Tribhuvan University
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A Study of Dynamical System with Some Applications

Thesis

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for the Master's Degree in Mathematics

By
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Student's Declaration

I hereby declare that I am the only author of this work and that no sources other than that listed here have been used in this work.

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Supervisor's Recommendation

I hereby recommend that this thesis prepared under my supervision by **Mr. Sabindra Giri** entitled **A Study of Dynamical System with Some Applications** during the period prescribed by the rules and regulations of this institute in partial fulfillment of the requirements for the degree of M.A./M.Sc. in Mathematics be processed for the evaluation.

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We certify that this research work by **Mr. Sabindra Giri** entitled **A Study of Dynamical System with Some Applications** is satisfactory in the scope and generality as a thesis in the partial fulfillment for the requirement of M.A./M. Sc. degree in Mathematics.

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Abstract

Dynamical system is mathematical formalization for any fixed rule which depends on time, these rules are described by differential equation and difference equation, which are also known as continuous and discrete time dynamical system.
This work consists the basic concept and the behaviors of dynamical system, which are *stability*, *bifurcation* and *chaos* and their examples. Most of the parts of this work contain a continuous time system. Also, the little work is done about a discrete time system with an example.

List of Symbols

$\mathbb{R}^n$	: n- dimensional Euclidean space
S	: Open subset of $\mathbb{R}^n$
X	: Initial position of dynamical system on $\mathbb{R}^n$
X_t	: General trajectory of X
$\mathbb{R}$	: Set of real numbers
t	: Continuous time
x_0	: Initial point
x, y	: Real number consists variables
k, n	: Discrete time
x^*	: Fixed point
$x(k), x(t)$	: Solution trajectories
η	: Difference between $x(k)$ and x^*
v	: Velocity
h	: Distance
g	: Gravitational acceleration
$\mathbf{x}$	: Vector consists variable
$f(\mathbf{x})$	: Vector consists function
L	: Length of pendulum
$\dot{x}, \dot{\mathbf{x}}$	: Differentiating with respect to t
A, B	: Square matrices
D	: Diagonal matrix
P, S	: Invertible matrices
M	: Real canonical matrix
I	: Identity matrix
Dig	: Diagonal
$\lambda, \lambda_1, \lambda_2$	: Eigenvalues
v, v_1, v_2	: Eigenvectors
x', y'	: Transformed variables
α, β	: Real and imaginary parts of eigenvalue $\lambda = \alpha + i\beta$
$\lambda, \overline{\lambda}$	: Complex conjugate eigenvalues
$w, \overline{w}$	: Complex conjugate eigenvectors
a, r, σ, b	: Parameters
Df	: Jacobian matrix
V	: Lyapunov function
∇V	: Gradient of a function from $\mathbb{R}^n$ to $\mathbb{R}$
p, b, c, k	: Constants
Δ	: Small change
$Q_\pm$	: Attracting fixed points of Lorenz System

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Chapter 1

Introduction

The purpose of this work is a study of the behavior of a dynamical system. A dynamical system studies the processes which are evolving in time, the description of these processes is given in terms of differential equation and difference equation or iteration of maps, which we call continuous and discrete time dynamical systems. Most of the part of this work contains the continuous time dynamical system.
The first chapter of the work contains *history*, *definition*, examples of *discrete* and *continuous* time dynamical system. Also, this chapter contains *linear* and *non-linear* dynamical systems and their examples and *flow on a line*. In chapter two, we have studied about the *fixed points and stability*, *nature of the fixed points of the linear system*, *stability of the linear system* and few methods to determine the stability of the non-linear system which are *Linearization* and *Lyapunov function*. The third chapter consists of basic concepts of *bifurcations*, like *saddle-node*, *transcritial* and *pitchfork*. Finally, the fourth chapter of the work contains *chaotic* behavior of the dynamical system with few examples which are *Lorenz system* and *logistic model of the population*.

1.1 Historical Background

The dynamical system began in the mid-1600s, when Newton discovered law of motion and Universal gravitation, and combined them to explain Keplar's laws of planetary motion with equation. It was necessary to invent calculus along the way, since the fundamental equation of motion involves velocities and acceleration which are derivatives of position. Specially, Newton solved the two body problem, the problem calculating the motion of the Earth around the Sun, giving the inverse square law of gravitational attraction between them. Newton's gravitational law of motion is, " *A force that is proportional to the product of masses and inversely proportional to the square of the distance between them.*" which specified with differential equation.
Subsequent generation of Mathematicians and Physicists tried to extend Newton's analytical methods to the three body problem (E.g. Sun, Earth and Moon) but the problem is

much more complicated to solve. After decades of effort, it was eventually realized that the three body problem was essentially impossible to solve. After 1800s, Henri Poincaré studied the qualitative behavior of the motion of the planets. For example, instead of asking for the exact position of the planets at all times, he asked, " *Is the solar system stable forever or will some planets eventually fly off to infinity ?* " Poincaré developed powerful geometric approach to analyzing such a question. That approach is useful to study the behavior of three body problem. Which is also known as the modern subject of dynamics, with applications far beyond celestial mechanics. Poincaré was the first person to glimpse the possibility of *chaos*, in which deterministic system exhibits aperiodic behavior that depends sensitively on the initial conditions, thereby rendering long term prediction impossible.
By the 1980s many people were working on dynamics, which is summarized as follows,

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytic studies of planetary motion
1890	Poincaré	Geometric approach, nightmare of chaos
1920-1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920-1960	Birkhoff Kolmogorov Arnol'd Moser	Complex Behavior in Hamiltonian Mechanics
1963	Lorenz	Strange Attractor in simple model of convection
1970s	Ruelle and Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universally and renormalization connection between chaos and phase transition
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators and their applications

1.2 Dynamical System

A dynamical system is a way of describing the passing in time of all points of a given space S, S is the space of states of some physical system. Mathematically, S might be an Euclidean space or an open subset of Euclidean space or some other space such as a surface in $\mathbb{R}^3$. When we consider dynamical systems that arise in mechanics, the space S will be the set of possible positions and velocities of the system. For the sake of simplicity, we will assume throughout that the space S is Euclidean space $\mathbb{R}^n$, although in certain cases the important dynamical behavior will be confined to a particular subset of $\mathbb{R}^n$.
Given an initial position $X \in \mathbb{R}^n$, a dynamical system on $\mathbb{R}^n$ tell us where X is located 1 unit of time later, 2 unit of time later, and so on. We denote these new positions of X by X_1, X_2, and so forth. At time zero, X is located at position X_0. One unit before time zero, X was at X_{-1}. Thus the dynamical system help us to describe the present state of the real world as the effect of its past and the causes of its future.
In general, the trajectory of X is given by X_t. If we measure the positions X_t using only integer time values is known as the discrete time dynamical system. If time is measured continuously with $t \in \mathbb{R}$, we have a continuous dynamical system. If the system depends on time in a continuously differentiable manner, we have a smooth dynamical system. And a continuous dynamical system is described by a differential equation and discrete dynamical system is described by a iteration mapping (difference equation).
Physically, a dynamical system is an object or collection of objects in the real world which evolves in time.

1. A fluid in a container subjected to stirring or external influences such as changes in temperature or pressure.
2. The population at time t of a certain species of animal or plant.
3. The current through a wire (motion of electrons).
4. The motion of an object suspended by a spring or rigid rod pendulum.
5. Molecules of a gas in a container.

Elementary examples of dynamical system

1.2.1 Discrete Dynamical System

Dynamical system is a rule which tells us how the system changes over time. In other words, if we are given the current state of the system, the rule tells us the state of the system in the next instant. Here, we discuss about discrete dynamical system with the following example.

Example

Consider a bank account opened with 100 units at 6 per interest compounded annually. The state of this system at any instant of time can be described by a single number: the balance in the account. In such case, time is discrete. That is to say, time is a sequence of separate chunks each following the next like beads on a string. For the bank account, it is easy to write down the rule which takes us from the state of the system at one instant to the state of the system in the next instant, namely,

$$x(k+1) = 1.06x(k) \tag{1.1}$$

where $x(k)$ denotes the state of the system at time k and we use the letter k to denote the discrete time. In this example (since interest is only paid once a year), time is always a whole number. A complete description of the system is

$$\begin{aligned} x(k+1) &= 1.06x(k) \\ x(0) &= 100. \end{aligned}$$

It is customary to begin time at 0, and to denote the initial state of the system by x_0. In this example, $x_0 = x(0) = 100$.

The state of the bank account in all future years can now be computed. We see that $x(1) = 1.06x(0) = 1.06 \times 100 = 106$, and then $x(2) = 1.06x(1) = 1.06 \times (1.06 \times 100) = 112.36$. Indeed, we see that

$$x(k) = (1.06)^k \times 100$$

or more generally,

$$x(k) = (1.06)^k x_0. \tag{1.2}$$

Therefore, $(1.06)^k x_0$ is a general formula for $x(k)$. However, we can verify that equation (1.2) is true by checking

1. that it satisfies the initial condition $x(0) = x_0$, and
2. that it satisfies equation (1.1).

That is,

$$x(0) = (1.06)^0 x_0 = x_0$$

and

$$x(k+1) = (1.06)^{k+1} x_0 = (1.06)(1.06)^k x_0 = 1.06x(k)$$

A Broader Context

Let us put this example into a broader context which is applicable to all discrete time dynamical systems. We have a state vector $\mathbf{x} \in \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$

$$\mathbf{x}(k+1) = f(\mathbf{x}(k)).$$

In this example, $n = 1$ (the bank account is described by a single number: the balance) and the function $f : \mathbb{R} \to \mathbb{R}$ is simply $f(x) = 1.06x$. Later, we consider more complicated function f. Once we are given that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(k+1) = f(\mathbf{x}(k))$, we can, in principal, compute all values of $\mathbf{x}(k)$, as follows:

$$\begin{array}{lll}
\mathbf{x}(1) & = f(\mathbf{x}(0)) & = f(\mathbf{x}_0) \\
\mathbf{x}(2) & = f(\mathbf{x}(1)) & = f(f(\mathbf{x}_0)) \\
\mathbf{x}(3) & = f(\mathbf{x}(2)) & = f(f(f(\mathbf{x}_0))) \\
\mathbf{x}(4) & = f(\mathbf{x}(3)) & = f(f(f(f(\mathbf{x}_0)))) \\
\vdots & & \\
\mathbf{x}(k) & = f(\mathbf{x}(k-1)) & = f(f(\ldots(f(\mathbf{x}_0))\ldots)),
\end{array}$$

where in the last line, we have f applied k times to $\mathbf{x}_0$. We need a notation for a repeated application of a function. Let us write $f^2(\mathbf{x})$ to mean $f(f(\mathbf{x}))$, $f^3(\mathbf{x}) = f(f(f(\mathbf{x})))$, and in general, write

$$f^k(\mathbf{x}) = f(f(\ldots(f(\mathbf{x}_0))\ldots)).$$

In general, for a discrete dynamical system, we denote time by k, and the system is specified by the equation

$$\begin{aligned} x_{k+1} &= f(x_k) \\ x(0) &= x_0 \end{aligned} \tag{1.3}$$

It thus follows that $x_k = f^k(x_0)$, where f^k denotes a k-fold applications of f to x_0, where $k \in \mathbb{Z}$.

Linear Stability Analysis

Suppose x^* satisfies $f(x^*) = x^*$, then x^* is a fixed point of (1.3), for if $x_k = x^*$ then $x_{k+1} = f(x_k) = f(x^*) = x^*$, hence the orbit remains at x^* for all future iterations. To determine the stability of x^*, we consider the nearby orbits $x_k = x^* + \eta_k$, substituting this in equation (1.3) and Taylor expanding about x^* yields

$$x^* + \eta_{k+1} = x_{k+1} = f(x^* + \eta_k) = f(x^*) + f'(x^*)\eta_k + O(\eta_k^2). \tag{1.4}$$

But, since $f(x^*) = x^*$, then the equation (1.4) reduces to

$$\eta_{k+1} = f'(x^*)\eta_k + O(\eta_k^2).$$

Suppose we can safely neglect the terms $O(\eta_k^2)$, then we obtain the *linearized map*

$$\eta_{k+1} = f'(x^*)\eta_k$$

with multiplier $\lambda = f'(x^*)$. The solution of this linear map can be explicitly written in a few terms as $\eta_1 = \lambda\eta_0$, $\eta_2 = \lambda\eta_1 = \lambda^2\eta_0$, and so in general $\eta_k = \lambda^k\eta_0$.

If $|\lambda| = |f'(x^*)| < 1$, then $\eta_k \to 0$ as $k \to \infty$ and the fixed point x^* is stable. Conversely, if $|\lambda| = |f'(x^*)| > 1$, the fixed point is unstable. Although these conclusions about local stability are based on linearization, they can be proven to hold for the original nonlinear map. But the linearization tells us nothing about the marginal case $|\lambda| = |f'(x^*)| = 1$, then the neglected $O(\eta_k^2)$ terms determine the local stability.
Cobwebs construction are useful because they allow us to see global behavior at a glance, thereby supplementing the local information available from the linearization. Cobwebs become even more valuable when linear analysis fails.
Given $x_{k+1} = f(x_k)$ and initial condition x_0, draw the vertical line until its intersection with graph of f; that height is the output x_1. At this stage, we could return to the horizontal axis and repeat the procedure to get x_2 from x_1. But it is more convenient simply to trace a horizontal line till it intersects the diagonal line $x_{k+1} = x_k$, and then move vertically to the curve again. Repeat this process k times to generate the first k points in the orbit. The cobwebs construction is shown in the figure (1.1).

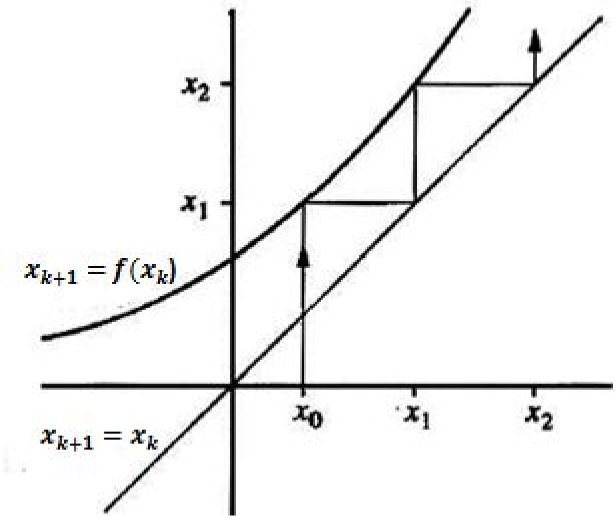

Figure 1.1: Cobwebs construction

1.2.2 Continuous Dynamical System

The other important type of a dynamical system is essentially the limit of discrete system with smaller and smaller updating times. The governing rule in that case becomes a set of differential equations, and the term sometimes called *continuous-time* dynamical system. Here, we try to study the continuous dynamical system with an example.

Example
A ball thrown upwards with velocity v reaches a hight h. If we know these two numbers h and v, the fate of the ball is completely determined. The pair (h, v) of numbers is a

vector $\mathbf{x}$ representing instantaneous status of the ball.
Instead of k in discrete case we use t to denote time, where $t \geq 0$, we start at time $t = 0$. Since we cannot write down a rule for the next instant of time, we instead describe how the system is changing at any given instant. First, if our ball has (upward) velocity v, then we know that $dh/dt = v$; this is the definition of the velocity. Second, gravity pulls down on the ball and we have $dv/dt = -g$ where g, is the positive constant. So the change in the system is described by

$$\dot{h}(t) = v(t) \tag{1.5}$$
$$\dot{v}(t) = -g \tag{1.6}$$

which can be expressed as

$$\begin{bmatrix} \dot{h}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -g \end{bmatrix}. \tag{1.7}$$

Since

$$\mathbf{x}(t) = \begin{bmatrix} h(t) \\ v(t) \end{bmatrix}$$

and

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}) \tag{1.8}$$

we have from (1.7)

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \tag{1.9}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ -g \end{bmatrix}.$$

Indeed, equation (1.8) is the form for all continuous time dynamical systems. A continuous time dynamical system has a state vector $\mathbf{x}(t) \in \mathbb{R}^n$ and we are given a function $f : \mathbb{R}^n \to \mathbb{R}^n$ which specifies how quickly each component of $\mathbf{x}(t)$ is changing, i.e. $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$, or compactly $\dot{\mathbf{x}} = f(\mathbf{x})$. Solving the equation we get

$$\begin{aligned} h(t) &= h_0 + v_0 t - \frac{1}{2} g t^2 \\ v(t) &= v_0 - gt \end{aligned}$$

describing the motion of the ball with initial conditions, $\mathbf{x}_0 = \begin{bmatrix} h_0 \\ v_0 \end{bmatrix}$.
In general, for a continuous dynamical system, we denote time by t, and the following equations specify the system:

$$\dot{\mathbf{x}} = f(\mathbf{x}) \text{ and } \mathbf{x}(0) = \mathbf{x}_0$$

1.3 Linear and Nonlinear Dynamical Systems

Dynamical system is the time evolution of some physical system, such as the motion of a few planets under the influence of their respective gravitational forces. Usually, we want to know the fate of the system for long times, for instance, will the planets eventually collide or will the system for persist for all times?
Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a point in n-dimensional space $\mathbb{R}^n$ that trace out a curve through time. We can describe this as

$$\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)), \text{ for } -\infty < t < \infty$$

The rate and direction of change of $\mathbf{x}(t)$ in some region of $\mathbb{R}^n$ is

$$\dot{\mathbf{x}} = f(\mathbf{x}), \ \mathbf{x} \in \mathbb{R}^n \tag{1.10}$$

where the dot indicates the derivative with respect to t, so $\dot{\mathbf{x}} = d\mathbf{x}/dt$. We always assume f has continuous partial derivatives. If we write these vector equations out in full, we get

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2, \ldots x_n) \\ \dot{x_2} &= f_2(x_1, x_2, \ldots x_n) \\ &\vdots \\ \dot{x_n} &= f_n(x_1, x_2, \ldots x_n). \end{aligned} \tag{1.11}$$

We call this a set of first-order ordinary differential equations in n unknowns. It is of the first order, because no derivatives higher than the first appears. It is ordinary as opposed to partial, because we want to solve for a function of the single variable t, as opposed to solving for a function of several variables. We call $\mathbf{x}(t)$ a dynamical system if it satisfies such a set of ordinary differential equations, in the sense that $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$ for t in some (possibly infinite) interval.
Here, the variables $x_1, x_2, \ldots, x_n$ may represents the concentration of the chemicals in a reactor, the population of the species in a ecosystem or the position and velocity of the planets in the solar system. Linear and non-linear problems are determined by the functions $f_1, f_2, \ldots, f_n$. For example, the damped oscillator,

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{1.12}$$

we introduce the new variables $x_1 = x$ and $\dot{x_1} = x_2$ then the equation (1.12) can be expressed in the form

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -\frac{b}{m}x_2 - \frac{k}{m}x_1, \end{aligned}$$

which is a linear system.
The swinging of a pendulum is governed by the nonlinear equation

$$\ddot{x} + \frac{g}{L}\sin x = 0$$

where x is the angle of the pendulum from the vertical, g is the acceleration due to the gravity, and L is the length of the pendulum. The equation is expressed as

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -\frac{g}{L}\sin x_1, \end{aligned}$$

which is a nonlinear system.
Nonlinearity makes the system very difficult to solve analytically. Considering small angle approximation, that is $\sin x \approx x$ for $(x << 1)$, we get the linear form, which can be solved exactly.

1.4 Flows on a Line

In (1.11), we introduce the general system and mention that its solutions could visualized as trajectories flowing through an n-dimensional phase space with coordinates $(x_1, x_2, \ldots, x_n)$. Here, we start simple one dimensional system $\dot{x} = f(x)$. Here $x(t)$ is a real valued function of time t, and $f(x)$ is a continuous real-valued function of x which is independent of time. Here the system means the dynamical system, not the set of differential equations.
Pictures are more helpful than the formulas for analyzing the nonlinear systems.

Example 1.4.1. *Let us consider the system*

$$\dot{x} = \sin x \tag{1.13}$$

having a solution

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right| \tag{1.14}$$

for $x = x_0$ at $t = 0$. To study the behavior of $x(t)$ it seems not easy as $t \to \infty$.

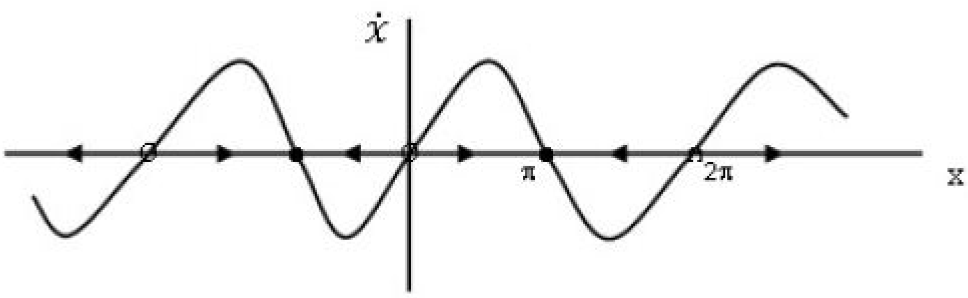

Figure 1.2: Flow on a line

We think of t as time, x the position of the imaginary particle moving along the real line,

and $\dot{x}$ as the velocity of the particle. Then the differential equation $\dot{x} = \sin x$ represents a *vector field* on the line: it dictates the velocity vector $\dot{x}$ at each x. In the figure, arrow points to the right for $\dot{x} > 0$ and left for $\dot{x} < 0$.
We imagine that the fluid is flowing steadily along the x-axis with a velocity that varies from place to place according to the rule $\dot{x} = \sin x$.
At the points where $\dot{x} = 0$, there is no flow, such points are called *fixed points*. In the figure, the solid black dots represent *stable* fixed points (also known as *attractors* or *sinks*) and the open circles represent *unstable* fixed points (also known as *repellers* or *sources*). In this way, the situation to the given differential equation can be described.

1.5 Some Definitions and Theorems

Flow
The set of functions $\phi_t = e^{tA}$ is called *flow* of the system $\dot{\mathbf{x}} = A\mathbf{x}$.

Fixed (Equilibrium) Point
A point x^* is called a fixed point of continuous time dynamical system $\dot{x} = f(x)$ where $x = x(t)$ and $t \in \mathbb{R}$ if $f(x^*) = 0$ such a point is known as a fixed point of continuous time dynamical system.

Isolated Point
A fixed point is called isolated if some neighborhood of it contains no other fixed point.

Matrix Exponential
Let A be an $n \times n$ matrix and $t \in \mathbb{R}$, we define a matrix exponential to be

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$$

Hyperbolic Fixed Point
An equilibrium point x^* of the system $\dot{x} = f(x)$ is called hyperbolic if all eigenvalues of the Jacobian matrix have non-zero real part.

Fundamental Existence and Uniqueness Theorem
Consider the initial value problem $\dot{x} = f(x)$ with $x(0) = x_0$ where $f : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that all partial derivatives in the Jacobian matrix are continuous for all x in the vicinity of the initial condition x_0, then there exists $\alpha > 0$ such that the initial value problem has a unique solution $x(t)$ in the interval $[-\alpha, \alpha]$.

Diagonalizable Matrix
A square matrix A is called diagonalizable, if B is a diagonal matrix then there exists an

invertible matrix S such that $A = SBS^{-1}$.

Eigenvalues and Eigenvectors
Suppose A is an $n \times n$ matrix. A scalar λ is called an eigenvalue for A, if there exists a non-zero vector v such that $Av = \lambda v$ and nonzero vector v is called an eigenvector of A corresponding to the eigenvalue λ.

Theorem
An $n \times n$ matrix A is diagonalizable if and only if there exists a set of n linearly independent eigenvectors for A.

Corollary
If an $n \times n$ matrix A has n distinct real eigenvalues, then A is diagonalizable.

Theorem
Suppose that $\dot{y} = Ay$ is a system of constant coefficients in ODEs and let $y_0 = y(0)$ be a vector of initial condition, then this initial value problem has exactly one solution, which is given by $y(t) = e^{tA}y_0$. Moreover, the solution exists for all times t.

Theorem
Suppose A is an $n \times n$ matrix with real eigenvalues, then A is diagonalizable if and only if the sum of the geometric multiplicities of the eigenvalues is equal to n.

Proposition
Suppose that $\dot{y} = Ay$ is a system of ODEs, where A is an $n \times n$ matrix and is diagonalizable. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the (possible repeated) eigenvalues and let $v_1, v_2, \ldots, v_n$ denote their corresponding eigenvectors. Then, the general solution of the system is given by

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$$

Generalized Eigenvectors
Suppose A is an $n \times n$ matrix and λ is eigenvalue of algebraic multiplicity $m \leq n$. Then for each $k = 1, 2, \ldots, m,$, any nonzero solution v of equation $(A - \lambda I)^k v = 0$ is called the generalized eigenvector of A.

Nilpotent Matrix
Let k be a positive integer, an $n \times n$ matrix N is called Nilpotent of order k if $N^{k-1} \neq 0$ but $N^k = 0$.

Simple Jordan Decomposition Theorem
Suppose that A is an $n \times n$ matrix with real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ repeated according to the multiplicity, then there exists a family of n generalized eigenvectors $v_1, v_2, \ldots, v_n$

such that

1. The matrix $P = [v_1|v_2|\dots|v_n]$ is invertible .
2. $A = S + N$, where N is nilpotent and S is diagonalizable. Specifically, if we define $D = \text{Dig}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, then $S = PDP^{-1}$
3. S and N commute: $SN = NS$

Definition
If A is a matrix with complex conjugate eigenvalues $\alpha \pm \beta i$, then the matrix

$$M = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

is called the real canonical form for A.

Proposition
Suppose

$$A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

where α and β are real numbers, then

$$e^{tA} = e^{\alpha t} \begin{bmatrix} \cos \beta t & -\sin \beta t \\ \sin \beta t & \cos \beta t \end{bmatrix}$$

Definition
Suppose A is $n \times n$ matrix, then the matrix A can be written as in the form $A = PMP^{-1}$, where P is invertible matrix and M has one of the followings

1. If A is diagonalizable, then
$$M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$
2. If A is non-diagonalizable and has a real repeated eigenvalue α, then
$$M = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$$
3. If A has a complex conjugate eigenvalues $\alpha \pm \beta i$, then
$$M = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

Caley-Hamiltion Theorem
Suppose A is an $n \times n$ matrix, and let $f(t)$ be the characteristics polynomial of A, then $f(A) = 0$ the $n \times n$ zero matrix.

Fixed Point
A point x^* is called a fixed point of a discrete time dynamical system $x_{n+1} = f(x_n)$, where $x_n = x(n)$ for all n is positive integer, if $f(x^*) = x^*$.

Hyperbolic Fixed Point
A fixed point x^* of the equation $x_{n+1} = f(x_n)$ is called hyperbolic if $|f'(x^*)| \neq 1$. Otherwise, the fixed point is called non-hyperbolic.

Theorem(Stability of a Fixed Point of Discrete System)
Suppose x^* is an isolated fixed point of the first order difference equation $x_{n+1} = f(x_n)$, where f is continuously differentiable. Then x^* is stable if $|f'(x^*)| < 1$ and is unstable if $|f'(x^*)| > 1$ and if $|f'(x^*)| = 1$ then the test is inconclusive.

Chapter 2

Stability in Dynamical System

2.1 Fixed Points and Stability

In example $(1.4.1)$ we developed the idea of fixed point and stability, which can be extended to any one dimensional system $\dot{x} = f(x)$ (see figure (1.2)) and define as: the imaginary fluid following along the real line with a local velocity $f(x)$ is called the *phase fluid* , and the real line is called the *phase space*. To find the solution to $\dot{x} = f(x)$ starting from an arbitrary initial condition x_0, we place an imaginary particle which is known as a *phase point* at x_0 and watch how it is carried along by the flow. As time goes on, the phase point moves along the x-axis according to some function $x(t)$ is called the *trajectory* based at x_0, and it represents the solution of the differential equation starting from the initial condition x_0. Figure (1.2) shows all the qualitatively different trajectories of the system and is called a *phase portrait*. Appearance of the phase portrait is controlled by the fixed points x^*, defined by $f(x^*) = 0$, from which we get fixed point or stagnation points of the flow. The solid black dot in figure (1.2) represents the *stable fixed point* (the local flow is toward it) and the open dot is an *unstable fixed point* (the flow is away from it).

Example 2.1.1. *Consider the simple equation,*

$$\dot{x}(t) = ax(t) \textit{ with } x(0) = x_0$$

where a is arbitrary constant

Solving we get $x(t) = x_0 e^{at}$. In particular, $x(t) = 0$ is a solution such that $f(0) = 0$. The behavior of the solution depends on a

1. If $a > 0$, then $x(t) \to \infty$ as $t \to \infty$.

2. If $a = 0$, then the solutions are all constant.

3. If $a < 0$, then $x(t) \to 0$ as $t \to \infty$.

The qualitative behavior is shown in the adjoining figure. The behavior of the solution is

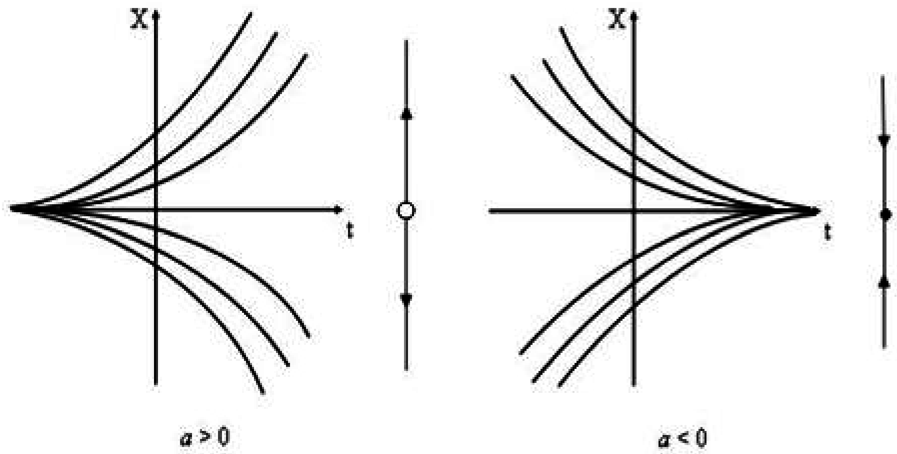

Figure 2.1: Qualitative behavior of the system

quite different for $a > 0$ and $a < 0$. When $a > 0$, all positive solutions tend away from the fixed point at 0 as t increases, whereas when $a < 0$, solution tends towards the fixed point. We say that the fixed point is a *source*, when nearby solution tend away from it. The fixed point is a *sink* when nearby solutions tend toward it. We also describe solutions by drawing them on the *phase line*. As $x(t)$ is a function of time, we may view $x(t)$ as a particle moving along the real line. At the fixed point, the particle remains rest (indicated by a solid dot), while any other solution moves up the x-axis is indicated by the arrows in the figures.
The equation $\dot{x} = ax$ is stable if $a \neq 0$. That is if a is replaced by another constant b whose sign is the same as a, then the qualitative behavior of the solution does not change. But if $a = 0$, the small change in a leads to a radical change in the behavior of the solutions. We therefore say that we have a *bifurcation* at $a = 0$ in the one parameter family of equation $\dot{x} = ax$.

2.1.1 Fixed Points Behavior

Suppose

$$\dot{x} = F(x, y) \;\text{ and }\; \dot{x} = G(x, y) \tag{2.1}$$

be a two dimensional autonomous dynamical system, then the point (x^*, y^*) is called a critical (fixed) point of a system if it satisfies $F(x^*, y^*) = G(x^*, y^*) = 0$. If (x^*, y^*) is a critical point of a system, then the constant valued function $x(t) \equiv x^*$ and $y(t) \equiv y^*$ satisfy the equation. Such a constant valued solution is called an equilibrium solution of the system.
A critical point (x^*, y^*) of a dynamical system (2.1) is called a node provided that

1. Either every trajectory approaches the fixed point (x^*, y^*) as $t \to \infty$ or every trajectory reduces from (x^*, y^*) as $t \to \infty$
2. Every trajectory is tangent at (x^*, y^*) to same straight line through the critical point.

Types of node are as follows:

Proper Node
A node is said to be proper provided that no two different pairs of opposite trajectories are tangent to the same straight line through the critical point. It might be called a star point. The critical point $(0, 0)$ is proper node which is illustrated in Figure (2.8).

Improper Node
If all trajectories except from a single opposite pair are tangent to a single straight line through a critical point, this type of node is said to be improper. The origin is improper node that is illustrated in Figure (2.2).

Saddle Point
There are two trajectories that approach the critical point $(0, 0)$, but all others are unbounded as $t \to \infty$. This type of critical point is called a saddle point, this is illustrated in Figure (2.6).

2.1.2 Stability of Linear System

Let us consider a dynamical system

$$\dot{\mathbf{x}} = A\mathbf{x} \text{ with } \mathbf{x}(0) = \mathbf{x}_0 \tag{2.2}$$

where A is an $n \times n$ matrix and $\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^n$, then the unique solution of (2.2) is

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 \tag{2.3}$$

and 0 is an equilibrium solution of the system. If A is diagonalizable matrix, then $A = PDP^{-1}$, where P is invertible matrix, the columns of P are the eigenvectors of A's and D is diagonal matrix whose entries $\{\lambda_k\}_{k=1}^n$ are exactly the eigenvalues of A. We have $A^k = (PDP^{-1})^k = PD^kP^{-1}$ and

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}.$$

Then we have $e^{tA} = Pe^{tD}P^{-1}$. By taking the power of the diagonal matrix,

$$e^{tA} = P \sum_{k=0}^{\infty} \frac{(tD)^k}{k!} P^{-1} = P \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

then the behavior of (2.3) will depend on the behavior of e^{tA}, which is described as follows:

1. If $Re(\lambda_k) = 0$ for all k, then all the solutions of the system are constant .
2. If $Re(\lambda_k) < 0$ for all k, then the solution (2.3) is stable.
3. If $Re(\lambda_k) > 0$ for some k, then the solution (2.3) is unstable.

If A is non-diagonalizable, then $e^{tA} = Pe^{tJ}P^{-1}$, where J is Jordan Canonical form. The general behavior of the system does not depend on the diagonalizability of the matrix A.

2.1.3 Behavior of Linear System for Different Cases of Eigenvalues

Let us consider a two dimensional linear system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{2.4}$$

with the constant coefficient matrix A. Let the eigenvalues λ_1 and λ_2 of A are the solution of a characteristics equation of A, that is

$$\begin{aligned} det(A - \lambda I) &= 0 \\ \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} &= 0 \\ (a-\lambda)(d-\lambda) - cb &= 0 \\ \lambda^2 - (a+d)\lambda + ad - cb &= 0 \end{aligned}$$

Let us assume that $(0, 0)$ is an isolated critical point of the system (2.4), so it follows that the determinant of coefficient matrix is $ad - bc \neq 0$. This implies that $\lambda \neq 0$. Hence, both eigenvalues of the system are non-zero.
The nature of the isolated fixed point $(0, 0)$ of the system is depended on two non-zero eigenvalues of the coefficient matrix A, which are described as follows,

Unequal Real Eigenvalues with the Same Sign

Let λ_1 and λ_2 are two real and unequal eigenvalues with same signs, then the matrix A is diagonalizable and has linearly independent eigenvectors v_1 and v_2 then, the system has a general solution $\mathbf{x}(t) = [x(t), y(t)]^T$ and takes the form

$$\mathbf{x}(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

And, change of the coordinates by taking invertible matrix P, whose columns are the eigenvectors of A as a new basis such that $A = PDP^{-1}$, then the system (2.4) can be transformed into diagonal form.

$$\dot{\mathbf{x}}' = Dx' \text{ where } D = \text{Dig}\{\lambda_1, \lambda_2\} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

The general solution of the transformed system is $\mathbf{x}'(t) = e^{Dt}\mathbf{x}_0$, where $\mathbf{x}(0) = \mathbf{x}_0$ that is

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix} \tag{2.5}$$

Therefore, the exponential solutions are

$$x'(t) = c_1 e^{\lambda_1 t} \text{ and } y'(t) = c_2 e^{\lambda_2 t} \tag{2.6}$$

which either grow or decay depending on the signs of λ_1 and λ_2. We sketch the phase portraits in the transformed coordinates by eliminating the parameter t. If $c_1 = 0$, the trajectory lies on the y'-axis and if $c_2 = 0$, the trajectory lies on x'-axis. Note that $x'^{\lambda_2} = c_1^{\lambda_2} e^{\lambda_1 \lambda_2 t}$ and $y'^{\lambda_1} = c_2^{\lambda_1} e^{\lambda_1 \lambda_2 t}$. If both are non-zero, then the parametric curves(2.6) takes the explicit form $y'(t) = Cx'^{\frac{\lambda_2}{\lambda_1}}$, where C is arbitrary constant. It may arise two cases to sketch the possible phase portrait.

Case1: Both λ_1 and λ_2 are Positive

If both λ_1 and λ_2 are positive, then $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ will increase as $t \to \infty$. And the solution trajectories point outwards from the origin. The possible phase portrait is shown in figure (2.2)

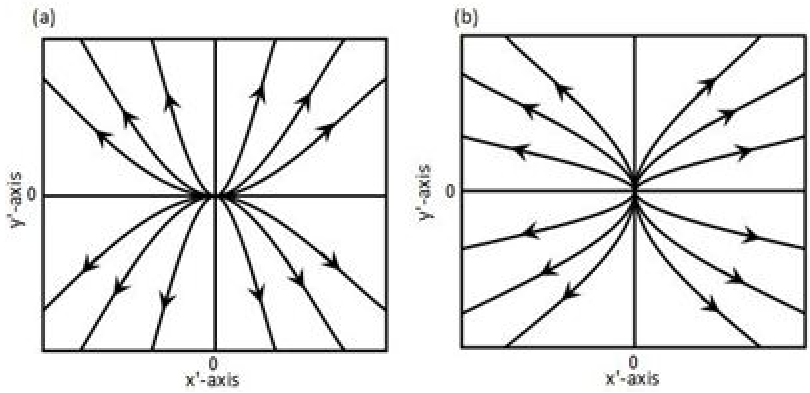

Figure 2.2: Phase portrait in which both λ_1 and λ_2 are positive (a) $0 < \lambda_1 < \lambda_2$, (b) $0 < \lambda_2 < \lambda_1$

Case2: Both λ_1 and λ_2 are Negative

If both λ_1 and λ_2 are negative, then $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ approach zero as $t \to \infty$. And the solution trajectories point towards the origin. The possible phase portrait is shown in

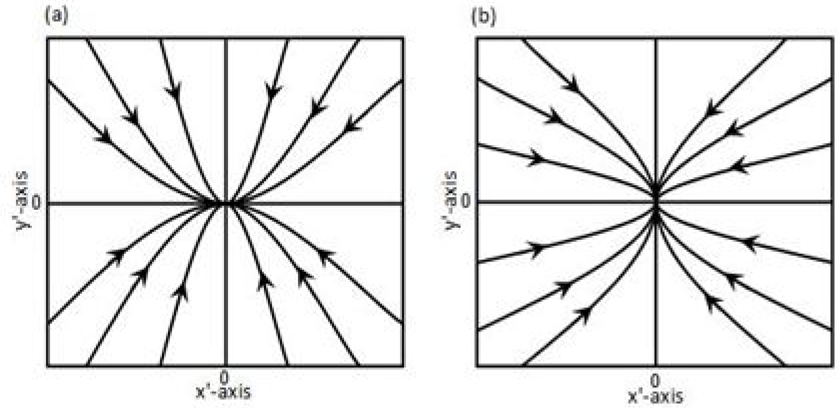

Figure 2.3: Phase portrait in which both λ_1 and λ_2 are negative (a) $\lambda_2 < \lambda_1 < 0$, (b) $\lambda_1 < \lambda_2 < 0$

figure (2.3). We see that by change of coordinates the original linear system (2.4) has two linearly independent solution $\mathbf{x_1}(t) = v_1 e^{\lambda_1 t}$ and $\mathbf{x_2}(t) = v_2 e^{\lambda_2 t}$ takes the form

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) \tag{2.7}$$

The solution simply we describe in the oblique $x'y'$-coordinates, where x'-axis and y'-axis are determined by the eigenvectors v_1 and v_2 respectively. Then the $x'y'$- coordinates function $x'(t)$ and $y'(t)$ of the moving point $\mathbf{x}(t)$ are simply its distance from the origin measured in the direction parallel to the vectors v_1 and v_2 as shown in the figure(2.4). So it follows from (2.7), the solution trajectory of the system is described by

$$x'(t) = x_0 e^{\lambda_1 t} \text{ and } y'(t) = y_0 e^{\lambda_2 t} \tag{2.8}$$

where $x(0) = x_0$ and $y(0) = y_0$. If $x_0 = 0$ and $y_0 = 0$ then the solution trajectory lies on y'-axis and x'-axis respectively. Otherwise if both are non-zero then the parametric curves (2.8) takes the explicit form $y' = Cx'^k$ where $k = \frac{\lambda_2}{\lambda_1} > 0$. These solution curves tangent to the x'-axis if $k > 1$ and tangent to the y'-axis if $0 < k < 1$. Therefore the critical point $(0, 0)$ is improper node.

Example 2.1.2. *Let us consider the matrix*

$$A = \begin{bmatrix} 6/5 & -4/5 \\ -1/5 & 9/5 \end{bmatrix}$$

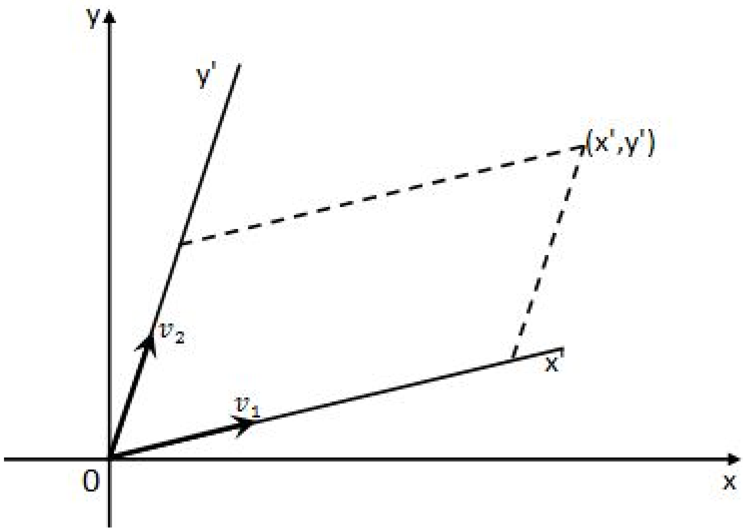

Figure 2.4: The oblique $x'y'$-coordinates determine by the eigenvectors v_1 and v_2.

The characteristics equation of the matrix A is

$$\begin{aligned} |A - \lambda I| &= 0 \\ \lambda^2 - 3\lambda + 2 &= 0 \end{aligned}$$

Solving we get the real eigenvalues of the matrix A are $\lambda_1 = 1$ and $\lambda_2 = 2$. Let $v \in \mathbb{R}^2$ be a eigenvector of A corresponding to the eigenvalue λ, then $(A - \lambda I)v = 0$. Solving this equation we get the eigenvectors corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$:

$$v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We consider the system $\dot{\mathbf{x}} = A\mathbf{x}$ and change the coordinates by taking

$$P = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix},$$

whose columns are the eigenvectors of the matrix A as the new basis of the system. We get the diagonal system with the diagonal entries $\lambda_1 = 1$ and $\lambda_2 = 2$, then the solution trajectories of the diagonal system are $x'(t) = c_1 e^t$ and $y'(t) = c_2 e^{2t}$. By changing the coordinates, the general solution of the original system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}.$$

The solution trajectories of the system are shown in the figure(2.5)

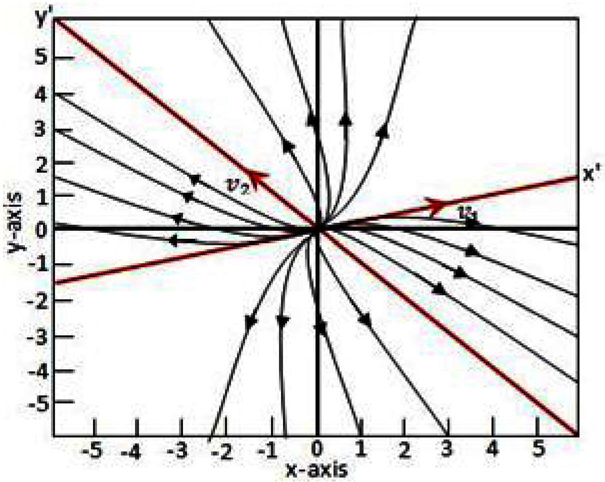

Figure 2.5: The improper nodal source in example $(2.1.2)$

Unequal Real Eigenvalues with the Opposite Signs

Let λ_1 and λ_2 are unequal real eigenvalues with opposite signs, then the matrix A is diagonalizable and has linearly independent eigenvectors v_1 and v_2, then the system has a general solution $\mathbf{x}(t) = [x(t), y(t)]^T$ and takes the form:

$$\mathbf{x}(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

and change of the coordinates by taking invertible matrix P, whose columns are the eigenvectors of A as a new basis such that $A = PDP^{-1}$. Then the system (2.4) can be transformed into diagonal form:

$$\dot{\mathbf{x}}' = Dx' \text{ where } D = \text{Dig}\{\lambda_1, \lambda_2\} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

From equation (2.6), we have the exponential solutions $x'(t) = c_1 e^{\lambda_1 t}$ and $y'(t) = c_2 e^{\lambda_2 t}$, In this case exponential growth for one of them and exponential decay for the other, therefore the explicit form $y'(t) = Cx'^{\frac{\lambda_2}{\lambda_1}}$ of the parametric curves (2.6) resembles the hyperbola and so the critical point $(0,0)$ is a saddle point. The possible phase portraits of the system in the transformed coordinates are shown in figure(2.6). By the change of coordinates, the original linear system (2.4) has the two linearly independent solutions $\mathbf{x_1}(t) = v_1 e^{\lambda_1 t}$ and $\mathbf{x_2}(t) = v_2 e^{\lambda_2 t}$.

Example 2.1.3. *Let us consider the matrix*

$$A = \begin{bmatrix} 5/4 & -3/4 \\ 3/4 & -5/4 \end{bmatrix}$$

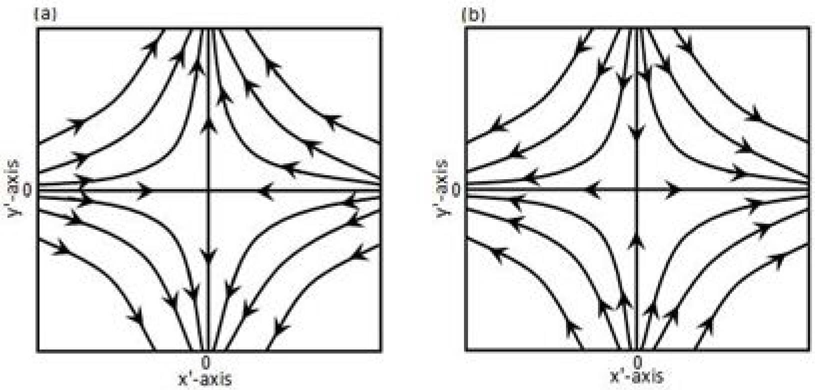

Figure 2.6: Phase portraits in which both λ_1 and λ_2 have different signs (a) $\lambda_1 < 0 < \lambda_2$, (b) $\lambda_2 < 0 < \lambda_1$

Then the eigenvalues of the matrix A are $\lambda_1 = 1$ and $\lambda_2 = -1$ and the eigenvectors of A corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$:

$$v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

We consider the system $\dot{\mathbf{x}} = A\mathbf{x}$, and change of the coordinates by taking

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix},$$

whose columns are the eigenvectors as the new basis of the system. We get the diagonal system with the diagonal entries $\lambda_1 = 1$ and $\lambda_2 = -1$. Then the solution trajectories of the diagonal system are $x'(t) = c_1 e^t$ and $y'(t) = c_2 e^{-t}$. By changing the coordinates, the general solution of the original system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

The solution trajectories of original system are described by the adjoining figure (2.7).

Real and Equal Eigenvalues

Suppose $\lambda_0 = \lambda_1 = \lambda_2$ and λ_0 be real and the characteristics polynomial of A be $(\lambda - \lambda_0)^2$. By the Cayley-Hamilton theorem $(A - \lambda_0 I)^2 = 0$, it may arise two cases

Case1: $A = \lambda_0 I$

In this case, the coefficients matrix A of (2.4) has two linearly independent eigenvectors v_1 and v_2. So, the matrix A is diagonalizable. Change of coordinates by taking the

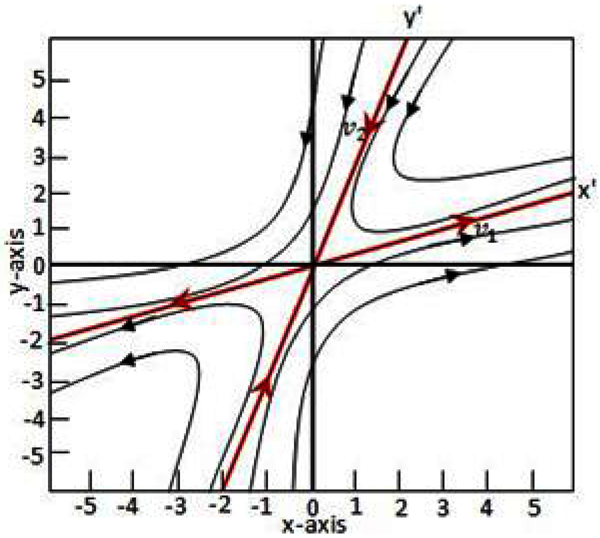

Figure 2.7: The saddle point in example (2.1.3)

invertible matrix P, whose columns are the eigenvectors of A such that $A = PDP^{-1}$, then we get the diagonal system $\dot{\mathbf{x}}' = D\mathbf{x}'$ from (2.6), the exponential $x'(t) = c_1 e^{\lambda_0 t}$ and $y'(t) = c_2 e^{\lambda_0 t}$ therefore the explicit form $y'(t) = Cx'(t)$ of the parametric curves (2.6) gives the straight lines through the critical point $(0, 0)$. Therefore the critical point $(0, 0)$ is proper node. If $\lambda_0 > 0$, the critical point is an unstable and if $\lambda_0 < 0$, the critical point is a stable. The possible phase portrait in the transformed coordinates of the system is shown [figure (2.8)]. We see that by change of coordinates the original linear system

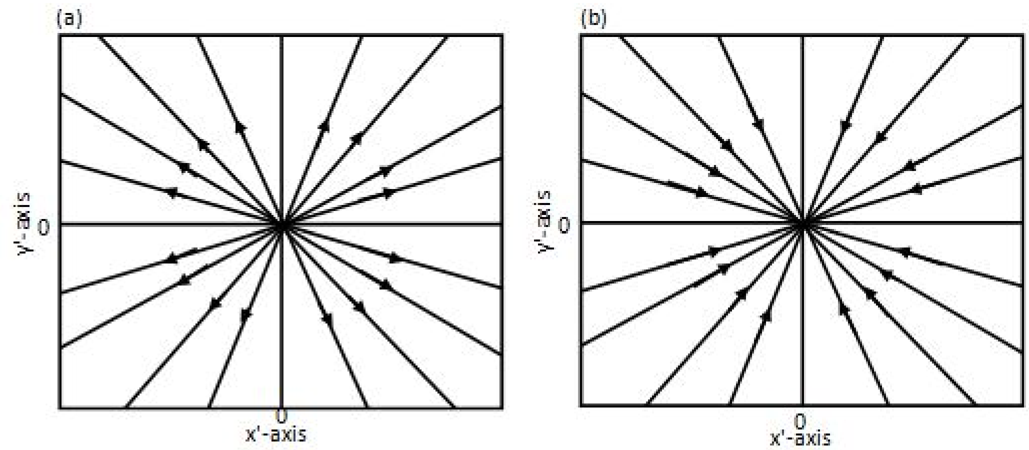

Figure 2.8: The phase portrait in which $\lambda_1 = \lambda_2 = \lambda_0$ (a) is positive, (b) is negative.

(2.4) has the two linearly independent solutions $\mathbf{x_1}(t) = v_1 e^{\lambda_0 t}$ and $\mathbf{x_2}(t) = v_2 e^{\lambda_0 t}$.

Case2: $A \neq \lambda_0 I$

Let $N = A - \lambda_0 I$. Since we have $N^2 = 0$, but $N \neq 0$, therefore N is nilpotent. Let v_1 be an eigenvector corresponding to the eigenvalue λ_0, we can find another eigenvector v_2 such that $v_1 = Nv_2 \neq 0$. Note that $Nv_1 = N^2 v_2 = 0$, we claim that v_1 and v_2 are linearly independent eigenvectors. For this suppose $c_1 v_1 + c_2 v_2 = 0$ for some scalars c_1 and c_2. Then, applying N both sides of this equation we get $c_1 N v_1 + c_2 N v_2 = 0$, which implies that $c_1 N v_1 + c_2 v_1 = 0$. Therefore $c_2 v_1 = 0$ implies $c_2 = 0$ since $v_2 \neq 0$ and this is in turn implies that $c_1 = 0$. Hence v_1 and v_2 are linearly independent. By using the definition of N, we see that

$$\begin{aligned} Av_1 &= \lambda_0 v_1 \\ Av_2 &= \lambda_0 v_2 + v_1 \end{aligned}$$

Thus, if we change to the canonical coordinate system given by $P = [v_1, v_2]$, a matrix of linear transformation given by a multiplication of A is

$$B = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}.$$

By changing the coordinates, the system (2.4) can be transformed to

$$\dot{\mathbf{x}}' = B\mathbf{x}'.$$

To solve the system we decompose the matrix B as the sum of diagonal(D) and nilpotent(N) matrix, that is

$$B = D + N = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Clearly N and D commute, since D is a scalar multiple of identity matrix. Note that $N \neq 0$, clearly $N^2 = 0$, so N is nilpotent. It follows that

$$\begin{aligned} e^{Bt} &= e^{Dt}[I + Nt] \\ &= \begin{bmatrix} e^{\lambda_0 t} & 0 \\ 0 & e^{\lambda_0 t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_0 t} & te^{\lambda_0 t} \\ 0 & e^{\lambda_0 t} \end{bmatrix} \end{aligned}$$

Thus the general solution trajectories in the canonical coordinates system are described by

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_0 t} & te^{\lambda_0 t} \\ 0 & e^{\lambda_0 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Therefore the exponential solutions are

$$x'(t) = c_1 e^{\lambda_0 t} + c_2 t e^{\lambda_0 t} \text{ and } y'(t) = c_2 e^{\lambda_0 t}$$

If $c_2 = 0$, then the solution trajectory lies along x'-axis, otherwise we have a non-linear trajectories with

$$\frac{dy'}{dx'} = \frac{dy'/dt}{dx'/dt} = \frac{\lambda_0 c_2 e^{\lambda_0 t}}{c_2 e^{\lambda_0 t} + \lambda_0(c_1 + c_2 t)e^{\lambda_0 t}} = \frac{\lambda_0 c_2}{c_2 + \lambda_0(c_1 + c_2 t)}$$

We see that $\frac{dy'}{dx'} \to 0$ as $t \to \infty$, so it follows that each trajectory is tangent to the x'-axis. Therefore the critical point $(0, 0)$ is an improper node. If $\lambda_0 < 0$, the critical point $(0, 0)$ is a stable and if $\lambda_0 > 0$, then the critical point $(0, 0)$ is an unstable. The possible phase portrait of the system is shown in figure (2.9). We see that by change of coordinates

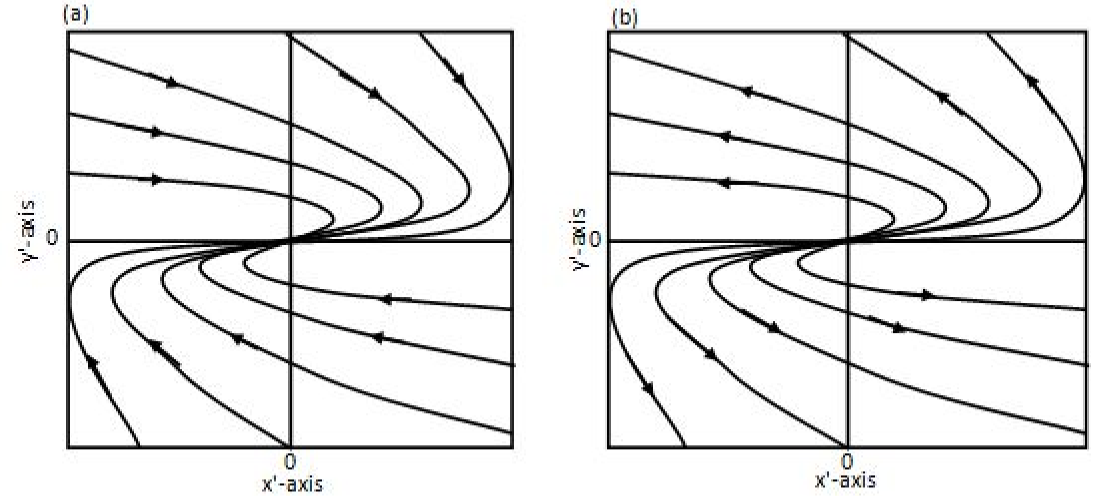

Figure 2.9: The phase portrait in which $\lambda_1 = \lambda_2 = \lambda_0$ (a) is negative, (b) is positive.

the original linear system (2.4) [where $A = PBP^{-1}$,and P is invertible matrix whose columns are eigenvectors of A] has the two linearly independent solutions

$$\mathbf{x_1}(t) = v_1 e^{\lambda_0 t} \text{ and } \mathbf{x_2}(t) = (v_2 + v_1 t)e^{\lambda_2 t}$$

Example 2.1.4. *Let us consider the matrix*

$$A = \begin{bmatrix} -11/8 & 9/8 \\ -1/8 & -5/8 \end{bmatrix}$$

The eigenvalue of the matrix A is $\lambda_0 = -1$ with the multiplicity 2, then the eigenvector of the matrix A corresponding to the eigenvalue $\lambda_0 = -1$ is $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Let $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ be the generalized eigenvector of A such that $(A - \lambda_0 I)v_2 = v_1$. Since v_1 and v_2 are two linearly independent eigenvectors, and change of coordinates by taking invertible matrix P such that $A = PBP^{-1}$ where,

$$B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Then, we get the canonical coordinates system whose solution trajectories are $x'(t) = (c_1 + c_2 t)e^{-t}$ and $y'(t) = c_2 e^{-t}$. By changing coordinates the general linearly independent solutions of the original system are $x_1(t) = v_1 e^{-t}$ and $x_2(t) = (v_2 + v_1 t)e^{-t}$. The solution trajectories of the system are shown in figure (2.10).

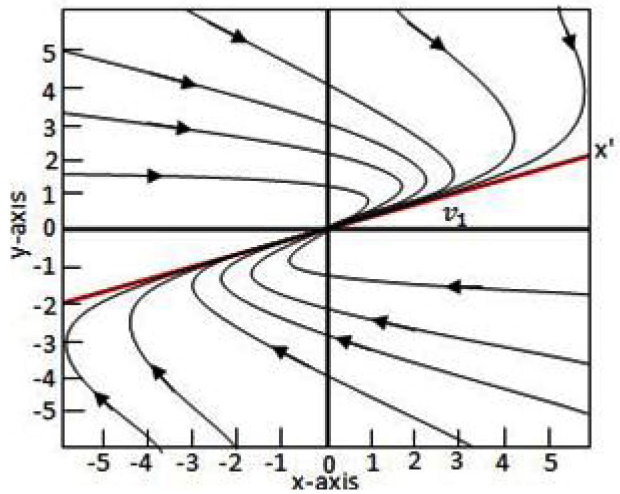

Figure 2.10: The improper node in the example $(2.1.4)$

Complex Conjugate Eigenvalues

Suppose $\lambda{=}\alpha{+}\beta i$ and $\overline{\lambda} = \alpha{-}\beta i$ be two complex conjugate eigenvalues where α and β are real, then the real canonical form for coefficient matrix A is,

$$M = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

change of coordinate by taking the invertible matrix P whose column are real eigenvectors of A such that $A = PMP^{-1}$ then the system (2.4) can be written as

$$\dot{\mathbf{x}}' = M\mathbf{x}'.$$

We can write

$$e^{Mt} = e^{\alpha t} \begin{bmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{bmatrix}.$$

The general solutions of real canonical system are

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{\alpha t}\cos\beta t - c_2 e^{\alpha t}\sin\beta t \\ c_1 e^{\alpha t}\sin\beta t + c_2 e^{\alpha t}\cos\beta t \end{bmatrix}$$

Therefore

$$x'(t) = c_1 e^{\alpha t}\cos\beta t - c_2 e^{\alpha t}\sin\beta t \;\text{ and }\; y'(t) = c_1 e^{\alpha t}\sin\beta t + c_2 e^{\alpha t}\cos\beta t \tag{2.9}$$

The equations (2.9) involve the combination of the exponential function $e^{\alpha t}$ with the periodic functions $\sin\beta t$ and $\cos\beta t$, therefore the solution trajectories $x_1(t)$ and $x_2(t)$ are either grow or decay that depends on the sign of α.

Case1: $\alpha<0$

In this case, exponential factors will go towards the origin as $t \to \infty$, while the other factor of (2.9) simply oscillate, therefore the possible phase portrait of the system spiral inwards to the origin. So the critical point $(0,0)$ is a stable spiral point. It is shown in figure (2.11).

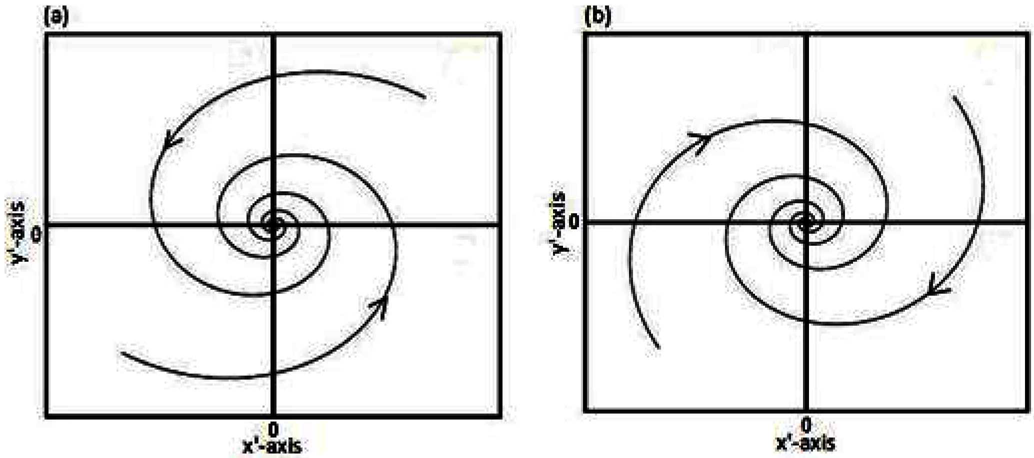

Figure 2.11: Phase portrait in which complex conjugate eigenvalues with $(\alpha < 0)$ (a) trajectories oriented counterclockwise$(\beta > 0)$, (b) trajectories oriented clockwise$(\beta < 0)$

Case2: $\alpha>0$

If $\alpha>0$, the exponential factor will grow from the origin as $t \to \infty$ while the other factors simply oscillate, therefore the possible phase portrait of the system simply spiral outwards from the origin so the critical point $(0,0)$ is an unstable spiral point. This can be shown in figure (2.12). Suppose $w = v_1 + iv_2$, where v_1 and v_2 are real eigenvectors of the complex eigenvector associate with eigenvalue λ=α+βi, then the complex valued solution of the original system $\dot{\mathbf{x}} = A\mathbf{x}$ associated with w and λ is

$$\mathbf{x}(t) = we^{\lambda t} = e^{\alpha}(v_1 + iv_2)(\cos\beta t + i\sin\beta t)$$

that is

$$\mathbf{x}(t) = e^{\alpha t}(v_1\cos\beta t - v_2\sin\beta t) + ie^{\alpha t}(v_1\sin\beta t + v_2\cos\beta t).$$

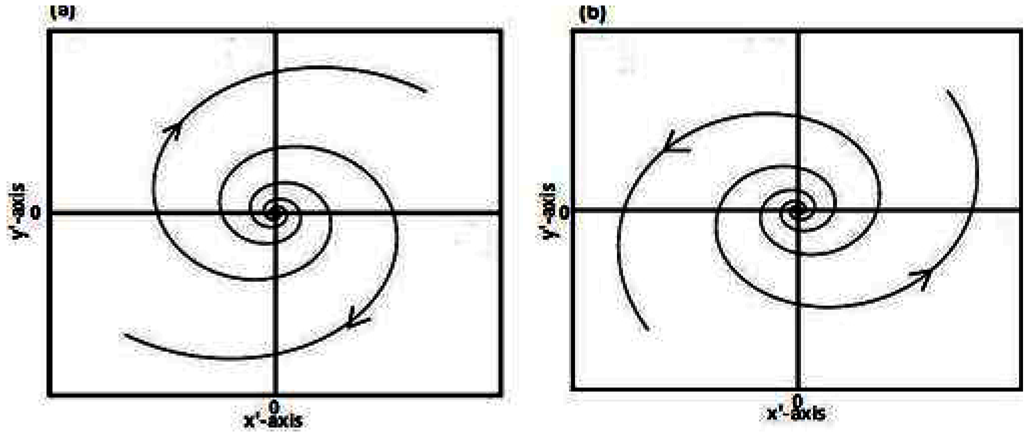

Figure 2.12: Phase portrait in which complex conjugate eigenvalues with $(\alpha > 0)$ (a) trajectories oriented clockwise $(\beta < 0)$, (b) trajectories oriented counterclockwise $(\beta > 0)$

Because the real and imaginary parts of the complex valued solutions are also the solution of the system. We thus get the two linearly independent real valued solutions of the system as follows:

$$\mathbf{x_1}(t) = e^{\alpha t}(v_1 \cos \beta t - v_2 \sin \beta t) \text{ and } \mathbf{x_2}(t) = e^{\alpha t}(v_1 \sin \beta t + v_2 \cos \beta t)$$

which associate with the complex conjugate eigenvalues. It is easy to check that the two same real valued solution resulting from taking the eigenvectors $\overline{w} = v_1 - iv_2$ corresponding to the eigenvalues $\overline{\lambda} = \alpha - \beta i$. Thus the components $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ of any solution $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$ oscillate between positive and negative values as $t \to \infty$, so the critical point $(0, 0)$ is a spiral point.

Example 2.1.5. *Let us consider the matrix*

$$A = \frac{1}{4}\begin{bmatrix} -10 & 15 \\ -15 & 8 \end{bmatrix}$$

Then the complex conjugate eigenvalues of the matrix A are $\lambda = -\frac{1}{4} + 3i$ and $\overline{\lambda} = -\frac{1}{4} - 3i$ and the complex conjugate eigenvectors of A corresponding to the eigenvalues λ and $\overline{\lambda}$ are

$$w = \begin{bmatrix} 3 \\ 5 \end{bmatrix} - i \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad \text{and} \quad \overline{w} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + i \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

respectively with the real eigenvectors $v_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. We consider the system $\dot{x} = Ax$ and change the coordinates by taking

$$P = \begin{bmatrix} 3 & 5 \\ 4 & 0 \end{bmatrix},$$

whose columns are the real eigenvectors v_1 and v_2 of the matrix A. We get the real canonical form $\dot{\mathbf{x}}\prime = M\mathbf{x}'$ with the real canonical matrix

$$M = \begin{bmatrix} -1/4 & -3 \\ 3 & -1/4 \end{bmatrix}$$

associated with the complex conjugate eigenvalues λ and $\overline{\lambda}$. Then the solution trajectories of the real canonical system are $x'(t) = c_1 e^{-\frac{1}{4}t}\cos 3t - c_2 e^{-\frac{1}{4}t}\sin 3t$ and $y'(t) = c_1 e^{-\frac{1}{4}t}\sin 3t + c_2 e^{-\frac{1}{4}t}\cos 3t$. By change of the coordinates of the system, two real linearly independent solutions of the original system are

$$\mathbf{x_1}(t) = e^{-\frac{1}{4}t}(v_1 \cos 3t - v_2 \sin 3t) \text{ and } \mathbf{x_2}(t) = e^{-\frac{1}{4}t}(v_1 \sin 3t + v_2 \cos 3t)$$

The solution trajectory of the system is shown in the figure (2.13).

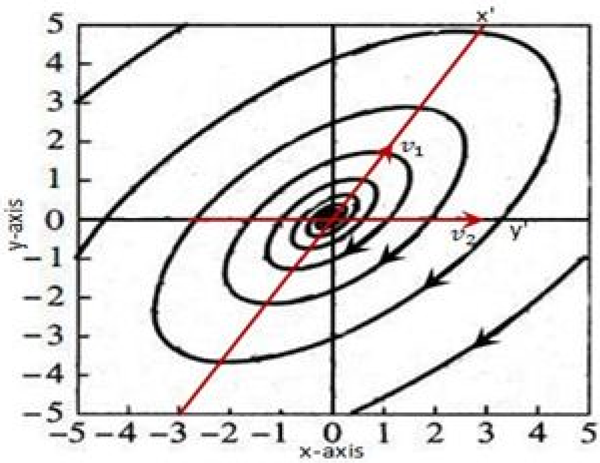

Figure 2.13: The spiral point in example $(2.1.5)$

Pure Imaginary Eigenvalues

If the matrix A has a pure imaginary conjugate eigenvalues say $\lambda = \beta i$ and $\overline{\lambda} = -\beta i$, then the general solution of the canonical system are

$$x'(t) = c_1 \cos \beta t - c_2 \sin \beta t \text{ and } y'(t) = c_1 \sin \beta t + c_2 \cos \beta t$$

which we get from (2.9) if we take $\alpha = 0$. The solutions $x_1(t)$ and $x_2(t)$ are stuck at a fixed point. Therefore the critical point $(0, 0)$ is a stable center. The possible phase portraits of the system are shown in figure (2.14). We see that the general solution of the original linear system (2.4) has the two linearly independent solutions.

$$\mathbf{x_1}(t) = (v_1 \cos \beta t - v_2 \sin \beta t) \text{ and } \mathbf{x_2}(t) = (v_1 \sin \beta t + v_2 \cos \beta t)$$

Thus the components $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ of any solution $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$ describe ellipse center at the origin. Therefore the critical point is a stable center.

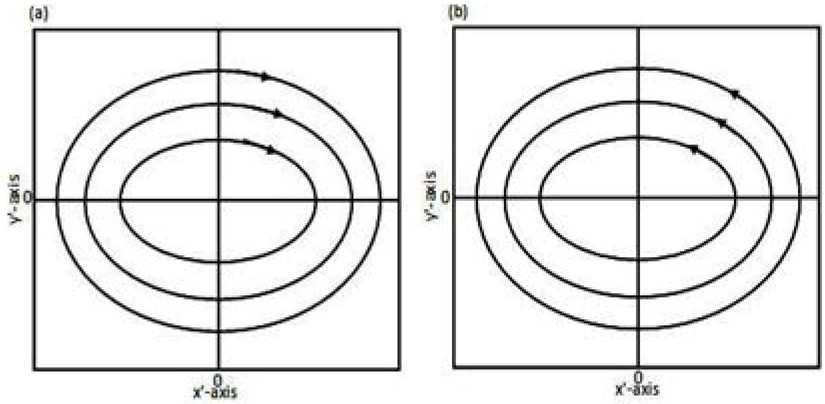

Figure 2.14: Phase portrait in which pure imaginary eigenvalues (a) trajectories oriented clockwise ($\beta < 0$), (b) trajectories oriented counterclockwise ($\beta > 0$)

Example 2.1.6. *Let us consider the matrix*

$$A = \frac{1}{4}\begin{bmatrix} -9 & 15 \\ -15 & 9 \end{bmatrix}$$

Then the complex conjugate eigenvalues with real part zero are $\lambda = 3i$ and $\overline{\lambda} = -3i$. And the complex conjugate eigenvectors of A corresponding to the eigenvalues λ and $\overline{\lambda}$ are

$$w = \begin{bmatrix} 3 \\ 5 \end{bmatrix} - i\begin{bmatrix} 4 \\ 0 \end{bmatrix} \text{ and } \overline{w} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + i\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

respectively with the real eigenvectors $v_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. We consider the system $\dot{x} = Ax$ and change the coordinates by taking

$$P = \begin{bmatrix} 3 & 5 \\ 4 & 0 \end{bmatrix}$$

whose columns are the real eigenvectors v_1 and v_2 of the matrix A. We get the real canonical form $\dot{\mathbf{x}}\prime = M\mathbf{x}'$ with the real canonical matrix

$$M = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$$

associate with the complex conjugate eigenvalues λ and $\overline{\lambda}$. Then the solution trajectories of the real canonical system are $x'(t) = c_1 \cos 3t - c_2 \sin 3t$ and $y'(t) = c_1 \sin 3t + c_2 \cos 3t$. By changing coordinates of the system, the two real linearly independent solutions of the original system are

$$\mathbf{x_1}(t) = v_1 \cos 3t - v_2 \sin 3t \text{ and } \mathbf{x_2}(t) = v_1 \sin 3t + v_2 \cos 3t$$

The solution trajectories of the system are shown in the figure (2.15).

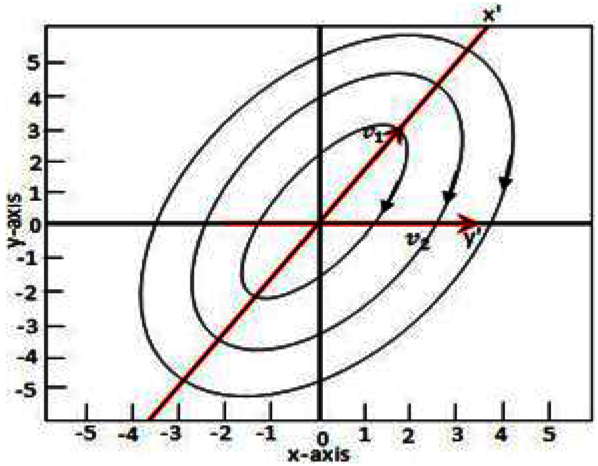

Figure 2.15: The stable point in example $(2.1.6)$

2.2 Linearization

Linearization helps us to describe the behavior of the nonlinear dynamical system

$$\dot{x} = f(x) \text{ with } x(0) = x_0 \tag{2.10}$$

Let $f : \mathbb{R} \to \mathbb{R}$ and x^* be an fixed point of(2.10). The derivatives of $f(x)$ at x^* determine the slopes of tangent of (2.10) which is a linear approximation of $f(x)$. The linear approximation of $f(x)$ near x^* is

$$\dot{x} = f(x) \approx f(x^*) + f'(x^*)(x - x^*) \tag{2.11}$$

Since x^* is a fixed point of the system (2.10) then $f(x^*) = 0$, and the equation (2.11) becomes

$$\dot{x} = f(x) \approx f'(x^*)(x - x^*) \tag{2.12}$$

let $y(t) = x(t) - x^*$ be the deviation of the solution from equilibrium. Since x^* is the fixed point then $\dot{y}(t) = \dot{x}(t)$, we rewrite the equation (2.12) as

$$\dot{y}(t) = ay(t) \tag{2.13}$$

where $a = f'(x^*)$ is the value of the deviation at the fixed point. Note that the original fixed point x^* corresponding to the fixed point $y^* = 0$ of the linearized equation (2.13). Then the original fixed point x^* is an asymptotically stable if and only if $a = f'(x^*) < 0$, that is if the fixed point of (2.13) is an asymptotically stable and if $a = f'(x^*) > 0$ and the original fixed point x^* is an unstable, that is if the fixed point $y^* = 0$ of the linearized equation (2.13) is an unstable.
More generally, suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ and $x^* \in \mathbb{R}^n$ be an fixed point of (2.10). In this

case, the linear approximation of $f(x) \in \mathbb{R}^n$ at x^* is given by

$$f_i(x_1, x_2, \dots, x_n) \approx f_i(x_1^*, x_2^*, \dots, x_n^*) + \frac{\partial f_i}{\partial x_1}(x_1^*, x_2^*, \dots, x_n^*)(x - x_1^*)$$
$$+ \frac{\partial f_i}{\partial x_2}(x_1^*, x_2^*, \dots, x_n^*)(x - x_2^*) + \dots + \frac{\partial f_i}{\partial x_n}(x_1^*, x_2^*, \dots, x_n^*)(x - x_n^*)$$

where $i = 1, 2, \dots, n$.
This equation can be written in a compact form using vector notation

$$f(x) \approx f(x^*) + Df(x^*)(x - x^*) \tag{2.14}$$

where

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \quad \text{and } Df(x^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=x^*}.$$

Here $Df(x^*)$ represents the $n \times n$ Jacobian matrix at fixed points x^*. Since x^* is a fixed point of the system (2.10), therefore $f(x^*) = 0$. Therefore the equation (2.14) becomes

$$\dot{x} = f(x) \approx Df(x^*)(x - x^*). \tag{2.15}$$

Let $y(t) = x(t) - x^*$ be the deviation of the solution from the fixed point. Since x^* be the fixed point, then $\dot{y}(t) = \dot{x}(t)$. We rewrite the equation (2.15) as

$$\dot{y}(t) = Ay(t) \ \text{ where } A = Df(x^*). \tag{2.16}$$

Here the fixed point of (2.16) is $y^* = 0$. The important result is that the original fixed point x^* is an asymptotically stable if all the eigenvalues of an $n \times n$ Jacobian matrix $Df(x^*)$ have negative real part, that is, the origin is an asymptotically stable for the linear system (2.16). And, the original fixed point x^* is an unstable if some eigenvalues of $Df(x^*)$ have a positive real part, that is, the origin is unstable for the linear system (2.16). Now, we state one of the most important theorems about the qualitative behavior of solution of non-linear dynamical system:

Theorem 2.2.1 (**Hartman-Grobman Theorem**). *Suppose x^* is an isolated and a hyperbolic equilibrium of a non linear system $\dot{x} = f(x)$. then the vicinity of x^*, the Linearization $\dot{x} = Df(x^*)(x - x^*)$ about that equilibrium has the same qualitative behavior for the original nonlinear system.*

The Hartman-Grobman theorem tells us that, the Linearization at the hyperbolic fixed point exhibits the same qualitative behavior as the original non-linear system. But this needs not be true for the non-hyperbolic fixed points.

2.3 Lyapunov Functions

Classifying non-hyperbolic fixed points $\mathbf{x}^*$ as stable, asymptotically stable, or unstable can be incredibly difficult (and often impossible). We now describe a classification technique that was originally proposed by Russian mathematician A.M. Lyapunov in his 1892s doctoral dissertation. Consider a dynamical system

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{2.17}$$

where f is continuously differential function . A solution $\mathbf{x}(t)$ of the equation can be written in term of its component functions $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)] \in \mathbb{R}^n$. Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a scalar valued function. Then by chain rule

$$\begin{aligned}
\frac{d}{dt}V(\mathbf{x}(t)) &= \frac{d}{dt}V[x_1(t), x_2(t), \ldots, x_n(t)] \\
&= \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt} \\
&= [\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}] \cdot [\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}] \\
&= \nabla V(\mathbf{x}) \cdot \dot{\mathbf{x}}(t) \\
&= \nabla V(\mathbf{x}) \cdot f(\mathbf{x}).
\end{aligned}$$

Suppose x^* is a fixed point of a dynamical system (2.17). A function V is said to be **Lyapunov Function** if it satisfies the following requirements:

1. V is continuous
2. $V(\mathbf{x})$ has a unique minimum at x^* with respect to all other points in $\mathbb{R}^n$ [i.e. $V(\mathbf{x}) > 0$ if $x \neq x^*$ and $V(x^*) = 0$]
3. The function $\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot f(\mathbf{x})$ satisfies $\dot{V}(\mathbf{x}) \leq 0$ for all $x \in \mathbb{R}^n$.

A function $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function and x^* is a fixed point of the system (2.17), then

1. The fixed point x^* is stable if $\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot f(\mathbf{x}) \leq 0$ for all $x \in \mathbb{R}^n$.
2. The fixed point x^* is asymptotically stable if $\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot f(\mathbf{x}) < 0$ which is possible for all $x \in \mathbb{R}^n - \{x^*\}$.
3. The fixed point x^* is unstable if $\dot{V}(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot f(\mathbf{x}) > 0$ which is possible for all $x \in \mathbb{R}^n - \{x^*\}$.

The function V can be visualized to construct the figure (2.16). The shape of a typical Lyaponov function for a two variables system of ODEs with an equilibrium at the

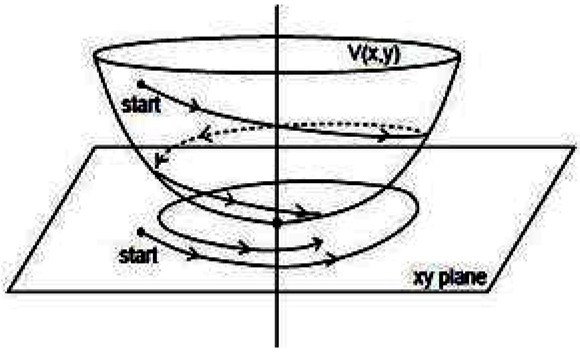

Figure 2.16: Illustration of the Lyapunov function for a two variable system with equilibrium at the origin

origin. In this case the graph of Lyapunoiv function $V(x, y)$ is a surface in $\mathbb{R}^3$, and is positive everywhere except at equilibrium. Any solution $(x(t), y(t))$ of the ODEs defined a parametrized curve in the $xy - plane$ is towards the origin, and we conclude that the equilibrium must be asymptotically stable. In general, finding a Lyapunov function is very difficult.

Example 2.3.1. *Let us consider a dynamical system*

$$\begin{aligned}\dot{x} &= -y - x(x^2 + y^2)\\ \dot{y} &= x - y(x^2 + y^2)\end{aligned}$$

To calculate the fixed point we set $\dot{x} = 0$ and $\dot{y} = 0$, then the system become

$$\begin{aligned}y &= -x(x^2 + y^2)\\ x &= y(x^2 + y^2)\end{aligned}$$

from these equation solving we get $x^* = (0, 0)$ which is unique. The Jacobian matrix associate with the system is

$$Df(x, y) = \begin{bmatrix}-3x^2 - y^2 & -1 - 2xy\\ 1 - 2xy & -x^2 - 3y^2\end{bmatrix}$$

Now, we claim that the equilibrium $x^* = (0, 0)$ is non-hyperbolic. At the fixed point yields

$$Df(0, 0) = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

The Linearization of our original system is

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

and the eigenvalues of the coefficient matrix are $\lambda = \pm i$. Both eigenvalues have zero real part, thus the origin is non-hyperbolic equilibrium. We also show that the origin is asymptotically stable. Let us consider a scalar valued function $V : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$V(x, y) = x^2 + y^2$$

we claim that V is the Lyapunov function for the equilibrium at the origin. V is continuous, clearly $V(0, 0) = 0$ and $V(x, y) > 0$ if $(x, y) \neq (0, 0)$. Moreover, note that $\nabla V(x, y) \cdot f(x, y)$ is given by

$$\begin{aligned}
\dot{V}(x, y) &= \nabla V(x, y) \cdot f(x, y) \\
&= (2x, 2y) \cdot (-y - x(x^2 + y^2), x - y(x^2 + y^2)) \\
&= -2xy - 2x^2(x^2 + y^2) + 2xy - 2y^2(x^2 + y^2) \\
\therefore \dot{V}(x, y) &= -2(x^2 + y^2)^2
\end{aligned}$$

Since $\dot{V} < 0$ except at $(0, 0)$, we conclude from Lyapunov criteria that the origin is asymptotically stable.

Chapter 3

Bifurcation in Dynamical System

All solutions of a dynamical system either settle down to equilibrium or head out to $\pm\infty$. Given the triviality of dynamical system, the most interesting thing about one-dimensional system is dependence on a parameters. Fixed points can be created or destroyed, or their stability can change as parameters are varied. These qualitative changes in the dynamics are called *bifurcation*, and the parameters value at which they occur are called *bifurcation points*.
Bifurcation are important sufficiently, because they provide models of transitions and instabilities as some control parameter is varied. For example, consider the buckling of a beam. If a small weight is placed on top of beam as in figure (3.1). The beam can support

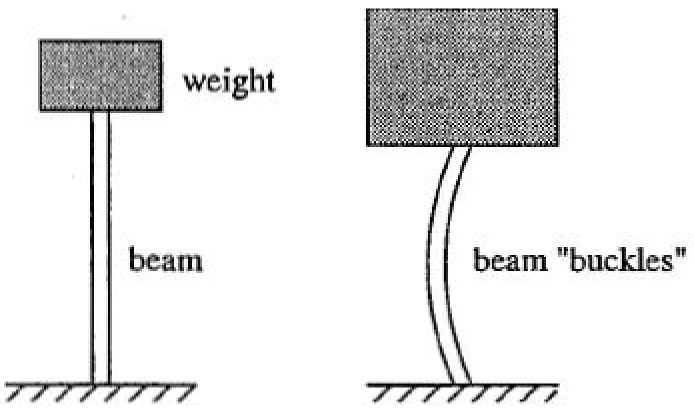

Figure 3.1: A weight being supported by a vertical beam

the load remain vertical. But, if the load is too heavy, the vertical position becomes an unstable and the beam may buckle. Here the weight plays the vital role of the control parameter, and the deflection of the beam from vertical plays the role of the dynamical variables x.
Here we discuss the most common types of bifurcation.

3.1 Saddle-node Bifurcation

Saddle-node bifurcation is the basic mechanism by which fixed points of a dynamical system are created and destroyed. As a parameter is varied, two fixed points move towards each other, collide, and mutually annihilate.

Example 3.1.1. *Let us consider the first order system*

$$\dot{x} = f(x, r) = r - x^2 \tag{3.1}$$

where r is a parameter, which may be positive, negative or zero

The figure (3.2) shows the different behaviors of the system (3.1) for different cases of parameter r. For $r > 0$ we have two fixed points, one stable and other unstable. As

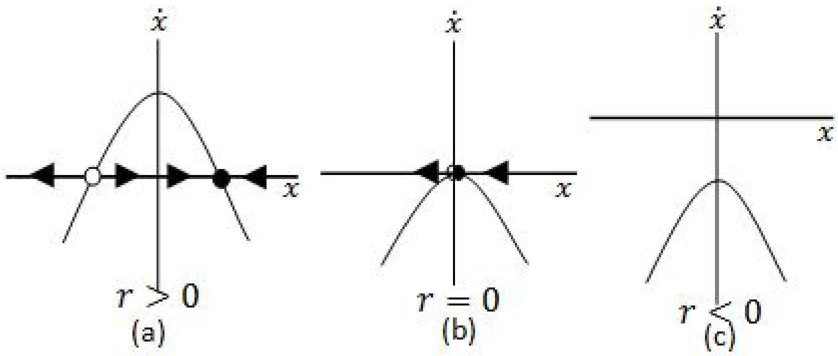

Figure 3.2: Vector fields of the system (3.1) for different cases of r

r approaches 0 from above, the parabola moves down and the two fixed points move towards each other. When $r = 0$, the fixed points coalesce into a half-stable fixed point

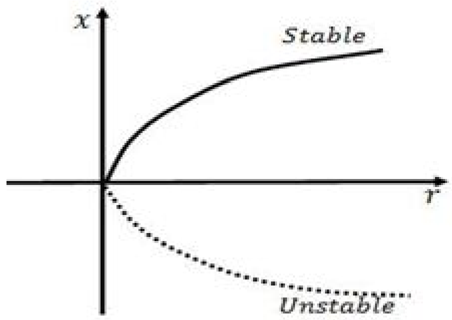

Figure 3.3: Saddle node bifurcation

at $x^* = 0$. This type of fixed point is extremely delicate-it vanished as soon as $r < 0$ and now there are no fixed points at all.

In this example, we say that a *bifurcation* occurs at $r = 0$, since the vector field for $r > 0$ and $r < 0$ are qualitatively different. The bifurcation diagram is shown in figure (3.3).$\dot{x} = 0$ gives the fixed points, for different values of r. Therefore, $x^2 = r$. The fixed points for $f(x) = r - x^2$ are given by $x^* = \pm\sqrt{r}$. To determine the linear stability, we compute $f'(x^*) = -2x^*$. Thus $x^* = \sqrt{r}$ is stable, since $f'(x^*) < 0$. Similarly, $x^* = -\sqrt{r}$ is unstable. At the bifurcation point $r = 0$, we find $f'(x^*) = 0$, the linearization vanishes when the fixed point vanishes.

3.2 Transcritical Bifurcation

Transcritical bifurcation is a standard mechanism by which fixed points of a dynamical system must exist for all values of the parameter and can never be destroyed. The fixed point may change its stability as the parameter is varied.

Example 3.2.1. *Let us consider a dynamical system with the parameter value* r

$$\dot{x} = f(x, r) = rx - x^2 \tag{3.2}$$

The figure (3.4) shows the vector field of (3.2) as r is varied. Note that there exist a fixed point at $x^* = 0$ for all values of r.
For $r < 0$, there is an unstable fixed point at $x^* = r$ and a stable fixed point at $x^* = 0$. As

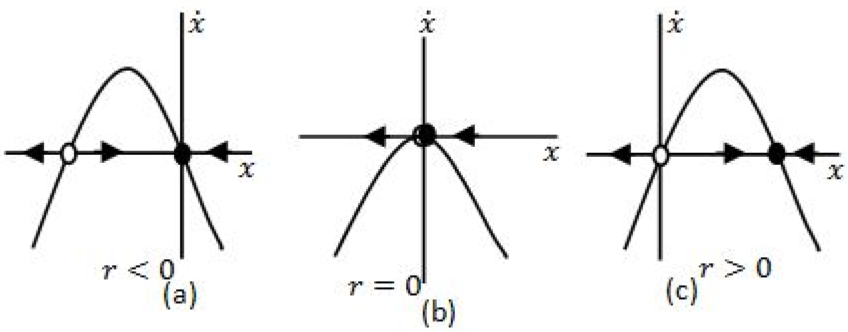

Figure 3.4: Vector fields as r is varied

r increases, the unstable fixed point approaches the origin, and coalesces with it when $r = 0$. Finally, when $r > 0$ the origin has become unstable, and $x^* = r$ is now stable. Figure (3.5) shows the bifurcation diagram for the transcritial bifurcation, where parameter r is regarded as the independent variable, and the fixed points $x^* = 0$ and $x^* = r$ are shown as the dependent variables.

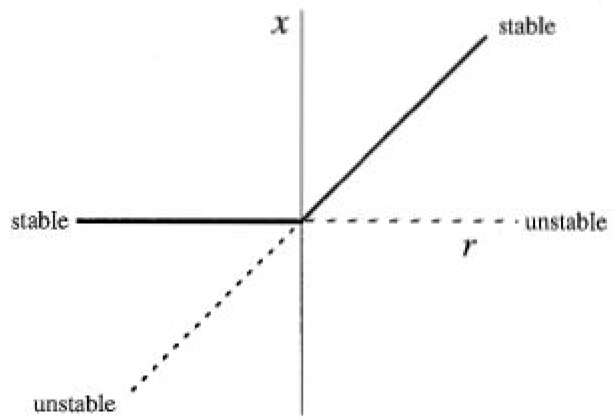

Figure 3.5: Transcritical bifurcation

3.3 Pitchfork Bifurcation

In a pitchfork bifurcation, fixed points tend to appear and disappear in symmetric pairs. For example in figure (3.1) the beam is stable in the vertical position if the load is small. In this case, there is a stable fixed point corresponding to zero deflection. But if the load exceeds the buckling threshold, the beam may buckle to either the left or the right. The vertical position has gone unstable, and two new symmetrical fixed points corresponding to left and right buckled configurations, have been born.
There are two different types of pitchfork bifurcation. The simpler type is called *super-critical* and another is *sub-critical*.

3.3.1 Supercritical Pitchfork Bifurcation

If the two new equilibriums occur after the bifurcation, then the point is called supercritical pitchfork bifurcation.

Example 3.3.1. *Let us consider a dynamical system with the parameter value* r

$$\dot{x} = f(x, r) = rx - x^3 \tag{3.3}$$

This equation is invariant under the change of variables $x \to -x$. This invariance is the mathematical expression of the left-right symmetry mentioned earlier. Figure (3.6) shows the vector field of different values of r. Note that there exists a fixed point at $x^* = 0$ for all values of r.
When $r < 0$, there is only the fixed point at $x^* = 0$, and it is stable. When $r = 0$, the origin is still stable, but much more weakly so, since the linearization vanishes. Finally, when $r > 0$, the origin has become unstable. Two stable fixed points appear on either side of the origin, symmetrically located at $x^* = \pm\sqrt{r}$ (we set $\dot{x} = 0$, for the fixed point). The

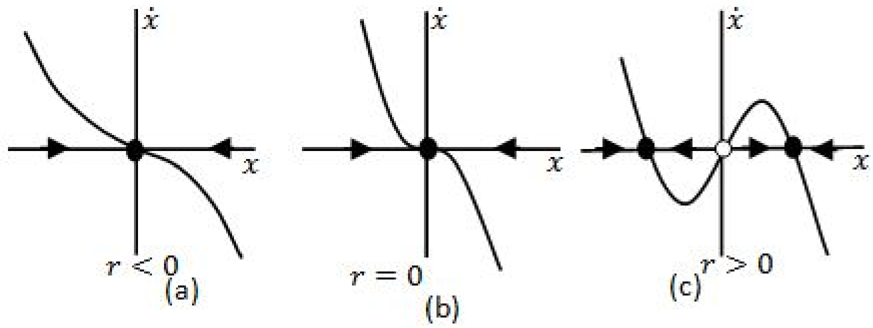

Figure 3.6: Vector fields of the system(3.3) for different values of r

reason for the term ,“ supercritical pitchfork” becomes clear when we plot the bifurcation diagram (figure (3.7)), which is better to say pitchfork trifurcation.

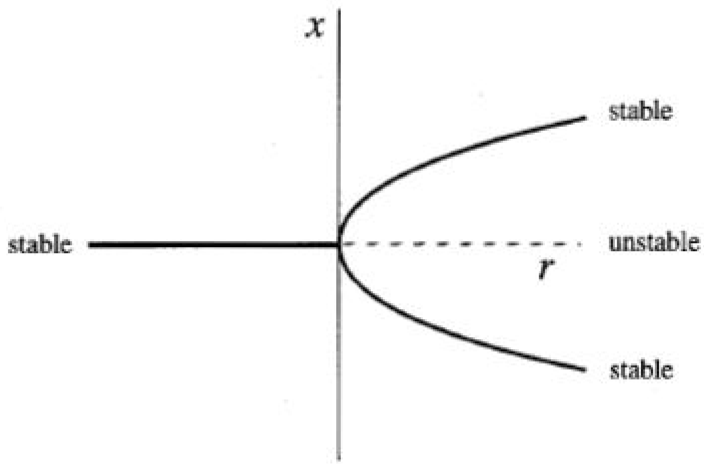

Figure 3.7: Supercritical pitchfork bifurcation

3.3.2 Sub-critical Pitchfork Bifurcation

If the two new equilibriums occur below the bifurcation point is called sub-critical pitchfork bifurcation.

Example 3.3.2.

$$\dot{x} = f(x, r) = rx + x^3 \tag{3.4}$$

where r is a parameter, which may be positive, negative or zero

This equation is invariant under the change of variable $x \to -x$. The vector fields of the different values of r is shown in the figure (3.8).

When $r > 0$ there is a only one fixed point at $x^* = 0$, which is unstable. When $r = 0$, the

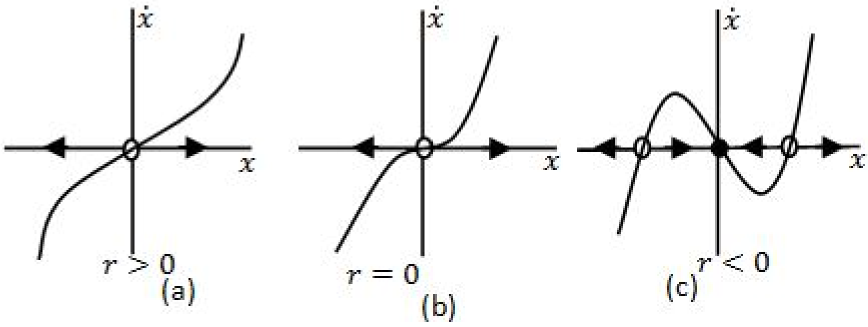

Figure 3.8: Vector fields

origin is still unstable. Finally, when $r < 0$, the origin has become stable. Two unstable fixed points appear, symmetrically located at $x^* = \pm\sqrt{-r}$, and exists only below the bifurcation $(r < 0)$, which motivates the term "sub-critical". Figure (3.9) shows the bifurcation diagram.

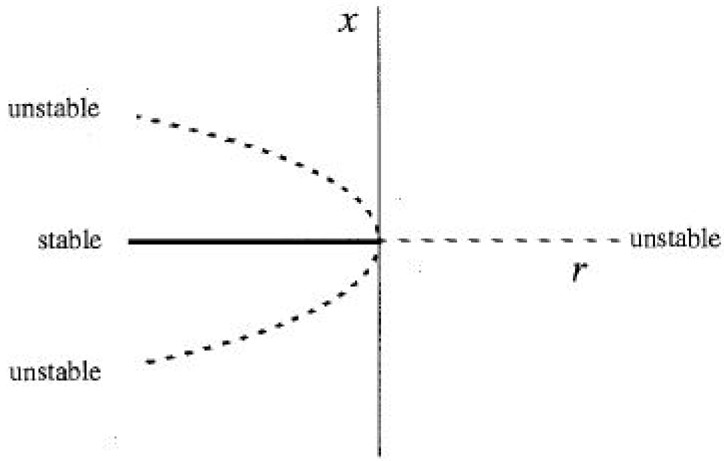

Figure 3.9: Sub-critical pitchfork bifurcation

3.4 Hopf-Bifurcation

Suppose two-dimensional system has a stable fixed point. The possible ways it could lose stability as a parameter r is varied, which depend on a eigenvalues of the Jacobin. If fixed point is stable, the real part of eigenvalues λ_1, λ_2 are negative i.e. Re$\lambda < 0$. Since the λ's satisfy a quadratic equation with real coefficients, then either the eigenvalues are both real

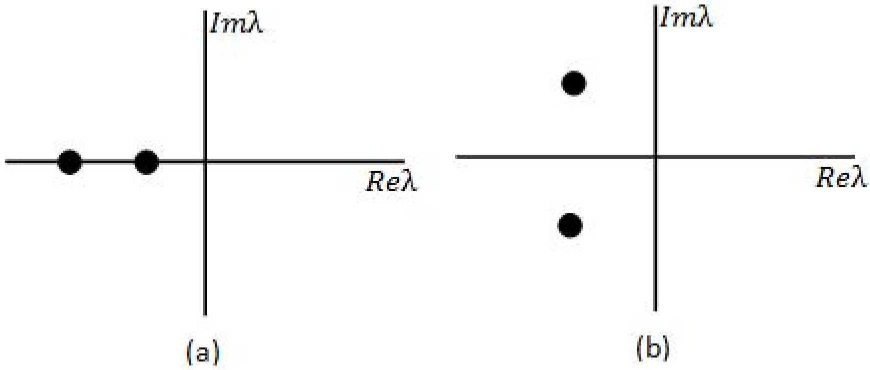

Figure 3.10: Possible picture of eigenvalues

and negative (Figure $(3.10(a))$) or they are complex conjugates (Figure $(3.10(b))$). The fixed points become unstable, if one or both of eigenvalues become positive as r varies. If the complex conjugate eigenvalues simultaneously passes through Re$\lambda = 0$, then we say that the system undergo a Hopf-bifurcation as r varies. That is the Hopf-bifurcation occur at $r = r_0$ when the pair of complex conjugate eigenvalues of Jacobin at the fixed points becomes the purely imaginary eigenvalues.

Chapter 4

Chaos in Dynamical System

Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.
There are three key phrases that appear in this definition. By *aperiodic long term behavior*, we typically mean that the system has solution that remain bounded but never convergent to a fixed point or periodic orbit. By *deterministic*, we mean that the system is not allowed to incorporate any randomness in its parameters and or inputs. The system must be able to produce erratic, aperiodic behavior on its own. Finally, *sensitive dependence on initial conditions* means that if we start from two different initial conditions these are "nearby", then the corresponding solutions trajectories will separate exponentially fast.
The chaotic behavior is described by following dynamical systems.

4.1 Lorenz System

The famous American Mathematician/meteorologist Edward N. Lorenz found the strange behavior of the differential equation that he used in his work in metrology. He noticed that a small changes in the initial condition led to drastic changes in the behavior of the solution at the later times. In 1963, Lorenz involve the set of differential equations to explain the unpredictable behavior of the weather such a system is called the Lorenz system of differential equations. The Lorenz original derivation of these equations are from a model for fluid flow of the atmosphere. Two dimensional fluid cell is warmed from below and cooled from above and resulting convective motion is modeled by a partial differential equation involving infinitely many variables and all except three of them are put zero. Then, the remaining independent variables we say x, which is proportional to intensity of the convection motion, y is proportional to the horizontal temperature variation and z is proportional to the vertical temperature variation. The resulting motion led to a three dimensional system of differential equations that involve three parameters the Prandtl number σ, Rayleigh number r and another parameter b have a physical interpretation, where σ is the quotient of the viscosity and the thermal conductivity r is essentially the temperature difference of the heated layer and b is related to the physical size of the system. When all of these simplifications were made the system of differential equations involve only two nonlinear terms that is the Lorenz system of differential equations, which was given in the form

$$\begin{aligned} \dot{x} &= \sigma(y - x) \qquad (4.1) \\ \dot{y} &= rx - xz - y \\ \dot{z} &= xy - bz \end{aligned}$$

In this system all three parameters are assumed to be positive, moreover $\sigma > b+1$. Lorenz derived this three-dimensional system from a drastically simplified model of convection rolls in the atmosphere. The same equation also arise in models of lasers and dynamos, they exactly describe the motion of a certain waterwheel.

4.1.1 Derivation of the Lorenz equation

The Lorenz equation is derived from the equation of motion of chaotic waterwheel and then performing the change of variables. A schematic diagram of a chaotic waterwheel is shown in the figure (4.1). The wheel sits on a top of the table and it rotates in a plane that is tilted slightly from the horizontal, water is pumped up into an overhanging manifolds and then sprayed out through the dozen of small nozzles. The nozzles direct the water into separate chambers around the rim of the wheel. The water leaks out through the small whole at the bottom of each chamber, and then collect underneath the wheel, where it is pumped back up through the nozzle. The system provides a steady inputs of water.
The parameters can be change in two ways. A break on the wheel can be adjusted to add more or less friction. The tilt of the wheel can be varied by turning a screw that props the wheel up; this alters the effective strength of gravity.
Let us define our variables and parameters and explain what each entails:

θ - The angle of the motion in the lab frame.

$\omega(t)$ - Angular velocity of the wheel increasing in the counterclockwise direction.

$m(\theta, t)$ - Mass distribution of the water around the rim of the wheel. The total mass of water between any two points θ_1 and θ_2 is given by $M(t) - \int_{\theta_1}^{\theta_2} m(\theta, t)\, d\theta$.

$Q(\theta)$ - Rate at which water is pumped into the chamber.

ρ - Radius of the wheel.

g - Gravity which can be changed by increasing or decreasing the tilt of the waterwheel.

K - Leakage rate.

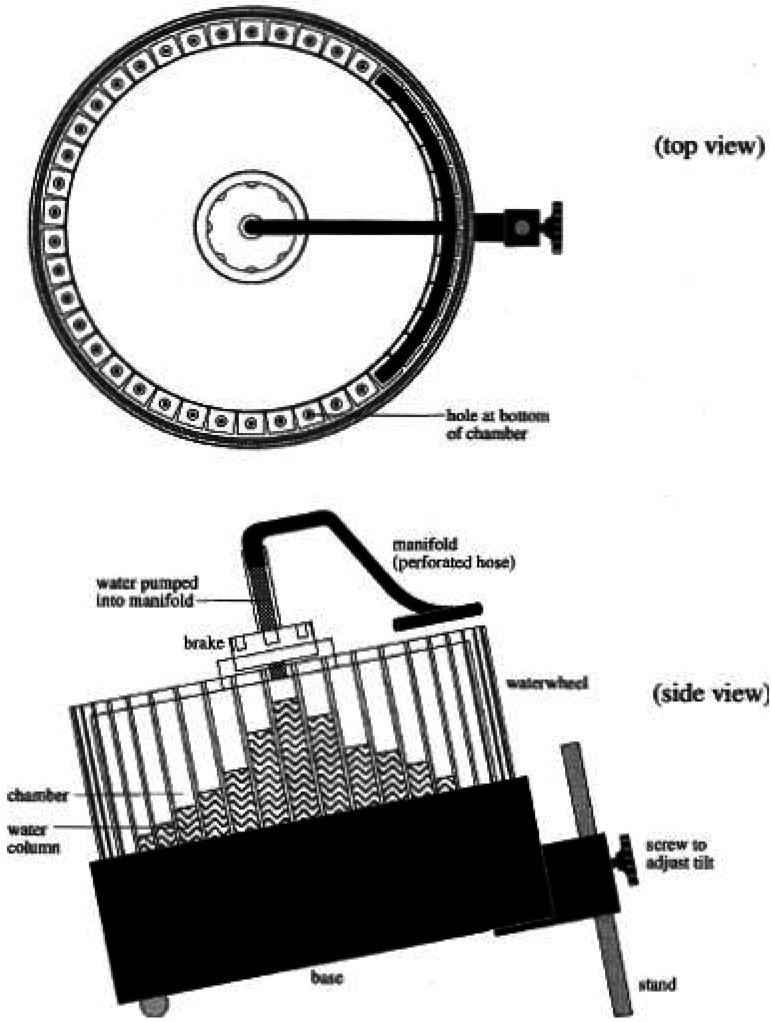

Figure 4.1: Illustration of the chaotic waterwheel

v - Rotational damping rate.

I - Momentum of inertia of the wheel.

Our goal is to derive the governing equations of $\omega(t)$ and $m(\theta, t)$ and then convert these equations into a system of ODEs. For this, we use *conservation of Mass*, *torque balance*, *the amplitude equation* etc.

Conservation of Mass:
Consider any section of the waterwheel in $[\theta_1, \theta_2]$ as in figure (4.2). Within any such section there are four contributions that govern how the mass ΔM changes in a time Δt.

1. The total mass pumped in by the nozzles is $\left[\int_{\theta_1}^{\theta_2} Q d\theta\right] \Delta t$.

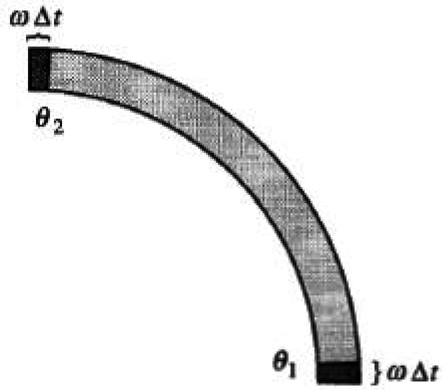

Figure 4.2: Section of the waterwheel

2. The mass of the water leaks from the system $\left[-\int_{\theta_1}^{\theta_2} Km\, d\theta\right]\Delta t$. Notice the factor of m in the integral implies that leakage occur at the rate proportional to the mass of water in the chamber, more water implies the large pressure head and therefore faster leakage. Although this is plausible physically, the fluid mechanics of leakage is complicated, and other rules are conceivable as well. The real justification for the rule above is that it agrees with direct measurements on the waterwheel itself to a good approximation.

3. The mass carried in by the rotating wheel is given by $m(\theta_1)\omega\Delta t$, where $m(\theta_1)$ is the mass per unit angle and $\omega\Delta t$ is the angular width, which is shown in figure (4.2).

4. Similarly, the mass carried out of the sector is $-m(\theta_2)\omega\Delta t$.

Combining these four we get,

$$\Delta M = \Delta t\left[\int_{\theta_1}^{\theta_2} Q d\theta - \int_{\theta_1}^{\theta_2} Kmd\theta\right] + \left[m(\theta_1) - m(\theta_2)\right]\omega\Delta t \tag{4.2}$$

We know

$$m(\theta_1) - m(\theta_2) = -\int_{\theta_1}^{\theta_2} \frac{\partial m}{\partial \theta} d\theta$$

Using this terms in equation (4.2) we get,

$$\begin{aligned}\Delta M &= \Delta t\left[\int_{\theta_1}^{\theta_2} Q d\theta - \int_{\theta_1}^{\theta_2} Kmd\theta\right] - \left[\int_{\theta_1}^{\theta_2} \frac{\partial m}{\partial \theta} d\theta\right]\omega\Delta t \\ &= \Delta t\left[\int_{\theta_1}^{\theta_2} (Q - Km - \omega\frac{\partial m}{\partial \theta})\right] d\theta\end{aligned}$$

Dividing both sides by Δt and taking $\Delta t \to 0$ we get,

$$\frac{dM}{dt} = \int_{\theta_1}^{\theta_2} (Q - Km - \omega\frac{\partial m}{\partial \theta}) d\theta \tag{4.3}$$

By the definition of M

$$\frac{dM}{dt} = \int_{\theta_1}^{\theta_2} \frac{\partial m}{\partial t} d\theta \tag{4.4}$$

Combining (4.3) and (4.4) we get,

$$\int_{\theta_1}^{\theta_2} \frac{\partial m}{\partial t} d\theta = \int_{\theta_1}^{\theta_2} (Q - Km - \omega \frac{\partial m}{\partial \theta}) d\theta$$

Since both hold for all θ_1 and θ_2, we have,

$$\frac{\partial m}{\partial t} = Q - Km - \omega \frac{\partial m}{\partial \theta}, \tag{4.5}$$

which is known as the *continuity equation*.

Torque Balance:
The rotation of the wheel is governed by Newton's law $F = ma$, expressed as a balance between the applied torque and the rate of change of angular momentum. Note that in general I depends on t, because the distribution of water does. But this complication disappears if we wait long enough: as $t \to \infty$, then $I \to constant$. Hence, after the transients decay, the equation of the motion will be

$I\dot{\omega}$ = damping torque + gravitational torque.

There are two sources of rotational damping, one is viscous damping caused by the break, which can be adjusted by the users and another is more inertial damping caused by spin-up effects, "*the water enters the wheel at zero angular velocities, but is spun up to angular velocity ω before it leaks out.*" Both of these effects produce *torques* proportional to ω, so we have,

damping torque = $-v\omega$, where $v > 0$.

The gravitational torque is like that of an inverted pendulum, since water is pumped in at the top of wheel as in figure (4.3). In the small sector $d\theta$, the mass $dM = md\theta$. There is a gravitational torque on the system (i.e. mass element produces a torque) given by

$$d\tau = (dM) g\rho \sin\theta = mg\rho \sin\theta \, d\theta.$$

Here g is effected gravitational constant, given by $g = g_0 \sin\alpha$, where g_0 is the usual gravitational constant and α is the tilt of the wheel from horizontal. Integrating over the total mass element gives,

$$\tau = g\rho \int_0^{2\pi} m(\theta, t) \sin\theta \, d\theta$$

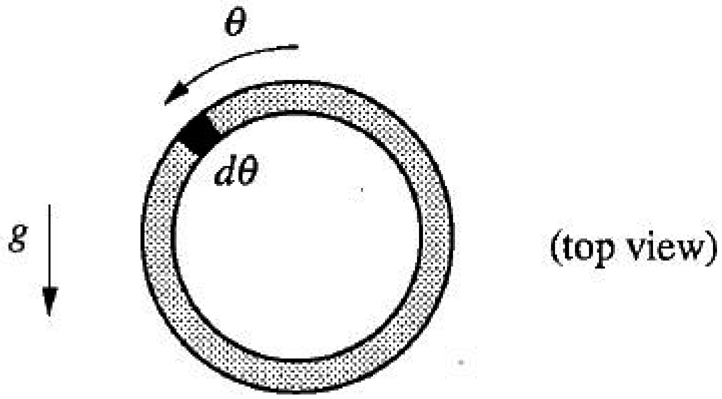

Figure 4.3: Waterwheel

Putting it all together, we obtain the torque balance equation

$$I\dot{\omega} = -v\omega + g\rho \int_0^{2\pi} m(\theta, t)\sin\theta \, d\theta. \tag{4.6}$$

This is called an *integro-differential equation*.

Amplitude Equation:
Equations (4.5) and (4.6) completely specify the evolution of the system, where (4.5) tells us how to update m and (4.6) tells us how to update ω. These two equations appear to be a set of very complicated partial differential equations. Since $m(\theta, t)$ periodic in θ, then use Fourier series, we can write

$$m(\theta, t) = \sum_{n=0}^{\infty} (a_n(t)\sin n\theta + b_n(t)\cos n\theta) \tag{4.7}$$

We also can write the inflow $Q(\theta)$ as a Fourier series

$$Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta, \tag{4.8}$$

where the $\sin$ term is neglected, since the inflow is symmetric about $\theta = 0$ (at the top of the waterwheel) and the terms $a_n(t)$, $b_n(t)$ and q_n are arbitrary constant. Now we substitute (4.7) and (4.8) in equation (4.5) and get

$$
\begin{aligned}
\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty}(a_n(t)\sin n\theta + b_n(t)\cos n\theta)\right] &= -\omega\frac{\partial}{\partial\theta}\left[\sum_{n=0}^{\infty}(a_n(t)\sin n\theta + b_n(t)\cos n\theta)\right] \\
&\quad + \sum_{n=0}^{\infty} q_n \cos n\theta \\
&\quad - K\left[\sum_{n=0}^{\infty}(a_n(t)\sin n\theta + b_n(t)\cos n\theta)\right] \\
\sum_{n=0}^{\infty}\left(\dot{a_n}\sin n\theta + \dot{b_n}\cos n\theta\right) &= -\omega\left[\sum_{n=0}^{\infty}(na_n\cos n\theta - nb_n\sin n\theta)\right] \\
&\quad + \sum_{n=0}^{\infty} q_n \cos n\theta \\
&\quad - K\left[\sum_{n=0}^{\infty}(a_n(t)\sin n\theta + b_n(t)\cos n\theta)\right] \\
\therefore\ \dot{a_n}\sin n\theta + \dot{b_n}\cos n\theta &= (n\omega b_n - Ka_n)\sin n\theta \\
&\quad + (q_n - Kb_n - n\omega a_n)\cos n\theta
\end{aligned}
$$

Equating the coefficients of $\sin n\theta$ and $\cos n\theta$ we get

$$\dot{a_n} = n\omega b_n - Ka_n \quad \text{and} \quad \dot{b_n} = q_n - Kb_n - n\omega a_n \tag{4.9}$$

which hold for all $n = 0, 1, \ldots$
Again using (4.8) in (4.6), we get

$$
\begin{aligned}
I\dot{\omega} &= -v\omega + g\rho\int_0^{2\pi}\left[\sum_{n=0}^{\infty}(a_n(t)\sin n\theta + b_n(t)\cos n\theta)\right]\sin\theta\, d\theta \\
&= -v\omega + g\rho\int_0^{2\pi} a_1\sin^2\theta\, d\theta \\
&= -v\omega + \pi g\rho a_1
\end{aligned}
$$

Hence, only a_1 enters the differential equation for $\dot{\omega}$. But (4.9) imply that a_1, b_1, $n = 1$ and ω from a closed system, these three variables are uncoupled from all other a_n, b_n and $n \neq 1$, therefore the resulting governing equation for our waterwheel are,

$$
\begin{aligned}
\dot{a_1} &= \omega b_1 - Ka_1 \\
\dot{b_1} &= -\omega a_1 - Kb_1 + q_1 \\
\dot{\omega} &= -\frac{v}{I}\omega + \frac{\pi g\rho}{I}a_1
\end{aligned}
\tag{4.10}
$$

which are known as the *amplitude equations*.

Change of Variables:
The system (4.10) is equivalent to the Lorenz equations, The change of variables by putting, $a_1 = \frac{Kv}{\pi\rho g}y$, $b_1 = -\frac{Kv}{\pi\rho g}z + \frac{q_1}{K}$ and $\omega = Kx$ in (4.9) where x, y and z are also functions of t. Now from the first equation of (4.9) we get

$$\begin{aligned}\frac{Kv}{\pi\rho g}\dot{y} &= Kx\left(-\frac{Kv}{\pi\rho g}z + \frac{q_1}{K}\right) - K\left(\frac{Kv}{\pi\rho g}y\right)\\ \therefore\ \dot{y} &= \frac{\pi g\rho q_1}{K^2 v}x - xz - y\end{aligned}$$

from the second equation of (4.9) we get

$$\begin{aligned}-\frac{Kv}{\pi rg}\dot{z} &= -Kx\left(\frac{Kv}{\pi\rho g}y\right) - K\left(\frac{Kv}{\pi\rho g}z + \frac{q_1}{K}\right) + q_1\\ -\dot{z} &= -Kxy + Kz\\ \therefore\ \dot{z} &= Kxy - Kz\end{aligned}$$

from the third equation of (4.9) we get

$$\begin{aligned}K\dot{x} &= -\frac{v}{I}Kx + \frac{\pi g\rho}{I}\left(\frac{Kv}{\pi\rho g}y\right)\\ \therefore\ \dot{x} &= \frac{v}{I}(y - x)\end{aligned}$$

Therefore the resulting Lorenz equations are:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - xz - y\\ \dot{z} &= xy - bz,\end{aligned}$$

where the Rayleigh number $r = \frac{\pi g\rho q_1}{K^2 v}$, the Prandtl number $\sigma = \frac{v}{KI}$. However, note that with this change of variables, the Lorenz equation parameter must be set as $b = K = 1$, meaning this the waterwheel is specific case of the broader Lorenz equation.

4.1.2 Properties of the Lorenz Equation

The Lorenz system (4.1) has a symmetry about $z-$axis and can be determined by replacing (x, y, z) with $(-x, -y, z)$. Therefore, the Lorenz system of the equation (4.1) can be written in the form:

$$\begin{aligned}-\sigma(-x) + \sigma(-y) &= \sigma(x) - \sigma(y) &&= -\dot{x}\\ r(-x) - (-y) - (-x)z &= -rx + y + xz = &&-\dot{y}\\ (-x)(-y) - bz &= xy - bz &&= \dot{z}\end{aligned}$$

So, if $(x(t), y(t), z(t))$ is the solution of the Lorenz equations then $(-x(t), -y(t), z(t))$ also. To find the fixed point of the Lorenz system (4.1) we set $\dot{x} = \dot{y} = \dot{z} = 0$, then we get

$$\begin{aligned} \sigma(y-x) &= 0 \\ rx - y - xz &= 0 \\ xy - bz &= 0 \end{aligned}$$

from first equation

$$y = x \ \ (\because \sigma \neq 0)$$

from third equation

$$x = \pm\sqrt{bz} \ \ (\because y = x)$$

from second equation

$$x(r-1-z) = 0 \ \ (\because y = x) \Rightarrow \text{either } x = 0 \ \text{ or } r-1-z = 0 \Rightarrow z = r-1.$$

If $x = 0$, then we get $y = z = 0$ and if $z = r-1$ then we get $x = y = \pm\sqrt{b(r-1)}$. So, the fixed points from solving this system are:

$$(0,0,0) \text{ and } Q_{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1).$$

If $0 \leq r \leq 1$,then there is only one fixed point at the origin $(0,0,0)$. If $r > 1$, then there exists two other fixed points Q_- and Q_+.
First we calculate the Jacobian matrix of the system, which is given by

$$Df = \begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix} \tag{4.11}$$

Linearization at the origin of the system is

$$\dot{Y} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} Y \qquad \text{[from(4.11)]}$$

The eigenvalues of the coefficient matrix A are given by

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ r & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{vmatrix} &= 0 \\ (\lambda+b)[(\lambda+\sigma)(\lambda+1) - r\sigma] &= 0 \\ (\lambda+b)(\lambda^2 + (\sigma+1)\lambda + \sigma(1-r)) &= 0. \end{aligned}$$

Therefore the eigenvalues of the coefficient matrix are

$$\lambda = -b \text{ and } \lambda_{\pm} = \frac{1}{2}(-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}). \tag{4.12}$$

If $r < 1$, all eigenvalues are negative, hence the origin is a stable. However, we can even do better let us construct the function

$$V(x, y, z) = rx^2 + \sigma y^2 + \sigma z^2$$

as a Lyapunov function at the origin, for this V is a continuous function, clearly $V(0, 0, 0) = 0$ and $V(x, y, z) > 0$ for $(x, y, z) \neq 0$. Moreover

$$\begin{aligned}
\dot{V}(x, y, z) &= 2rx\dot{x} + 2\sigma y\dot{y} + 2\sigma z\dot{z} \\
&= 2rx(\sigma(y-x)) + 2\sigma y(rx - y - xz) + 2\sigma z(xy - bz) \\
&= -2\sigma(r(x-y)^2 + (1-r)y^2 + bz^2) \\
\therefore \dot{V} &= -2\sigma(r(x-y)^2 + (1-r)y^2 + bz^2) < 0 \text{ for } r \leq 1
\end{aligned}$$

Hence $\dot{V} < 0$ for $r \leq 1$ except origin, this all shows that V is a strict Lyapunov function. Hence by the Lyapunov criteria, all solutions of the Lorenz system tends to an origin, therefore an origin is globally stable.
We see from (4.12) all eigenvalues are negative for $r < 1$ and if $r > 1$ not all eigenvalues are negative so the origin looses the stability and, two new fixed points $Q_{\pm}$ are born, we see that the bifurcation occurs at $r = 0$.
Also the linearization of the system at the point $Q_{\pm}$ is given

$$\dot{Y} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{bmatrix} Y \quad [\text{from } (4.11)]$$

then, the eigenvalues of the coefficient matrix B are given by

$$\begin{aligned}
|B - \lambda I| &= 0 \\
\begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b-\lambda \end{vmatrix} &= 0 \\
(\sigma+\lambda)(\lambda^2 + (b+1)\lambda + br) - \sigma(\lambda + 2b - br) &= 0 \\
\lambda^3 + (\sigma+b+1)\lambda^2 + b(\sigma+r)\lambda + 2\sigma b(r-1) &= 0.
\end{aligned}$$

Therefore, the eigenvalues of the system at $Q_{\pm}$ are given by the roots of polynomial

$$f_r(\lambda) = \lambda^3 + (\sigma+b+1)\lambda^2 + b(\sigma+r)\lambda + 2\sigma b(r-1).$$

If $r = 1$ the polynomial f_1 has a three distinct roots $-b$, $-\sigma-1$ and 0. These roots are distinct since $\sigma > b + 1$, so that

$$-\sigma - 1 < -\sigma + 1 < -b < 0.$$

For r is closed to but greater than one, f_r has three distinct real roots close to these values. Note that $f_r(\lambda) > 0$ for $\lambda \geq 0$ and $r > 1$. This follows that at least for r close to 1, the three roots of $f_r(\lambda)$ must be real and negative. Therefore, these two new fixed points $Q_\pm$ are stable but it does not remain long time when r is increased.
For the lowest value of r, which $f_r(\lambda)$, has an eigenvalues with zero real part. Note that, these eigenvalues must in fact be of the form $\pm i\omega$ with $\omega \neq 0$, since $f_r(\lambda)$ is a real polynomial that has no real roots equal to zero when $r > 1$. Now, we have to solve $f_r(i\omega) = 0$, that is

$$\begin{aligned}
(i\omega)^3 + (\sigma + b + 1)(i\omega)^2 + b(\sigma + r)i\omega + 2\sigma b(r-1) &= 0 \\
-i\omega^3 - (\sigma + b + 1)\omega^2 + b(\sigma + r)i\omega + 2\sigma b(r-1) &= 0 \\
(2\sigma b(r-1) - (\sigma + b + 1)\omega^2) + i(\sigma b + rb - \omega^2)\omega &= 0
\end{aligned}$$

Equating the real and imaginary parts to zero, we get

$$2\sigma b(r-1) - (\sigma + b + 1)\omega^2 = 0 \text{ and } (\sigma b + rb - \omega^2)\omega = 0$$

from first equation

$$\omega^2 = \frac{2\sigma b(r-1)}{(\sigma + b + 1)}$$

and from second equation

$$\begin{aligned}
\sigma b + rb - \omega^2 &= 0 \ (\because \omega \neq 0) \\
\sigma b + rb - \frac{2\sigma b(r-1)}{(\sigma + b + 1)} &= 0 \\
(\sigma + b + 1)(\sigma b + rb) - 2\sigma b(r-1) &= 0 \\
\therefore r = \frac{\sigma(\sigma + b + 1)}{(\sigma - b - 1)} = r^*(say) &
\end{aligned}$$

Therefore, $f_r(\lambda)$ gives the pure imaginary eigenvalues when $r^* = \frac{\sigma(\sigma+b+1)}{(\sigma-b-1)}$, this all shows that $f_r(\lambda)$ gives the eigenvalues with negative real part for $1 < r < r^*$. Hence the fixed points $Q_\pm$ are stable for $1 < r < r^*$. For $r > r^*$ the polynomial $f_r(\lambda)$ gives the eigenvalues with positive real part, this tell us that the fixed points $Q_\pm$ are unstable. We remark that a Hopf bifurcation is known to occur at r^* with $\sigma > b + 1$. The bifurcation diagram of the system is shown in the figure (4.4).
We set $\sigma = 10$, $b = 8/3$ and $r = 28$ in the Lorenz system (4.1) we get the three fixed point origin $O(0, 0, 0)$ and $Q_\pm = (\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$ and all fixed points are unstable since $r > r^* = 24.7$. Let us begin the system with $x_0 = [0.3, 0.3, 0.3]$, then the behavior of the Lorenz system is shown in the figure (4.5). We see from the figure (4.5) the variables $x(t)$, $y(t)$ and $z(t)$ fluctuate as time t progress. The behavior of $x(t)$, $y(t)$ and $z(t)$ are seen to be bounded, but aperiodic. Thus this system shows the different behaviors than the two dimensional continuous system, which must blow up, converge to a fixed points, or become a periodic. The behavior of the Lorenz system is better to see in the

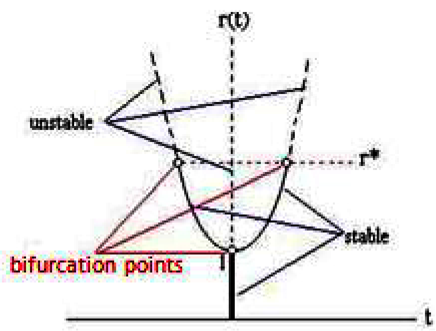

Figure 4.4: Bifurcation diagram for Lorenz's system

three dimensional figure (4.6). The trajectory $x(t)$, $y(t)$ and $z(t)$ spiral out around the fixed points Q_+ and Q_- one loop of the diagram for a while and then suddenly jumps and spins around the other before returning to the first at seemingly random intervals. For the values $r > r^*$ the system shows the chaotic behavior.

MATLAB SYNTEX used to draw the figure 4.5 and 4.6:

```
function xdot=lorenz(t,x)
% sigma=11; r=25; b=8/3;
sigma=10; r=28; b=8/3;
xdot=[-sigma*(x(1)-x(2));
r*x(1)-x(2)-x(1)*x(3);
-b*x(3)+x(1)*x(2)];
close all;
clear all;
clc;
ti=0;
tf=150;
tspan=[ti tf];
x0=[0.3 0.3 0.3]';
[t,x]= ode23('lorenz',tspan,x0);
figure
subplot(3,1,1), plot(t,x(:,1),'r'),grid on;
title('Lorenz Attractor'),ylabel('x(t)');
subplot(3,1,2), plot(t,x(:,2),'b'),grid on;
ylabel('y(t)');
subplot(3,1,3), plot(t,x(:,3),'g'),grid on;
ylabel('z(t)');xlabel('t')
figure
plot3(x(:,1),x(:,2),x(:,3)),grid on;
```

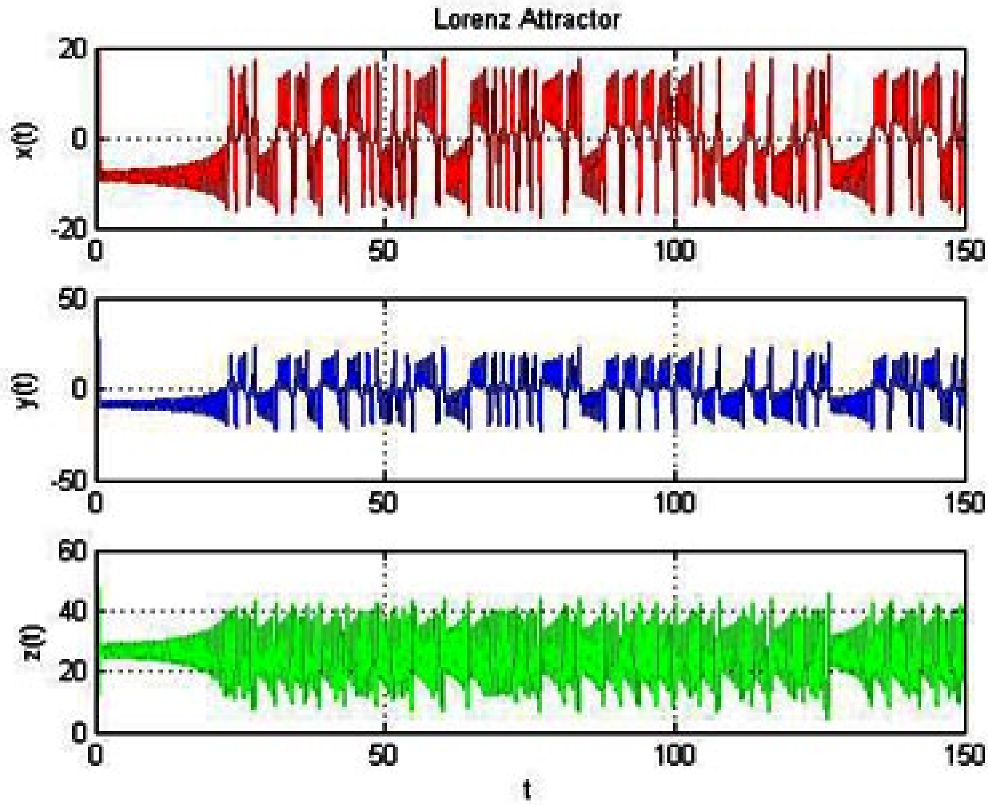

Figure 4.5: Plot of $x(t)$,$y(t)$ and $z(t)$ for the Lorenz system

```
xlabel('x');ylabel('y');zlabel('z')
title('Lorenz Attractor');
```

4.2 Logistic Population Model

Suppose that the population $p(t)$ changes only by the occurrence of births and deaths, there is no immigration or emigration from outside the country or environment under consideration. Let $\beta(t)$ is the births rate and $\delta(t)$ is the deaths rate. The number of births and deaths that occur during the time interval $[t, t + \Delta t]$ is given by,

$$\text{Births; } \beta(t) \cdot p(t) \cdot \Delta t, \quad \text{deaths; } \delta(t) \cdot p(t) \cdot \Delta t$$

Hence the change of the population in $[t, t + \Delta t]$ is

$$\Delta p = \beta(t) \cdot p(t) \cdot \Delta t - \delta(t) \cdot p(t) \cdot \Delta t.$$

Dividing both sides by Δt and taking limit as $\Delta t \to 0$, we get

$$\frac{dp}{dt} = (\beta - \delta)p. \tag{4.13}$$

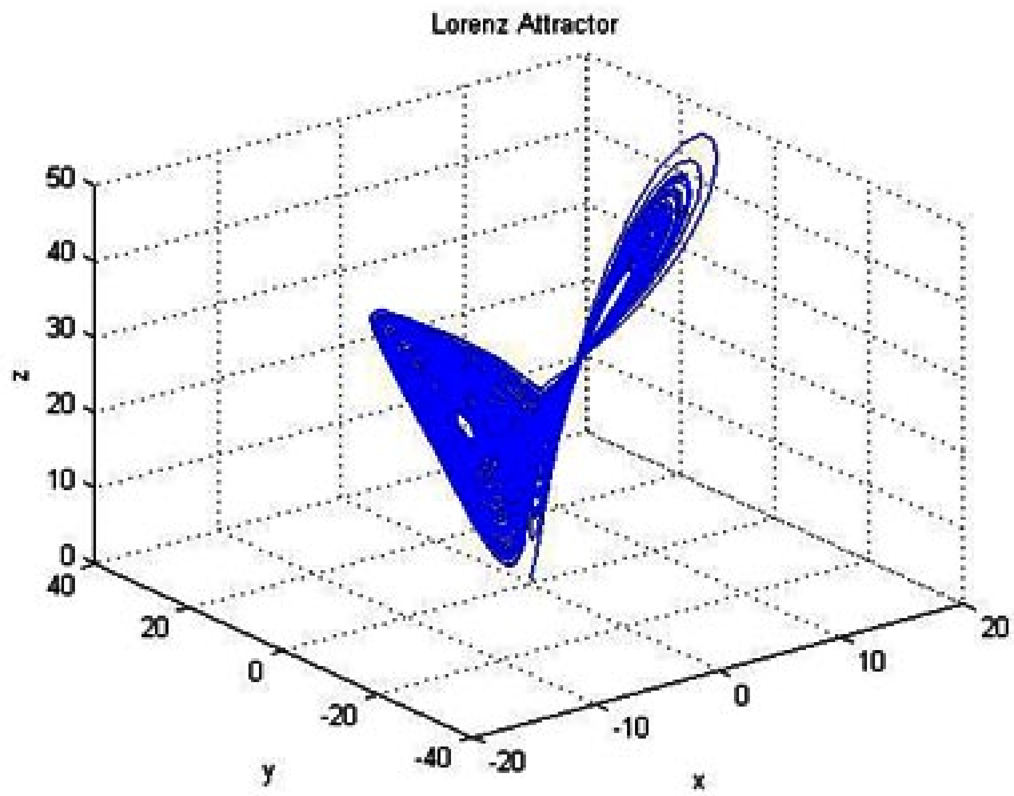

Figure 4.6: Three dimensional plot of the trajectory of the Lorenz system

For the case of fruit fly population in a closed container, it is often observed that the birth rate decreases as the population itself increases, because of a limited food supply. The birth rate β is linearly decreasing function of the population size p so that $\beta = \beta_0 - \beta_1 p$, where β_0 and β_1 are positive constants. If the death rate $\delta = \delta_0$ remains constant then equation (4.13) takes the form

$$\frac{dp}{dt} = (\beta_0 - \beta_1 p - \delta_0)p$$

that is

$$\frac{dp}{dt} = ap - bp^2$$

where $a = \beta_0 - \delta_0$ and $b = \beta_1$, both a and b are positive. This equation is called the logistic population model.

4.2.1 Logistic Equation and Logistic Map

The Logistic equation is the simple model for the bounded population of the single species that is given by

$$\frac{dp}{dt} = ap - bp^2 \quad a, b > 0 \tag{4.14}$$

with the initial condition $p(t_0) = p_0$, which is better to write in the form

$$\frac{dp}{dt} = kp(M - p) \text{ with } p(t_0) = p_0 \tag{4.15}$$

where $M = a/b$ and $k = b$. Solving equation(4.15) we get

$$p(t) = \frac{Mp_0}{p_0 + (M - p_0)e^{-kMt}}$$

the population $p(t)$ satisfies (4.14), then $p(t)$ approaches the limiting population $M = a/b$ as $t \to \infty$. In order to solve (4.14) numerically, we first choose a fixed step size $h > 0$ and consider the sequence of discrete times

$$t_0, t_1, t_2, t_3, \ldots, t_n, t_{n+1}, \ldots$$

where $t_{n+1} = t_n + h$ for each n. Beginning with initial values $p(t_0) = p_0$, we then calculate the approximations

$$p_1, p_2, \ldots, p_n, p_{n+1} \cdots \tag{4.16}$$

to the true values $p(t_1), P(t_2), p(t_3), \ldots$ of the actual populations $p(t)$. For instance Euler's methods for the logistic equation in (4.14) consist of the calculating the approximations (4.16) iteratively by means of the formula

$$p_{n+1} = p_n + (ap_n - bp_n^2) \cdot h \tag{4.17}$$

If h is the interval between successive breeding seasons, then the population p_n during one breeding seasons may depend only on the population p_{n-1} during the previous seasons, and p_n may completely determine the population p_{n+1} during the next breeding season. Let us assume that the successive values $p_n = p(t_n)$ of the population are given by the equation (4.17). Thus we replace the original differential equation in (4.14) with a difference equation

$$\Delta P_n = (ap_n - bp_n^2)\Delta t \tag{4.18}$$

that gives the population difference $\Delta p_n = p_{n+1} - p_n$ in terms of the time difference $h = \Delta t$ and the preceding population p_n. We can rewrite (4.17) as in the logistic difference form

$$p_{n+1} = rp_n - sp_n^2 \tag{4.19}$$

where $r = 1 + ah$ and $s = bh$. Now we substitute $p_n = \frac{r}{s}x_n$, then (4.19) takes the form

$$x_{n+1} = f(x_n) = rx_n(1 - x_n) \tag{4.20}$$

which is the simple discrete model for the growth of the population of the single species. Here x_n is the population of the n generation. In general, positive value of x_n may map to negative value of x_{n+1} which would not make a sense when using a logistic map as a population. Therefore, the system is bounded in the interval $[0, 1]$ and r is the real number with $1 \leq r \leq 4$ because we restric the larger value of r to make our problem easy. If the initial condition $x_0 \in [0, 1]$, then all subsequent iterations $x_1, x_2, \ldots$ will remain in the interval $[0, 1]$ corresponding to the successive times $t_1, t_2, \ldots$. The value of r is grater than 1 since $r = 1 + ah$ and if $r \geq 1$, then the roots of $f(x)$ are at $x = 0$ and $x = 1$ and the maximum value occurs at $x = 1/2$ and the maximum value of $f(x)$ is $r/4$. Thus, if $1 \leq r \leq 4$ and $0 \leq x \leq 1$ then $0 < f(x) \leq 1$. The behavior of the logistic map as shown in the figure (4.7).

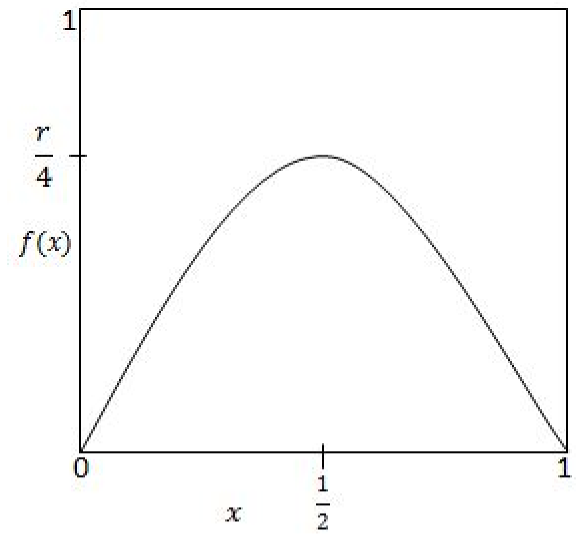

Figure 4.7: The behavior of the logistic map

4.2.2 Fixed Point and Periodic Doubling

Now we have to calculate the fixed point of the system (4.20) setting $f(x_n) = x_n = x$, we obtain

$$x = rx(1 - x)$$

Solving, we get $x = 0$ which is a fixed point independent in r and the another fixed point is $x = \frac{r-1}{r}$ which is changed as the parameter r is changed. The right hand side of the logistic equation (4.20) is defined by the function $f(x) = rx(1-x)$. Taking the derivative

$f'(x) = r - 2rx$, we compute that $|f'(0)| = |r|$, since $1 < r \leq 4$, then $|f'(0)| = |r| > 1$. Therefore, the fixed point $x = 0$ is an unstable. For the another fixed point $x = \frac{r-1}{r}$, we compute that $|f'(\frac{r-1}{r})| = |2 - r|$, the fixed point is a stable provided that$|f'(\frac{r-1}{r})| < 1$ if $1 < r < 3$ and is unstable provided that $|f'(\frac{r-1}{r})| > 1$ if $r > 3$.
For $r > 3$, both fixed points $x = 0$ and $x = (r-1)/r$ are an unstable and they act as the solution of the system goes to infinity. However, we start from an initial condition $x_0 \in [0, 1]$. Then all subsequent iterations x_n are bounded inside the interval $[0, 1]$ if $r > 3$, the system never convergent to the fixed point but the system bounded inside the interval $[0, 1]$, therefore the fixed point cannot solve the system then the 2-period orbits appear when the fixed point looses the stability. In the system (4.20) the periodic two orbit is born when r is increased from 3. Because, if $r > 3$, the fixed point $x = (r-1)/r$ looses the stability of the system. Now we determine the 2-period solution of the system if the system (4.20) satisfy $x_{n+2} = x_n$. Moreover

$$\begin{aligned}
x_{n+2} &= f(x_{n+1}) = f(f(x_n)) = f^2(x_n) \\
x_{n+2} &= f(rx_n(1-x_n)) \\
&= r[rx_n(1-rx_n)][1-rx_n(1-rx_n)] \\
&= -r^3x_n^4 + 2r^3x_n^3 - r^3x_n^2 - r^2x_n^2 + r^2x_n
\end{aligned}$$

$$\therefore x_{n+2} = f^2(x_n) = -r^3x_n^4 + 2r^3x_n^3 - r^3x_n^2 - r^2x_n^2 + r^2x_n \tag{4.21}$$

Suppose x is the periodic point of the system then it must be satisfied

$$\begin{aligned}
f^2(x) &= x \\
-r^3x^4 + 2r^3x^3 - r^3x^2 - r^2x^2 + r^2x &= x \\
rx(r^2x^3 - 2r^2x^2 + r^2x + rx - r - \frac{1}{r}) &= 0 \\
x(r^2x^3 - 2r^2x^2 + r^2x + rx - r - \frac{1}{r}) &= 0 \because r \neq 0 \\
x(r^2x^3 - 2r^2x^2 + r^2x + rx - r - \frac{1}{r}) &= 0 \\
x(x - \frac{r-1}{r})\{r^2x^2 - r(1+r)x + (1+r)\} &= 0
\end{aligned}$$

Solving, this equation we get the four roots, first two are $x = 0$ and $x = \frac{r-1}{r}$, which are also the fixed points of the logistic map (4.20), therefore the fixed points are also periodic point, another two roots are

$$x_{\pm} = \frac{(1+r) \pm \sqrt{(r+1)(r-3)}}{2r}$$

which are real numbers provided that $r > 3$. Both of these are fixed points of the second iteration mapping but not of the logistic equation (4.20) itself. If r is slightly grater than 3 the iteration of logistic equation settle into an alternating patterns, if we start from any initial condition that is near either x_+ or x_- the iteration of logistic will always converge to the periodic two solutions, that is alternating between x_+ and x_-. So $f(x_-) = x_+$ or $f(x_+) = x_-$. Now to test the stability of the 2-period solution, first we define $g(x) = f(f(x))$. We take, x_- is the fixed point of (4.21), then the stability criterion would require that $|g'(x_-)| < 1$. By chain rule

$$\begin{aligned} g'(x_-) &= f'(f(x_-))f'(x_-) \\ &= f'(x_+)f'(x_-) \because [f(x_-) = x_+] \end{aligned}$$

It follows that the 2-period of the discrete logistic equation is stable if $|f'(x_+)f'(x_-)| < 1$. Now, we compute

$$\begin{aligned} f'(x_+)f'(x_-) &= \Big[r - (1 + r + \sqrt{(r+1)(r-3)})\Big] \\ &\quad \Big[r - (1 + r - \sqrt{(r+1)(r-3)})\Big] \\ &= (1 + \sqrt{r^2 - 2r - 3})(1 - \sqrt{r^2 - 2r - 3}) \\ &= -r^2 + 2r + 4 \\ \therefore |f'(x_+)f'(x_-)| &= |r^2 - 2r - 4| \end{aligned}$$

Note that, we assume $r^2 - 2r - 4 = 1$ which implies that $r^2 - 2r - 5 = 0$, which is a quadratic equation in r, factorized we get $r = 1 \pm \sqrt{6}$, which only possible $r = 1 + \sqrt{6}$, since $r > 1$. The 2-period points $x_\pm$ are a stable provided that $|f'(x_+)f'(x_-)| = |r^2 - 2r - 4| < 1$, if and only if $3 < r < 1 + \sqrt{6}$, and the 2-period points $x_\pm$ become an unstable if $r > 1 + \sqrt{6}$. When 2-period solution looses the stability then there occur 4-period orbits if r is increased from $1 + \sqrt{6}$ and take over stability and same way if the 4-period solution looses the stability there may occur 8-period orbits and so on, this is known as the periodic doubling and, as r increase leads to chaotic behavior. The periodic doubling process as a bifurcation diagram of the logistic map is shown in figure (4.8).

MATLAB SYNTEX used to draw the figure 4.8:

```
for r=1:0.005:4
x=rand(1);
for j=1:1000
x=r*x*(1-x);
end
xout=[];
for j=1:4000
x=r*x*(1-x); xout=[xout x];
end
plot(r*ones(size(xout)),xout,'.','MarkerSize',3)
axis([0 4 0 1]), hold on, pause(0.01);
xlabel('r');ylabel('x')
end
```

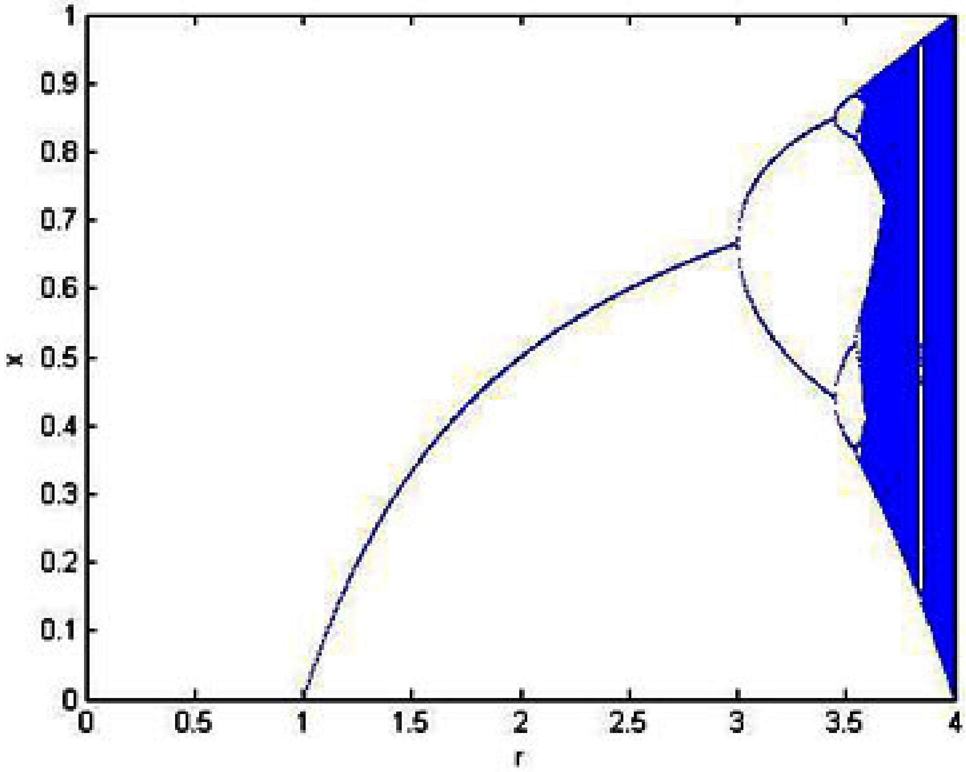

Figure 4.8: Bifurcation of the logistic map

We see from the figure (4.8), for $1 < r < 3$, the points in the bifurcation diagram follow the curve $x = (r-1)/r$, since $(r-1)/r$ is the stable fixed point. At $r = 3$, the curve branches into two curves which persists until $r = 1+\sqrt{6} = 3.44948974$ and for $3 < r < 3.44948974$, the long term behavior of the iteration alternating between the two branches corresponding to the values $x_{\pm}$. For $x > 3.44948974$, there are four branches in the bifurcation diagram and the iteration of the logistic map cycle repeatedly through the four different values indicated by the branch. For most choice of $r > 3.569946$, the branch in the bifurcation diagram becomes a blur. Apparently, the logistic map exhibits the erratic behavior.

Conclusion

In this work, we have studied the basic concepts of dynamical system with some examples. Also, we have studied the behaviors of the dynamical systems which are *stability*, *bifurcations* and *chaos*, with some examples, Lorenz curve and discrete logistic map are used to describe the chaotic behavior.
This is not a sufficient study for the dynamical system, it is a very basic study about the system. In most of the parts of this work, we have tried to study the continuous time dynamical system. Also, a little work is done about the discrete time system with an example. Those who are interested in this work may go for future research on it as the work we have accomplished is not complete in its own.

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