite-Hadamard type for* s *-geometrically convex functions*, Abstr. Appl. Anal, (2012), 10.1155/2012/560586.