SOME COMMON FIXED POINT RESULTS IN PROBABILISTIC METRIC SPACE AND ITS APPLICATIONS

A DISSERTATION

SUBMITTED IN FULFILLMENT OF THE REQUIREMENTS FOR
THE DOCTOR OF PHILOSOPHY (Ph.D.) DEGREE
IN MATHEMATICS

BY

AJAY KUMAR CHAUDHARY
M.Sc.(T.U.), M.Phil.(K.U.)

DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE
KATHMANDU UNIVERSITY
DHULIKHEL, NEPAL

JANUARY, 2023

Dedication

To

My Beloved Parents

Mrs. Kailash Devi, Mr. Shivjee Chaudhary

And

My Wife

Mrs. Shobha Chaudhary

Declaration

I **Ajay Kumar Chaudhary**, hereby declare that, the research work **Some Common Fixed Point Results in Probabilistic Metric Space and Its applications** submitted here for the fulfillment of Doctor of Philosophy **(Ph.D.)** degree in mathematics to the Department of Mathematics, School of Science, Kathmandu University, Nepal in January 2023, is a genuine work done originally by me and has not been published, or submitted elsewhere for the requirement of a degree program. Any literature, data or works done by others and cited within this thesis has been given due acknowledgment and listed in the reference section.

Ajay Kumar Chaudhary
KU Regd. No.: **016816-14**
Department of Mathematics
School of Science,
Kathmandu University
Dhulikhel, Kavre, Nepal.

Date:

Certification

This is to certify that the thesis entitled **Some Common Fixed Point Results in Probabilistic Metric Space and Its applications** which is being submitted by Mr. Ajay Kumar Chaudhary in the fulfillment for the award of Doctor of Philosophy (**Ph.D.**) degree in mathematics of Kathmandu University, Nepal is a record of his own work carried out by him under our guidance and supervision.

The matter embodied in this thesis has not been submitted for the award of any degree.

(**Kanhaiya Jha, Ph.D.**)
Professor
Department of Mathematics
School of Science
Kathmandu University
Dhulikhel, Kavre, Nepal
Date:

(**K. B. Manandhar, Ph.D.**)
Assistant Professor
Department of Mathematics
School of Science
Kathmandu University
Dhulikhel, Kavre, Nepal
Date:

Approved by:

I hereby declare that the candidate qualifies to submit this work as PhD dissertation to the Examination Committee.

Head of the Department
(**Associate Prof. Rabindra Kaystha, Ph.D.**)
Department of Mathematics, School of Science
Kathmandu University
Date:

Acknowledgments

I gratefully acknowledge my Supervisor **Prof. Dr. Kanhaiya Jha** and Co-supervisor **Asst. Prof. Dr. K. B. Manandhar** for their academic greatness, valuable advice, kind guidance, continuous interest, and assistance to complete this research work.

I offer my sincere thanks to **Prof. Dr. Janardan Lamichhane**, Dean, and Associate Dean Prof. Dr. Bed Mani Dahal, School of Science, Kathmandu University for providing me the opportunity to be a part of Kathmandu University and permitting me to use K.U. facilities.

I am equally thankful to **Associate Prof. Dr. Rabindra Kayastha**, Head of Department of Mathematics, Kathmandu University for helpful comments and suggestions.

I express my sincere thanks to **Prof. Dr. Dil Bahadur Gurung** and **Prof. Dr. Ram Prasad Ghimire** for their encouragement and inspiration. Also, I would like to express my warm regards and thanks to **Prof. Dr. Bhandra Man Tuladhar**, faculty members Prof. Dr. Jyoti Upadhyaya, Associate Prof. Dr. Samir Shrestha, Asst. Prof. Dr. Ganga Ram Phaijoo, Asst. Prof. Dr. Khim Bdr. Khattri, Asst. Prof. Dr. Gokul K. C, , Asst. Prof. Kiran Kumar Shrestha, Asst. Prof. Dr. Saraswati Acharya of Department of Mathematics, Kathmandu University, Dhulikhel for their suggestions and help as per need during the entire research work. I extend my sincere thanks to the staffs of Dean office K. U., Kathmandu University library and Tribhuvan University library for their resourceful support.

I remember to acknowledge Prof. Dr. Tanka Nath Dhamla, Prof. Dr. Chet Raj Bhatta, Prof. Dr. Narayan Pahari, Prof. Dr. Binod Dhakal, Prof. Dr. Ajaya Singh, Associate Prof. Dr. Shree Ram Khadka of Central department of Mathematics, Tribhuvan University and Associate Prof. Dr. Dinesh Panthi of Sanskrit University T. U. Nepal for their instant support.

I would also like to appreciate and thank Prof. Dr. P. P. Murthy, Guru Ghasidas Vishwavidyalaya, Bilaspur (CG), India, Prof. Dr. H. K. Pathak, Pt. Ravishankar Shukla University, India, Prof. Dr. Dorel Mihet, West University of Timisoara, Romania, Prof. Dr. Poom Kumam, King Mongkuts University of Technology Thonburi, Thailand, Prof. Dr. Yeol Je Cho, Gyeongsang National University, Korea, Prof. Dr. Erdal Karapinar, Cankaya University, Ankara, Turkey, Asst. Prof. Dr. Javid Ali, Aligarh Muslim University, India, and Asst. Prof. Dr. Izhar Uddin, Jamia Millia Islamia, New Delhi, India for their help by providing research papers and for moral supports.

I am thankful to **Trichandra Multiple Campus, Tribhuvan University** for providing me with Ph. D. study leave, and i am equally thankful to **University Grant Commission, Nepal** for Ph. D. fellowship grant, Ph.D.-76/77-S and T-02. I am also thankful to Nepal Mathematical Society, Nepal for awarding me by NMS Fellowship award-2020. I would like to thank all mathematics faculty members and staffs of Trichandra Multiple Campus, specially my friend cum brother Nandu Prasad Kushwaha, my M. Phil. batch mates, and also to our Fixed point research team.

I heartily acknowledge the great patience, belief, continuous encouragement and support of my wife **Shobha Chaudhary**, elder son Aman Kumar Chaudhary and younger son Ayush Kumar Chaudhary, without whom i would not have been able to complete my studies at this level. Also, I want to thank my elder brother Arun Kumar Chaudhary, younger brother Bijay Kumar Chaudhary and younger sister Renu Chaudhary for their direct and indirect support. I thank my close friends for enabling a synergistic space of growth.

Finally, I would like to thank everyone who helped me directly or indirectly in my research work.

Thanks a lot again to all.

Ajay Kumar Chaudhary

Tri Chandra Multiple Campus,

Tribhuvan University,

Kathmandu, Nepal.

Abstract

Fixed point theory is considered as a core subject of non-linear analysis and enters in the mainstream of mathematics due to its applications in diverse fields. The roots of fixed point theory lie in the method of successive approximations for proving the existence of solutions of differential equations. The abstraction of this classical theory is the pioneering work of the great mathematician S. Banach[10] which provides a constructive method to find fixed points of a function in 1922.

In 1942, Karl Menger [130] introduced the notion of a probabilistic metric space (PM-space) as a generalization of metric space. Menger Theory explained how to replace the numerical distance between two points x and y by a distribution function $F_{x,y}$ whose value $F_{x,y}(t)$ at the real number t is interpreted as the probability that the distance between x and y is less than t. B. Schweizer and A. Sklar [182] took up the work initiated by K. Menger and developed the theory of probabilistic metric space.

V. M. Sehgal and A. T. Bharucha Reid [185] introduced fixed point in probabilistic metric space by establishing a fixed point theorem in complete Menger space. And S. N. Mishra [137] initiated the compatibility in this space.

The existence and uniqueness of common fixed point results in metric space and its generalized form like PM space for some contractive conditions are the open areas of a research domain. Based on the results established by H. K. Pathak, Y. J. Cho, S. S. Chang and S. M. Kang [154], K. Tas, M. Telci, and B. Fischer [206] and by K. Jha, V. Popa, and K. B. Manandhar [94], we have extended some notions in probabilistic metric space.

In this thesis, we have introduced new notions compatible mappings of type (P), weakly compatible mappings of type (P) and compatible mappings of type (K) in Menger space. And established some common fixed point theorems under compatible mappings for two pairs and three pairs of self-mappings in this space. We have obtained relevant examples to verify our established result.

Keywords: Fixed Point, common fixed point, probabilistic metric space, T- norm, Menger space, compatible mappings, weakly compatible mappings

Outline of the Dissertation

Chapter wise brief sketch of a dissertation is as follows:

Chapter 1 is introductory in nature. In this chapter, we have given a historical overview of fixed point theory and probabilistic metric space in brief. Also, we have defined some basic definitions and theorems needed for the development of fixed point theory in probabilistic metric spaces.

Chapter 2 deals with compatible mappings study in probabilistic metric spaces with fundamental concepts and fixed point theorems. We have introduced and established a common fixed point theorem in complete Menger space by using Meir Keeler contraction with (K) compatible mappings and verified our result with examples.

In **Chapter 3**, we have introduced the notion of compatible mappings of type (P) in probabilistic metric space and used this compatible mapping to establish a common fixed point theorem in complete Menger space. Also, we introduced another notion of weakly compatible mappings of type (P) in Menger space and obtained fixed point results in this space.

In **Chapter 4**, we have studied occasionally weakly compatible mappings in probabilistic metric space and obtained some common fixed point theorems in complete Menger space in four and six self-mappings. This result generalizes and improves other similar results in the literature. This chapter also includes conclusions and some future scopes.

The list of literature consulted has been placed at the end of the thesis as **References**.

Our original contributions have been included in chapters 2, 3, and 4. A part of the research work contained in this thesis has been already published in international peer-reviewed journals [38, 39, 40, 41].

Contents

Abbreviations

PMS	Probabilistic Metric Space
SMS	Semi Metric Space
MS	Metric Space
M- Space	Menger Space
OWC	Occasionally Weakly Compatible Mappings
WRC	Weakly reciprocal Continuous
PSM	Probabilistic Semi Metric
Sup	Supremum
Inf	Infimum
Max	Maximum
Min	Minimum
Lim	Limit
B-contraction	Bharucha and Sehgal contraction
C-contraction	Hick's contraction
C[0,1]	The space of all continuous function on $[0, 1]$
T-norm	Triangular Norm
Fix(T)	Fixed point of function T

Symbols

$:$	such that
$\neq$	not equal to
$\mathbb{R}$	set of Real number
$\mathbb{R}^+$	set of positive Real number
N	set of natural number
ϕ	phi
$\in$	belongs to
$\notin$	not belongs to
$<$	less than
$>$	greater than
$\leq$	lessthan or equal to
$\geq$	greaterthan or equal to
$\subset$	proper subset
$\subseteq$	subset
$\forall$	forall
ϵ	epsilon
λ	lambda
$\triangle$	delta
ξ	xi
Ψ	Shi
$\Rightarrow$	Implies
$\Leftrightarrow$	Implies and implied by
X	Cartesian Product
$\exists$	there exists
I	Identity Map
Σ	Summation
$\rightarrow$	tends to

$f^2 = fof$	f composite f
$[\quad]$	big bracket
$\{\quad\}$	middle bracket
$(\quad)$	small bracket
α	alpha
β	beta
γ	gamma
δ	delta
∞	infinity
$\lvert\quad\rvert$	modulus
$\lVert\quad\rVert$	norm
τ	tau
(X, F, T)	Menger space
T_M	Minimum triangular norm
T_P	Product triangular norm
T_L	Lukasewicz triangular norm
T_D	Drastic product triangular norm

Chapter 1

Introduction

The aim of this chapter is to discuss the historical overview of fixed point theory on metric space, and on probabilistic metric space with basic definitions. Also, it deals with some fixed point theorems using contraction and compatible mappings in Menger probabilistic metric space via metric space as a concept that will be used in the entire thesis together with some applications.

1.1 Introduction

1.1.1 A historical overview of fixed point theory

In the 17th century, the analysis in mathematics was created independently as a part of a scientific revolution. Its development was aided by mathematicians like Johannes Kepler (1571-1630), Galileo Galilei (1564-1642), Rene Descartes (1596-1650), Pierre de Fermat (1607-1665), Christian Huygens (1629-1695), Sir Isaac Newton (1643-1727), and Gottfried Wilhelm Leibniz (1646-1716), to name a few.

As the combinations, functional and variational analysis were developed as an independent branch of mathematics, called nonlinear analysis, named by famous mathematician Andrew Browder (1931-2019) in the mid of twentieth century. Its nonlinear results have a wide range of applications in subjects like physics, chemistry, biology, and also in economics, which leads to nonlinear models. The fixed point theory has been considered and developed as one of the dynamical research domains of nonlinear analysis in the process of advancement of the same and is a nice combination of analysis, topology, and geometry. Over the past 80 years, fixed point theory has established itself as a most powerful and considerable tool in the study of nonlinear phenomena. The point at which function $y = f(x)$ and the line $y = x$ intersects gives the solution of the curve, and the point of intersection is the fixed point of the curve. The usefulness of concrete applications has increased enormously due to the development of accurate techniques for computing fixed points. Fixed point theory is rapidly moving into the mainstream of mathematics because of its applications in diverse fields which include numerical methods like the Newton-Raphson method, establishing Picards existence theorem [158], the existence of solution of integral equations, and a system of linear equations.

Definition 1.1.1. *[35]*
Let X be a non-empty set and $f : X \to X$ be a mapping. Then, a solution of $f(x) = x$ is called a fixed point of f. In other words, a point that does not change under a certain map is a fixed point.

Example 1.1.1. *[37]*

(i) A translation mapping $f(x) = x + 5$ *for all* $X \in \mathbb{R}$ *has no fixed point.*

(ii) A mapping $f : [0, 6] \to [0, 6]$ *defined by* $f(x) = \frac{x^2+6}{7}$ *where* x *is a positive integer. Then* $f(1) = 1$ *and* $f(6) = 6$*. So,* 1 *and* 6 *are fixed points of* f*.*

(iii) A mapping $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x$*, has infinitely many fixed points, i.e., every point of the domain is a fixed point of* f*.*

There are several methods to find the solution of equation $f(x) = 0$. Some of these methods are used to find exact solutions and some are used to find approximate solutions. And fixed point theory is also well-suited to tackle such problems. For example, $f(x) = x^2 - 5x + 6 = 0$ then clearly we get the roots of this quadratic equation as $x = 2$ and $x = 3$. Again, we rewrite the given equation in the following way: $x = (x^2 + 6)/5 = f(x)$ then we again find $x = 2$ and $x = 3$ as fixed points of f. So, we conclude that problem of finding the solution to the functional equation $g(x) = 0$ is the same as finding the fixed points of function $f(x)$ where $g(x) = f(x) - x$.

Example 1.1.2. *(a)* $f(x) = x^2 + 7x + 9 = 0$*. Suppose* r *be fixed point of a given function then* $r^2 + 6r + 9 = 0$*. So,* $r = -3$ *is a fixed point of* f *which is shown in figure* (1.1).

(b) $f(x) = \log(x)$ *has no fixed point shown in figure* (1.2).

(c) $f(x) = -\frac{1}{x}$ *has no fixed point shown in figure* (1.3).

It is observed that

- A function can have more than one fixed point or none at all.
- The set of fixed points is always closed.
- Theorems concerning the existence of fixed points and their properties are called fixed point theorems.

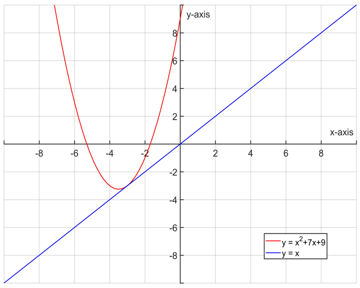

Figure 1.1: Graph of fixed point

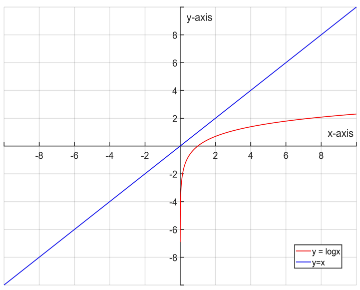

Figure 1.2: Graph of no fixed point

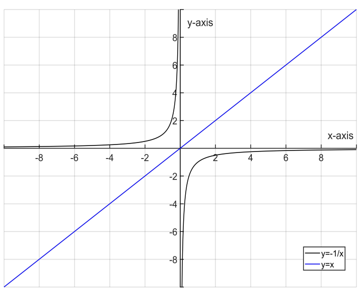

Figure 1.3: Graph of no fixed point

Historically, the origin of fixed point theory was effectively applied to establish the existence and uniqueness of solutions to differential equations via the method of successive approximations by eminent mathematicians namely: Joseph Liouville (1809-1882) [119], Rudolf Otto Sigismund Lipschitz (1832-1903) [120] Baron Augustin-Louis Cauchy (1789-1857)[28], Jules Henri Poincare (1854-1912) [159], Giuseppe Peano (1858-1932) [157] and Charles Emile Picard (1856-1941)[158] at the end of the nineteenth century. However, it was formally started in 1886 as an important part of the analysis of pioneering work of the great mathematician H. Poincare [159].

There are mainly two types of fixed point theorems: Constructive fixed point theorems and non-constructive fixed point theorems. Constructive fixed point theorems are those theorems that provide the existence of a solution as well as yield an algorithm such as Banach's fixed point theorem. On the other hand, non-constructive theorems define estimates of the number of fixed points such as Brouwers and Tarski's fixed point theorem. By nature, fixed point theory has divided into the following three major areas:

1. Topological fixed point theory
2. Metric fixed point theory
3. Discrete fixed point theory

An extensive study of fixed point theory and its related results, one can be referred to the classical books of K. Goebel and W. A. Kirk [69], W. A. Kirk and B. Sims [112], M. A. Khamsi and W. A. Kirk [108], I. A. Rus [174], R. P. Agarwal, D. O. Regan, and D. R. Sahu [5], W. A. Kirk and N. Shahzad [111], S. Carl and S. Heikkila [27], S. P. Singh, B. Watson, and P. Srivastava [199], J. Dugundji and A. Granas [58], D. R. Smart [201] and V. I. Istratescu [83].

Topological Fixed Point Theory was initiated by French mathematician H. Poincare [159] in 1886. He introduced topological methods in the study of non-linear problems of analysis. He applied a combinatorial approach to topology and proved a result like the Poincare-Bendixson theorem using triangulation of geometrical structure into simplexes. In 1912, Dutch mathematician L. E. J. Brouwer(1881-1966) [23] who was inspired by H. Poincare's idea, introduced and used the topological concepts of homotopy and degree of mapping in the context of a fixed point. In addition, he also established the fixed point results for a sphere (and a square), one of the earliest results that came to be known as the "Brouwer fixed point theorem" and whose extension to n-dimensional Euclidean space as follows:

Theorem 1.1.1. ***Brouwer fixed point theorem*** *[23]*
Every continuous mapping defined on a closed unit ball in $\mathbb{R}^n$ *has a fixed point.*

Brouwer's fixed point theorem is existential by nature. But this theorem just gave information about the solution that exists but no more information about the uniqueness and determination of the solution. It also does not give any information about the location of the fixed point.
Furthermore, when $n = 1$ then the above theorem becomes the "intermediate value theorem". In 1904, Piers Bohl (18651921) [16] established the equivalent of the above theorem, whereas, in 1909, Brouwer proved the theorem (1.1.1) for $n = 3$. Jacques Salomon Hadamard (18651963) [74] proved the general case in 1910, and thereafter L.E.J. Brouwer [23] independently in 1912. The "Brouwer fixed point theorem" was extended to an infinite dimensional space in 1930 by Juliusz Pawe Schauder (18991943) [178] and also proved the following theorem in Banach space:

Theorem 1.1.2. ***Schauder fixed point theorem*** *[178]*
Every continuous mapping defined on a compact convex subset of a Banach space has a fixed point.

The "Schauder fixed point theorem" was subsequently expanded to a locally convex topological vector space by Andrey Nikolayevich Tikhonov (1906-1993) [207] in 1935 and established following theorem:

Theorem 1.1.3. ***Tychonoff fixed point theorem*** *[207]*
Every continuous mapping defined on a compact convex subset of a locally convex topological vector space has a fixed point.

It is noted that all earlier proofs were indirect and non-constructive. Herbert Scarf (1930-2015) [177] had given the first constructive proof of L. E. J. Brouwer's fixed point theorem in 1967. The results of topological fixed points for multi-valued mappings are seen in the literature. Shizuo Kakutani (19112004), Henri Frederic Bohnenblust (19062000), and Samuel Karlin (19242007), to mention a few, generalized the Brouwer fixed point theorem. Ky Fan (19142010) and Irving Leonard Glicksberg (19142010) extended the Tychonoff fixed point theorem for multi-valued mappings, and several authors also extended the aforementioned theorems in multi-functions. Band Karlin provided the multi-functions of the Schauder fixed point theorem.

The root of **Metric fixed point theory** is the classical Banach contraction principle, which appeared in the 1920 doctoral dissertation of the eminent Polish mathematician **Stefen Banach** (March 30, 1892- August 31, 1945). In 1922, this classic work was published in the form of a research article [10]. Although the idea of successive approximations in some concrete situations (solving differential and integral equations) appears in some works of E. Picard (1856-1941), R. Caccioppoli (1904-1959), it was Banach who placed it in the right abstract setting, making it is suitable for a wide range of applications.

Theorem 1.1.4. ***Banach fixed point theorem*** *[10]*
Every contraction self-mapping f defined on a complete metric space (X, d) has a unique fixed point.

Above theorem is a fundamental tool of non-linear analysis. Here, we mention some relevant ones, R. Kannan [107], [169], S. K. Chatterjea [33], S. Reich [165], G. E. Hardy and T. D. Rogers [76], L. B. Ciric [49], D. W. Boyd and J. S. W. Wong [20], R. M. T. Bianchini [12], A. Meir and E. Keeler [126], J. Caristi [26], M. Geraghty [66], F. E. Browder [24]. For comprehensive analysis also we refer to B. E. Rhoades [166], [166], [167], [168], I. A. Rus [173], [174], J. Jachymski [87], W. A. Kirk and B. Sims [112], S. Park [150], D. R. Smart [201] and V. I. Istratescu [83].
Banach fixed point theorem guarantees a unique solution but the disadvantage of this classical result is the condition of contraction mapping is very restrictive. In 1961, M. Edelstein [60] established the "Edelstein fixed point theorem" on compact metric space, embodying this circumstance.

Theorem 1.1.5. ***Edelstein fixed point theorem*** *[60]*
Every contractive self-mapping f defined on a compact metric space (X, d) has a unique fixed point.

Then, in 1968, R. Kannan [107], [169] gave a turn to the metric fixed point theory by investigating a new class of maps, not necessarily continuous, possessing the fixed point property in an ordinary metric space.
Here, we list some improvements and generalizations of Banach's principle in the literature:
1. Schauder theorem (1930);
2. Schauder-Tychonoff theorem (1935);

3. Markov-Kakutani fixed point theorem (1936);
4. Krasnoselskii theorem (1955);
5. Browder-Gohde-Kirk theorem (1965); and
6. Caristi's fixed point theorem (1976).

Bronisaw Knaster (1893-1980) [114] introduced **discrete (or algebraic) fixed point theory** in which the order-preserving mapping is defined on a complete lattice in 1928. Alfred Tarski (1901-1983) [205] improved B. Knasters [114] result in 1955. And the earliest discrete fixed point result is often referred to as the "Knaster-Tarski fixed point theorem."

Theorem 1.1.6. ***Tarski or (Knaster-Tarski) fixed point theorem*** *[205]*
Every order-preserving self-mapping defined on a complete lattice has a fixed point. Moreover, the fixed points set is also a complete lattice.

Nicolas Bourbaki and Ernst Witt [19] established the Bourbaki-Witt fixed point theorem as a generalization of The Knaster-Tarski fixed point theorem without lattice structure. Numerous researchers proved remarkable results in this direction. For reference: we can consider S. Abian and A. B. Brown [4], R. DeMarr [56], J. S. Vandergrift [208], J. S. W. Wong [213], A. Abian [3], A. Pasini[152] , D. Kurepa [118], H. Amann [9], A. Brondsted [22], P. Cousot and R. Cousot [156], A. Bjorner [14], J. Sun and Y. Sun [204].

In 1968, K. Goebels [68] first studied two self-mappings for common fixed points by defining coincidence points and obtained a coincidence point theorem. Also, G. Jungck [95] established the first common fixed point theorem for a pair of two commuting maps in a complete metric space. Then, the flow of common fixed point results appeared in the literature, some of the details see in [[97], [54], [96], [98], [102], [81], [30], [184], [63], [43], [53],

[109]]. A new dimension has been added to the fixed point theory as a result of the invention of computers and the creation of new software for quick and efficient computing. Algorithms, numerical analysis, and applied mathematics are some of the new fields of study that have emerged. With the introduction of G. Jungck's fixed point theorem on commutative maps and the subsequent relaxing of the constraint of commutativity by weakly commutative results of S. Sessa, followed by the work and related concepts, a new turn in the development of fixed point theory occurs. With the work of L. B. Ciric, followed by the work of B. E. Rhoades, and W. A. Krik on non-expansive mappings, significant changes were made. In addition, the work of S. Park [150] and B. Sadovski [175] has made an important contribution by taking new sorts of mapping circumstances into consideration. The work of W. R. Mann [124] and S. Ishikawa [82] in the area of fixed point theory changed the approach to the approximation of fixed points and the convergence of iterative sequences. Moreover, fixed points can be controlled either by altering the nature of mappings or by placing more emphasis on research into the topological characteristics and spatial structure. Fixed point theory is therefore a key field for fresh research and developments.

1.1.2 A historical overview of probabilistic metric space

In the 19^{th} century, which is considered to represent the beginning of the modern era of science, saw significant development in the art of management. These developments sparked comparable worries about the errors they brought with them. However, it was still believed that any measurement error could be made arbitrarily tiny with proper planning and adequate data until the early decades of this century. This idea was disproved with the development of quantum mechanics since, in this theory, measurement uncertainties are inherent to the measurement process itself and, in theory, cannot be eliminated.

The reality of such inherent uncertainties and thresholds is widely acknowledged in the fourth quarter of the 20th century. This is true not just for physics but also for fields like psychometric [203], communication theory [188], [21], and pattern recognition [57], [214], [163]. Additionally, it serves as the foundation for several branches of mathematics, including cluster analysis [88], [192] and interval analysis [138]. The measurement in question is presumed to be done with regard to a rigid reference frame in almost all mathematical models developed to characterize these diverse scenarios. Several remarks are scattered throughout the literature [160], [161], [79], [55], [15], [212], [140] that suggest this assumption might not be realistic and that some of the uncertainties should be incorporated into the geometry. In this direction, there are also some significant efforts [59], [170], [171], [64]. The theory of probabilistic metric space is also a direct outgrowth of one of these attempts.

Here, we need to share H. Poincares essays on the philosophy of science, characterizing the difference between mathematics and physics. In mathematics, mathematical equality is a transitive relation, which means that if the quantity X equals the quantity Y and the quantity Y equals the quantity Z, then the quantity X equals the quantity Z. Although "equal" in the observable physical continuum refers to indistinguishably, in this continuum, if the quantity X is equal to the quantity Y and the quantity Y is equal to the quantity Z, it may not follow that the quantity X is equal to the quantity Z. In the language of psychologists Weber and Fechner [62], X may be within the threshold of Y, and Y may be within the threshold of Z, even though X is not within the threshold of Z. According to Poincare, the relation $X = Y$, $Y = Z$, $X < Z$, is frequently referred to as the formula for the physical continuum. It, therefore, seems more promising to keep the transitive relation in mathematics and introduce a probability, that is, a number lying between 0 and 1, for the distinction of physical and psychological quantities. This is instead of making a distinction between a transitive mathematical and an intransitive physical relation of equality. This concept is developed to give rise to the idea of a space where every pair of elements is assigned a distribution function rather than a specific number. The distance between two points is the number related to two points in metric space, however, it is not always possible to forecast the distance between two points with an exact number, thus we provide possible answers. So, the distribution function is introduced in place of the distance function to overcome such situations answer [130]. The distribution function associated with two numbers of statistical metric space might be said to give, for every x, the probability that the distance between two points in question does not exceed x. Such a statistical generalization of metric spaces appears to be well adapted for the investigation of the physical quantities and psychological thresholds. The idealization of local behaviors of rods and boards, implied by this statistical approach, differs radically from that of Euclid. In spite of this fact, or perhaps just because of it, the statistical approach may provide a useful means for geometrizing the physics of microcosm [182].

Historically, French mathematician **Rene Maurice Frechet** (1878-1973) creates a notion of distance in 1906 [65]. This notion now known as metric space, named by F. Hausdorff in 1914 [77]. This approach provides the widespread idealization of numerous mathematical, physical, and other scientific constructs where the concept of "distance" arises. The objects under evaluation could be extremely diverse. They could be sets, points, functions, or even purely irrational sensations. It is important that each ordered pair of items in a given set can have a non-negative real number as-

sociated with it, and that this number fulfills specific requirements for pairs and triples of such elements. When the theory of metric spaces is used, this identification of a single number with a pair of elements is typically an over-idealization. Even when measuring an ordinary length, the number provided as the distance between two points is frequently the average of a number of measurements rather than the result of a single measurement. In fact, in this instance and many similar situations, it is reasonable to see the distance idea as a statistical rather than a determinate one.
More precisely, Austrian mathematician **Karl Menger** (1902-1985) in 1942 [130] [also refer Menger paper [127], [128], [129], [132]] introduced a probabilistic version of metric space as a generalization of metric space, called statistical metric space (or probabilistic metric space shortly PMS, the name change in adjective from "statistical " to "probabilistic" from 1964). Menger proposed to replace the distance function in metric space with a distribution function to overcome the uncertainty case of the distance between two points in space i.e. replaced number $d(u, v)$ by a real function $F_{u,v}$ whose value $F_{u,v}(t)$, for any real number t, is interpreted as the probability that the distance between u to v less than t [181].

K. Menger provided the distribution function $F_{u,v}$ postulates in the original paper [130] where generalized triangle inequality was also included. In addition, K. Menger created a theory of betweenness and indicated possible application areas.
In 1943, A. Wald [210] questioned Menger's generalized triangle inequality and also proposed an alternative inequality. A. Wald created a theory of **betweenness** with certain advantages over Menger's theory on the basis of this new inequality [211].
In 1951, K. Menger [131] continued his research into statistical metric space that was devoted to a review of earlier work where they developed a number

of examples in detail and further considerations of the possible applications of the theory. In this paper, Menger adopted Walds version of the triangle inequality.

In 1956, when both K. Menger and A. Wald were constrained by the problems posed by Menger inequality, work was begun on developing the probabilistic metric:

$$F_{p,r}(x+y) \geq T(F_{p,q}(x), F_{q,r}(y)) \tag{1.1}$$

for all p, q, r in S and all real numbers x and y, T is a function from the closed unit square [0, 1] × [0, 1] into the closed unit internal [0, 1] satisfies certain conditions.

$$(i) \qquad T(a,b) = T(b,a) \tag{1.2}$$

$$(ii) \qquad T(a,b) \leq \text{T(c, d) whenever } a \leq c, b \leq \text{d and} \tag{1.3}$$

$$(iii) \qquad T(a,1) > 0 \text{ whenever } a > 0 \text{ and } T(1,1) = 1 \tag{1.4}$$

Wald suggested replacing (1.1) by the inequality

$$F_{pr} \geq (F_{pq} * F_{qr})(x)$$

When $*$ denotes convolution, so that

$$(F_{pq} * P_{qr})(x) = \int_{-\infty}^{\infty} F_{pq}(x-y)dF_{qr}(y) = \int_{0}^{x} F_{pq}(x-y)dF_{qr}(y)$$

A. Wald and K. Menger collaboration "break through came in 1957 when, in the course of studying certain particular spaces rediscovered Menger inequality (1.1) - not at first in its general form, but rather via particular functions T satisfying conditions (1.2) to (1.4.). Among these were the following:

$$
\begin{aligned}
&W(a,b) = Max.(a+b-1,0),\\
&\pi(a,b) = ab,\\
&M(a,b) = Min(a,b),\\
&M^*(a,b) = Max(a,b),\\
&\pi^*(a,b) = a+b-ab,\\
&W^*(a,b) = Min(a+b,1)
\end{aligned}
\tag{1.5}
$$

If T is taken to be anyone of the functions, then (1.1) not only tells us that the probability $F_{pr}(x+y)$ depends in some monotonic way on the probabilities $F_{pq}(x)$ and $F_{qr}(y)$, but also makes the nature of this dependence precise.

Example 1.1.3. *If $T = \pi$, then* (1.1) *states that the probability that the distance from p to r is less than $x + y$ is at least as large as the joint probability that independently, the distance from p to q is less than x and the distance from q to r less than y. The other function in* (1.5) *yield corresponding interpretation.*

In 1962, A.N. Serstnev [186] introduced an inequality that includes all those previously proposed as special cases and is without doubt the appropriate probabilistic generalization of the ordinary triangle inequality

$$d(p,r) \leq d(p,q) + d(q,r) \ \forall \ p,q,r \text{ in } S.$$

After 1963, the probabilistic metric space grew rapidly [[139], [179], [84], [86], [85]]. And many authors studied the geometrical and topological properties of probabilistic metric spaces based on Menger's paper and developed the probabilistic operator theory.

This space becomes famous when B. Schweizer and A.Sklar [182] studied the properties of spaces introduced by K. Menger and gave some basic results. They studied topology, the convergence of sequences, the continuity of mappings, defined the completeness of these spaces. Following A. N. Serstnev [186], H. Sherwood [193] gave a notion of probabilistic metric spaces and also proved a theorem of characterization of a nested, closed sequence of nonempty sets in complete probabilistic metric space.

Researchers take interest after great creation of contraction mapping on probabilistic metric spaces, by V.M. Sehgal and A.T. Bharucha in 1972 [185]. They introduced fixed point theory in probabilistic metric spaces by establishing theorem **Every such mapping on a complete Menger probabilistic metric space (S,F,min) has a unique fixed point.** Immediately, T.L. Hicks [78] considered another notation of contraction mapping and showed that **Every such contraction mapping on a complete Menger metric space** (S, F, min) **has a unique fixed point.**

In 1975, L. J. Ciric [47], [48] introduced the notion of generalized contraction on a PMS and V. I. Istratescu[83] in 1981 developed some interesting problems in this space. For details, one can refer books in PM space: Probabilistic metric spaces by B. Schweizer and A. Sklar [182], Elements of probabilistic analysis with applications' by G.Constantin and I.

Istractescu [51], "Lectures on probabilistic Analysis by V. Radu [164], Fixed point theory in PM Space by O. Hadzic and E. Pap [75], Triangular Norms by E. P. Klement, R. Mesiar and E. Pap [113], Non-linear operator theory in PMS by S. S. Chang, and Y. J. Cho and S. M. Kang [31].

The Fixed point and common fixed point properties for a mapping defined on probabilistic space has been the center of research domain and have been studied by many authors. Most of the properties provide the existence of fixed points and common fixed points are of linear contractive type conditions.

Above are the fundamental mathematicians in the development of Menger probabilistic metric spaces. After that this space has become very popular among mathematics researchers and introduced many contractions and compatible mappings as an extension of Banach contraction in Menger space and established fixed point theorems in single mapping, two pair, three pairs self mappings in this space. For details, one can refer some references of main contributors in this domain by G. Jungck [95], [96], S. Sessa [187], B. D. Pant [141], S. N. Mishra [137], H. K. Pathak [154], S. L. Singh and B. D. Pant [198], B. Singh and S. Jain [196], Y.J. Cho, P. P. Murthy and M. Stojakovic [45], K. Jha, V. Popa and K. B. Manadhar [94], G. Jungck and B. E. Rhodes [104], Al-Thagafi and N. Shahzad [6], H. Chandra and A. Bhatt [29], M. Aamri and D. E. Moutawakil [1], I. Kubiaczyk and S. Sharma [116].

Recently in 2021, A. K. Chaudhary, K. Jha, K. B. Manandhar and P. P. Murthy introduced compatible mapping of type (P) in Menger space [40] and introduced and established common fixed point theorem using compatible mappings of type (K) [38] and also have introduced weakly compatible mappings of type (P) [41] in 2022 and established theorems on occasionally compatible mapping [39] in Menger space.

1.1.3 Some applications of fixed point theory

Fixed point theories are used to demonstrate the existence and uniqueness of many mathematical models like differential, integral and partial differential equations and variational inequalities, etc., to represent phenomena arising in fields like steady state temperature distribution, chemical equations, neutron transport theory, economic theories, epidemics and flow of fluids. It is also useful in the field of computer science, image processing, artificial intelligence, decision-making, population dynamics, operational research, industrial engineering, pattern recognition, medicine, group health underwriting, management, and many others. Equilibrium and stability are fundamental ideas that can be represented in the way of fixed points in many different fields. Additionally, many compilers that convert the natural language to programming language use fixed-point computations for program analysis. Taking data flow analysis as an example. Followings are very some applications of fixed point theory as an examples :

1.2.1 Application to differential equation:[194]

Consider the differential equation:

$\frac{dy}{dx} = f(x, y), y(x_0) = y_0.$ $\quad (DE)$

Using Banach fixed point theorem to show following differential equation:

Theorem 1.1.7. ***(Picard Theorem) [194]:*** *Let $f(x, y)$ be continuous function on $X = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ satisfies Lipschitz conditions of the form: $|f(x, y) - f(x, y')| \leq \alpha |y - y'|$ for all $y, y' \in [c, d]$. Let (x_0, y_0) be an interior point of X. Then, the differential equation (DE) with the given initial condition has a unique solution.*

1.2.2 Application to integral equation:[44]

Consider $F(x, y)$ be a continuous function on $[a, b] \times [a, b]$ and $\phi(x)$ be a continuous function on $[a, b]$. Consider the following integral equation:

$f(x) = \phi(x) + \lambda \int_a^x F(x, y) f(y) dy$ $\quad (VE)$

for all $x \in [a, b]$, where λ is parameter, is a Volterra equation. By using Banach fixed point theorem, showing following theorem:

Theorem 1.1.8. *For each $\lambda \in \mathbb{R}$, the Volterra equation (VE) has a unique solution f which is continuous on $[a, b]$.*

1.2.3 Application to matrix equation:[90]

By using Banach fixed point theorem, we may find unique solutions of systems of n linear algebraic equations with n unknowns.

1.2.4 Application to image compression :[197] The proper way to save an image in memory is to preserve each pixel's color. The two issues with this approach are as follows::

a) It requires a huge amount of memory.

b) If we try to enlarge the image, such as for using it in a large poster, then the pixels will enlarge into larger squares and we will be missing information on how to fill in the details in these squares.

It is necessary to encode less information than the original image so that the eye cannot see that image observed is determined. The demand for effective image compression has increased due to the internet. Indeed, images significantly slow down web navigation. Therefore, it is best to have images encoded in files that are as small as possible for internet navigation. The process of compressing image files can be done in several ways. The JPEG format and the GIF format are typically the two more high-quality compressed graphic image formats for use on the internet. The GIF format is typically used for line art and other images with relatively simple geometric shapes, while the JPEG method is more frequently used for photographs. Wavelets and fractals are two additional methods of image compression. As of the time of this writing, these techniques are not widely accepted for use on the internet. There is another approach, which has remained more experimental. This approach is an Iteration function system given by Barnsley. The main motive behind this method is to approximate an image with geometric objects. Geometric objects include fractal objects like the fern, the Sierpinski Carpet, the Menger Sponge, and many others in addition to straight lines and smooth curves.

1.2.5 Application to Page Rank algorithm :[197] Incredible use of Banach's fixed point theorem is the Page Rank algorithm. Google search uses an algorithm called Page rank to rank websites in their search engine results. Page rank is a concept that Larry Page created. Additionally, he established Google. In 1998, he created it. In order to determine a web page's importance, Page rank looked at the pages that linked to it. It is a method of determining a website's importance. In the Page rank algorithm, the contraction-based fixed point of a linear operator $\mathbb{R}_n$ is calculated, and this fixed point yields the ordering of pages. Google is successful because a search engine arrives from its algorithm: the page rank algorithm. In this algorithm, one can determine a fixed point of a linear operator on Euclidean space which is a contraction, and this fixed point yields the arranging of the pages.

1.2.6 Application in Game Theory: [153]

Consider a game with $n \geq 2$ players, under the assumption that the players do not cooperate among themselves. Each player pursues a strategy, in dependence of the strategies of the other players. Denoting the set of all possible strategies of the Kth player by K_k, and set $K = K_1 \times K_2 \times K_3 \times ... \times K_n$. An element $x \in K$ is called strategy profile. For each k, let $f_k : K \to \mathbb{R}$ be the loss function of Kth player. If $\sum_1^n f_k(x) = 0, \forall x \in K$ then the game is said to be Zero- sum. The aim of each player is to minimize his loss, or equivalently, to maximize his gain. Following theorems helps to overcome on game theory:

Definition 1.1.2. *A Nash equilibrium is a strategy profile with the property that no player can benefit by changing his strategy, while all other players keep their strategies unchanged. In formulas, it is an element*
$\bar{x} = (\bar{x_1}, \bar{x_2}, \bar{x_3}, ...\bar{x_n}) \in K$ *such that* $f_k(\bar{x}) \leq f_k\{\bar{x_1}, ..., \bar{x_{k-1}}, \bar{x_k}, \bar{x_{k+1}}, ..., \bar{x_n}\}$,
$\forall x \in K_k$ *, for every* $k = 1, 2, 3, ..., n$.

Nash equilibrium suggests a convenient cautious strategy to be adopted by each player in the game. We said a strategy rather than the strategy since a Nash equilibrium (if it exists) might not be unique. We need of course further hypotheses on the sets K_k and on the maps f_k. It is reasonable to assume that, with all the other strategies fixed, the loss function f_k has a small variation in correspondence of a small variation of x_k. Also, loosely speaking, it is assumed that the average of losses corresponding to two different strategies of the k th player is greater than the loss corresponding to the "average" strategy. Convexity can suitably translate this issue. The fundamental result of game theory is the following:

Theorem 1.1.9. ***(Nash)*** *For every $k = 1, 2, 3, ..., n$, let K_k be non void, compact and convex subset of locally convex space X_k. Assume that for every k, the loss function f_k is continuous on K. In addition, for every fixed $x_j \in K_j$ with $j \neq k$, the map $f_k\{\bar{x_1}, ..., \bar{x_{k-1}}, \bar{x_k}, \bar{x_{k+1}}, ..., \bar{x_n}\} : K_k \to \mathbb{R}$ is convex. Then there exist $\bar{x} \in K$ satisfying above Nash equilibrium equation.*

Theorem 1.1.10. ***(von Neumann)*** *Let $K_1 \subset X_1$ and $K_2 \subset X_2$ be as in theorem 1.1.13. Let $\Psi : K_1 \times K_2 \to \mathbb{R}$ be such that*

(a) $\Psi(., x_2)$ is lower semi continuous and convex, $\forall x_2 \in K_2$;

(b) $\Psi(x_2, .)$ is upper semi continuous and concave, $\forall x_1 \in K_1$.

Then, there exists a Nash equilibrium $(x_1, x_2) \in K_1 \times K_2$.

This theorem is called minimax theorem in literature.

1.2.7 Application to solve Kepler's equation: [25]

The contraction principle is an abstract version of the successive approximation method; this method has been used empirically from antiquity, in order to solve numeric equations, and it has been successfully used, for instance, to solve Keplers equation, $E = M + esinE$, in order to determine the position of the planets in orbit ($E_0 = M, E_1 = M + esin(E_0)..., E_n = M + esin(E_{n-1})$. Keplers equations are used to calculate the position of objects from our Solar System, by using the eccentricity e of the orbit and the mean anomaly M. E represents the eccentric anomaly.

1.2 Basic Definitions

Metric space generalize and clarify the notion of distance in real line.
M. Frechet[65] introduced metric on set as:

Definition 1.2.1. *Let X be a non empty set and d be a real function from $X \times X \to \mathbb{R}^+$ such that for all $u, v, w \in X$, we have*

1. $d(u, v) > 0$, *(Positivity)*
2. $d(u, v) = 0 \iff u = v$, *(Identity)*
3. $d(u, v) = d(v, u)$ *(Symmetry) and*
4. $d(u, w) \leq d(u, v) + d(v, w)$, *(Triangle Inequality)*

*then d is called a **metric** or distance function and the pair (X, d) is called a **metric space**.*

Definition 1.2.2. *[35] A sequence $\{u_n\}$ in a metric space (X, d) is said to be **Cauchy sequence** if for given $\epsilon > 0$, there corresponds $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, we have $d(u_m, u_n) < \epsilon$.*

Definition 1.2.3. *[35] A sequence $\{u_n\}$ in metric space is said to be **convergent** to a point $z \in X$ if for given $\epsilon > 0$, there exists a positive number $n_0 \in \mathbb{N}$ such that $d(u_n, z) < \epsilon$. In this case, z is called limit point of $\{u_n\}$ and we write: $u_n \to z$.*

Definition 1.2.4. *[94] A metric space (X, d) is said to be **complete metric space** if every Cauchy sequence in it is convergent to a point in X.*

Definition 1.2.5. *[94] A function* $f : X \to Y$ *from a metric space* (X, d_x) *to a metric space* (Y, d_y) *is said to be* ***continuous*** *at* $u_0 \in X$ *if whenever* $u \to u_0$ *in X,* $(f(x)) \to f(u_0)$ *in* Y*. We say f is continuous if it is continuous at each* $u_0 \in X$ *.*

Definition 1.2.6. *[94] A metric space X is said to be* ***compact*** *if every sequence in it has a convergent subsequence.*

Definition 1.2.7. *[94] A subset* f *of metric space X is called a* ***closed set*** *if it contains each of its limit points.*

Definition 1.2.8. *[35] Let* $y = f(x)$ *be a function on interval* (a, b)*.*
If for any two points $x_1, x_2 \in (a, b)$ *such that* $x_1 < x_2$*, there holds the inequality* $f(x_1) \leq f(x_2)$*, the function is called* ***increasing (or non-decreasing)*** *in this interval. And if* $f(x_1) < f(x_2)$ *then function* $y = f(x)$ *is said to be strictly increasing on the interval* (a, b)*.*

Definition 1.2.9. *[35] A set* $A \subset \mathbb{R}$ *of real numbers is bounded from above if there exists a real number* $b \in \mathbb{R}$*, called an* ***upper bound*** *of A such that* $x \leq b$ *for every* $x \in A$*. Similarly, A is said to be bounded from below if there exists* $c \in R$ *called a* ***lower bound*** *of A such that* $x \geq c$ *for every* $x \in A$*. The set A is said to be bounded if it is bounded both from above and below.*

Definition 1.2.10. *[35] Let* $A \subset \mathbb{R}$ *is a set of real number. If* $b \in \mathbb{R}$ *is an upper bound of A such that* $b \leq b'$ *for every upper bound* b' *of A, then b is called* ***supremum*** *of A, denoted by* ***sup*** $A = b$*. If* $c \in R$ *is lower bound of A such that* $c \geq c'$ *for every lower bound* c' *of A, then c is called* ***infimum*** *of A, denoted as* ***Inf*** $A = c$*.*

Definition 1.2.11. *[90] Let* (X, d) *be a metric space and let* $f : X \to X$ *be a mapping. Then,*

(i) f *is called* ***contraction*** *if there exists a number* $\alpha \in [0, 1)$*, such that*

$d(f(x), f(y)) \leq \alpha\, d(x, y), \;\; \forall x, y \in X.$

So, contraction shrinks distances by a uniform factor $\alpha < 1$ *for all pair of points.*

(ii) f *is called* ***contractive*** *or strict contractive if*

$$d(f(x), f(y)) < d(x, y), \;\; \forall x, y \in X, x \neq y$$

(iii) f *is called* ***non-expansive*** *if*

$$d(f(x), f(y)) \leq d(x, y), \;\; \forall x, y \in X.$$

(iv) f *is called* ***isometry*** *if*

$$d(f(x), \; f(y)) = d(x, y), \;\; \forall \; x, y \in X.$$

(v) f *is called* ***Lipschitzian continuous*** *(or L - Lipschitzian) if there exists* $\alpha > 0$ *such that*

$$d(f(x), f(y)) \leq \alpha\, d(x, y).$$

So, we find that:

Contraction ⇒ Contractive ⇒ Non expansive ⇒ Lipschitz Continuous.

Example 1.2.1. *a) Consider the usual metric space* $(\mathbb{R}, d)$*, and function*

$$f(x) = \frac{x}{b} + c, \;\; \textit{for all } x \; \in \mathbb{R}$$

Then, f *is contraction on* $\mathbb{R}$ *if* $b > 1$ *and the solution of the equation*

$$x - f(x) = 0 \;\; \textit{is } x = \frac{bc}{b-1}$$

b) Consider the Euclidean metric space $(\mathbb{R}^2, d)$. *Define*

$$f(x,y) = (\frac{x}{a} + b, \frac{y}{c} + b), \text{ for all } (x,y) \in \mathbb{R}^2.$$

Then, f *is contraction on* $\mathbb{R}^2$ *if* $a, c > 1$. *Now, solving the equation* $f(x,y) = (x,y)$ *for a fixed point, we get*

$$x = \frac{ab}{a-1} \text{ and } y = \frac{cb}{c-1}.$$

c) Let $f: [0,2] \to [0,2]$ *be defined by*

$$f(x) = \begin{cases} 0, & x \in [0,1], \\ 1, & x \in (1,2]. \end{cases}$$

Then, $f^2(x) = 0$ *for all* $x \in$*[0,2], and so,* f^2 *is a contraction on* $[0,2]$ *but* f *is not continuous and thus not a contraction map.*

d) Let $T : X \to X$ *be a mapping defined by* $Tx = x^2$, *where* $X = [0, \frac{1}{3}]$ *then* T *is a contraction mapping on* X *with the usual metric* d.

Solution: $\forall\ x, y \in X$,

$$\begin{aligned} d(Tx,Ty) =& d(x^2,y^2) \\ =& |x^2 - y^2| \\ =& |x-y||x+y| \\ <& \frac{2}{3}|x-y| \\ \because& \quad |x| < \frac{1}{3} \\ \therefore& \quad d(Tx,Ty) < \frac{2}{3}\ d(x,y). \end{aligned}$$

Therefore, T *is a contraction mapping on* X *with* $h = \frac{2}{3}$.

Example 1.2.2. *[90]*

i) Let $T : [1/2, 2] \to [1/2, 2]$ *be a defined as* $T(x) = 1/x$*, then* T *is 4-Lipschitzian with* $Fix(T) = \{1\}$*, where* $Fix(T)$ *denotes fixed point of the mapping* T*.*

ii) $T : \mathbb{R} \to \mathbb{R},\ T(x) = x/2+3,\ x \in \mathbb{R}$*. Obviously* T *is a Banach contraction and* $Fix(T) = \{6\}$*.*

iii)$Tx = 1 - x, x \in R$ *is non expansive and* $Fix(T) = \{1/2\}$

iv) $T : [1, \infty] \to [1, \infty],\ T(x) = x + 1/x$*, is contractive and* $Fix(T) = \phi$

v) $T(x) = x + 2$*, then* $Fix(T) = \phi$ *is isometry.*

In 1976, G. Jungck [95] introduced commuting mapping in metric space as:

Definition 1.2.12. *[95] Two self mappings* Q *and* R *of a metric space* (X, d) *are said to be* ***commuting*** *if,*

$$Q(R(x)) = R(Q(x)),\ \forall x \in X.$$

We shall also denote it by $QRx = RQx$. And, Q and R are said to be non commuting if there is no such point x in X where Q and R commute.

In 2002, Z. Lu [121] defined following definition in metric space:

Definition 1.2.13. *[121] Two mappings* $Q, R : X \to X$ *are said to be* ***conversely commuting mappings*** *in metric space* (X, d) *if for all* $x \in X$, $QRx = RQx$ *implies* $Qx = Rx$*.*

In 1982, S. Sessa [187] introduced weakly commuting mappings as:

Definition 1.2.14. *[187] Two self mappings* Q *and* R *of a metric space* X *are said to be* ***weakly commuting*** *if*

$$d(QRx, RQx) \leq d(Qx, Rx), \forall x \in X.$$

In 1986, G. Jungck [96] introduced compatible mappings in metric space as:

Definition 1.2.15. *[96] Two self mappings Q and R of a metric space (X, d) are called* ***compatible mappings*** *if*

$$\lim_{n\to\infty} d(QRx_n, RQx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = t,$$

for some $t \in X$.

In metric space the following implication holds :
Commutativity $\Longrightarrow$ Weak commutativity $\Longrightarrow$ Compatibility $\Longrightarrow$ Weak compatibility but not conversely.

Example 1.2.3. *[145] Let (X, d) be a usual metric where $X = [0, 5]$ and defining function $Q, R : X \to X$ by $Q(x) = \frac{x}{x+2}$ and $R(x) = \frac{x}{2}$ then (Q, R) is weakly commuting but not commuting.*

Example 1.2.4. *[45] Let (X, d) be a usual metric where $X = \mathbb{R}$ and defining function $Q, R : X \to X$ by $Q(x) = x^3$ and $R(x) = 2x^3$ then (Q, R) is compatible but not weakly commuting.*

Example 1.2.5. *[93] Let (X, d) be a metric space where $X = [1, 8]$ and $Q, R : X \to X$ is defined by*

$$Q(x) = \begin{cases} 2 & \text{for } x = 2 \quad \text{or} \quad x \in (5, 8] \\ 7 & \text{for } x \in [1, 2) \quad \text{or} \quad x \in (2, 5] \end{cases}$$

and

$$R(x) = \begin{cases} x & \text{for } x \in [1, 5] \\ x - 3 & \text{for } x \in (5, 8] \end{cases}$$

Then Q and R are weakly compatible on X but not compatible.

Definition 1.2.16. *[96] Two self mappings Q and R of a metric space X are said to be* ***non -compatible mappings*** *if they are not compatible mappings.*

Definition 1.2.17. *[99] Two self mappings Q and R of a metric space (X, d) are said to be* ***weakly compatible mappings*** *if they commute at their coincident point;*
that is, $Qx = Rx$ for some $x \in X$ implies $QRx = RQx$.

Example 1.2.6. *[104] Let $X = [0, \infty)$ be endowed with usual metric and $Q, R : X \to X$ such that $Qx = x^3$ and $Rx = 2x^3$. Then $QRx \neq RQx$. so, Q and R are not commuting on X and $d(QRx, RQx) > d(Qx, Rx)$. Therefore, Q and R are not weakly commuting as well on X. However, $\lim_{n\to\infty} d(Qx, Rx) = 0 \in X$ and it implies that $\lim_{n\to\infty} d(QRx, RQx)$.*
Q and R are compatible mappings.

In 2008, M. A. Thapagi and N. shahzad [6] introduced following definition in metric space:

Definition 1.2.18. *[6] Two self mappings Q and R of a metric space (X, d) are* ***occasionally weakly compatible (owc)*** *if and only if there is a point x in X which is a coincident point of Q and R at which Q and R commute.*

In 1993, G. Jungck, P. P. Murthy and Y. J. Cho [101] introduced following definition:

Definition 1.2.19. *[101] Two self mappings Q and R of a metric space (X, d) are said to be* ***compatible mappings of type*** *(A) if*

$$\lim_{n\to\infty} d(RQx_n, QQx_n) = 0 \quad and \quad \lim_{n\to\infty} d(QRx_n, RRx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = t, \ for\, some\ t \in X.$$

In 1996, H. K. Pathak, Y. J. Cho, S. S. Chang and S. M. Kang [154] introduced compatible mapping of type (P) in metric space as:

Definition 1.2.20. *[154] Two self mappings Q and R of a metric space (X, d) are said to be **compatible mappings of type** (P) if*

$$\lim_{n\to\infty} d(QQx_n, RRx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = t, \quad for\, some \quad t \in X.$$

In 2007, M. R. Singh and Y. R. Singh [172] introduced following theorem:

Definition 1.2.21. *[172] Two self mappings Q and R of a metric space (X, d) are said to be **compatible mappings of type (E)**, if*

$$\lim_{n\to\infty} QQx_n = \lim_{n\to\infty} QRx_n = R(t), \text{ and}$$

$$\lim_{n\to\infty} RRx_n = \lim_{n\to\infty} RQx_n = Q(t),$$

whenever $\{x_n\}$ is a sequence in X such that
$lim_{n\to\infty}Qx_n = lim_{n\to\infty}Rx_n = t$ for some $t \in X$.

In 2014, K. Jha, V. Popa and K. B. Manandhar [94] introduced compatible mappings of type (K) in metric space as:

Definition 1.2.22. *[94] Two self mappings Q and R of a metric space (X, d) are said to be **compatible mappings of type (K)** if*

$$\lim_{n\to\infty} QQx_n = Rt \quad and \quad \lim_{n\to\infty} RRx_n = Qt,$$

whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = t$ for some t in X.

Example 1.2.7. *[94] Let* $X = [0, 2]$ *with the usual metric* $d(x, y) = |x - y|$*.*
We define self-mappings Q *and* R *as:*
$Qx = 2,\ Rx = 0$ *for* $x \in [0, 1] - 1/2$*,* $Qx = 0, \quad Rx = 2$ *for* $x = \frac{1}{2}$ *and*
$Qx = \frac{2-x}{2}$*,* $Rx = \frac{x}{2}$ *for* $x \in (1, 2]$*.*
Then, Q *and* R *are compatible of type (K).*

In1942, K. Menger [130] defined following definition:

Definition 1.2.23. *[130] A function* $F : \mathbb{R} \to \mathbb{R}^+$ *is said to be a* ***distribution function*** *if*

(i) F *is non-decreasing;*

(ii) F *is left continuous, and*

(iii) $\inf_{x \in \mathbb{R}} F(x) = 0$ *and* $\sup_{x \in \mathbb{R}} F(x) = 1$*.*

If X is a non empty set, $F : X \times X \to \triangle$ is called probabilistic distance on X and $F(x, y)$ is usually denoted by F_{xy}.
We will denote by $\triangle$ the family of all distribution function on $(-\infty, \infty)$ and $\triangle^+$ on $[0, \infty)$.

Example 1.2.8. *[182] Let* $H(x)$ *denote the specific distribution function defined by*

$$H(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

In 1942, K. Menger [130] defined probabilistic metric space as:

Definition 1.2.24. *[130] A* ***probabilistic metric space*** *(briefly, PM-space) is an order pair* (X, F) *where* X *is a non-empty set and* F *is a function defined by* $F : X \times X \to \triangle^+$ *(the set of all distribution functions)* *that is F associates a distribution function $F(u,v)$ with every pair (u,v) of points in X. The distribution function $F(u,v)$ is denoted by $F_{u,v}$, whence the symbol $F_{u,v}(x)$ will represent the value of $F_{u,v}$ at $x \in \mathbb{R}$. And the function $F_{u,v}$, for $u,v \in X$ are assumed to satisfy following conditions:*

(i) $F_{u,v}(0) = 0$;

(ii) $F_{u,v}(x) = F_{v,u}(x)$

(iii) $F_{u,v}(x) = 1$, *for every* $x > 0 \Leftrightarrow u = v$.

(iv) For every $u, v, w \in X$ and for every

$$x,\ y > 0,\ F_{u,v}(x) = 1,\ F_{v,w}(y) = 1 \ \Rightarrow F_{u,w}(x+y) = 1.$$

The interpretation of $F_{u,v}(x)$ as the probability that the distance from u to v is less than x, we know that PM condition (iii), (i) and (ii) are staright forward generalizations of the corresponding metric space conditions (i), (ii) and (iii). The PM condition (iv) is a 'minimal' generalization of the triangle inequality of metric space condition (iv). If it is certain that the distance of u and v is less than x, and like wise certain that the distance of v and w is less than y, then it is certain that the distance of u and w is less than $x + y$. The PM condition (iv) is always satisfied in metric spaces, where it reduces to the ordinary triangle inequality.

Example 1.2.9. *[36] Let* $X = \mathbb{R}$ *be set of real number and let distribution function be defined by*

$$F_{u,v}(t) = \begin{cases} e^{-\frac{d(u,v)}{t}} & for\ t > 0 \\ 0 & for\ t = 0 \end{cases}$$

for all $u, v \in X$*, Then* (X, F) *is PM space.*

In 1996, E. Pap and O. Hadzic [149] defined following definition:

Definition 1.2.25. *[149] Let* X *be a non empty set and* F *be a distribution function. Then, an ordered pair* (X, F) *is said to be* ***probabistic semi metric space (PSM)*** *if it holds the following inequality:*

(i) $F_{u,v}(x) = 1$, *for every* $x > 0 \Leftrightarrow u = v$, *and*

(ii) $F_{u,v}(x) = F_{v,u}(x)$

for every $u, v \in X$.

K. Menger [130] also defined triangular norm in 1942 as:

Definition 1.2.26. *[130] A mapping* $T : [0, 1] \times [0, 1] \to [0, 1]$ *is called a* ***triangular norm*** *(shortly t-norm) if for all* $a, b, c, d, \in [0, 1]$ *the following conditions are satisfied :*

(i) $T(a, 1) = a$ *for every* $a \in [0, 1]$, *(Neutral Element)*

(ii) $T(a, b) = T(b, a)$ *for every* $a, b \in [0, 1]$, *(Commutativity)*

(iii) $T(a, b) \leq T(c, d)$ *whenever* $a \leq c$ *and* $b \leq d$ *(Monotonicity)*

(iv) $T(a, T(b, c)) = T(T(a, b), c))$ $(a, b, c \in [0, 1])$. *(Associativity)*

Example 1.2.10. *[75]*

$$T(a, b) = max\{(a + b) - 1, 0\} \text{ and } T(a, b) = min\{a,\ b\}$$

The four basic t-norms are [75]:

(i) The Minimum t-norm, T_M*, is defined by* $T_M(x, y) = min\{x, y\}$,

(ii) The Product t-norm, T_p*, is defined by* $T_P(x, y) = x.y$,

(iii) The Lukasiewicz t-norm, T_L*, is defined by* $T_L(x, y) = max\{x+y-1, 0\}$,

(iv) The Weakest t-norm, the drastic product, T_D*, is defined by*

$$T_D(x, y) = \begin{cases} min(x, y) if & max(x, y) = 1, and \\ 0 & otherwise \end{cases}$$

With references to the point wise ordering, we have the following inequalities $T_D < T_L < T_P < T_M$.

We have obtained following example in Triangular norm:

Example 1.2.11.

$$T(a, b) = \begin{cases} 0 & for \quad a = b = 0, \\ a \;\; or \; b & for \quad b = 1 \;\; or \; a = 1 \\ \frac{ab}{a+b} & otherwise \end{cases}$$

In 2001, O. Hadzic and E. Pap [75] defined following definition:

Definition 1.2.27. *[75] A mapping* $T : [0, 1] \times [0, 1] \to [0, 1]$ *is called a* ***continuous t-norm*** *if it satisfies the following conditions:*

(i) T *is associative and commutative,*

(ii) T *is continuous,*

(iii) $T(a, 1) = a$ *for all* $a \in (0, 1)$*, and*

(iv) $T(a, b) \leq T(c, d)$ *wherever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0, l]$.

Example 1.2.12. $T(a, b) = ab$ *for all* $a, b \in [0, 1]$ *is a continuous t-norm.*

K. Menger [130] introduced Menger space in 1942 as:

Definition 1.2.28. *[130]* ***A Menger probabilistic metric space*** *(briefly, Menger PM-space) is a triple* (X, F, T), *where* (X, F) *is a probabilistic metric space, T is a triangular norm (abbreviated t-norm) which satisfies the following conditions, for all* $u, v, w \in X$ *and* $x, \ y > 0$,
(V) $F_{u,w}(x + y) \geq T(F_{u,v}(x), \ F_{v,w}(y))$.

This is the extension of triangle inequality. This inequality is called Menger's triangle inequality.

Example 1.2.13. *[37] Let* $X = \mathbb{R}, a * b = min\ (a, b)\ \forall\ a, b \in (0, 1)$ *and*

$$F_{u,v}(x) = \begin{cases} H(x) & for\ u \neq v \\ 1 & for\ u = v \end{cases}$$

where

$$H(x) = \begin{cases} 0 & if\ x \leq 0 \\ x, & if\ 0 < x \leq 1 \\ 1, & if\ x > 1 \end{cases}$$

then $(X, F, *)$ *is Menger Space.*

Definition 1.2.29. *[45] Let* (X, d) *be a metric space then the metric d induces a mapping from* $X \times X \rightarrow L$ *(is the set of all distribution functions), defined by* $F_{x,y}(t) = H(t - d(x, y)), \forall x, y \in X$ *and* $t \in \mathbb{R}$.
Further if the t -norm t : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ *is defined by* $t(a, b) = min\{a, b\}$ *then* (X, F, t) *is called induced Menger space.*

Proposition 1.2.1. *[45] Let t be a T-norm defined by* $t(a, b) = min\{a, b\}$. *Then, induced Menger space* (X, F, t) *is complete if a metric space* (X, d) *is complete.*

Definition 1.2.30. *[40] Let (X, F, T) be a **Menger space** and T be a continuous t-norm. Then,*

*(1) A sequence $\{x_n\}$ in X is said to be **converge** to a point x in X (written $x_n \to x$) iff for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq N$.*

*(2) A sequence $\{x_n\}$ in X is called a **Cauchy** if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq N$.*

*(3) Every Cauchy sequence in X is convergent in a point X is said to be **complete Menger space**.*

Definition 1.2.31. *[130]Let (X, F, T) be a Menger space. Then, a mapping $f : X \to X$ is continuous at $x \in X$, if for each $0 < \lambda < 1$, there exist a real number $0 < \delta < 1$ satisfies condition:*

$F_{x,y}(t) \geq 1 - \delta$ implies $F_{fx,fy}(t) \geq 1 - \lambda$.

The definition of contraction mapping was first suggested and studied by V.M. Seghal [183] in his doctoral dissertation at Wayne state university, which is probabilistic version of the notion of Banach contraction in metric space.

Definition 1.2.32. *[183] Let (X, F) be a probabilistic metric space. A mapping $T : X \to X$ is a **contraction Mapping** on (X, F) if and only if there is an $k \in (0, 1)$ such that*

$$F_{Tp,Tq}(kt) \geq F_{p,q}(t),$$

where $p, q \in X$ and $t > 0$.

It is interpreted that probability that the distance between the image of Tp and Tq is less than kt is at least as large as the probability that the distance between p and q is less than t.

In 1972, V. M. Sehgal and A. T. Bharucha-Reid [185] asserted that any contraction mapping on a complete Menger space in which the triangle inequality is formulated under the strongest triangular norm T_M has a unique fixed point.

In 1983, T.L. Hicks [78] introduced another contraction in PM space as:

Definition 1.2.33. *[78] Let (X, F) be a probabilistic metric spaces and a mapping $T : X \to X$. The mapping T is called* ***Hicks C-contraction (Or, C-contraction)*** *if there exists $k \in (0, 1)$ such that the following implication holds for every $p, q \in S$: and for every $t > 0$*

$$F_{pq}(t) > 1 - t \Rightarrow F_{T(p)T(q)}(kt) > 1 - kt.$$

By weakening the contraction (C), D. Mihet [135] in 2005 introduced the following w-H contraction:

Definition 1.2.34. *Let X be a nonempty set and F be a probabilistic distance on X. A mapping $f : X \to X$ is a said to be a weak - Hicks contraction* ***(w-H contraction)*** *if there exists $k \in (0, 1)$ such that, for all $p, q \in X$.*

$$(w - H) : t \in (0, 1), F_{pq}(t) > 1 - t \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt.$$

Example 1.2.14. *[135] Let $X = [0, \infty)$ and*

$$F_{xy}(t) = \frac{min(x, y)}{max(x, y)}, \ \forall \ t \in (0, \infty), \ \forall \ x, y \in X, \ x \neq y.$$

It is known that (X, F, T) is a complete Menger space under the triangular norm $T = T_p > T_L$.

Also, it can be seen that the mapping $g : X \to X$ defined by

$$g(x) = \begin{cases} 0, & if x = 0 \\ 1, & if x > 0 \end{cases}$$

is a w-H contraction for every $k \in (0, 1)$.

Indeed, if $F_{g(x),g(y)}(kt) \leq 1 - kt$ then (exactly) one of the numbers x and y is 0, which implies $F_{xy}(t) = 0 < 1 - t$.

It is noted that this mapping has two fixed points viz $x = 0$ and $x = 1$.
In 1976, G. Jungck [95] defined following definition:

Definition 1.2.35. *[95] Two mappings $Q, R : X \to X$ are said to be **commuting mappings** in probabilistic metric space (X, F) if $QRt = RQt$ for each t in X*

In 1982, S. Sessa [187] initiated weakly commuting in metric space and S. L. Singh and B. D. Pant [144] extended it into PM space as:

Definition 1.2.36. *Two mappings $Q, R : X \to X$ are said to be **weakly commuting mappings** in probabilistic metric space (X, F) if*

$F_{QRx,RQx}(t) \geq F_{Qx,Rx}(t)$ *for each x in X and $t > 0$*

It is noted that every pair of commuting self mappings is weakly commuting but converse is not true.
In 1991 S. N. Mishra [137] had introduced following compatible mapping in Menger probabilistic metric space as the extension of compatible mapping in metric space introduced by G. Jungck in 1986 [96].

Definition 1.2.37. *[137] Two mappings $Q, R : X \to X$ are said to be **compatible mappings** in Menger space (X, F, t) iff*

$$\lim_{n\to\infty} F_{QRx_n,RQx_n}(t) = 1 \quad \text{for all} \quad t > 0$$

whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$ *for some z in X.*

Y. J. Cho, P. P. Murthy and M. Stojakovic [45] introduce the compatible mapping of type(A) in Menger space which is the extension of compatible mapping of type (A) in metric space [101].

Definition 1.2.38. *[45] Two mappings $Q, R : X \to X$ are said to be **compatible mappings of type** (A) in Menger space (X, F, t) iff*

$\lim_{n\to\infty} F_{RQx_n,QQx_n}(t) = 1$ *and* $\lim_{n\to\infty} F_{QRx_n,RRx_n}(t) = 1, \forall\, t > 0$
whenever $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$
for some z *in* X*.*

The notion of non-compatible mappings in metric space introduced by R. P. Pant in 1994 [145] and extended this concept in PM space by J. Ali, M. Imdad, and D. Bahuguna [7] as follows:

Definition 1.2.39. *[7] Two mappings* $Q, R : X \to X$ *are said to be* ***non-compatible Mappings*** *in probabilistic metric space* (X, F) *if there exist at least one sequence* x_n *in* X *such that whenever*
$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = t$ *for some* t *in* X *then we get*
$\lim_{n\to\infty} F_{QRx_n,RQx_n}(x) < 1$ *or non existent for all* $x > 0$.

Y. J. Cho, B. K. Sharma and D. R. Sahu [46] introduced the non-symmetric concept of semi-compatible mappings in a $d-$ topological space in 1995 and B. Singh and S. Jain [195] extended the notion of semi-compatibility in PM-space as follows:

Definition 1.2.40. *[195] Two mappings* $Q, R : X \to X$ *are said to be* ***semi -compatible mappings*** *in PM space* (X, F) *iff*

$$\lim_{n\to\infty} F_{QRx_n,Rx}(t) = 1 \quad \text{for all} \quad t > 0$$

whenever $\{x_n\}$ *is a sequence in* X *such that*
$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = x$ *for some* x *in* X*.*

The following example shows that semi- compatible need not to be a compatible mappings in PM space:

Example 1.2.15. *[195] Let* (X, d) *be a metric space where* $X = [0, 1]$ *and let*

$$F_{x,y}(t) = \begin{cases} t/t + d(x, y) & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

for all $x, y \in X$*.Then* (X, F) *is a PM Space. Also, let* $Q, R : X \to X$ *be defined by*

$$Q(x) = \begin{cases} x & \text{for } 0 \leq x < 1/2 \\ 1 & \text{for } (\frac{1}{2}) \leq x < 1 \end{cases}$$

and

$$R(x) = \begin{cases} 1 - x & \text{for } 0 \leq x < 1/2 \\ 1 & \text{for } (\frac{1}{2}) \leq x < 1 \end{cases}$$

Taking sequence $x_n = \frac{1}{2} - \frac{1}{n}$.
Then, we get $Qx_n = \frac{1}{2} - \frac{1}{n}$, $Rx_n = \frac{1}{2} + \frac{1}{n}$.
So, $Qx_n \to \frac{1}{2}$, $Rx_n \to \frac{1}{2}$ *as* $n \to \infty$.
Further, $QRx_n = 1$, $RQx_n = \frac{1}{2} + \frac{1}{n}$.
Now, $\lim_{n\to\infty} F_{QRx_n,RQx_n}(t) = \lim_{n\to\infty} F_{1,\frac{1}{2}+\frac{1}{n}}(t) = \dfrac{t}{t+\frac{1}{2}} = \dfrac{2t}{2t+1} < 1$ *for all* $t > 0$.
So, (Q, R) *is not compatible.*
Also, $\lim_{n\to\infty} F_{QRx_n,\,Rx}(t) = \lim_{n\to\infty} F_{1,1}(t) = 1$.
So, it is clear that (Q, R) *is semi- compatible.*
On the other hand,
$\lim_{n\to\infty} F_{RQx_n,\,Qx}(t) = \lim_{n\to\infty} F_{1/2,1}(t) = \dfrac{t}{t+\frac{1}{2}} = \dfrac{2t}{2t+1} < 1.$
That is (R, Q) *is not semi- compatible.*

In 2005, B. Singh and S. Jain [196] extended the notion of weakly compatible mappings in PM space as follows:

Definition 1.2.41. *[196] Two mappings* $Q, R : X \to X$ *are said to be* ***weakly compatible mappings*** *in Menger space* (X, F, t) *if they commute at their coincidence points i.e.* $Qx = Rx$ *for some* $x \in X$*, then* $QRx = RQx$.

Example 1.2.16. *[41] Let* (X, d) *be a metric space where* $X = [0, 2]$ *and* (X, F, t) *be induced Menger space with* $F_{x,y}(t) = H(t - d(x, y))$*,* $x, y \in X$

and $t > 0$. Defining $Q, R : X \to X$ as:

$$Q(x) = \begin{cases} 1 - x & \text{for } 0 \leq x < 1/2 \\ 1 & \text{for } 1/2 \leq x \leq 2 \end{cases}$$

and

$$R(x) = \begin{cases} x & \text{for } 0 \leq x < 1/2 \\ 1 & \text{for } 1/2 \leq x \leq 2 \end{cases}$$

Taking $x_n = \frac{1}{2} - \frac{1}{n}$. Then, $Qx_n = Q(\frac{1}{2} - \frac{1}{n}) = 1 - (\frac{1}{2} - \frac{1}{n}) = \frac{1}{2} + \frac{1}{n}$.
So, $Qx_n \to \frac{1}{2}$.
$Rx_n = R(\frac{1}{2} - \frac{1}{n}) = \frac{1}{2} - \frac{1}{n}$. So, $Rx_n \to \frac{1}{2}$.
Further, $QRx_n = \frac{1}{2} + \frac{1}{n}$ and $RQx_n = 1$.
Now, $\lim_{n\to\infty} F_{QRx_n,RQx_n}(t) = \lim_{n\to\infty} F_{\frac{1}{2}+\frac{1}{n},1}(t) = F_{\frac{1}{2},1}(t) = H(t - \dfrac{1}{2}) \neq 1$ for $0 < t \leq \frac{1}{2}$. Therefore, (Q, R) is not compatible.
Again, since coincidence of Q and R are in $[\frac{1}{2}, 2]$.
so, for any $x \in [\frac{1}{2}, 2]$, $Qx = Rx = 1$ and $QR(x) = Q(1) = 1 = RQ(x)$.
Hence, (Q, R) is weakly compatible.

In 2008, I. Kubiaczyk and S. Sharma [116] extended the notion of (E.A) property to PM-space as follows:

Definition 1.2.42. *[116] Two mappings $Q, R : X \to X$ are said to satisfy **E.A. Property** in PM space (X, F) if their exists a sequence x_n in X such that $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = x$ for some x in X.*

The notions of weakly compatible mappings and (E.A) property are independent of each other.
In 2008, S. Kumar and B. D. Pant [117] extended the notion of reciprocal continuity to PM-spaces as follows:

Definition 1.2.43. *[117] Two mappings $Q, R : X \to X$ are said to be **reciprocal continuous** in PM space (X, F) if $\lim_{n\to\infty} QRx_n = Qx$ and $\lim_{n\to\infty} RQx_n =$*

Rx whenever $\{x_n\}$ in X such that $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = x$ for some x in X.

It is noted that if Q and R are both continuous self-mappings then they are obviously reciprocally continuous but the converse is not true. This fact can be understood by the following example

Example 1.2.17. *[117] Let (X, d) be a metric space where $X = [2, 20]$ and $Q, R : X \to X$ be defined by*

$$Q(x) = \begin{cases} 2 & \text{for } x = 2 \\ 3 & \text{for } x > 2 \end{cases}$$

and

$$R(x) = \begin{cases} 2 & \text{for } x = 2 \\ 6 & \text{for } x > 2. \end{cases}$$

Then, (Q, R) are reciprocal continuous mappings but not continuous.

In 2007, Kohli and Vasistha[115] extended the concepts of R-weak commutativity to PM space which is introduced by Pant [145] in metric space as follows:

Definition 1.2.44. *A pair of self-mappings (f, g) of a probabilistic metric space (X, F) is said to be*
*(1) **Weakly commuting** if $F(fgx, gfx, t) \geq F(fx, gx, t)$,*
*(2) **R-weakly commuting** if there exists some $R > 0$ such that $F(fgx, gfx, t) \geq F(fx, gx, t/R)$,*
*(3) **R-weakly commuting mappings of type (1)** if there exists some $R > 0$ such that $F(gfx, ffx, t) \geq F(fx, gx, t/R)$,*
*(4) **R-weakly commuting mappings of type (2)** if there exists some $R > 0$ such that*

$$F(fgx, ggx, t) \geq F(fx, gx, t/R),$$

*(5)**R-weakly commuting mappings of type (3)** if there exists some $R > 0$ such that*

$$F(ffx, ggx, t) \geq F(fx, gx, t/R), for\, all\, x \in X \;\; and\, t > 0.$$

Moreover, such mappings commute at their coincidence points.

1.3 Some common fixed point theorems in metric and probabilistic metric space

1.3.1 Some common fixed point theorems in metric space

In this section, we state some common fixed point theorems in metric and PM space with necessary proof relevant to our results.
Stefan Banach [10] established following benchmark theorem which ensures existence and uniqueness of a fixed point in metric space under appropriate conditions, called Banach fixed point theorem or Banach contraction principle.

Theorem 1.3.1. *[10] Let (X, d) be a complete metric space, then each contraction map $f : X \to X$ has a unique fixed point.*

This theorem has enormous applications inside as well as outside mathematics. And this theorem has been generalized and improved in different directions such as metric space, 2-metric space, PM-space, 2-PM-space, fuzzy metric space, D-metric space, probabilistic D-metric space, Non-archimedean Menger probabilistic metric space, intuitionistic Fuzzy metric space, intuitionistic Menger space, complex-valued metric space [127, 181, 99, 91, 123, 122].

In 2014 F. Khojasteh, M. Abbas and S. Costache [110] established the following fixed point theorem in single-valued mappings in complete metric spaces.

Theorem 1.3.2. *[110] Let (X, d) be a complete metric space. And mapping $f : X \to X$ satisfies following condition:*
$d(fx,\ fy) \leq \{\frac{d(x,\ fy) + d(y,\ fx)}{d(x,\ fx) + d(y,\ fy) + 1}\}\ d(x,\ y)$, *for all* $x, y \in X$. *Then,*

(i) f *has at least one fixed point* $x_1 \in X$ *;*

(ii) $(f^n x)$ *converges to a fixed point, for all* x *in* X*; and*

(iii) if x_1, y_1 *are two distinct fixed points of* f*, then* $d(x_1,\ y_1) > 1/2$.

In 1976, G. Jungck [95] proved the following theorem as a generalization of Bancah contraction principle as:

Theorem 1.3.3. *[95] Let T be a continuous mappings of a complete metric space (X, d) into itself. Then, T has a fixed point in X if there exists $\alpha \in$ (0,1) and a mapping $S : X \to X$ which commutes with T and satisfies $S(X) \subset T(X)$ and $d(Sx, Sy) \leq \alpha\ d(Tx, Ty)$, for all $x, y \in X$.*

In 2008, M. Imdaad and J. Ali [80] established fixed point theorems in two self mappings in metric space as:

Theorem 1.3.4. *Let T and I be self mappings of a metric space (X, d) such that:*

(i) T *and* I *satisfy* $(E.A)$ *property;*

(ii) for all x *in* X *and* $\psi \in \Psi$,
$\Psi\{d(Tx, Ty), d(Ix, Iy), d(Ix, Tx), d(Tx, Ty), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\} \leq 0$;

(iii) $I(X)$ *is complete subspace of* X.

Then, (T, I) has a point of coincidence and the pair (T, I) has a common fixed point provided it is weakly compatible.

In 2012, V. Parbaneh, G. G. Branach and G. Gharab [151] established following theorem in three self mappings in metric space:

Theorem 1.3.5. *[151] Let (X, d) be a complete metric space and E be a nonempty closed subset of X. Let $T, S : E \to E$ be such that $d(Sx, Ty) \leq 1/2\{d(Rx, Sy) + d(Ry, Tx) - (d(Rx, Sy), d(Ry, Tx))\}$, for every $(x, y) \in X \times X$, where ψ: $[0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$ and $R : E \to X$ satisfying following hypotheses:*

(i) $T(E) \subseteq R(E)$ and $S(E) \subseteq R(E)$;

(ii) Pairs (T, R) and (S, R) are weakly compatible.

In addition, if $R(E)$ is closed subset of X then T, R, and S have a unique common fixed point.

In 1986, G. Jungck [96] established the first common fixed point theorem by using compatible mapping in complete metric space:

Theorem 1.3.6. *[96] Let S and T be self-maps of a metric space (X, d) and let A and B be (ϵ, δ) - S, T- contractions such that pairs (A, S), (B, T) are compatible. Let $x_0 \in X$ and let $\{y_n\}$ be any S, T- contractions of x_0 under A and B. If $\{y_n\}$ has a cluster point z in X, then $\{y_n\}$ converges to z and Tz is the unique common fixed point of A, B, S and T provided these functions are continuous at z.*

Theorem 1.3.7. *[96] Let S and T be continuous self-maps of a complete metric space (X, d) and let A and B be (ϵ, δ) - S, T- contractions such that pairs (A, S), (B, T) are compatible. If δ is lower semi-continuous, then A, B, S and T have a unique common fixed point.*

In 2004, R. P. Pant, V. Pant and V. P. Pandey [162] established following theorems in metric space:

Theorem 1.3.8. *[162] Let (A, S) and (B, T) be compatible pairs of self-mappings of a complete metric space (X,d) such that*

1. $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

2. *Given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $x, y \in X$,
   $\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax,\ By) \leq \epsilon$
   where $M(x,\ y) = max\left\{d(Sx,\ Ty), d(Ax,\ Sx), d(By,\ Ty), \frac{d(Ax,\ Ty) + d(Sx,\ By)}{2}\right\}$,

3. $d(Ax,\ By) < max\left\{d(Sx,\ Ty), k\frac{d(Ax,\ Sx) + d(By,\ Ty)}{2}, \frac{d(Ax,\ Ty) + d(Sx,\ By)}{2}\right\}$
   for $0 < k \leq 2$

Suppose that the mappings in one of the pair (A, S) or (B, T) are reciprocal continuous. Then A, B, S and T have unique common fixed point.

In 2005, K. Jha, R.P. Pant and S. L. Singh [93] established following theorems for compatible pairs of self mappings in complete metric space.

Theorem 1.3.9. *[93] Let (A, S) and (B, T) be compatible pairs of self-mappings of a complete metric space (X, d) such that*

1. $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

2. *Given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $x, y \in X$,
   $\epsilon \leq M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) < \epsilon$

3. $d(Ax, By) < [k[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Ax, Ty)]$
   for $0 \leq k_1 < 1/3$.
   where, $M(x,\ y) = max\left\{d(Sx,\ Ty), d(Ax,\ Sx), d(By,\ Ty), \frac{d(Sx, By) + d(Ax, Ty)}{2}\right\}$

If one of the mappings A, B, S and T is continuous then A, B, S and T have unique common fixed point.

In 2014, K. Jha, R. P. Pant and K. B. Manandhar [92] established following theorem:

Theorem 1.3.10. *[92] Let (A, S) and (B, T) be compatible and reciprocally continuous pairs of self-mappings in a complete metric space (X,d) such that*

1. $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

2. *Given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x, y \in X$, *we have* $\epsilon \leq M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) < \epsilon$,
   and $d(Ax, By) < M(x, y)$ *where*
   $M(x,\ y) = max\left\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \dfrac{d(Sx, By) + d(Ax, Ty)}{2}\right\}$,
   and

3. $d(Ax, By) < max\,[k_1[d(Sx, Ty)+d(Ax, Sx)+d(By, Ty)], k_2\dfrac{d(Sx, By) + d(Ax, Ty)}{2}]$
   for $0 \leq k_1,\ k_2 \leq 1$.

Then, the mappings A, B, S and T have unique common fixed point.

In 2014, K. Jha, V. Popa and K. B. Manandhar have established following theorem for two pairs of self mappings in metric space:

Theorem 1.3.11. *[94] Let (X, d) be a complete metric space and A, B, S and T be a self mappings of X satisfying the following conditions:*

1. $A(X) \subset T(X), B(X) \subset S(X)$,

2. *Given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all x, y in X*
   $\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) \leq \epsilon$ *and* $d(Ax, By) < M(x, y)$
   where $M(x, y) = Max\ [d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty)]$

3. *S and T are continuous.*

If (A, S) and (B, T) compatible mappings of type (K), then A, B, S and T have a unique common fixed point.

1.3.2 Some common fixed point theorem in Menger probabilistic metric space

In 1972, V. M. Sehgal and A. T. Bharucha-Reid [185] established following fixed point theorem in Menger space as a first generalization of Banach contraction principle on complete Menger space as:

Theorem 1.3.12. *[185] Let* $(X, F, \triangle)$ *be a complete Menger space, where* $\triangle$ *is a continuous function satisfying* $\triangle(x, x) \geq x$ *for each* $x \in [0, 1]$, *if* T *is any contraction mapping of* X *into itself, then there is unique* $p \in X$ *such that* $T_p = p$. *Moreover,* $T^n q \to p$ *for each* $q \in X$.

In 1966, H. Sherwood [193] established the following theorem in Menger space:

Theorem 1.3.13. *[193] Every Menger space with a continuous t-norm has a completion that is unique up to isometry.*

In 1983, T. L. Hicks [78] studied probabilistic metric space and established following theorems in Menger space :

Theorem 1.3.14. *[78] Let* (X, F, min) *be a complete Menger probabilistic metric space,*
$T : X \to X$ *be a contraction mapping satisfying the condition for every* $p, q \in S$, $k \in (0, 1)$ *and every* $x > 0$
$F_{p,q}(x) > 1 - x \Rightarrow F_{Tp.Tq}(kx) > 1 - kx.$
Then, T *has a fixed point in* X.

In 2005, D. Mihet [135] introduced weak-Hicks contraction and established fixed point theorems in Probabilistic Metric Space:

Theorem 1.3.15. *[135] The mapping* $f : X \to X$ *is an* ***H-contraction*** *on the PM space* (X, F, τ) *with*

$$\tau \geq \tau_M$$

if and only if there is a $\gamma \in (0, 1)$ *such that*

$$\beta(f_p, f_q) \leq \gamma\ \beta(p, q) \quad \textit{for all} \quad p, q \in S.$$

Theorem 1.3.16. *[134] Let* (X, F, T) *be a complete Menger sapce where either* $\triangle \geq \triangle_L$ *or* T *is of* ***Hadzic type*** *and* $f : X \to X$ *be a weaks-Hicks contraction. If* $F_{p,f(p)}(1) > 0$ *for some* $p \in S$*, then* f *has a fixed point.*

In 2008, I. Kubiaczyk and S. Sharma [116] established following theorem:

Theorem 1.3.17. *[116] Let* $(X, F, \triangle)$ *be a Menger space with* $\triangle(x, y) = min\{x, y\}$ $\forall x, y \in [0, 1]$ *and* S *and* T *be weakly compatible mappings of* X *into itself such that* S *and* T *satisfy the property (E.A), there exists a number* $k \in (0, 1)$ *such that*

$$F_{Tu,Tv}(kx) \geq min\ \{F_{S_u,S_v}(x), F_{S_u,T_u}(x), F_{S_v,T_v}(x), F_{S_u,T_u}(x), F_{S_u,T_v}(x)\}.$$

for all $u, v \in X$*, and* $T(X) \subset S(X)$*. If* $S(X)$ *or* $T(X)$ *be a closed subset of* X*, then* S *and* T *have a unique common fixed point.*

In complete Menger space, K. P. R. Sastry, G. A. Naidu, P. V. S. Prasad and S. S. A. Sastri [176] established following common fixed point theorems:

Theorem 1.3.18. *[176] Let* $(X, F, \triangle)$ *be a Menger space with* $\triangle(x, y) = min\{x, y\}$, $\forall x, y \in [0, 1]$. S *and* T *be weakly compatible mappings of* X *into itself such that*
i) S *and* T *satisfy the property (E.A), and*
ii) there exists a number $k \in (0, 1)$ *such that*

$$F_{Tu,Tv}(kx) \geq min\{F_{S_u,S_v}(x), F_{S_u,T_u}(x), F_{S_v,T_v}(x), F_{S_v,T_u}(x), F_{S_u,T_v}x\}$$

for all $u, v \in X$ *and for* $x > 0$.
If $S(X)$ *be a closed subset of* X*, then* S *and* T *have a unique common fixed point.*

Theorem 1.3.19. *[176] Let* $(X, F, \triangle)$ *be a Menger space with* $\triangle(x, y) = min\{x, y\}$ *for all* $x, y \in [0, 1], S$ *and* T *be occasionally weakly compatible maps of* X *satisfying there exists a number* $k \in (0, 1)$*such that*

$$F_{Tu,Tv}(kx) \geq min\{ F_{S_u,S_v}(x),\ F_{S_u,T_u}(x),\ F_{S_v,T_v}(x),\ F_{S_v,T_u}(x),\ F_{S_u,T_v}(x)\}.$$

for all $u, v \in X$.
Then S *and* T *have a unique common fixed point.*

Theorem 1.3.20. *[176] Let* $(X, F, \triangle)$ *be a Menger space with* $\triangle(x, y) = min\{x, y\}$ *for all* $x, y \in [0, 1], S$ *and* T *be self mappings of* X *into itself such that*
i. S *and* T *satisfy the property (E.A), and*
ii. there exist a number $k \in (0, 1)$ *such that*

$$F_{Tu,Tv}(kx) \geq min\{F_{S_u,S_v}(x), F_{S_u,T_u}(x), F_{S_v,T_v}(x), F_{S_v,T_u}(x), F_{S_u,T_v}(x)\}.$$

for all $u, v \in X$
If $S(X)$ *be a closed subset of* X, *then* S *and* T *have a coincidence point.*

In 2013, S. Mantro, S. Kumar, S. S. Bhatia and S. M. Kang [125] established following theorem:

Theorem 1.3.21. *[125] Let* f *and* g *be* ***weakly reciprocally continuous*** *self-mappings of a complete Menger space* $(X, F, \triangle)$ *satisfying*
(i) $f(X) \subset g(X)$.
(ii) there exists a number $k \in (0, 1)$ *such that*

$$F(fx, fy, kt) \geq min\{F(gx, gy, t), F(Fx, gx, t), F(fy, gy, t)\}$$

for any $x,\ y \in X$ *and* $t > 0$. *If* f *and* g *are either compatible or R-weakly commuting of type* (A_g) *or R-weakly commuting of type* (A_f) *or R-weakly commuting of type (P), then* f *and* g *have a unique common fixed point.*

In 2007, J. K. Kohli and S. Vashistha [115] proved the following common fixed point results in Menger Space:

Theorem 1.3.22. *[115] Let f and g be R-weakly commuting self mappings of a Menger probabilistic metric space $(X, F, \triangle)$, where $\triangle$ denotes a continuous t-norm, satisfying conditions*
(i) $f(X) \subset g(x)$
(ii) $F(fx, fy, t) \geq r(F(gx, gy, t))$, for all x, y in X,
where $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1, r(0) = 0$ and r(1) = 1.
(iii) If there exist sequence $\{x_n\}$ and $\{y_n\}$ in X such that $x_n \to x, y_n \to y$ and $t > 0$, then $F(x_n, y_n, t) \to F(x, y, t)$.

If one of the mapping f and g is continuous, then the mappings f and g have a unique common fixed point.

Theorem 1.3.23. *[115] Let $(X, F, \triangle)$ be a complete Menger space and let A and S be self-mappings of X. Let the pair (A, S) be occasionally weakly-compatible. If there exists a point $k \in (0, 1)$ for all $x, y \in X$ and $t > 0$, such that*

$$F(Sx, Sy, kt) \geq a\,F(Ax, Ay, t) + b\ min\{F(Ax, Ay, t), F(Sx, Ax, t), F(Sy, Ay, t)\}$$

where $a,\ b > 0,\ a + b\ > 1$.

Then A and S have a unique common fixed point.
In 1991, S.N. Mishra [137] established following common fixed point theorem by using compatible mapping in Menger space as:

Theorem 1.3.24. *[137] Let (X, F, t) be a complete Menger space with $t(x, x) \geq x$,for all $x \in [0, 1]$ and $A, B, S, T : X \to X$ be a mapping such that*

1) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$*

2) the pairs (A, S) and (B, T) are compatible mapping.

3) one of S and T are continuous, and

4) there exists a constant $k \in (0, 1)$ such that

$$F_{Ap,Bq}(kx) \geq t(F_{Ap,Sp}(x), t(F_{Bq,Tq}(x), t(F_{Sp,Tq}(x), t(F_{Ap,Tq}(\alpha x), F_{Bq,Sp}(2-\alpha)x)))))$$

for all $p, q \in X$, $\alpha \in (0, 2)$ and $x > 0$.

Then, A, B, S, T have a unique common fixed point in X.

In 1992, Y. J. Cho, P. P. Murthy and M. Stojakovic [45] have introduced compatible mappings of type (A) established following common fixed point theorem in Menger space as:

Theorem 1.3.25. *[45] Let (X, F, t) be a complete Menger space with $t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$ and $A, B, S, T : X \to X$ be a mapping such that*

1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

2) the pairs (A, S) and (B, T) are compatible mappings of type (A),

3) one of A, B, S and T is continuous.

4) there exists a constant $k \in (0, 1)$ such that

$$F_{Ap,Bq}(kx) \geq \min\{F_{Sp,Tq}(x)^2, F_{Sp,Ap}(x), F_{Tq,Bq}(x), F_{Sp,Bq}(2x), F_{Tq,Ap}(x), F_{Tq,Ap}(x)F_{Sp,Ap}(x), F_{Sp,Bq}(2x)F_{Tq,Bq}(x)\}$$

for all $p, q \in X$, and $x > 0$.

Then, A, B, S, T have a unique common fixed point in X.

In 2018, S. Gupta, S. Kumar and G. K. Soni [71] established common fixed point theorem in five self mappings as:

Theorem 1.3.26. *[71] Let A, B, S, T and P be self maps on complete Menger space$(X, F, *)$ with $t * t \geq t$ for all $t \in [0, 1]$, satisfying:*

1) $P(X) \subseteq AB(X)$, $P(X) \subseteq ST(X)$;
2) *there exists a constant $k \in (0, 1)$ such that*
$$M_{Px, Py}(kt) \geq M_{ABx, Px}(t) * M_{Px, STy}(t) * M_{ABx, STy}(t) * \frac{M_{Px, ABx}(t) * M_{Px, STy}(t)}{M_{STy, ABx}(t)} * M_{ABx, Py}(3-\alpha)t$$
for all $x, y \in X$, $\alpha \in (0, 3)$ and $t > 0$;
3) $PB = BP$, $PT = TP$, $AB = BA$, $ST = TS$;
4) *A and B are continuous ; and*
5) *the pair (P, AB) is compatible(if compatible then it is weak compatible).*

Then, A, B, S, T and P have a unique common fixed point in X.

In 2005, B. Singh and S. Jain [196] have established common fixed point theorem in Menger space by using weakly compatible mappings for three pairs of self mappings as:

Theorem 1.3.27. *[196] Let (X, F, t) be a complete Menger space with $t(a, a) \geq a$,for all $a \in [0, 1]$ and $A, B, S, T, L, M : X \to X$ be a mapping such that*

1) $L(X) \subset ST(X)$ *and* $M(X) \subset AB(X)$
2) $AB = BA, ST = TS, LB = BL, MT = TM$
3) *the pairs (L, AB) is compatible mappings and (M, ST) is weakly compatible mappings.*

4) Either AB or L is continuous, and

5) there exists a constant $k \in (0,1)$ such that

$$F_{Lp,Mq}(kx) \geq \min\{(F_{ABp,Lp}(x), F_{STq,Mq}(x), F_{STq,Lp}(\alpha x), F_{ABp,Mq}(2-\alpha)x), F_{ABp,STq}(x)\}$$

for all $p, q \in X$, $\alpha \in (0,2)$ and $x > 0$.

Then, A, B, S, T, L and M have a unique common fixed point in X.

In 2020, V. Gupta, M. S.Khan, B. Singh and S. Kumar [72] have established followings fixed point theorems for Meir -Keeler Contractive conditions in Menger Space :

Theorem 1.3.28. *[72] Let (X, F, t) be a complete Menger space with $t(a,b) = \min\{a,b\}$ for all $a, b \in [0,1]$ and $A, B, S, T : X \to X$ be a mapping such that*

1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

2) for $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ such that for all x, y in X,
$\epsilon - \delta < m(x,y,t) \leq \epsilon \Rightarrow F(Ax, By, t) > \epsilon$
where $m(x,y,t) = min\{F(Sx,Ty,t), F(Ax,Sx,t), F(By,Ty,t)\}$,

3) one of A, B, S or T is complete subspace of X.

Then $Av = z = Sv$ and $Bw = z = Tv$. Also, if the pair (A, S) as well as (B, T) are weakly compatible, then $AZ = Bz = Sz = Tz = z$, and z is unique in X.

Theorem 1.3.29. *[72] Let (X, F, t) be a complete Menger space with $t(a,b) = \min\{a,b\}$ for all $a, b \in [0,1]$ and $A, B, S, T : X \to X$ be a mapping such that*

1) for $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ such that for all x, y in X,
$\epsilon - \delta < m(x,y,t) \leq \epsilon \Rightarrow F(Ax, By, t) > \epsilon$
where $m(x,y,t) = min\{F(Sx,Ty,t), F(Ax,Sx,t), F(By,Ty,t)\}$,

2) pairs (A, S) and (B, T) holds E.A. common property,

3) S or T is closed in X.

Then $Au = r = Su$ and $Bv = r = Tv$. Also, if the pair (A, S) as well as (B, T) are weakly compatible, then $Ar = Br = Sr = Tr = r$ and r is unique in X.

In 2021, I. H. Jebril, S. K. Datta, S. Sarkar and N. G. Biswas [89] have established followings common fixed point theorems in probabilistic metric space using Lukasiecz t-norm and product t-norm:

Theorem 1.3.30. *[89] Let $A, B, P, Q : X \to X$ be a self mapping in probabilistic metric space satisfying*

1) $P(X) \subset B(X)$ and $Q(X) \subset A(X)$

2) $F_{Px,Qy}(kt) \geq \max\{F_{Ax,By}, \triangle_L(F_{Px,Ax}(t), F_{Qx,Bx}(t))\}$ for all $x, y \in X, k \in (0, 1)$ and $t > 0$. where $\triangle_L$ is Lukasiecz t-norm ;

*3) If one of P, B, Q and A is complete subset of X then (**a**) P and A have a coincident point, (**b**) Q and B have a coincident point; and*

4) pairs $(P, A)and(Q, B)$ are weakly compatible.

Then, A, B, P and T have a unique common fixed point in X.

Theorem 1.3.31. *[89] Let $(X, F, \triangle_M)$ be a complete Menger space. Moreover, let (A, S) and (B, T) be point wise R weakly commuting self pairs of mappings of X satisfying*

1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

2) $\triangle_P(F_{Au,Bv}(ht), F_{Bv,Tv}(ht)) \geq \triangle_P(F_{Su,Tv}(t), F_{Au,Su}(t))$

for all $u, v \in X, h \in (0, 1)$ and $t > 0$ and P is the product $t-$ norm on $\triangle$. If one of the mappings in compatible pair (A, S) or (B, T) is continuous , then A, B, S, T have a unique common fixed point.

1.3.3 Interrelation between contraction mappings in probabilistic metric space

In Metric space, If (X, d) be a metric space and $f : X \to X$. Then, there exists $b \in (0, 1)$ such that

$$d(f_p, f_q) \leq b\, d(p, q), \quad for\, every \quad p, q \in X, \tag{1.6}$$

f is a contraction mapping, say, $b-$ contraction. And for every $b-$ contraction mapping on complete metric space has one and only one fixed point which is called Banach contraction principle or b- contraction principle [37]. As its generalization in probabilistic metric space, V. M. Sehgal and A. T. Bharucha Reid [185] introduced b- contraction:
A mapping $f : X \to X$ is a contraction mapping (or a SB - Contraction mapping or b- contraction) on probabilistic metric space (X, F) if and only if there is $b \in (0, 1)$ such that

$$F_{fp,fq}(x) \geq F_{p,q}(x/b),\ forevery\ \ p, q \in X \quad and \quad x \in \mathbb{R} \tag{1.7}$$

The inequality (1.7) is a generalization of inequality (1.6). In order to prove that (1.6) implies (1.7). We just recall that every metric space (X, d) is also a Menger space (X, F, T_m), if F is defined in the following way:

$$F_{p,q}(x) = \begin{cases} 1 & \text{if } d(p, q) < x, \\ 0 & \text{if } d(p, q) \geq x \qquad \text{for } x \in \mathbb{R}. \end{cases} \tag{1.8}$$

Suppose that $f : X \to X$ is such that (1.6) holds and prove that (1.7) is satisfied i.e., for every $x > 0$, we have

$$F_{p,q}\ (\frac{x}{b}) = 1 \Rightarrow F_{f_p,\ f_q}(x) = 1.$$

If $F_{p,q}(\frac{x}{b}) = 1$, then $d(p,q) < \frac{x}{b}$ and (1.6) implies

$$d(f_p, f_q) < b.\frac{x}{b} = x,$$

which means that

$$F_{f_p,f_q}(x) = 1.$$

[34]

In 2001, Dorel Mihet [133] defined following contractions in his Doctoral thesis:

Definition 1.3.1. *[133] Let (X, F) be probabilistic metric space and $(q_n), n \in \mathbb{N}$ an increasing sequence from $(0,1)$ such that $lim_{n\to\infty} q_n = 1$. A mapping $f : X \to X$ is a strict $(q_n)-$* ***probabilistic contraction*** *if there exist $b = b_n \in (0,1)$, we have*
$F_{p,\,q}(x) > q_n \implies F_{fp,\,fq}(b_n x) > q_n$, for all $p,\, q \in X, \forall x > 0$.

The class of q_n- probabilistic contraction is strictly larger than the class of probabilistic b-contraction.

Theorem 1.3.32. *[133] Let (X, F, t) be a complete Menger space, t be a T-norm of type H and $f : X \to X$ a strict q_n- probabilistic contraction. Then, there exist a unique fixed point $x \in X$ of the mapping f and $x = \lim_{n\to\infty} f^n p$ for every $p \in X$.*

In 1970, L. F. Guseman [73] defined following locally contraction in PM space:

Definition 1.3.2. *[73] Let (X, F) be probabilistic metric space . A mapping $f : X \to X$ is a probabilistic $(q, n)-$ locally contraction $q \in (0,1)$, if for every $x \in X$ there exists $n(x) \in \mathbb{N}$ such that for every $y \in X$ and $t \in \mathbb{R}$:*
$F_{f^{n}(x)\,x,\, f^{n}(x) y}(qt) \geq F_{x,\,y}(t)$.

In 1996, S. S. Chang, B. S. Lee, Y. J. Cho, Y. Q. Chen, S. M. Kang and J. S. Jung [32] defined generalized contraction mapping and established following theorem:

Definition 1.3.3. *[32] Let (X, F) be probabilistic metric space and $f : X \to X$. Then, f is generalized b- contraction if for every $x, y \in X$ and every $\alpha, \beta > 0$:*
$F_{x,y}(\alpha) > 0,\ F_{x,y}(\beta) < 1 \implies F_{fx,fy}(t) \geq F_{x,y}(\frac{t}{L(\alpha,\beta)})$*, for every* $t > 0$*, where* $L : \mathbb{R}^+ \times \mathbb{R}^+ \to (0, 1)$*.*

Theorem 1.3.33. *[32] Let (X, F, T_m) be a complete Menger space and $f : X \to X$ be such that following implication holds for every $x, y \in X$ and every $\alpha, \beta > 0$:*
$F_{x,y}(\alpha) > 0,\ F_{x,y}(\beta) < 1 \implies F_{fx,fy}(t) \geq F_{x,y}(\frac{t}{L(\alpha,\beta)})$*, for every* $t > 0$*, where* $L : \mathbb{R}^+ \times \mathbb{R}^+ \to (0, 1)$*. Then, f has a unique fixed point.*

As a generalization of the notion of a probabilistic B-contraction, the notion of a probabilistic (m, k) - B-contraction where $m \geq 1$ and $k \in (0, 1)$ is

Definition 1.3.4. *[136] If (S, φ) is a PSM-space, $m \geq 1$ and $k \in (0, 1)$, a function $f : S \to S$ is called probabilistic (m, k)-B-contraction if for any $p, q \in S$ there is an i with $1 \leq i \leq m$ such that for every $t > 0$,*

$$F_{f^i, f^i q}(k^i t) \geq F_{p,q}(t).$$

If $m = 1$ and $k \in (0, 1)$ then a probabilistic $(1 - k) - B$-contraction f is a probabilistic B-contraction.

As a generalization of C-contraction, we have

Definition 1.3.5. *[75] A mapping $f : X \to X$ is a generalized C- contraction of Krasnoselski's type if for each pair of real numbers (a, b), with $0 < a < b$, there exists $L(a, b) \in (0, 1)$ such that for every $x, y \in X$ if*

$a \leq 1 - F_{x,y}(a)$ and $1 - F_{x,y}(b^+) \leq b$, then following implications holds for every $t > 0$:, $F_{x,y}(t) > 1 - t \implies F_{fx,fy}(L(a,b)t) > 1 - L(a,b)t$

Theorem 1.3.34. *[75] Let (X, F, T) be a complete Menger space, where $T \geq T_L$. If $f : X \to X$ is a generalized C -contraction of Krasnoselski's type, then there exists a unique fixed point of the mapping f which is globally attractive.*

D. Mihet [135] weakens the Hick's contraction as:

Definition 1.3.6. *[135] Let S be a nonempty set and F be a probabilistic distance on S. A mapping $f : S \to S$ is a said to be a weak - Hicks contraction* ***(w-H contraction)*** *if there exists $k \in (0,1)$ such that, for all $p, q \in S$.*

$$(w - H) : t \in (0,1), F_{pq}(t) > 1 - t \Rightarrow F_{f(p)f(q)}(kt) > 1 - kt.$$

The notion of (Ψ, C)-contraction, where $\Psi : [0,\infty) \to [0,\infty)$ is a multi valued generalization of C contraction defined as:

Definition 1.3.7. *[149] Let (X, F) be probabilistic metric space and $f : X \to 2^X$. The mapping f is called a (Ψ, C)-contraction, where $\Psi : [0,\infty) \to [0,\infty)$, if for every $x, y \in X$ and every $t > 0$ the following implications holds:*
$F_{x,y}(t) > 1 - t \implies (\forall u \in fx)(\exists v \in fy)F_{u,v}(\Psi(t)) > 1 - \Psi(t)$.
If $\Psi(t) = kt, t > 0, k \in (0,1)$, then (Ψ, C)-contraction $f : X \to X$ is a C-contraction.

Definition 1.3.8. *[136] If (S, φ) is a PSM - space, m$\geq$ 1 and $k \in (0,1)$, a function $f : S \to S$ is called a (m,k)-C-contraction if for any $p, q \in S$ there is an i with $l \leq i \leq m$ such that for every $t > 0$, we have*

$$F_{p,q}(t) > 1 - t \Rightarrow F_{f^i p, f^i q}(k^i t) > 1 - k^i t.$$

If $m = 1$ and $k \in (0,1)$ then a probabilistic $(1,k) - C$-contraction f is a probabilistic C-contraction.

In 2004, I. Goelet [70] introduced g- contraction mapping as generalization of C- contraction as:

Definition 1.3.9. *[70] Let $f,\ g$ be two mappings defined on a Menger space (S, F, T) with values into itself and let us suppose that g is bijective. The mapping f is called a probabilistic g-contraction with a constant $k \in (0, 1)$ if*

$$t > 0 \text{ and } F_{g(x),\, g(y)}(t) > 1 - t \text{ implies } F_{f(x),\, f(y)}(kt) > 1 - kt.$$

The notion of g-contraction is justified because the images of two points x, y under the function f are nearer than images of the same points under the functions g.

Theorem 1.3.35. *[70] Let g be a injective mapping defined on a Menger space (S, F, T) into self. Then the following statements are true:*
a) The mapping F^g defined on $S \times S$ with values in D^+, by $F^g(x, y) = F_{g(x),g(y)}$ is a probabilistic metric on S, that is, (S, F^g, T) is a Menger space under the same t-norm T.
b) If $S_1 = g(s)$ and (S_1, F, T) is a complete Menger space then (S, F^g, T) is also a complete Menger space .
c) If (S_1, F, T) is compact, then (S, F^g, T) is also compact.

Theorem 1.3.36. *[70] Let g be a injective mapping defined on a Menger space (S, F, T) into itself. Then the following statements are true:*
a) The mapping F^g defined on $S \times S$ with values in D^+, given by $F^g(x, y) = F_{g(x),g(y)}$ is a probabilistic metric on S, that is, (S, F^g, T) is a Menger space under the same t-norm T.
b) If $S_1 = g(s) and (S_1, F, T)$ is a complete Menger space then (S, F^g, T) is also a complete Menger space .
c) If (S_1, F, T) is compact, then (S, F^g, T) is also compact.

Theorem 1.3.37. *[70] If f is a probabilistic g-contraction, then we have:*
a) f is a continuous mapping on (S, F^g, T) with values in(S, F, T).
b) $g^{-1}of$ is a continuous mapping on (S, F^g, T) with values into itself.

The above concept of probabilistic g-contraction is a generalization of Hick's probabilistic contraction [78] which can be obtained when g is considered to be an identity on the Menger space (X, F, T).

Theorem 1.3.38. *[70] If f and g are two mappings defined on a complete Menger space (S, F, T) with values into itself, g is bijective and f is a g-contraction, there exists a unique point $p \in S$ such that $f(p) = g(p)$, (p is considered a fixed point of the probabilistic g- contraction f). Moreover, $p = \lim_{n\to\infty} x_n$, where the sequence $(x_n)_{n\geq 1}$ is defined by recurrence relation $g(x_{n+1}) = f(x_n)$.*

Theorem 1.3.39. *[70] If f and g are two mappings defined on a complete metric space (S, d) with values into itself, g is bijective and f is a g-contraction, that is, there exists a constant $k \in (0, 1)$ such that*

$$d(f(x), f(y)) \leq k\, d\,(g(x)), g(y)),$$

for every $x, y \in S$, then there exists a unique point $p \in S$ such that $f(p) = g(p)$.

As an interrelation between contractions on PM space, contractions B- and C-contractions are independent. B. Schweizer, H. Sherwood and R.M. Tardff [180] shows that an Hicks- contraction (C) need not be a B -contraction.

Example 1.3.1. *[180] Let* $S = \{0, 1, 2, ...\}$ *and, for* $p \neq q, we define$ $F : S \times S \to \triangle^+ where \triangle^+$ *is all set of distribution function by*

$$F_{pq}(x) = F_{qp}(x) = \begin{cases} 0 & x \leq 2^{-min(p,q)} \\ 1 - 2^{-min(p,q)} & 2^{-min(p,q)} < x \leq 1 \\ 1, & 1 < x. \end{cases}$$

Then, (S, F, τ_M) *is PM - space. We define* $f : S \to S$ *such that* $f(r) = r + 1$ *then since*

$$\tau_M \geq \tau W \, and \, \beta \left(f_p, f_q \right) = \frac{1}{2} \beta(p, q),$$

so f *is H-contraction.*

Next, let γ *be any number in* $(0, 1)$ *and we choose* $x \in (1, 1/\gamma)$*. Then* $\gamma x < 1$ *such that*

$$F_{f(0)\, f(1)}\left(y(x) \right) = F_{12}(\gamma\, x) \leq \frac{1}{2} < 1 = F_{01}(x),$$

where f *is not a B-contraction on* (S, F).

It is noted that M is the strongest t- norm. The following lemma [180] shows that notion of a B-contraction is some times stronger than that of an H-contraction in PM space.

Lemma 1.3.1. *[180] If* f *is a B-contraction on a probabilistic metric space* (S, F) *and if the distribution function* $F(f_p, f_q)$ *is strictly increasing on* $[0, 1]$*, then*

$$\beta(f_p, f_q) < \beta(p, q).$$

Proof. We find η such that

$$0 < \eta < \frac{1 - \gamma}{\gamma} \beta(p, q).$$

Then, we have

$$\beta(p, q) > \gamma \left[\beta(p, q) + \eta \right].$$

Since $F_{fp,fq}$ is strictly increasing on [0,1], so $0 \leq \beta(p,q) \leq 1$, and since f is a B-contraction, so we have

$$\beta(fp, fq) = F_{fp,fq}\ (\beta(p,q)) > \ F_{fp,fq}\ (\gamma\ [\beta(p,q)+\eta]) \geq F_{pq}\ (\beta(p,q)+n > 1-\beta(p,q).$$

This implies that $\beta(fp, fq) < \beta(p,q)$. This completes the proof. □

Following examples show that every B- contraction is not an H- contraction in general:

Example 1.3.2. *[180]. Let $S = \{p,q,r\}$ and let $F : S \times S \to \triangle^+$ where $\triangle^+$ is all set of distribution function, defined by*

$$F_{pr}(x) = F_{rp}(x) = F_{rq}(x) = F_{qr}(x) = \begin{cases} 0 & if x \leq 0 \\ 1/2 & if 0 < \ x \leq 2 \\ 1, & if x > 2 \end{cases}$$

and

$$F_{pq}(x) = F_{qp}(x) = \begin{cases} 0 & if x \leq 0 \\ 1/2 & if 0 < x \leq 3/2 \\ 1, & if x \ > 3/2 \end{cases}$$

It is again straightforward to show that (S, F, τ_M) is a PM space. we define f as follows: $f(p) = f(q) = p$ and $f(r) = q$. Since, $F_{pq}(3x/4) = F_{pr}(x)$ for all x, it follows at once that f is a B- contraction on (S, F). By theorem, f is not an H-contraction because $\beta\{fp, fr\} = \beta\{p,q\} = 1/2 = \beta\{p,r\}$.
The difference between this type of contractions and Hicks C-contractions has nicely been illustrated by V. Radu [164] in the following example:

Example 1.3.3. *[164] Let X be a set containing at least two elements. Consider the discrete Menger space under T_M determined (for $x \neq y$) by* *the probabilistic metric defined by*

$$f_{xy}(t) = \begin{cases} 0, & if t {\leq} 1 \\ 1, & if t > 1. \end{cases}$$

Then any mapping $A : X \to X$ is a weak-Hicks contraction, while the only Hicks C-contraction on (X, F, T_M) are the constant mappings.

The Probabilistic **g**-contraction is Hicks **C**-contraction when $\mathbf{g} = \mathbf{I}$ an identity mapping. Since **C**-contraction need not be **B**-contraction. So, Probabilistic **g**-contraction need not be **B**-contraction. Moreover, **C**-contraction is an extension of **B**- contraction in probabilistic metric space.

The generalizations of B and C- contractions and their interrelation and fixed point results in single mapping in probabilistic metric space have been published in following journals:

(i) **American Journal of Mathematics and Statistics 2019, 9(5): 199-202.**

(DOI: 10.5923/j.ajms.20190905.03) and

(ii) **International Journal of Mathematics And its Applications, 7(4)(2019), 23-30.**

(Available Online: http://ijmaa.in/)

Chapter 2

Compatible mappings of type (K) and Common fixed point theorems in probabilistic metric space

This chapter includes the notion of compatible mappings of type (K) in Menger probabilistic metric space and also some common fixed point theorems for two pairs of self mappings using Meir-Keeler contractive condition in this space.

2.1 Introduction

The notion of compatible mappings is the generalization of commutative mappings. This concept was introduced by G. Jungck [96] in metric space and established common fixed point theorems. In 1991, S. N. Mishra [137] introduced the notion of compatible mappings in Menger probabilistic metric and established a common fixed point theorem in this space.

And gradually different compatible mappings types like A, B, C, E, P, and R have introduced and established common fixed point theorems by using these compatible mappings. In 2014, K. Jha, V. Popa, and K. B. Manandhar introduced new compatible mappings of type (K) in metric [94] and in fuzzy metric space [123], [122] and also established some common fixed point theorem in four self-mappings.
Recently, in 2020, V, Gupta, and B. Singh, M. S. Khan and S. Kumar [72] and in 2021, R. K. Bisht and V. Rakocevic [13] have established common fixed point theorems using A. Meir and E. Keeler contractive condition in Menger space.
In this chapter, we extend compatible mappings of type (K) in Menger space and have established a new common fixed point theorem using Meir -Keeler contractive condition in four self-mappings with example.

2.2 Basic Definitions

In 1969, A. Meir and E. Keeler [126] introduced following contraction in metric space:

Definition 2.2.1. *[126] Let (X, d) be a metric space and $Q : X \to X$ be self mapping. Then mapping Q is said to be* ***Meir- Keeler or*** (ϵ, δ) ***contraction*** *mapping if for every $\epsilon > 0$ there exists $\delta > 0$ such that*
$\epsilon \leq d(x, y) < \epsilon + \delta \implies d(Qx, Qy) < \epsilon$.

In 1999, R. P. pant defined reciprocal continuous in metric space as:

Definition 2.2.2. *[146] Two self mappings Q and R of a metric space (X, d) are said to be* ***reciprocally continuous*** *on X if*

$$\lim_{n\to\infty} QRx_n = Qt \quad \text{and} \quad \lim_{n\to\infty} RQx_n = Rt$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = t$ for some t in X.

In 2007, M. R. Singh and Y. R. Singh [172] introduced following compatible mapping in metric space:

Definition 2.2.3. *[172] Two self mappings Q and R of a metric space (X, d) are said to be* ***compatible mappings of type (E)****, if*

$$\lim_{n\to\infty} QQx_n = \lim_{n\to\infty} QRx_n = R(t) \quad and$$

$$\lim_{n\to\infty} RRx_n = \lim_{n\to\infty} RQx_n = Q(t),$$

whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = t$ *for some* $t \in X$.

The new notion of compatible mappings of type (K) in metric space was introduced by K. Jha, V.Popa, and K.B. Manandhar [94]in 2014.

Definition 2.2.4. *[94] Two self mappings Q and R of a metric space (X, d) are said to be* ***compatible mappings of type (K)*** *if*

$$\lim_{n\to\infty} QQx_n = Rt \quad and \quad \lim_{n\to\infty} RRx_n = Qt,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = t$ for some t in X.

Example 2.2.1. *[94] Let $X = [0, 2]$ with the usual metric $d(x, y) = |x - y|$. We define self-mappings Q and R as $Qx = 2$, $Rx = 0$ for $x \in [0, 1] - 1/2$, $Qx = 0$, $Rx = 2$ for $x = \frac{1}{2}$*
and $Qx = \frac{2-x}{2}$, $Rx = \frac{x}{2}$ for $x \in (1, 2]$.
Then, Q and R are not continuous at $x = 1, 1/2$.
So, (Q, R) is compatible mappings of type (K).

In 2005, B. Singh and S. Jain [196] introduced weakly compatible mappings in PM space as:

Definition 2.2.5. *[196] Two mappings $Q, R : X \to X$ are said to be* ***weakly compatible mappings*** *in PM space (X, F) if they commute at their coincidence points i.e. $Qx = Rx$ for some $x \in X$, then $QRx = RQx$.*

In 2008, I. Kubiaczyk and S. Sharma [116] defined following property:

Definition 2.2.6. *[116] Two mappings $Q, R : X \to X$ are said to satisfy* ***E. A. Property*** *in PM space (X, F) if their exists a sequence x_n in X such that $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = x$ for some x in X.*

Definition 2.2.7. *[37] Let (X, F, T) be a* ***Menger space*** *and T be a continuous t-norm. Then,*

(1) A sequence $\{x_n\}$ in X is said to be ***converge*** *to a point x in X (written $x_n \to x$) iff for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq N$.*

(2) A sequence $\{x_n\}$ in X is said to be a ***Cauchy*** *if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq N$.*

(3) A Menger space in which every Cauchy sequence is convergent is said to be ***complete Menger space****.*

We have extended the following definition of compatible mappings of type (K) in Menger space.

Definition 2.2.8. *[38] Two mappings $Q, R : X \to X$ are said to be* ***compatible mappings of type*** (K) *in Menger space (X, F, t) iff*

$$\lim_{n\to\infty} F_{QQx_n,Rz}(t) = 1 \quad \textit{and} \quad \lim_{n\to\infty} F_{RRx_n,Qz}(t) = 1 \quad \textit{for all} \quad t > 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$ for some z in X.

Example 2.2.2. *[38] Let (X, d) be a metric space where $X = [0, 2]$ and let (X, F, t) be Menger space with*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

for all $x, y \in X$. Let $Q, R : X \to X$ be defined by

$$Q(x) = \begin{cases} 2 & \text{for } x \in [0,1] - \frac{1}{2} \\ 0 & \text{for } x = \frac{1}{2} \\ \frac{2-x}{2} & \text{for } x \in (1,2] \end{cases}$$

and

$$R(x) = \begin{cases} 0 & \text{for } x \in [0,1] - \frac{1}{2} \\ 2 & \text{for } x = \frac{1}{2} \\ \frac{x}{2} & \text{for } x \in (1,2] \end{cases}$$

Taking sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{n}, n \in N$.

Then, it is neither compatible mappings of type (A) nor compatible mappings of type (P) but (Q, R) is compatible mappings of type (K).

2.3 Some common fixed point theorem in Menger probabilistic metric space

In 1969, A. Meir and E. Keeler [126] proved the following very interesting fixed point theorem, which is one of the generalizations of the Banach contraction principle [10].

Theorem 2.3.1. *[126] Let* (X, d) *be a complete metric space and* A *be a* ***Meir- Keeler or*** (ϵ, δ) ***contraction*** *on* X*, i.e., if for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*
$\epsilon \leq d(x, y) < \epsilon + \delta \implies d(Ax, Ay) < \epsilon$. *Then, A has a unique fixed point.*

In 2004, R. P. Pant, V. Pant and V. P. Pandey [162] established following theorems:

Theorem 2.3.2. *[162] Let* (A, S) *and* (B, T) *be compatible pairs of self-mappings of a complete metric space* (X, d) *such that*

1. $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$*,*
2. *Given* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $x, y \in X$*,*
   $\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) \underline{\leq} \epsilon$
   where $M(x, y) = max \left\{ d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{d(Ax, Ty)+d(Sx, By)}{2} \right\}$,
3. $d(Ax, By) < max \left\{ d(Sx, Ty), k\frac{d(Ax, Sx)+d(By, Ty)}{2}, \frac{d(Ax, Ty)+d(Sx, By)}{2} \right\}$
   for $0 < k \leq 2$

Suppose that the mappings in one of the pair (A, S) *or* (B, T) *are reciprocal continuous. Then A, B, S and T have unique common fixed point.*

In 2005, K. Jha , R. P. Pant and S. L. Singh[93] established following theorems for compatible pairs of mappings in complete metric space.

Theorem 2.3.3. *[93] Let* (A, S) *and* (B, T) *be compatible pairs of self-mappings of a complete metric space* (X, d) *such that*

1. *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*

2. *Given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$,*
   $\epsilon \le M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) < \epsilon$

3. $d(Ax, By) < [k[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Ax, Ty)]$
   for $0 \le k_1 < 1/3$.
   where, $M(x, y) = max\left\{d(Sx,\ Ty), d(Ax,\ Sx), d(By,\ Ty), \frac{d(Sx+By)\ +d(Ax\ +Ty)}{2}\right\}$

If one of the mappings A, B, S and T is continuous then A, B, S and T have unique common fixed point.

In 2014, K. Jha, V. Popa and K. B. Manandhar [94] have established following theorem using Compatible mappings of type (K) in metric space as:

Theorem 2.3.4. *[94] Let (X, d) be a complete metric space and A, B, S and T be a self mappings of X satisfying the following conditions:*

1. *$A(X) \subset T(X)$, $B(X) \subset S(X)$,*

2. *Given $\epsilon > 0$ there exists $\delta > 0$ such that for all x, y in X,*
   $\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) \le \epsilon$ *and*
   $d(Ax, By) < M(x, y)$
   where $M(x, y) = Max\ [d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty)]$

3. *S and T are continuous.*

If (A, S) and (B, T) compatible mappings of type (K), then A, B, S and T have a unique common fixed point.

In 2019, The following theorem established by R. P. Pant, A. Pant, M. Nikoli and S. N. Jevic [148] in Menger space:

Theorem 2.3.5. *[148] Let (X, F, t) be a complete Menger space,and let f be self-mappings of X satisfying one of the following conditions:*

(i) *for every* $\epsilon \in (0,1)$, *there exists* $\delta \in (0,\epsilon]$ *such that*
$\epsilon - \delta < \min\{(F_{x,gx}(t), F_{y,gy}(t)\} < \epsilon$ *implies* $F(gx, gy, t) \geq \epsilon$

(ii) $F_{gx,gy}(t) > \min\{(F_{x,gx}(t), F_{y,gy}(t)\}$
or,

(i) *for every* $\epsilon \in (0,1)$, *there exists* $\delta \in (0,\epsilon]$ *such that*
$\epsilon - \delta < \min\{(F_{x,gx}(t), F_{y,gy}(t)\} < \epsilon$ *implies* $F(gx, gy, t) > \epsilon$

for all $x, y \in X$. *Then, for any* $x \in X$ *the sequence of iterates* $\{g^n x\}, n \in N$ *is a cauchy sequence and there exists a point* z *in* X *such that* $lim_{n\to\infty} g^n x = z$ *for each* $x \in X$.

Using Meir- Keeler contractive condition, V. Gupta, B. Singh, M.. S. Khan and S. Kumar [72]established followings common fixed point theorem in Menger space in 2020.

Theorem 2.3.6. *[72] Let* (X, F, t) *be a complete Menger space with* $t(a, b) = min(a, b)$ *for all* $a, b \in [0, 1]$ *and* $A, B, S, T : X \to X$ *be four self mappings such that*

(i) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$;

(ii) *for every* $\epsilon > 0$, *and for all* $x, y \in X$, *there exists* $\delta \in (0, \epsilon]$
such that $\epsilon - \delta < m(x, y, t) \leq \epsilon$ *implies* $F(Ax, By, t) \geq \epsilon$, *where*
$m(x,y,t) = \min\{(F(Sx, Ty, t), F(Ax, Sx, t), F(By, Ty, t)\}$
for all $t > 0$, *and*

(iii) *one of* $A(X), B(X), S(X)$ *or* $T(X)$ *is complete subspace of* X.

Then, $Av = z = Sv$ *and* $Bw = z = Tv$. *Also, if the pair* (A, S) *as well as* (B, T) *are weakly compatible then* $Az = Bz = Sz = Tz = z$, *and* z *is unique in* X.

Theorem 2.3.7. *[72] Let* (X, F, t) *be a complete Menger space with* $t(a, b) = min(a, b)$ *for all* $a, b \in [0, 1]$ *and* $A, B, S, T : X \to X$ *be four self mappings such that*

(i) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$*;*

(ii) for every $\epsilon > 0$*, and for all* $x, y \in X$*, there exists* $\delta \in (0, \epsilon]$ *such that* $\epsilon - \delta < m(x, y, t) \leq \epsilon$ *implies* $F(Ax, By, t) \geq \epsilon$*, where*

$m(x, y, t) = \min\{(F(Sx, Ty, t), F(Ax, Sx, t), F(By, Ty, t)\}$
for all $t > 0$*;*

(iii) (A, S) *and* (B, T) *holds E.A. common property, and*

(iv) $S(X)$ *or* $T(X)$ *are closed in* X*.*

Then, $Au = r = Su$ *and* $Bv = z = Tv$*. Also, if the pair* (A, S) *as well as* (B, T) *are weakly compatible then* $Ar = Br = Sr = Tr = r$*, and* r *is unique in* X*.*

2.3.1 Common fixed point theorem using compatible mapping of type (K) in Menger probabilistic metric space

We now introduce new theorem in Menger space by using compatible mapping of type (K) under Meir- Keeler contractive condition. The following lemma need to prove our main result:

Lemma 2.3.1. *[38] Let* $\{x_n\}$ *be a sequence in Menger space* $(X,\ F,\ t)$*, where* t *is continuous* $T-$*norm and* $t(x,\ x)\ \geq x$ *for all* $x \in [0,\ 1]$*. If there exists a constant* $k \in (0,\ 1)$ *such that* $\lim_{n\to\infty} F_{x_n, x_n+1}(kt) \geq F_{x_n-1, x_n}(x)$*, for all* $t > 0$ *and* $n \in N$*, then* $\{x_n\}$ *is a Cauchy sequence in* X*.*

Lemma 2.3.2. *[196] Let (X, F, t) be a Menger space. If there exists $k \in (0, 1)$ such that for all $p, q \in X$, $F_{p,q}(kt) \geq F_{p,q}(t)$ then $p = q$.*

Now, we prove our main theorem by using compatible mappings of type (K) in complete Menger space:

Theorem 2.3.8. *Let (X, F, t) be a complete Menger space with $t(a, b) = min(a, b)$ for all $a, b \in [0, 1]$ and $A, B, S, T : X \to X$ be four self mappings such that*

(2.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(2.2) the pairs (A, S) and (B, T) are compatible mappings of type (K),

(2.3) S and T be continuous, and

(2.4) there exists a constant $k \in (0, 1)$ and for every $\epsilon \in (0, 1)$, there exists $\delta \in (0, \epsilon]$ such that $\epsilon - \delta < M_{x,y} < \epsilon$ implies $F_{Ax, By}(kt) \geq \epsilon$ and $F_{Ax, By}(kt) \geq M_{x,y}(t)$ where

$M_{x,y}(t) = \min\{(F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Ax,Ty}(t)\}$

for all $x, y \in X$ and $t > 0$.

Then, A, B, S and T have a unique common fixed point in X.

Proof. Consider $x_0 \in X$. From condition (2.1), we have
$A(X) \subset T(X)$and $B(X) \subset S(X)$. So if there exists a point x_1 in X such that $Ax_0 = Tx_1$. And for $x_1 \in X$ there exist $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. And inductively, we may construct sequence $\{y_n\}$ in X such that
$y_{2n-1} = Ax_{2n-2} = Tx_{2n-1}$,
and $y_{2n} = Bx_{2n-1} = Sx_{2n}$, for $n = 1, 2, 3...$
Putting $x = x_{2n}$ and $y = x_{2n+1}$ in condition (2.4), We get

$F_{y_{2n+1},y_{2n+2}}(kt) = F_{Ax_{2n},Bx_{2n+1}}(kt)$

$\geq \min\{(F_{Sx_{2n},Tx_{2n+1}}(t), F_{Ax_{2n},Sx_{2n}}(t), F_{Bx_{2n},Tx_{2n+1}}(t), F_{Ax_{2n},Tx_{2n+1}}(t)\}$

$\geq \min\{(F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n}}(t), F_{y_{2n+2},y_{2n+1}}(t), F_{y_{2n+1},y_{2n+1}}(t)\}$

$\geq \min\{(F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n+2}}(t)\}$

So, $F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n},y_{2n+1}}$

Therefore, for every $n \in N$, $F_{y_n,\, y_{n+1}}(kt) \geq F_{y_{n-1},\, y_n}$

So, by Lemma (2.3.1), $\{y_n\}$ is Cauchy sequence in X.

Since the Menger space (X, F, t) is complete, so $\{y_n\}$ converges to a point z in X and consequently the sub sequences $\{Ax_{2n-2}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}\, and\, \{Tx_{2n-1}\}$, also converges to z.

Since S and T are continuous and (A, S) and (B, T) are compatible mappings of type (K). So, we have

$AAx_{2n-2} \to Sz$ and $SSx_{2n} \to Az$. Also $BBx_{2n-1} \to Tz$

and $TTx_{2n-1} \to Bz$. .... (2.1)

Also, from condition (2.4), we get

$F_{AAx_{2n-2},BBx_{2n-1}}(kt)$

$\geq \min\{(F_{SAx_{2n-2},TBx_{2n-1}}(t), F_{AAx_{2n-2},SAx_{2n-2}}(t), F_{BBx_{2n-1},TBx_{2n-1}}(t), F_{AAx_{2n-2},TBx_{2n-1}}(t)\}$

As $n \to \infty$ and by using equation (2.1), we have

$F_{Sz,Tz}(kt) \geq \min\{(F_{Sz,Tz}(t), F_{Sz,Sz}(t), F_{Tz,Tz}(t), F_{Sz,Tz}(t)\}$

or, $F_{Sz,Tz}(kt) \geq F_{Sz,Tz}(t)$

From Lemma (2.3.2), we get

$Sz = Tz$ ....(2.2)

Again, from (2.4) condition, we have

$F_{Az,BBx_{2n-1}}(kt) \geq \min\{(F_{Sz,TBx_{2n-1}}(t), F_{Az,Sz}(t), F_{Bz,TBx_{2n-1}}(t), F_{Az,TBx_{2n-1}}(t)\}$

Taking $n \to \infty$ and using equation (2.1) and (2.2), we get

$F_{Az,Tz}(kt) \geq \min\{(F_{Sz,Sz}(t), F_{Az,Tz}(t), F_{Tz,Tz}(t), F_{Az,Tz}(t)\}$

or, $F_{Az,Tz}(kt) \geq F_{Az,Tz}(t)$

From Lemma (2.3.2) , we get

$$Az = Tz \text{ ....(2.3)}$$

Then, from equation (2.2) and (2.3), we obtain

$F_{Az,Bz}(kt) \geq \min\{(F_{Sz,Tz}(t), F_{Az,Sz}(t), F_{Bz,Tz}(t), F_{Az,Tz}(t)\}$

or, $F_{Az,Bz}(kt) \geq \min\{(F_{Az,Az}(t), F_{Az,Az}(t), F_{Bz,Az}(t), F_{Az,Az}(t)\}$

or, $F_{Az,Bz}(kt) \geq F_{Az,Bz}(t)$. So, we get

$$Az = Bz \text{ ....(2.4)}$$

From equation (2.2), (2.3) and (2.4), we get

$$Az = Bz = Tz = Sz \text{ ....(2.5)}$$

Now, we have to show that $Az = z$.

From condition (2.4), we have

$F_{Az,Bx_{2n-1}}(kt) \geq \min\{(F_{Sz,Tx_{2n-1}}(t), F_{Az,Sz}(t), F_{Bx_{2n-1},Tx_{2n-1}}(t), F_{Az,Tx_{2n-1}}(t)\}$

Taking $n \to \infty$ and using equation (2.2) and (2.3), we get

$F_{Az,z}(kt) \geq \min\{(F_{Sz,z}(t), F_{Az,Sz}(t), F_{z,z}(t), F_{Az,z}(t)\}$

$\geq \min\{(F_{Az,z}(t), F_{Az,Az}(t), F_{z,z}(t), F_{Az,z}(t)\}$

or, $F_{Az,z}(kt) \geq F_{Az,z}(t)$

or, by Lemma (2.3.2) $Az = z$

Hence, from equation (2.5), we get

$z = Az = Bz = Tz = Sz$ and z is a common fixed point of A, B, S, and T.

Uniqueness: Suppose $w \neq z$ is another common fixed point of A, B, S, and T. Then $Aw = Bw = Sw = Tw = w$.

Therefore, from condition (2.4), we have

$F_{z,w}(kt) = F_{Az,Bw}(kt) \geq \min\{(F_{Sz,Tw}(t), F_{Az,Sz}(t), F_{Bw,Tw}(t), F_{Az,Tw}(t)\}$

or, $F_{z,w}(kt) \geq \min\{(F_{z,w}(t), F_{z,z}(t), F_{w,w}(t), F_{z,w}(t)\}$

or, $F_{z,w}(kt) \geq F_{z,w}(t)$

By Lemma (2.3.2), we get $z = w$. Hence, $z = Az = Bz = Tz = Sz$ and z is unique in X. This completes the proof. □

We verify our main Theorem (2.3.8) through following example:

Example 2.3.1. *Let (X, F, t) be a complete Menger space with $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ where $X = [1, 10]$ with metric d defined by $d(x, y) = |x - y|$ and F is defined by*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+d(x,y)} & for\ t > 0 \\ 0 & for\ t = 0 \end{cases}$$

for all $x, y \in X$. We define $A, B, S, T : X \to X$ as:

$$A(x) = \begin{cases} 1 & for\ x \leq 4 \\ 2 & for\ x > 4, \end{cases}$$

$$B(x) = \begin{cases} 1 & for\ x \leq 5 \\ 2 & for\ x > 5 \end{cases}$$

and $S(x) = T(x) = x$ for all $x \in X$. Taking sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{n}, n \in N$. Then, the mappings A, B, S and T satisfy all the conditions of the above Theorem (2.3.8) and have a unique common fixed point at $x = 1$.

Our main result help to prove following corollary:

If we take in theorem (2.3.8) $A = B, T = S$ then we obtain following result:

Corollary 2.3.1. *Let A and S be self-mappings in complete Menger space (X, F, t) with continuous $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ satisfying the following conditions:*

(2.1) $A(X) \subset S(X)$

(2.2) the pairs (A, S) *be compatible mappings of type* (K),

(2.3) S *be continuous, and*

(2.4) there exists a constant $k \in (0,1)$ *and for every* $\epsilon \in (0,1)$, *there exists* $\delta \in (0, \epsilon]$ *such that* $\epsilon - \delta < M_{x,y} < \epsilon$ *implies* $F_{Ax,Ay}(kt) \geq \epsilon$ *and* $F_{Ax,Ay}(kt) \geq M_{x,y}(t)$
where $M_{x,y}(t) = \min\{(F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t), F_{Ax,Sy}(t)\}$

for all $x, y \in X$ *and* $t > 0$. *Then,* A *and* S *have a unique common fixed point in* X.

Remarks: Our result Theorem (2.3.8) generalize the results of Jungck et.al [101], Meir Keeler [126] and K. Jha et al [94]. Also, this result improves other similar results in the literature.

The above Theorem (2.3.8) has been published in
Advances in Mathematics: Scientific Journal 11 (2022), no.10, 883-892.
(https://doi.org/10.37418/amsj.11.10.6)

Chapter 3

Compatible and Weakly compatible mappings of type (P) in probabilistic metric space

This chapter includes compatible mappings of type (P) in Menger probabilistic metric space with common fixed point theorem. Also, we study the notion to weakens compatible mapping of type (P) in Menger space that is introducing weakly compatible mapping of type (P) in Menger probabilistic metric space and established new common fixed point theorem in four self mappings with verification through example.

3.1 Introduction

In 1982, S. Sessa [187] introduced the notion of weakly commuting mappings in metric space in 1982. In 1986, G. Jungck [96] enlarged this concept to compatible mappings. The notion of compatible mappings in Menger space has been introduced by S. N. Mishra [137]. Also, G. Jungck and B. E. Rhoades [102] introduced the notion of weak compatibility and showed that compatible mappings are weakly compatible mappings but the reverse is always not true. B. Singh and S. Jain [196] generalized the notion of compatible maps by introducing the notion of weakly compatible mappings and established a common fixed point theorem for six self-mappings.

Here, we extends compatible mappings of type (P) in Menger probabilistic metric space which is introduced by H. K. Pathak, Y. J. Cho, S. S. Chang, and S. M. Kang [154] in metric space. Also, we give new notion of weakly compatible mapping of type (P) in Menger space with establishing common fixed point theorem in this section.

3.2 Basic Definitions

In 1991, S. N. Mishra [137] introduced compatible notion in Menger space:

Definition 3.2.1. *[137] Two mappings $Q, R : X \to X$ are said to be **compatible mappings** in Menger space (X, F, t) iff*

$$\lim_{n\to\infty} F_{QRx_n,RQx_n}(t) = 1 \quad \textit{for all} \quad t > 0$$

whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$ *for some z in X.*

In 2005, B. Singh and S. Jain [196] introduced following weakly compatible notion in Menger space:

Definition 3.2.2. *[196] Two mappings $Q, R : X \to X$ are said to be **weakly compatible mappings** in Menger space (X, F, t) if they commute at their coincidence points i.e. $Qx = Rx$ for some $x \in X$, then $QRx = RQx$.*

In 1992, Y. J. Cho, P. P. Murthi and M. Stojakovic [45] extended compatible Mappings of type (A) in Menger space as:

Definition 3.2.3. *[45] Two mappings $Q, R : X \to X$ are said to be* ***compatible Mappings of type*** *(A) in Menger space (X, F, t) iff*

$$\lim_{n\to\infty} F_{QRx_n,RRx_n}(t) = 1$$

and

$$\lim_{n\to\infty} F_{RQx_n,QQx_n}(t) = 1$$

for all $t > 0$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$ *for some* z *in* X.

In 1995, H. K. Pathak, S. M. Kang and J. H. Beak [155] introduced weakly compatible mappings of type (A) in Menger space as :

Definition 3.2.4. *[155]Two mappings $Q, R : X \to X$ are said to be* ***Weak compatible Mappings of type*** *(A) in Menger space (X, F, t) iff*

$$\lim_{n\to\infty} F_{QRx_n,RRx_n}(t) \geq \lim_{n\to\infty} F_{RQx_n,RRx_n}(t)$$

and

$$\lim_{n\to\infty} F_{RQx_n,QQx_n}(t) \geq \lim_{n\to\infty} F_{QRx_n,QQx_n}(t)$$

for all $t > 0$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$ *for some* z *in* X.

In 1996, H. K. Pathak, Y. J. Cho, S. S. Chang, and S. M. Kang [154] introduced compatible mappings of type (P) in metric space as:

Definition 3.2.5. *[154] Two self mappings $Q, R : X \to X$ in metric space (X, d) are said to be* ***Compatible mappings of type*** *(P)*
if $\lim_{n\to\infty} d(QQx_n, RRx_n) = 0$
whenever $\{x_n\}$ *is a sequence in* X *such that* $lim_{n\to\infty} Qx_n = lim_{n\to\infty} Rx_n = z$ *for some* z *in* X.

We have introduced and extended the above definition in Menger probabilistic metric space as:

Definition 3.2.6. *Two mappings $Q, R : X \to X$ are said to be **compatible mappings of type** (P) in Menger space (X, F, t) iff*

$$\lim_{n\to\infty} F_{QQx_n,RRx_n}(t) = 1 \quad \text{for all} \quad t > 0$$

whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$ *for some z in X.*

Example 3.2.1. *Let (X, d) be a metric space where $X = [0, 2]$ with usual metric $d(x, y) = |x - y|$ and let (X, F) be PM space with*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

for all $x, y \in K$. Let $Q, R : X \to X$ be defined by

$$Q(x) = \begin{cases} 1 & \text{for } x \in [0, 1) \\ 2 - x & \text{for } x \in [1, 2] \end{cases}$$

and

$$R(x) = \begin{cases} 1 & \text{for } x \in [0, 1) \\ x & \text{for } x \in [1, 2] \end{cases}$$

Taking sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{n}$.
Then, $Qx_n = \lim_{n\to\infty} Q(1 + (\dfrac{1}{n})) = 2 - (1 + \dfrac{1}{n}) = 1 = z.$
$Rx_n = \lim_{n\to\infty} R(1 + (\dfrac{1}{n})) = (1 + \dfrac{1}{n}) = 1 = z.$
Also, $QQx_n = Q(Q((1 + \frac{1}{n}))) = Q(1) = 1 = z.$
and $RRx_n = R(R((1 + \frac{1}{n}))) = R(1) = 1 = z.$
So that $\lim_{n\to\infty} F_{QQx_n,RRx_n}(t) = \lim_{n\to\infty} F_{1,1}(t) = \frac{t}{t+|1-1|} = 1$ *for all $t > 0$.*
So, (Q, R) are compatible mappings of type (P).

Example 3.2.2. *Let* (X, d) *be a metric space where* $X = [0, 2]$*,* $d(x, y) = |x - y|$ *and* (X, F, t) *be induced Menger space with* $F_{x,y}(t) = H(t - d(x, y))$*,* $x, y \in X$ *and* $t > 0$*.*
Defining $Q, R : X \to X$ *as:*

$$Q(x) = \begin{cases} 2 - x & \text{for } 0 \leq x < 1 \\ 1 & \text{for } 1 < x \leq 2 \end{cases}$$

and

$$R(x) = \begin{cases} x & \text{for } 0 \leq x < 1 \\ 1 & \text{for } 1 < x \leq 2 \end{cases}$$

Taking $x_n = 1 - \frac{1}{n}$*. As* $n \to \infty$*. Then,* $Qx_n = Q(1 - \frac{1}{n}) = 2 - (1 - \frac{1}{n}) = 1 + \frac{1}{n}$*.*
So, $Qx_n \to 1$*.*
Since, $Rx_n = R(1 - \frac{1}{n}) = 1 - \frac{1}{n}$*. So,* $Rx_n \to 1$*.*
Further, $QQx_n = Q(1 + \frac{1}{n}) = 1$ *and* $RRx_n = 1 - \frac{1}{n}$*.*
Now, $\lim_{n\to\infty} F_{QQx_n, RRx_n}(t) = H(t - \dfrac{1}{n}) \to H(t) \to 1$ *as* $n \to \infty$*.*
So, $\lim_{n\to\infty} F_{QQx_n, RRx_n}(t) = 1$*.*
Hence, (Q, R) *are compatible mappings of type* (P)*.*

Example 3.2.3. *Let* (X, d) *be a metric space where* $X = [0, \infty)$ *and* $t(a, b) = ab$ *be t- norm. We define distribution function as :*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t + |x - y|} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

for all $x, y \in X$*. Then,* (X, F, t) *be Menger space.*
Let $Q, R : X \to X$ *be defined by*

$$Q(x) = \begin{cases} 5 & \text{for } x \in [0, 1) \\ x & \text{for } x \in [1, \infty) \end{cases}$$

and

$$R(x) = \begin{cases} 1 & \text{for } x \in [0, 1) \\ 1/x & \text{for } x \in [1, \infty). \end{cases}$$

Taking sequence $\{x_n\}$ *where* $x_n = 1 + \frac{1}{n}$.
Then, $Qx_n = \lim_{n\to\infty} Q(1 + (\frac{1}{n})) = (1 + \frac{1}{n}) = 1$ *as* $n \to \infty$.
$Rx_n = \lim_{n\to\infty} R(1 + (\frac{1}{n})) = 1/(1 + \frac{1}{n}) = 1,$ *as* $n \to \infty$.
Now, $QQx_n = Q(Q((1 + \frac{1}{n}))) = 1,$ *as* $n \to \infty$
and $RRx_n = R(1/(1 + \frac{1}{n})) = 1,$ *as* $n \to \infty$
So that $\lim_{n\to\infty} F_{QQx_n,RRx_n}(t) = \lim_{n\to\infty} F_{1+\frac{1}{n},1}(t) = \frac{t}{t+\frac{1}{n}} \to 1,$ *as* $n \to \infty$.
So, (Q, R) *are compatible mappings of type* (P).
Again, $QRx_n = 5$ *and* $RQx_n = 1/(1 + \frac{1}{n}) = n/(n + 1) = 1 - 1/(n + 1)$
So, $F_{QRx_n,RQx_n}(t) = F_{5,1-1/(n+1)}(t) = \frac{t}{t+|5-(1-1/(n+1))|} = \frac{t}{t+4-1/n+1} \to \frac{t}{t+4} \neq 1,$ *as* $n \to \infty$.
So, $(Q,\ R)$ *are not compatible mappings.*

Example 3.2.4. *Let* (X, d) *be a metric space where* $X = [0, 2]$ *and* $t(a, b) = ab$ *be t- norm. Define distribution function as :*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

for all $x, y \in X$. *Then,* (X, F, t) *be Menger space.*
Let $Q, R : X \to X$ *be defined by*

$$Q(x) = \begin{cases} 1 - x & \text{for } x \in [0, 1/2] \\ 1 & \text{for } x \in (1/2, 2] \end{cases}$$

and $R(x) = x$, *for all* $x \in X$.
Taking sequence $\{x_n\}$ *where* $x_n = |\frac{1}{2} - \frac{1}{n}|$.
Then, $Qx_n = RQx_n = QRx_n = 1 - |\frac{1}{2} - \frac{1}{n}|$
, $Rx_n = RRx_n = |\frac{1}{2} - \frac{1}{n}| \to \frac{1}{2}$ *and* $QQx_n = Q(1 - |\frac{1}{2} - \frac{1}{n}|) \to 1,$ *as* $n \to \infty$
. *Since,* $QRx_n = RQx_n$, $\lim_{n\to\infty} F_{QRx_n,RQx_n}(t) = 1 \implies (Q, R)$ *are compatible mappings.*
But $\lim_{n\to\infty} F_{QQx_n,RRx_n}(t) = \lim_{n\to\infty} F_{1,\frac{1}{2}}(t) = \frac{t}{t+\frac{1}{2}} = \frac{2t}{2t+1} \neq 1$.
Therefore, (Q, R) *are not compatible mappings mappings of type* (P).

We also prove that compatible mappings and compatible mappings of type P are equivalent under conditions as follows:

Proposition 3.2.1. *Let (X, F, t) be Menger space and also let self mappings Q and R be continuous. Then, Q and R are compatible if and only if they are compatible mappings of type P.*

Proof. Consider a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Qx_n = Rx_n = z$, for some $z \in X$. Since, Q and R are continuous, so we may write
$\lim_{n\to\infty} QQx_n = \lim_{n\to\infty} QRx_n = Qz$ and
$\lim_{n\to\infty} RQx_n = \lim_{n\to\infty} RRx_n = Rz$
Since as $n \to \infty$, $QQx_n, QRx_n \to Qz$, so we write
$F_{QQx_n,QRx_n}(t) \to 1$ as $n \to \infty$, for all $t > 0$.
Consider Q and R are compatible mappings. Then,
$F_{QRx_n,RQx_n}(t) \to 1$ as $n \to \infty$.
By definition of Menger space,

$$\begin{aligned} F_{QQx_n,RRx_n}(t) &\geq t\{F_{QQx_n,QRx_n}(t/2), F_{QRx_n,RRx_n}(t/2)\} \\ &\geq t\{F_{QQx_n,QRx_n}(t/2), t\{F_{QRx_n,RQx_n}(t/4), F_{RQx_n,RRx_n}(t/4)\} \end{aligned}$$

So, $F_{QQx_n,RRx_n}(t) \to t\{1, t(1,1)\} = 1$, as $n \to \infty$. Hence, Q and R are compatible mappings of type P.
Conversely, Suppose that Q and R are compatible mappings of type P.
So, $F_{QQx_n,\,RRx_n}(t) \to 1$, as $n \to \infty$.
Again from Menger space definition, we have

$$\begin{aligned} F_{QRx_n,RQx_n}(t) &\geq t\{F_{QRx_n,QQx_n}(t/2), F_{QQx_n,RQx_n}(t/2)\} \\ &\geq t\{F_{QRx_n,QQx_n}(t/2), t\{F_{QQx_n,RRx_n}(t/4), F_{RRx_n,RQx_n}(t/4)\} \end{aligned}$$

When $n \to \infty$, $F_{QRx_n,RQx_n}(t) \to 1$.
Therefore, Q and R are compatible mappings. Hence, proved. □

Now, we have introduced weakly compatible mappings of type (P) in Menger space with an example as follows:

Definition 3.2.7. *Two mappings $Q, R : X \to X$ are said to be* ***Weakly compatible Mappings of type*** (P) *in Menger space* (X, F, t) *if and only if* $\lim_{n\to\infty} F_{QQx_n,RRx_n}(t) \geq F_{Qx_n,Rx_n}(t)$ *for all* $t > 0$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$ *for some* z *in* X*.*

Example 3.2.5. *Let* (X, d) *be a metric space where* $X = [0, 2]$ *with usual metric* $d(x, y) = |x - y|$ *and let* (X, F) *be PM space with*

$$F_{x,y}(t) = \begin{cases} e^{\frac{d(x,y)}{t}} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

for all $x, y \in K$*. Let* $Q, R : X \to X$ *be defined by*

$$Q(x) = \begin{cases} 1 - x & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 2] \end{cases}$$

and

$$R(x) = \begin{cases} x & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 2] \end{cases}$$

Taking sequence $\{x_n\}$ *where* $x_n = \frac{1}{2} - \frac{1}{n}$*.*
Then, $Qx_n = \lim_{n\to\infty} Q(1 - (\dfrac{1}{2} - \dfrac{1}{n})) = \dfrac{1}{2} + \dfrac{1}{n} = \dfrac{1}{2} = k,$ *and*
$Rx_n = \lim_{n\to\infty} (\dfrac{1}{2} - \dfrac{1}{n}) = \dfrac{1}{2} = k.$
Also, $QQx_n = Q((\frac{1}{2} + \frac{1}{n})) = 1$
and $RRx_n = R((\frac{1}{2} - \frac{1}{n})) = \frac{1}{2} - \frac{1}{n} = \frac{1}{2}.$
So that $\lim_{n\to\infty} F_{QQx_n,RRx_n}(t) = \lim_{n\to\infty} F_{1,\frac{1}{2}}(t) = e^{\frac{d(1,1/2)}{t}} = e^{\frac{1}{2t}} > 1 \neq 1$ *for all* $t > 0$ *and* $\lim_{n\to\infty} F_{Qx_n,Rx_n}(t) = \lim_{n\to\infty} F_{1,1}(t) = e^{\frac{d(1,1)}{t}} = 1$
for all $t > 0$*.*
Therefore, we have $\lim_{n\to\infty} F_{QQx_n,RRx_n}(x) \geq F_{Qx_n,Rx_n}(x)$ *for all* $x > 0$*.*
Hence, $(Q,\ R)$ *are weakly compatible mappings of type* (P) *but it is neither compatible mappings of type* (P) *nor compatible mappings.*

3.3 Some common fixed point theorem in Menger space

The following theorems established by H. K. Pathak, Y. J. Cho, S. S. Chang and S. M. Kang [154] in metric space:

Theorem 3.3.1. *[154] Let* $X, d)$ *be a complete metric space and* A, B, S, T :$X \to X$ *be four self mappings such that*

(1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

(2) the pairs (A, S) *and* (B, T) *are compatible mappings of type* (P),

(3) one of A, B, S *and* T *be continuous, and*

(4) there exists $\phi \in \mathbb{G}$ *such that*

$$d(Ax, By) \leq \phi\{d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx), d(Sx, Ty)\}$$

for all $x, y \in X$. *Here,* $\mathbb{G}$ *be the family of mappings* $\phi : (R^+)^5 \to \mathbb{R}^+$ *such that* ϕ *is upper semi continuous and non decreasing in each coordinate variable. Then,* $A, B, S,$ *and* T *have a unique common fixed point in* X.

In 2003, S. Sharma and B. Despandey [191] established following theorem in Menger space:

Theorem 3.3.2. *[191] Let* (X, F, t) *be a complete Menger space where* t *is continuous with* $t(x, x) \geq x$ *for all* $x \in [0, 1]$ *and* $A, B, S, T : X \to X$ *be four self mappings such that*

(1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

(2) there exists a constant $k \in (0, 1)$ *such that*

$$F_{Au,Bv}(kx) \geq t\{(F_{Au,Su}(x), t\{F_{Bv,Tv}(x), t\ F_{Au,Tv}(\alpha x), F_{Bv,Su}((2-\alpha)x)\}$$

for all $u, v \in X, \alpha \in (0, 2)$ *and* $x > 0$, *if*

(3) one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is complete subspace of X, then

(i) A and S have coincidence point and

(ii) B and T have a coincidence point. Further if

(4) the pairs (A, S) and (B, T) are weakly compatible mappings

Then, $A, B, S,$ and T have a unique common fixed point in X.

In 2003, S. Sharma and K. Dubey [190] established following theorem:

Theorem 3.3.3. *[190] Let (X, F, t) be a complete Menger space where t is continuous with $t(x, x) \geq x$ for all $x \in [0, 1]$ and $A, B, S, T : X \to X$ be four self mappings such that*

(1) $A(X)$, $B(X)$ are closed sets of X and $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(2) there exists a constant $k \in (0, 1)$ such that
$F_{Au,Bv}(kx) \geq t\{(F_{Au,Su}(x), t\{F_{Bv,Tv}(x), t\, F_{Au,Tv}(\alpha x), F_{Bv,Su}((2 - \alpha)x)\}$
for all $u, v \in X, \alpha \in (0, 2)$ and $x > 0$, and

(3) (A, S) and (B, T) are weakly compatible mapping,

Then, A, B, S, and T have a unique common fixed point in X.

By using weak compatible mappings of type (A), H. K. Pathak, S. M. Kang, and J. H. Beak [155] established following theorem in Menger space:

Theorem 3.3.4. *[155] Let (X, F, t) be a complete Menger space where t is continuous with $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ and A, B, S, $T : X \to X$ be four self mappings such that*

(1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(2) (A,S) *and* (B,T) *are weak compatible mappings of type* (A)

(3) One of A,B,S,T *be continuous , and*

(4) there exists a constant $k\in(0,1)$ *such that*
$[F_{Au,Bv}(kx)]^2$
$\geq \min\{[F_{Su,Tv}(x)]^2, F_{Su,Au}(x).F_{Tv,Bv}(x), F_{Su,Tv}(x).F_{Su,Au}(x), F_{Su,Tv}(x).F_{Tv,Bv}(x),$
$F_{Su,Tv}(2x).F_{Su,Bv}(x), F_{Su,Tv}(x).F_{Tv,Au}(x), F_{Su,Bv}(2x).F_{Tv,Au}(x), F_{Su,Au}(x).F_{Tv,Au}(x),$
$F_{Su,Bv}(2x).F_{Tv,Bv}(x)\}$
for all $u,v\in X$*, and* $x>0$.

Then, A, B, S, and T have a unique common fixed point in X.

In 2021, T. N. C. Raju, V. Nagaraju, and M. Venkatakrishna established common fixed point theorem for five self mappings in complete Menger space as:

Theorem 3.3.5. *Let* (X,F,t) *be a complete Menger space with* $t(a,b)=min(a,b)$ *for all* $a,b\in[0,1]$ *and* $A,B,S,T,P:X\to X$ *be four self mappings such that*

(1) $P(X)\subseteq AB(X)$ *and* $P(X)\subseteq ST(X)$*,*

(2) the pairs (P,AB) *is compatible mappings,*

(3) $PB=AB, PT=TP, AB=BA, ST=TS$

(4) A and B be continuous, and

(5) there exists a constant $k\in(0,1)$ *such that*

$F_{Px,\,Py}(kt)\geq F_{ABx,\,Px}(t)*F_{Px,\,STy}(t)*F_{ABx,\,STy}(t)*\frac{F_{Px,\,ABx}(t)*F_{Px,\,STy}(t)}{F_{STy,\,ABx}(t)}*$
$F_{ABx,\,Py}\,((3-\alpha)t)$ *for all* $x,\,y\in X,\alpha\in(0,\,3)$ *and* $t>0$.

Then, A, B, S, T and P have a unique common fixed point in X.

3.3.1 Common fixed point theorem in Menger space with compatible mappings of type (P)

We need following theorem, lemma and propositions to establish main theorem in Menger space:

Theorem 3.3.6. *[40] Let (X, F, t) be Menger space with the continuous $T-norm$ t and $Q : X \to X$ be self mapping. Then, Q is continuous at a point $x \in X$ if and only if for every sequence $\{x_n\}$ in X converging to a point x, then sequence $\{Qx_n\}$ converges to the point Qx, i.e. if $\{x_n\} \to x$ then it implies $Qx_n \to Qx$.*

Lemma 3.3.1. *[196] Let (X, F, t) be a Menger space. If there exists $k \in (0, 1)$ such that for all $p, q \in X$, $F_{p,q}(kx) \geq F_{p,q}(x)$ then $p = q$.*

We introduce following propositions to prove our main results:

Proposition 3.3.1. *Let (X, F, t) be a Menger Space such that the $T-norm$ t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Q, R : X \to X$ be mappings. Let Q and R are compatible mappings of type(P) and $Qz = Rz$ for some $z \in X$, then, $QQz = QRz = RQz = RRz$.*

Proof. Suppose $\{x_n\}$ is a sequence in X defined by $x_n = z$
where $n = 1, 2, 3, ...$ for some $z \in X$ and $Qz = Rz$.
Then, we have $Qx_n, Rx_n \to Qz$ as $n \to \infty$.
Since, Q and R are compatible mappings of type (P), then for every $\epsilon > o$
$F_{QQz,RRz}(\epsilon) = \lim_{n\to\infty} F_{QQx_n,RRx_n}(\epsilon) = 1.$
Therefore, $QQz = RRz$. But $Qz = Rz$
implies $QQz = QRz = RQz = RRz$. □

Proposition 3.3.2. *Let (X, F, t) be a Menger Space such that the $T-$ norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Q, R : X \to X$ be* *mappings. Let Q and R are compatible mappings of type(P) and $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$ for some $z \in X$. Then, we have*

(i) $\lim_{n\to\infty} RRx_n = Qz$ *if Q is continuous at z;*

(ii) $\lim_{n\to\infty} QQx_n = Rz$ *if R is continuous at z; and*

(iii) $QRz = RQz$ *and* $Qz = Rz$ *if Q and R are continuous at z.*

Proof. **(i)** Suppose that Q is continuous at z. Since, we have

$\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} Rx_n = z$ for some $z \in X$.

So, $\lim_{n\to\infty} QQx_n = Qz$

Again, since Q and R are compatible mappings of type(P).

Therefore, $\lim_{n\to\infty} F_{QQx_n,RRx_n}(\epsilon) = 1$, for all $\epsilon > o$.

So, $F_{RRx_n,Qz}(\epsilon) \geq t(F_{RRx_n,QQx_n}(\epsilon/2), F_{QQx_n,Qz}(\epsilon/2))$

$\geq t(1, F_{Qz,Qz}(\epsilon/2))$

$\geq t(1, 1)$

So, $F_{RRx_n,Qz}(\epsilon) = 1$

Therefore, $lim_{n\to\infty} RRx_n = Qz$.

(ii): We may prove (ii) as we prove (i).

(iii): Suppose that $Q, R : X \to X$ are continuous at z.

Since, $\lim_{n\to\infty} Rx_n = z$ and Q is continuous at z.

So, by propostion (3.3.2) (i), $\lim_{n\to\infty} RRx_n = Qz$,as $n \to \infty$.

On the other hand, since , $\lim_{n\to\infty} Rx_n = z$ and R is continuous at z.

So, $\lim_{n\to\infty} RRx_n = Rz$. Thus, we have $Qz = Rz$ by uniqueness of limit and by proposition (3.3.1), $RQz = QRz$. This completes the proof. □

Lemma 3.3.2. *[41] Let $\{x_n\}$ be a sequence in Menger space (X, F, t), where t is continuous $T-$norm and $t(x, x) \geq x$ for all $x \in [0, 1]$. If there exists a constant $k \in [0, 1]$ such that $lim_{n\to\infty}$ $F_{x_n, x_n+1}(kx) \geq F_{x_n-1, x_n}(x)$, for all $x > 0$ and $n \in N$, then $\{x_n\}$ is a Cauchy sequence in X.*

Now, we prove our **Main Theorem** for compatible mappings of type (P) in complete Menger space:

Theorem 3.3.7. *Let (X, F, t) be a complete Menger space with $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ and $Q, S, R, T : X \to X$ be mappings such that*

(1) $Q(X) \subset T(X)$ and $S(X) \subset R(X)$,

(2) the pairs (Q, R) and (S, T) are compatible mappings of type (P),

(3) One of Q, S, R, T be continuous, and

(4) there exists a constant $k \in (0, 1)$ such that

$$F_{Qx,Sy}(kt) \geq \min\{(F_{Rx,Qx}(t), F_{Ty,Qx}(\alpha t), F_{Rx,Sy}((2-\alpha)t), F_{Rx,Ty}(t)\}$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$.

Then, Q, S, R, T have a unique common fixed point in X.

Proof. Consider $u_0 \in X$. Since $Q(X) \subset T(X)$, so there exists a point u_1 in X such that $Qu_0 = Tu_1 = v_0$. Again, since $S(X) \subset R(X)$, so for u_1, we may choose u_2 in X such that $Su_1 = Ru_2 = v_1$ and so on.

And inductively, we may construct sequence $\{u_n\}$ and $\{v_n\}$ in X such that

$Qu_{2n} = Tu_{2n+1} = v_{2n}$, and $Su_{2n+1} = Ru_{2n+2} = v_{2n+1}$, for $n = 0, 1, 2, ...$ Putting $x = u_{2n}$ and $y = u_{2n+1}$ for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$ in theorem (3.3.7) condition (4), We get

$$F_{Qu_{2n},Su_{2n+1}}(kt) \geq \min\{(F_{Ru_{2n},Qu_{2n}}(t), F_{Tu_{2n+1},Su_{2n+1}}(t), F_{Tu_{2n+1},Qu_{2n}}((1-q)t),$$
$$F_{Ru_{2n},Su_{2n+1}}(1+q)t), F_{Ru_{2n},Tu_{2n+1}}(t)\}$$

or, $$F_{v_{2n},v_{2n+1}}(kt) \geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n}}((1-q)t),$$
$$F_{v_{2n-1},v_{2n+1}}(1+q)t), F_{v_{2n-1},v_{2n}}(t)\}$$

$$F_{v_{2n},v_{2n+1}}(kt) \geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), 1, F_{v_{2n-1},v_{2n+1}}(1+q)t), F_{v_{2n-1},v_{2n}}(t)\}$$

$$\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(qt), F_{v_{2n-1},v_{2n}}(t)\}$$

$$\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n+1}}(qt)\}$$

As $q \to 1$, we obtain

$$F_{v_{2n},v_{2n+1}}(kt) \geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n+1}}(t)\}$$

$$\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t)\}$$

Hence, we get $F_{v_{2n},v_{2n+1}}(kt) \geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t)\}$

i.e. $F_{v_{2n},v_{2n+1}}(kt) \geq F_{v_{2n-1},v_{2n}}(t)$

Similarly, we obtain

$$F_{v_{2n+1},v_{2n+2}}(kt) \geq F_{v_{2n},v_{2n+1}}(t)$$

Therefore, for every $n \in N$, $F_{v_n,v_{n+1}}(kt) \geq F_{v_{n-1},v_n}(t)$.

So, using Lemma (3.3.2), $\{v_n\}$ is a Cauchy sequence in X. Since the Menger space (X, F, t) is complete, so $\{v_n\}$ converges to a point z in X and consequently the sub sequences $\{Qu_{2n}\}, \{Su_{2n+1}\}, \{Ru_{2n}\}, \{Tu_{2n+1}\}$, of $\{v_n\}$ also converges to z.

Now, suppose that T is continuous. Then, since S and T are compatible mappings of type (P) then by proposition (3.3.2), SSu_{2n+1} , $TSu_{2n+1} \to Tz$ as $n \to \infty$. Putting $x = u_{2n}$ and $y = Su_{2n+1}$ in theorem (3.3.7) condition (4), we get

$$F_{Qu_{2n},SSu_{2n+1}}(kt) \geq \min\{(F_{Ru_{2n},Qu_{2n}}(t), F_{TSu_{2n+1},SSu_{2n+1}}(t), F_{TSu_{2n+1},Qu_{2n}}((\alpha)t),$$

$$F_{Ru_{2n},SSu_{2n+1}}(2-\alpha)t), F_{Ru_{2n},TSu_{2n+1}}(t)\}$$

Taking $n \to \infty$, we have

$$F_{z,Tz}(kt) \geq \min\{(F_{z,z}(t), F_{Tz,Tz}(t), F_{Tz,z}((\alpha)t), F_{z,Tz}(2-\alpha)t), F_{z,Tz}(t)\}$$

Letting $\alpha = 1 - q$ with $q \in (0, 1)$ then

$$F_{z,Tz}(kt) \geq \min\{(F_{Tz,z}((1-q)t), F_{z,Tz}((1+q)t), F_{z,Tz}(t)\}$$

or, $F_{z,Tz}(kt) \geq \min\{(F_{Tz,Tz}((1-q+1+q)t), F_{z,Tz}(t)\}$

$\geq \min\{F_{z,Tz}(t)\}$

Therefore, $F_{z,Tz}(kt) \geq F_{z,Tz}(t)$

which implies $z = Tz$ by Lemma (3.3.1)

Similarly, replacing x by u_{2n} and y by z in condition (4), we have

$F_{Qu_{2n},Sz}(kt) \geq \min\{(F_{Ru_{2n},Qu_{2n}}(t), F_{Tz,Sz}(t), F_{Tz,Qu_{2n}}((\alpha)t), F_{Ru_{2n},Sz}(2-\alpha)t), F_{Ru_{2n},Tz}(t)\}$

Taking $n \to \infty$, we get

$F_{z,Sz}(kt) \geq \min\{(F_{z,z}(t), F_{z,Sz}(t), F_{z,z}((\alpha)t), F_{z,Sz}(2-\alpha)t), F_{z,z}(t)\}$

$F_{z,Sz}(kt) \geq \min\{(F_{z,Sz}(t), , F_{z,Sz}(2-(1-q)t)\}$

$\geq \min\{(F_{z,Sz}(t), , F_{z,Sz}(1+q)t)\}$

$\geq \min\{(F_{z,Sz}(t), F_{z,z}(t), F_{z,Sz}(qt)\}$

$\geq \min\{(F_{z,Sz}(t), F_{z,Sz}(t)\}$ as $q \to 1$

So that $F_{z,Sz}(kt) \geq \{(F_{z,Sz}(t)\}$

which implies $z = Sz$. Since, $S(K) \subset R(K)$, so there exists a point w in X such that $Sz = Rw = z$.

By using theorem (3.3.7) condition (4) with $x = w, y = z$, we have

$F_{Qw,z}(kt) \geq \min\{(F_{Rw,Qw}(t), F_{Tz,Sz}(t), F_{Tz,Qw}(\alpha t), F_{Rw,Sz}(2-\alpha)t), F_{Rw,Tz}(t)\}$

$\geq \min\{(F_{z,Qw}(t), F_{Tz,z}(t), F_{z,Qw}((1-q)t), F_{Rw,z}(1+q)t), F_{z,Tz}(t)\}$

$\geq \min\{(F_{z,Qw}(t), F_{Tz,z}(t), F_{Qw,z}((1-q)t), F_{Rw,z}(1+q)t), F_{z,Tz}(t)\}$

$\geq \min\{(F_{z,Qw}(t), F_{z,z}(t), F_{Qw,Rw}((1-q+1+q)t)\}$

$\geq \min\{(F_{z,Qw}(t), F_{Qw,z}(2t)\}$

Therefore, $F_{Qw,z}(kt) \geq F_{z,Qw}(t) = F_{Qw,z}(t)$

which implies $Qw = z$, by Lemma (3.3.1)

Again, since Q and R are compatible mappings of type (P) and $Qw = Rw = z$, by proposition (3.3.1), we have for every $\epsilon > 0$

$1 = F_{QQw,RRw}(\epsilon)$

Hence, $Qw = QQw = RRw = Rw$.

Finally, by theorem (3.3.7) condition (4) with $x = z, y = Sz = z$, we have $F_{Qz,z}(kt) = F_{Qz,Sz}(kt) \geq \min\{(F_{Rz,Qz}(t), F_{Tz,z}(t), F_{Tz,Qz}(\alpha t), F_{Rz,z}(2 - \alpha)t), F_{Rz,Tz}(t)\}$

$$\geq \min\{(F_{Qz,Qz}(t), F_{z,z}(t), F_{z,Qz}(\alpha t), F_{Qz,z}(2-\alpha)t), F_{Qz,z}(t)\}$$

$$\geq \min\{(F_{Qz,z}(\alpha t), F_{z,Qz}(2-\alpha)t), F_{Qz,z}(t)\}$$

$$\geq \min\{(F_{Qz,Qz}(\alpha t + 2t - \alpha t), F_{Qz,z}(t)\}$$

or, $F_{Qz,z}(kt) \geq F_{Qz,z}(t)$

Therefore, $Qz = z$. Hence, $Qz = Sz = Rz = Tz = z$. That is, z is common fixed point of given mappings Q, S, R and T.

Uniqueness: Suppose z_1 be other point in X such that $z_1 = Qz_1 = Sz_1 = Rz_1 = Tz_1$. Then, putting $x = z$ and $y = z_1, \alpha = 1$ in theorem (3.3.7) condition (4), we get

$F_{Qz,Sz_1}(kt) = F_{z,z_1}(kt) \geq \min\{(F_{Rz,Qz}(t), F_{Tz_1,Sz_1}(t), F_{Tz_1,Qz}(t), F_{Rz,Sz_1}(t), F_{Rz,Tz_1}(t)\}$

or, $F_{z,z_1}(kt) \geq \min\{(F_{z,z_1}(t), F_{z,z}(t)\}$

or, $F_{z,z_1}(kt) \geq F_{z,z_1}(t)$

By Lemma (3.3.1), $z = z_1$. Hence, $z = Qz = Sz = Rz = Tz$ and z is unique common fixed point for Q, S, R, T in X.

This completes the proof. □

We verify our main Theorem with following example:

Example 3.3.1. *Let (X, F, t) be a complete Menger space with $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ where $X = [1, 10]$ with usual metric*

$d(x, y) = |x - y|$ and F is defined by

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & for\ t > 0 \\ 0 & for\ t = 0 \end{cases}$$

for all $x, y \in K$. Define $Q, S, R, T : X \to X$ as below

$$Q(x) = \begin{cases} 1 & for\ x \leq 4 \\ 2 & for\ x > 4; \qquad and \end{cases}$$

$$S(x) = \begin{cases} 1 & for\ x \leq 5 \\ 2 & for\ x > 5 \end{cases}$$

$R(x) = T(x) = x$, for all $x \in X$.
Taking sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{n}, n \in N$. Then, Q, S, R, T satisfy all condition of above theorem (3.3.7) and have a unique common fixed point at $x = 1$.

In the Theorem (3.3.7), if we take $Q = S, T = R$, then we have

Corollary 3.3.1. *Let Q and R be self maps in complete Menger space (X, F, t) with continuous $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ satisfying following conditions $Q, R : X \to X$ be mappings such that*

(i) $Q(X) \subset R(X)$,

(ii) the pairs (Q, R) be compatible mappings of type (P),

(iii) R be continuous, and

(iv) there exists a constant $k \in (0, 1)$ such that

$$F_{Qx,Qy}(kt) \geq \min\{(F_{Rx,Qx}(t), F_{Ry,Qy}(t), F_{Ry,Qx}(\alpha t), F_{Rx,Qy}(2-\alpha)t), F_{Rx,Ry}(t)\}$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$.

Then, Q and R have a unique common fixed point in X.

In the Theorem (3.3.7), if we take $S = Q = g$, $R = T = I_K$, an identity mapping on X. Then we have following result:

Corollary 3.3.2. *Let g be self maps in complete Menger space (X, F, t) with continuous $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ if there exists a constant $k \in (0, 1)$ such that*

$$F_{gx,gy}(kt) \geq F_{x,y}(t) = \min\{(F_{x,gx}(t), F_{y,gy}(t), F_{y,gx}(\alpha t), F_{x,gy}(2-\alpha)t), F_{x,y}(t)\}$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$.
Then, Q and R have a unique common fixed point in X.

Which is the probabilistic version of the Banach contraction theorem established by V. M. Sehgal and A. T. Bharucha Reid [185] in 1972.
Our result theorem (3.3.7) extends and generalizes the results of G. Jungck [96], [101] in metric space. Also, this result improves other similar results in literature.
This result have been published in International Journal of Mathematical Sciences and Engineering Applications (IJMSEA), Vol. 15 (2) (2021), pp. 59-70.

3.3.2 Common fixed point theorem in Menger space with weakly compatible mapping of type (P)

We introduce another notion weakly compatible mapping of type (P) and establish common fixed point theorem in complete Menger space. We also introduce the following propositions for the establishment of our main result in the Menger space.

Proposition 3.3.3. *Let (X, F, t) be a Menger Space such that the $T-$norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Q, R : X \to X$ be continuous mappings. Then, Q and R also written as (Q, R), are weakly compatible mappings of type (P) if they are compatible mappings of type(P).*

Proof. Suppose Q and R be compatible mappings of type (P) and $\{k_n\}$ be a sequence in X. Then, we have, $1 = \lim_{n\to\infty} F_{QQk_n,\, RRk_n}(x) \geq F_{Qk_n,Rk_n}(x)$.
So, (Q, R) be weakly compatible mappings of type(P). □

Proposition 3.3.4. *Let (X, F, t) be a Menger space such that the $T-$ norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Q, R : X \to X$ be continuous mappings. Then, Q and R are compatible mappings of type (P) if they are weakly compatible mappings of type (P).*

Proof. Let $\{k_n\}$ be a sequence in X and since Q and R be continuous mappings. Then, by Theorem (3.3.6), we have $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ for some k in K. If Q and R are weakly compatible mappings of type (P).
Then, we have $\lim_{n\to\infty} F_{QQk_n,RRk_n}(x) \geq F_{Qk_n,Rk_n}(x) = F_{k,k}(x) = 1$, for all $x > 0$.
So, (Q, R) be compatible mappings of type (P). □

Proposition 3.3.5. *Let (X, F, t) be a Menger space such that the $T-$norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Q, R : X \to X$ be mappings. If Q and R are weakly compatible mappings of type (P) and $Qk = Rk$ for some $k \in X$ then, $QQk = QRk = RQk = RRk$.*

Proof. Suppose $\{k_n\}$ is a sequence in X defined by $k_n = k$ where $n = 1, 2, 3,$ for some $k \in X$ and $Qk = Rk$. Then, we have $Qk_n, Rk_n \to$ Qk as $n \to \infty$. Since Q and R are weakly compatible mappings of type (P), then for every $\epsilon > O$

$$F_{QQk,RRk}(\epsilon) = lim_{n\to\infty}\ F_{QQk_n,RRk_n}(\epsilon) \geq F_{Qk_n,Rk_n}(\epsilon)$$
$$= F_{Qk,Rk}(\epsilon) = F_{Qk,Qk}(\epsilon) = 1.$$

So, $QQk = RRk$, since $Qk = Rk$ implies $QQk = QRk = RQk = RRk$. □

Proposition 3.3.6. *Let (X, F, t) be a Manger space such that the $T-norm$ t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$ and $Q, R : X \to X$ be mappings. Let Q and R be weakly compatible mappings of type (P) and $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ for some k in X. Then We have,*

(i) $\lim_{n\to\infty} RRk_n = Qk$, *if Q is continuous at k,*

(ii) $\lim_{n\to\infty} QQk_n = Rk$, *if R is continuous at k,*

(iii) $QRk = RQk$ *and* $Qk = Rk$ *if Q and R are continuous at k.*

Proof. **(i)** Suppose that Q is continuous at k. Since, we have $\lim_{n\to\infty} Qk_n = \lim_{n\to\infty} Rk_n = k$ for some k in K. So, $\lim_{n\to\infty} QQk_n = Qk$, as $n\to\infty$. Again, since Q and R are weakly compatible mappings of type (P), So for every $\epsilon > 0$, $F_{QQk_n,\ RRk_n}(\epsilon) \geq F_{Qk_n,Rk_n}(\epsilon)$. Therefore, we have

$F_{RRk_n,Qk}(\epsilon) \geq t(F_{RRk_n,QQk_n}(\frac{\epsilon}{2}), F_{QQk_n,Qk}(\frac{\epsilon}{2}))$, by definition of Menger space

or, $F_{RRk_n,Qk}(\epsilon) \geq t(F_{Qk_n,Rk_n}(\frac{\epsilon}{2}), F_{Qk,Qk}(\frac{\epsilon}{2}))$,

$$\geq t(F_{k,k}(\tfrac{\epsilon}{2}), 1)),$$
$$\geq t(1, 1))$$

This implies that $F_{RRk_n,Qk}(\epsilon) = 1$. So, $\lim_{n\to\infty} RRk_n = Qk$.

(ii) We may prove (ii) as we prove (i)

(iii) Suppose that $Q, RX \to X$ are continuous at k. So, by (i) , $RRk_n \to Qk$ as $n \to \infty$.

On the other hand, since $\lim_{n\to\infty} Qk_n \to k$, as $n \to \infty$ and R is continuous at k. So, by proposition (3.3.6) (ii), we get $\lim_{n\to\infty} QQk_n = Rk$. Thus, we have $Qk = Rk$ by the uniqueness of the limit and so by preposition (3.3.5), we get $QRk = RQk$.

This completes the proof. □

Now, we prove our main theorem for weakly compatible mappings of type (P) in complete Menger space:

Theorem 3.3.8. *Let (X, F, t) be a complete Menger space with $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ and $Q, S, R, T : X \to X$ be mappings such that*

(1) $Q(X) \subseteq T(X)$ and $S(X) \subseteq R(X)$,

(2) the pairs (Q, R) and (S, T) are weakly compatible mappings of type (P),

(3) One of Q, S, R, T be continuous, and

(4) there exists a constant $k \in (0, 1)$ such that

$$F_{Qx,Sy}(kt) \geq \min\{(F_{Rx,Qx}(t), F_{Ty,Sy}(t), F_{Ty,Qx}(\alpha t), F_{Rx,Sy}((2-\alpha)t), F_{Rx,Ty}(t)\}$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$.

Then, Q, S, R, T have a unique common fixed point in X.

Proof. Consider $u_0 \in X$. Since $Q(X) \subseteq T(X)$, so there exists a point u_1 in X such that $Qu_0 = Tu_1 = v_0$. Again, since $S(X) \subseteq R(X)$, so for u_1, we may choose u_2 in X such that $Su_1 = Ru_2 = v_1$ and so on.

And inductively, we may construct sequence $\{u_n\}$ and $\{v_n\}$ in X such that

$Qu_{2n} = Tu_{2n+1} = v_{2n}$,

and $Su_{2n+1} = Ru_{2n+2} = v_{2n+1}$, for $n = 0, 1, 2, ...$

Putting $x = u_{2n}$ and $y = u_{2n+1}$ for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$ in theorem (3.3.8) condition (4). We get

$$F_{Qu_{2n}, Su_{2n+1}}(kt) \geq \min\{(F_{Ru_{2n}, Qu_{2n}}(t), F_{Tu_{2n+1}, Su_{2n+1}}(t), F_{Tu_{2n+1}, Qu_{2n}}((1-q)t), F_{Ru_{2n}, Su_{2n+1}}(1+q)t), F_{Ru_{2n}, Tu_{2n+1}}(t)\}$$

or, $F_{v_{2n}, v_{2n+1}}(kt) \geq \min\{(F_{v_{2n-1}, v_{2n}}(t), F_{v_{2n}, v_{2n+1}}(t), F_{v_{2n}, v_{2n}}((1-q)t), F_{v_{2n-1}, v_{2n+1}}(1+q)t), F_{v_{2n-1}, v_{2n}}(t)\}$

$$F_{v_{2n},v_{2n+1}}(kt) \geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), 1, F_{v_{2n-1},v_{2n+1}}(1+q)t), F_{v_{2n-1},v_{2n}}(t)\}$$

$$\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(qt), F_{v_{2n-1},v_{2n}}(t)\}$$

$$\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n+1}}(qt)\}$$

As $q \to 1$, we obtain

$$F_{v_{2n},v_{2n+1}}(kt) \geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t), F_{v_{2n},v_{2n+1}}(t)\}$$

$$\geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t)\}$$

Hence, we get $F_{v_{2n},v_{2n+1}}(kt) \geq \min\{(F_{v_{2n-1},v_{2n}}(t), F_{v_{2n},v_{2n+1}}(t)\}$

i.e. $F_{v_{2n},v_{2n+1}}(kt) \geq F_{v_{2n-1},v_{2n}}(t)$

Similarly, we obtain

$F_{v_{2n+1},v_{2n+2}}(kt) \geq F_{v_{2n},v_{2n+1}}(t)$

Therefore, for every $n \in N$, $F_{v_n,v_{n+1}}(kt) \geq F_{v_{n-1},v_n}(t)$.

So, using Lemma (3.3.2), $\{v_n\}$ is a Cauchy sequence in K. Since the Menger space (X, F, t) is complete, so $\{v_n\}$ converges to a point z in X and consequently the sub sequences $\{Qu_{2n}\}, \{Su_{2n+1}\}, \{Ru_{2n}\}, \{Tu_{2n+1}\}$, of $\{v_n\}$ also converges to z.

Now, suppose that T is continuous. Then, since S and T are weakly compatible mappings of type (P) then by proposition (3.3.6), SSu_{2n+1}, $TSu_{2n+1} \to Tz$ as $n \to \infty$. Putting $x = u_{2n}$ and $y = Su_{2n+1}$ in theorem

(3.3.8) condition (4), we get

$$F_{Qu_{2n},\ SSu_{2n+1}}(kt) \geq \min\{(F_{Ru_{2n},Qu_{2n}}(t), F_{TSu_{2n+1},SSu_{2n+1}}(t), F_{TSu_{2n+1},Qu_{2n}}((\alpha)t),$$

$$F_{Ru_{2n},SSu_{2n+1}}(2-\alpha)t), F_{Ru_{2n},TSu_{2n+1}}(t)\}$$

Taking $n \to \infty$, we have

$$F_{z,Tz}(kt) \geq \min\{(F_{z,z}(t), F_{Tz,Tz}(t), F_{Tz,z}((\alpha)t), F_{z,Tz}(2-\alpha)t), F_{z,Tz}(t)\}$$

Letting $\alpha = 1-q$ with $q \in (0,1)$ then

$F_{z,Tz}(kt) \geq \min\{(F_{Tz,z}((1-q)t), F_{z,Tz}((1+q)t), F_{z,Tz}(t)\}$

or, $F_{z,Tz}(kt) \geq \min\{(F_{Tz,Tz}((1-q+1+q)t), F_{z,Tz}(t)\}$

$\geq \min\{F_{z,Tz}(t)\}$

Therefore, $F_{z,Tz}(kt) \geq F_{z,Tz}(t)$

which implies $z = Tz$ by Lemma (3.3.1)

Similarly, replacing x by u_{2n} and y by z in condition (4), we have

$F_{Qu_{2n},Sz}(kt) \geq \min\{(F_{Ru_{2n},Qu_{2n}}(t), F_{Tz,Sz}(t), F_{Tz,Qu_{2n}}((\alpha)t), F_{Ru_{2n},Sz}(2-\alpha)t), F_{Ru_{2n},Tz}(t)\}$

Taking $n \to \infty$, we get

$F_{z,Sz}(kt) \geq \min\{(F_{z,z}(t), F_{z,Sz}(t), F_{z,z}((\alpha)t), F_{z,Sz}(2-\alpha)t), F_{z,z}(t)\}$

$F_{z,Sz}(kt) \geq \min\{(F_{z,Sz}(t), , F_{z,Sz}(2-(1-q)t)\}$

$\geq \min\{(F_{z,Sz}(t), , F_{z,Sz}(1+q)t)\}$

$\geq \min\{(F_{z,Sz}(t), F_{z,z}(t), F_{z,Sz}(qt)\}$

$\geq \min\{(F_{z,Sz}(t), F_{z,Sz}(t)\}$ as $q \to 1$

So that $F_{z,Sz}(kt) \geq \{(F_{z,Sz}(t)\}$

which implies $z = Sz$. Since, $S(K) \subseteq R(K)$, so there exists a point w in K such that $Sz = Rw = z$.

By using theorem (3.3.8) condition (4) with $x = w, y = z$, we have

$F_{Qw,z}(kt) \geq \min\{(F_{Rw,Qw}(t), F_{Tz,Sz}(t), F_{Tz,Qw}(\alpha t), F_{Rw,Sz}(2-\alpha)t), F_{Rw,Tz}(t)\}$

$$\geq \min\{(F_{z,Qw}(t), F_{Tz,z}(t), F_{z,Qw}((1-q)t), F_{Rw,z}(1+q)t), F_{z,Tz}(t)\}$$

$$\geq \min\{(F_{z,Qw}(t), F_{Tz,z}(t), F_{Qw,z}((1-q)t), F_{Rw,z}(1+q)t), F_{z,Tz}(t)\}$$

$$\geq \min\{(F_{z,Qw}(t), F_{z,z}(t), F_{Qw,Rw}((1-q+1+q)t)\}$$

$$\geq \min\{(F_{z,Qw}(t), F_{Qw,z}(2t)\}$$

Therefore, $F_{Qw,z}(kt) \geq F_{z,Qw}(t) = F_{Qw,z}(t)$

which implies $Qw = z$, by Lemma (3.3.1)

Again, since Q and R are weakly compatible mappings of type (P) and $Qw = Rw = z$, by proposition (3.3.5), we have for every $\epsilon > 0$

$1 = F_{QQw,RRw}(\epsilon) \geq F_{Qw,Rw}(\epsilon)$

Hence, $Qw = QQw = RRw = Rw$.

Finally, by theorem (3.3.8)condition (4) with $x = z$, $y = Sz = z$, we have

$F_{Qz,z}(kt) = F_{Qz,Sz}(kt) \geq \min\{(F_{Rz,Qz}(t), F_{Tz,z}(t), F_{Tz,Qz}(\alpha t), F_{Rz,z}(2-\alpha)t), F_{Rz,Tz}(t)\}$

$$\geq \min\{(F_{Qz,Qz}(t), F_{z,z}(t), F_{z,Qz}(\alpha t), F_{Qz,z}(2-\alpha)t), F_{Qz,z}(t)\}$$

$$\geq \min\{(F_{Qz,z}(\alpha t), F_{z,Qz}(2-\alpha)t), F_{Qz,z}(t)\}$$

$$\geq \min\{(F_{Qz,Qz}(\alpha t + 2t - \alpha t), F_{Qz,z}(t)\}$$

or, $F_{Qz,z}(kt) \geq F_{Qz,z}(t)$

Therefore, $Qz = z$.

Hence, $Qz = Sz = Rz = Tz = z$. That is, z is common fixed point of given mappings Q, S, R and T.

Uniqueness: Suppose z_1 be other point in X such that $z_1 = Qz_1 = Sz_1 = Rz_1 = Tz_1$. Then, putting $x = z$ and $y = z_1, \alpha = 1$ in (4), we get

$F_{Qz,Sz_1}(kt) = F_{z,z_1}(kt) \geq \min\{(F_{Rz,Qz}(t), F_{Tz_1,Sz_1}(t), F_{Tz_1,Qz}(t), F_{Rz,Sz_1}(t), F_{Rz,Tz_1}(t)\}$

or, $F_{z,z_1}(kt) \geq \min\{(F_{z,z_1}(t), F_{z,z}(t)\}$

or, $F_{z,z_1}(kt) \geq F_{z,z_1}(t)$

By Lemma (3.3.1), $z = z_1$. Hence, $z = Qz = Sz = Rz = Tz$ and z is unique common fixed point for Q, S, R, T in X.
This completes the proof. □

We verify our above main theorem with following example:

Example 3.3.2. *Let (X, d) be metric space where $X = [0, 2]$ with usual metric $d(x, y) = |x - y|$ and distribution function F is defined by*

$$F_{x,y}(t) = \begin{cases} e^{\frac{d(x,y)}{t}} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

for all $x, y \in X$. Let $Q, R, S, T : X \to X$ be defined by

$$Q(X) = \begin{cases} 1 - x & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 2] \end{cases}$$

,

$$S(X) = \begin{cases} \frac{1}{2} - x & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 2] \end{cases}$$

$$R(X) = \begin{cases} x & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 2] \end{cases}$$

and

$$T(X) = \begin{cases} 0 & \text{for } x \in [0, \frac{1}{2}) \\ 1 & \text{for } x \in [\frac{1}{2}, 2] \end{cases}$$

we take sequence $\{k_n\}$ in X where $k_n = \frac{1}{2} - \frac{1}{n}$, $n \in N$
Then Q, S, R and T satisfy all the conditions of the above Theorem (3.3.8) and have a unique common fixed point $x = 1$ in K.

Our established Theorem (3.3.8) may apply in consequences results in metric space in four self mappings and also may use to prove following corollaries: In the Theorem (3.3.8), if we take $Q = S$,$T = R$, then we have

Corollary 3.3.3. *Let (X, F, t) be a complete Menger space with continuous $t(x, y) = min(x, y)$ for all $x, y \in [0, 1]$ and $Q, R : X \to X$ be mappings such that*

(1) $Q(X) \subseteq R(X)$,

(2) the pairs (Q, R) be weakly compatible mappings of type (P),

(3) R be continuous, and

(4) there exists a constant $k \in (0, 1)$ such that

$$F_{Qx,Qy}(kt) \geq \min\{(F_{Rx,Qx}(t), F_{Ry,Qy}(t), F_{Ry,Qx}(\alpha t), F_{Rx,Qy}(2-\alpha)t), F_{Rx,Ry}(t)\}$$

for all $x, y \in K, \alpha \in (0, 2)$ and $t > 0$.

Then, Q and R have a unique common fixed point in X.

Our result Theorem (3.3.8) extends and generalize the results of G. Jungck, K. B. Moon, S. Park and B. E. Rhodes [100], G. Jungck, P. P. Murthy, Y. J. Cho [101] and of A. K. Chaudhary, K. B. Manandhar, K. Jha and P. P. Murthy [40]. This result also generalizes and improve the result of H. K. Pathak, Y. J. Cho, S. S. Chang and S. M. Kang [154], Y. J. Cho, P. P. Murthy, and M. Stojavic [45] and other similar results in literature.

The above result have been published in Advances in Mathematics: Scientific Journal 11 (2022), no.11, 1019-1031.
(https://doi.org/10.37418/amsj.11.11.3)

Chapter 4

Some common fixed point theorems for occasionally weakly compatible mappings in Menger space

In this chapter, we have studied some common fixed point theorems for occasionally weakly compatible mappings in Menger space and established common fixed point theorem in two pairs and in three pairs of self mappings in this space.

4.1 Introduction

In 2008, Al-Thagafi and N. Shahzad [6] introduced the notion of occasionally weakly compatible mappings (shortly owc) in metric spaces, while H. Chandra and A. Bhatt [29] extended the notion of owc in probabilistic settings. It is worth mentioning that every pair of weakly commuting self-mappings is compatible, each pair of compatible self-mappings is weakly

compatible and each pair of weak compatible self-mappings is owc but the reverse is not always true. And the notion of occasionally weakly compatible mappings is more general than all the commutativity concepts. In historical development, it is seen that S. Sessa [187] weakened the condition of commutativity to that of pairwise weakly commuting. Jungck generalized the notion of weak commutativity to that of pairwise compatible and then pairwise weakly compatible mappings. G. Jungck and B. E. Rhodes [103] introduced the concept of occasionally weakly compatible mappings. Also, M. Abbas and B. E. Rhodes [2] generalized the concept of weak compatibility in the setting of single and multi-valued maps by introducing the notion of owc. Many authors proved a number of fixed point theorems using the notion of owc mappings on different spaces (we refer for references [2, 8, 6, 11, 18, 42, 50, 105, 142, 143, 148, 209]).
Here, we have established common fixed point results in Menger space for two pairs and three pairs of self-mappings using occasionally weakly compatible mappings and verified it by examples.

4.2 Basic Definitions

Definition 4.2.1. *[200] Let $Q, R : X \to X$ be two self-mappings in Menger Space (X, F, t). Then, $t \in X$ is said to be a coincidence point of Q and R iff $Qt = Rt = w$ for some $t \in X$. And $w \in X$ is called a point of coincidence of Q and R.*

In 2005, B. Singh and S. Jain introduced coincidentally commuting mapping as:

Definition 4.2.2. *[196] Two mappings $Q, R : X \to X$ are said to be* ***weakly compatible mappings or coincidentally commuting*** *in Menger space*

(X, F, t) if they commute at their coincidence points i.e. $Qx = Rx$ for some $x \in X$, then $QRx = RQx$.

Example 4.2.1. *[39] Let $X = [0, 5]$ equipped with the usual metric $d(x, y) = |x - y|$ and define self-maps $Q, R : [0, 5] \to [0, 5]$ by*

$$Q(x) = \begin{cases} 5 - x, & \text{if } 0 \leq x < 3, \\ 5, & \text{if } 3 \leq x \leq 5, \end{cases}$$

and

$$R(x) = \begin{cases} x, & \text{if } 0 \leq x < 3, \\ 5, & \text{if } 3 \leq x \leq 5, \end{cases}$$

Then, we see that for any $x \in [3, 5]$, x is a coincidence point and $QRx = RQx$, showing Q and R are weakly compatible.

In 2008, M.A. AI-Thapagi and N. Shahzad [6] had given the concept of occasionally weakly compatible mappings and introduced it in metric space and extended it in Menger space by S. Chauhan, S. Kumar and B.D. Pant [42] as:

Definition 4.2.3. *[42] Two self-mappings $Q, R : X \to X$ are said to be* ***occasionally weakly compatible mapping*** *(shortly owc) in Menger space (X, F, t) iff there is a point t in K which is a coincidence point of Q and R at which Q and R commute.*

It is remarked that every weakly compatible mappings is occasionally weakly compatible mappings but the converse is not true.

Example 4.2.2. *Let (X, F, t) be Menger space, where $X = \mathbb{R}$ and F is defined by*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t + |x - y|} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

Define two self-mappings Q and R by $Q(x) = 2x$ and $R(x) = x^2$ for all $x \in \mathbb{R}$. We see here that $Qx = Rx$ for $0, 2$. And $QR0 = RQ0$ but $QR2 \neq RQ2$. So, Q and R are not weakly compatible but occasionally weakly compatible.

4.3 Some common fixed point theorem in Menger probabilistic metric space

In 2008, Al-Thagafi and N. Shahzad [6] proved the following theorem:

Theorem 4.3.1. *[6] Let D be a subset of a metric space (X, d), I and T be self mappings of D , $\overline{T(D)} \subseteq I(D)$, and $\overline{T(D)}$ is complete. Suppose that I and T are occasionally weakly compatible and $d(Tx, Ty) \leq k\ \psi_{I,T}(x, y)$ for all $x, y \in D$ and some $k \in [0, 1)$. Then I and T have a unique common fixed point.*

In 2012, R. P. Pant and R. K. Bisht established following theorem in Metric space:

Theorem 4.3.2. *[147] Let I and T be occasionally weakly compatible self-mappings of a metric space (X, d) satisfying*
$d(Tx, T^2x) \neq max\{d(Ix, ITx), d(Ix, Tx), d(ITx, TTx), d(Ix, TTx), d(ITx, Tx)\}$,
whenever $Tx \neq T^2x$. Then I and T have a common fixed point.

In 2019, V. Srinivas and S. Ravi established following theorem in metric space:

Theorem 4.3.3. *[202] Let $P, Q, S,$ and T be self- mappings from a complete metric space (X, d) into itself satisfying the following conditions:*

(i) $S(X) \subset Q(X)$ *and* $T(X) \subset P(X)$,

(ii) $(S(x), T(y) \leq \alpha \dfrac{d(Q(y), T(y)[1 + d(P(x), S(x))]}{1 + d(P(x), Q(y))} + \beta\, d(P(x), Q(y))$
for all $x, y \in X$ where $\alpha, \beta \geq 0$, $\alpha + \beta < 1$.

(iii) *one of* $P, Q, S,$ *and* T *be continuous, and*

(iv) *the pairs* (S, P) *and* (T, Q) *are occasionally weakly compatible mappings on* X*.*
Further associated sequence relative to four self mappings $P, Q, S,$ *and* T *such that the sequence* $Sx_0, Tx_1, Sx_2, Tx_3, \cdots, Sx_{2n}, Tx_{2n+1}, \cdots$ *converges to* z *as* $n \to \infty$*.*

Then, $P, Q, S,$ *and* T *have a unique common fixed point in* X*.*

In 2011, B. D. Pant and S. Chauhan [143] established following theorem in Menger space:

Theorem 4.3.4. *[143] Let* (X, F, t) *be Menger space. Further let* (Q, R) *and* (S, T) *are occasionally weakly compatible mappings on* X *satisfying*
$min\{F_{Qx,Sy}(kt), F_{Ty,Qx}(kt)\} + \gamma F_{Ty,Sy}(kt) \geq [\alpha F_{Rx,Qx}(t) + \beta F_{Rx,Ty}(t)]$
for all $x, y \in X$*,* $k \in (0, 1)$ *and* $t > 0$ *where* $0 < \alpha$ *,*$\beta < 1$*, and* $0 < \gamma < 1$ *such that* $\alpha + \beta - \gamma = 1$*. Then,* $Q, R, S,$ *and* T *have a unique common fixed point in* X*.*

In 2012, The following theorem established by S. Kumar and A. Rani [200]:

Theorem 4.3.5. *[200] Let* (X, F, t) *be complete Menger space and* $A, B, S,$ *and* T *be self mappings of* X*. Let the pairs* (A, S) *and* (B, T) *are occasionally weakly compatible mappings and* $k \in (0, 1)$ *such that*
$F_{Ax,By}(kt) \geq \min\{(F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Sx,By}(2t), F_{Ax,Ty}(t)\}$
for all $x, y \in X$*, and* $t > 0$ *there exists a unique point* $w \in X$ *such that* $Aw = Sw = w$ *and a unique point* $z \in X$ *such that* $Bz = Tz = z$*. Moreover,* $z = w$*, so that there is a unique common fixed point of* $A, B, S,$ *and* T*.*

In 2013, S. Dalal established following theorem:

Theorem 4.3.6. *[52] Let* (X, F, t) *be Menger space with* $A, B : X \to X$ *and* $S, T : X \to \kappa(X)$ *be mappings satisfying the followings* *(i) the pairs (A, S) and (B, T) are occasionally weakly compatible mappings,*

(ii) $\gamma\big(\int_0^{F^{\triangle}(Ty,By,t)} \psi(t)dt\big)^p + w\Big\{\big(\int_0^{F^{\triangle}(Ax,By,t)} \psi(t)dt\big)^p, \big(\int_0^{F^{\triangle}(Ty,By,t)} \psi(t)dt\big)^p, \big(\int_0^{F^{\triangle}(Ty,Ax,t)} \psi(t)dt\big)^p, \big(\int_0^{F^{\triangle}(Sx,Ax,t)} \psi(t)dt\big)^p, \big(\int_0^{F\triangledown(Sx,Ty,t)} \psi(t)dt\big)^p\Big\},$
$\geq \alpha\big(\int_0^{F^{\triangle}(Sx,Ax,t)} \psi(t)dt\big)^p + \beta\big(\int_0^{F\triangledown(Sx,Ty,t)} \psi(t)dt\big)^p$

for all $x, y \in X$, $0 < \alpha$, $\beta < 1$, and $0 < \gamma < 1$ such that $\alpha + \beta - \gamma = 1$ and $\psi : \mathbb{R}^+ \to \mathbb{R}$ is a Lebseque- Integrable mapping which is summable, non-negative and such that $\int_0^{F^\epsilon} \psi(t)dt > 0$ for each $\epsilon > 0$. Then, $A, B, S,$ and T have a unique common fixed point in X.

4.3.1 Some common fixed point theorems for occasionally weakly compatible mappings in Menger space

The followings lemmas help us to prove main theorems:

Lemma 4.3.1. *[42] Let Q and R be occasionally weakly compatible self-mappings of Menger space (K, F, t). If Q and R have a unique point of coincidence, $w = Qt = Rt$ then w is a unique common fixed point of Q and R.*

Lemma 4.3.2. *[196] Let (K, F, t) be a Menger space. If there exists $k \in (0, 1)$ such that for all $u, v \in K$, $F_{u,v}(kx) \geq F_{u,v}(x)$ then $u = v$.*

Now we prove our main theorem for four self-mappings by using occasionally weakly compatible mappings in Menger Space:

Theorem 4.3.7. *Let (K, F, t) be complete Menger space and Q, S, R, T: $K \to K$ be four mappings. Let (Q, R) and (S, T) be occasionally weakly compatible mappings. And if there exists a constant $k \in (0, 1)$ such that*

(1) $[F_{Qx,\,Sy}(kt)]^2 \geq c_1 \min\{[F_{Rx,\,Qx}(t)]^2, [F_{Ty,\,Sy}(t)]^2, [F_{Rx,\,Ty}(t)]^2\}$
$+c_2 \min\{F_{Rx,\,Qx}(t)F_{Rx,\,Sy}(t)F_{Qx,\,Ty}(t)F_{Sy,\,Ty}(t)\}+c_3F_{Sy,\,Rx}(t)F_{Ty,\,Qx}(t)$

for all x, $y \in K$, and $t > 0$, c_1, c_2, $c_3 > 0$ and $c_1 + c_2 + c_3 > 1$. Then there exists a unique point $w \in K$ such that $Qw = Rw = w$ and a unique point $k \in K$ such that $Sk = Tk = k$. Moreover, $k = w$ so that there is a unique common fixed point of Q, S, R and T.

Proof. Consider (Q, R) and (S, T) be occasionally weakly compatible mappings. Then there exist $x, y \in K$ such that $Qx = Rx$ and $Sy = Ty$.
We claim $Qx = Sy$.
Suppose, $Qx \neq Sy$ then by Theorem (4.3.7) condition (1)

$$[F_{Qx,Sy}(kt)]^2 \geq c_1 \min\{[F_{Qx,Qx}(t)]^2, [F_{Sy,Sy}(t)]^2, [F_{Qx,Sy}(t)]^2\}$$
$$+ c_2 \min\{F_{Qx,Qx}(t)F_{Qx,Sy}(t)F_{Qx,Sy}(t)F_{Sy,Sy}(t)\} + c_3F_{Sy,Qx}(t)F_{Sy,Qx}(t)$$

or, $$[F_{Qx,Sy}(kt)]^2 \geq c_1 \min\{1, 1, [F_{Qx,Sy}(t)]^2\}$$
$$+ c_2 \min\{1.F_{Qx,Sy}(t)F_{Qx,Sy}(t).1\} + c_3F_{Sy,Qx}(t)F_{Sy,Qx}(t)$$

or, $$[F_{Qx,Sy}(kt)]^2 \geq c_1[F_{Qx,Sy}(t)]^2 + c_2[F_{Qx,Sy}(t)]^2 + c_3[F_{Qx,Sy}(t)]^2$$

or, $$[F_{Qx,Sy}(kt)]^2 \geq (c_1 + c_2 + c_3)[F_{Qx,Sy}(t)]^2$$

Which is a contradiction because $c_1 + c_2 + c_3 > 1$.
So, by Lemma (4.3.2), we have
$Qx = Sy$, that is $Qx = Rx = Sy = Ty$. Let z be another point such that $Qz = Rz$,then by theorem (4.3.7) condition (1) $Qz = Rz = Sy = Ty$. So, $Qx = Qz$ and $w = Qx = Rx$ is the unique point of coincidence of Q and R. By Lemma (4.3.1), w is the only common fixed point of Q and R. That is $w = Qw = Rw$. Similarly, there is a unique point $z \in K$ such that $z = Sz = Tz$.

Assume that $w \neq z$. Then by theorem (4.3.7) condition (1), we have

$$[F_{w,z}(kt)]^2 = [F_{Qw,Sz}(kt)]^2 \geq c_1 \min\{[F_{Rw,Qw}(t)]^2, [F_{Tz,Sz}(t)]^2, [F_{Rw,Tz}(t)]^2\}$$
$$+ c_2 \min\{F_{Rw,Qw}(t)F_{Rw,Sz}(t)F_{Qw,Tz}(t)F_{Sz,Tz}(t)\}$$
$$+ c_3 F_{Sz,Rw}(t)F_{Tz,Qw}(t)$$

$$\text{or, } [F_{w,z}(kt)]^2 = [F_{Qw,Sz}(kt)]^2 \geq c_1 \min\{[F_{w,w}(t)]^2, [F_{z,z}(t)]^2, [F_{w,z}(t)]^2\}$$
$$+ c_2 \min\{F_{w,w}(t)F_{w,z}(t)F_{w,z}(t)F_{z,z}(t)\}$$
$$+ c_3 F_{z,w}(t)F_{z,w}(t)$$

$$[F_{w,z}(kt)]^2 = [F_{Qw,Sz}(kt)]^2 \geq c_1 \min\{1, 1, [F_{w,z}(t)]^2\} + c_2 \min\{1.[F_{w,z}(t)]^2.1\}$$
$$+ c_3[F_{w,z}(t)]^2$$
$$\geq c_1 \min\{[F_{w,z}(t)]^2\} + c_2 \min\{[F_{w,z}(t)]^2\} + c_3[F_{w,z}(t)]^2$$
$$\geq (c_1 + c_2 + c_3)[F_{w,z}(t)]^2$$

Which is a contradiction because $c_1 + c_2 + c_3 > 1$.
Therefore, by Lemma (4.3.2) $z = w$ and z is a common fixed point of Q, S, R and T.

Uniqueness:

Let z_1 ($z_1 \neq z$) be another common fixed point of Q, S, R and T.
Taking $x = z$ and $y = z_1$ then from (1), we have
$[F_{Qz,Sz_1}(kt)]^2 \geq c_1 \min\{[F_{Rz,Qz}(t)]^2, [F_{Tz_1,Sz_1}(t)]^2, [F_{Rz,Tz_1}(t)]^2\}$
$+ c_2 \min\{F_{Rz,Qz}(t)F_{Rz,Sz_1}(t)F_{Qz,Tz_1}(t)F_{Sz_1,Tz_1}(t)\} + c_3 F_{Sz_1,Rz}(t)F_{Tz_1,Qz}(t)$

On simplification, $[F_{z,z_1}(kt)]^2 \geq (c_1 + c_2 + c_3)[F_{z,z_1}(t)]^2$
Which is a contradiction because $c_1 + c_2 + c_3 > 1$. Thus, by Lemma (4.3.2), $z = z_1$ and hence uniqueness of common fixed point. □

The following example verify our main Theorem (4.3.7)

Example 4.3.1. *Let* $K = [0, 20]$ *with metric* d *defined by* $d(x, y) = |x - y|$ *and* F *is defined by*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

for all $x, y \in K$. *Then,* (K, F, t) *be a complete Menger Space with* $t(a, b) = min(a, b)$ *for all* $a, b \in [0, 1]$. *Let* $Q, R, S, T : K \to K$ *be defined by*

$$Q(K) = \begin{cases} 0 & \text{for } x = 0 \\ 4 & \text{for } 0 < x \leq 20 \end{cases}$$

,

$$S(K) = \begin{cases} 0 & \text{for } x = 0 \\ 7 & \text{for } 0 < x \leq 20 \end{cases}$$

$$R(K) = \begin{cases} 0 & \text{for } x = 0 \\ 11 - x & \text{for } 0 < x \leq 11 \\ x - 7 & \text{for } 11 < x \leq 20 \end{cases}$$

and

$$T(K) = \begin{cases} 0 & \text{for } x = 0 \\ 11 - x & \text{for } 0 < x \leq 11 \\ x - 4 & \text{for } 11 < x \leq 20 \end{cases}$$

Then Q, S, R *and* T *satisfy all the conditions of the above Theorem* (4.3.7) *with* $k \in (0, 1)$ *and have a unique common fixed point* $x = 0$ *in* K. *That is,* $Q(0) = 0 = R(0)$,$QR(0) = 0 = RQ(0)$. *And* $S(0) = 0 = T(0)$, $ST(0) = 0 = TS(0)$. *Also,* Q *and* S *as well as* R *and* T *are occasionally weakly* *compatible. Hence,* 0 *is unique common fixed point of* Q *,* S*,* R*, and* T*. Also, these maps are discontinuous at* $x = 0$*.*

If we put $Q = S$ and $R = T$ in Theorem (4.3.7) then, we obtain

Corollary 4.3.1. *Let* (K, F, t) *be Menger Space and* $Q, R : K \to K$ *be mappings. Let* (Q, R) *be occasionally weakly compatible mappings. And if there exist a constant* $k \in (0, 1)$ *such that*

$$[F_{Qx,Qy}(kt)]^2 \geq c_1 \min\{[F_{Rx,Qx}(t)]^2, [F_{Ry,Qy}(t)]^2, [F_{Rx,Ry}(t)]^2\}$$
$$+ c_2 \min\{F_{Rx,Qx}(t)F_{Rx,Qy}(t)F_{Qx,Ry}(t)F_{Qy,Ry}(t)\} + c_3 F_{Qy,Rx}(t)F_{Ry,Qx}(t)$$

for all $x, y \in K$*,and* $t > 0$ *,* c_1*,* c_2*,* $c_3 > 0$ *and* $c_1 + c_2 + c_3 > 1$*.*
Then, there is a unique common fixed point of Q *and* R*.*

Introducing our another result in Menger space in six self-mappings.

Theorem 4.3.8. *Let* (K, F, t) *be a complete Menger Space with* $t(x, y) = min(x, y)$ *for all* $x, y \in [0, 1]$ *and*
$P, Q, R, S, T, U : K \to K$ *be mappings such that*

(i) the pairs (PQ, T) *and* (RS, U) *are occasionally weak compatible,*

(ii) there exists a constant $k \in (0, 1)$*such that*

$$F_{PQx,RSy}(kt) \geq \min\{(F_{Tx,Uy}(t), \frac{1}{2}[F_{PQx,Tx}(t) + F_{RSy,Uy}(t)],$$
$$\frac{1}{2}[F_{PQx,Uy}(t) + F_{RSy,Tx}(t)]\}$$

for all $x, y \in K$ *and* $t > 0$*.*

Then, PQ*,* RS*,* T*,* U *have a unique common fixed point in* K*. Furthermore, if the pairs* (P, Q) *and* (R, S) *are commuting pairs of mappings then* P*,* Q*,* R*,* S*,* T*,* U *have a unique common fixed point.*

Proof. Here, (PQ, T) and (RS, U) are occasionally weak compatible. So, for all $x, y \in K$, we have $PQx = Tx$ and $RSy = Uy$. We claim $RSy = Uy$. From condition (ii), we have

$$F_{PQx,RSy}(kt) \geq \min\{(F_{PQx,RSy}(t), \frac{1}{2}[F_{PQx,PQx}(t) + F_{RSy,RSy}(t)], \frac{1}{2}[F_{PQx,RSy}(t) + F_{RSy,PQx}(t)]\}$$

$$\geq \min\{(F_{PQx,RSy}(t), 1, \frac{1}{2}[F_{PQx,RSy}(t) + F_{RSy,PQx}(t)]\}$$

$$\geq \min\{(F_{PQx,RSy}(t), 1, 1\}$$

$$\geq F_{PQx,RSy}(t)$$

From lemma (4.3.2), we get , $PQx = RSy$. So,

$$PQx = Tx = RSy = Uy \tag{4.1}$$

Moreover, if there is another coincidence point z such that $PQz = Tz$ then from condition (ii), we get

$$PQz = Tz = RSy = Uy \tag{4.2}$$

Also, from relations (4.1)and (4.2) it follows that
$PQx = PQz \implies z = x$.
Hence, $w = PQx = Tx$ for $w \in K$ is the unique point of coincidence of PQ and T.
By lemma (4.3.1), w is unique common fixed point of PQ and T.
Hence, $PQw = Tw = w$. Similarly, there is unique common fixed point $u \in K$ such that $u = RSu = Uu$.
Uniqueness: Suppose that $u \neq w$ then by condition (ii)
$F_{w,u}(kt) = F_{PQw,RSu}(kt) \geq \min\{(F_{Tw,Uu}(t), \frac{1}{2}[F_{PQw,Tw}(t) + F_{RSu,Uu}(t)], \frac{1}{2}[F_{PQw,Uu}(t) + F_{RSu,Tw}(t)]\}$

$\geq \min\{(F_{w,u}(t), \frac{1}{2}[F_{w,w}(t) + F_{u,u}(t)], \frac{1}{2}[F_{w,u}(t) + F_{u,w}(t)]\}$

$\geq \min\{(F_{w,u}(t), 1, \frac{1}{2}[F_{w,u}(t) + F_{w,u}(t)]\}$

$\geq \min\{(F_{w,u}(t), 1, F_{w,u}(t)\}$

$\geq F_{w,u}(t)$

By lemma (4.3.2), $w = u$.

Hence, w is unique common fixed point of PQ,RS,T, U. Finally, we have to show that w is only the common fixed point of P, Q, R, S, T and U. If the pairs $(P, Q)$and (R, S) are commuting pairs.

We may write, $Pw = P(PQw) = P(QPw) = PQ(Pw) \implies Pw = w$. Also, $Qw = Q(PQw) = QP(Qw) = PQ(Qw) \implies Qw = w$. Similarly, we have $Rw = w$ and $Sw = w$.

Hence, then P, Q, R, S, T and U have a unique common fixed point. □

As the consequences of Theorem (4.3.8), we have following theorem in metric space:

Theorem 4.3.9. *Let (K, d) be a complete metric space and P, R, T and U be self-mappings in K such that*

(i) (P, T) *and* (R, U) *are occasionally weakly compatible.*

(ii) $d(Px, Ry) \leq \phi(\max\{(d(Tx, Ty), \frac{1}{2}[d(Px, Tx) + d(Ry, Uy)], \frac{1}{2}[d(Px, Uy) + d(Ry, Tx)]\})$, *for all* $x, y \in K$.

Then, P, R, T and U have a unique common fixed point.

Our Theorem (4.3.7) generalizes the theorems of B. K. Sharma and N. K. Shahu [189] and K. Tas, M. Telci and B. Fisher [206]. And theorem (4.3.8) extends and generalize theorem K. Jha, M. Imdaad and U. Rajopadhayaya [91] and G. Jungck and B. E. Rhodes [105]. Our result also generalizes and improves other similar results in literature.

The above result have been published in Journal of Computer Science

and Applied Mathematics, Vol. 3, no.2, (2022), 6576.
(https://doi.org/10.37418/jcsam.3.2.3)

Conclusions:

In conclusions, we have introduced **three** new notions namely compatible mappings of type (K), compatible mappings of type (P), and weakly compatible mappings of type (P) in Menger probabilistic metric space. We have established **six** original common fixed-point theorems for four and six self-mappings in Menger pm space with the help of our notions and our propositions and have five corollaries. We have obtained **eleven** examples for the verification of new results.

Our results generalize, extend, and improve other similar results in the literature. Our work has definitely helped in the knowledge increment in the domain of fixed point theory and applications

Future Work:

The future research scope for further study in fixed point theory may be considered as follows:

- In this dissertation, compatible and weakly compatible mappings of type -(P), of type- (K), in Menger probabilistic metric space have been formulated. So, by using these compatible mappings, one can establish common fixed point theorem in another generalized spaces of metric space and also can be extended in other spaces too.
- The introduced compatible mappings can be used for sequence of self mappings in Menger space.
- To Study the interrelationship between various type of contractive mappings in generalized forms of probabilistic metric space.
- Our results can be used for further study for applications in different fields.

Articles in Peer-Reviewed Journals:

1. A common fixed point theorem in Menger space with compatible mapping of type (K),*Advances in Mathematics: Scientific Journal* , **11**(10) (2022), 883-892.
   https://doi.org/10.37418/amsj.11.10.6.

2. A common fixed point theorem in Menger space with weakly compatible mapping of type (P), *Advances in Mathematics: Scientific Journal* , **11**(11) (2022), 1019-1031.
   https://doi.org/10.37418/amsj.11.11.3.

3. Some common fixed point theorems for occasionally weakly compatible mappings in Menger space, *Journal of Computer Science and Applied Mathematics* , **3**(2) (2022), 65-76.
   https://doi.org/10.37418/jcsam.3.2.3.

4. A common fixed point theorem in Menger space with compatible mapping of type (P), *International Journal of Mathematical Sciences and Engineering Applications (IJMSEA)*, **15**(2) (2021), 59-70.

5. Contraction conditions in probabilistic metric spaces, *American Journal of Mathematics and Statistics*, **9**(5) (2019), 199-202.
   https: DOI: 10.5923/j.ajms.20190905.03.

6. Fixed point results for single self maps in probabilistic metric space, *International Journal of Mathematics and its Applications*, **7**(4) (2019), 23-30.
   http://ijmaa.in/.

Articles Published in Proceedings

1. On Probabilistic Metric Space, Proceedings of 7th National Conference of Mathematics and its Applications-2019 (NCMA-2019): Published by Nepal Mathematical Society (NMS), 128- 133.

2. On contraction conditions and fixed point results in probabilistic metric space Abstract accepted and published in 2019 Abstract book of Spring Topology and Dynamic Conference March 14-16, 2019, the University of Alabama at Birmingham, USA.

Participated in Workshops

1. Participated in International Workshop on Applied Nonlinear Analysis 2022 (IWANA-2022) organized by King Mongkut's University of Technology Thonburi (KMUTT) held at Bangsaen Heritage Hotel, Bangsaen Beach, Chonburi, Thailand during September 5-6, 2022.

2. Participated on Capacity Building Workshop for Mathematics Faculties (CBWMF-2021) organized by Nepal Mathematical Society from June 19 - 26, 2021.

3. Participated on Capacity Building Workshop on Research and Evaluation in Mathematics(CBWREM-2020) organized by Nepal Mathematical Society during October 10-11 and 14-15, 2020.

4. Participated in International e- conference on fixed Point theory and its applications to real world problem organized by Department of Mathematics, Government Post Graduate college, Maldevta, Raipur, Dehradun, India on date June 27,2020.

5. Participated in National Workshop on Project work writing in Mathematics on date September 13-16, 2019 organized by Nepal Mathematical Society at Nagarkot, Bhaktapur.

6. Participated in International Workshop on Fixed Point Theory and its Application organized by the Department of Mathematics Jamia Millia Islamia, New Delhi, India on the date 15-17, March 2019 A.D.

Oral Paper Presentations in International and Conferences, Seminars and Workshops

1. September 5 - September 6, 2022: **FiXED POINT RESULTS BY USING COMPATIBLE MAPPINGS OF TYPE (P) IN MENGER SPACE**, International Workshop on Applied Nonlinear Analysis 2022 (IWANA-2022) organized by King Mongkut's University of Technology Thonburi (KMUTT) held at Bangsaen Heritage Hotel, Bangsaen Beach, Chonburi, Thailand.

2. June 26 - June 28, 2022: **COMPATIBLE MAPPINGS OF TYPE (P) IN MENGER SPACE**, 9th National Conference on Science and Technology organized by Nepal Academy of Science and Technology, Kathmandu, Nepal.

3. April 9 - April 11, 2021: **ON COMPATIBLE MAPPINGS IN PROBABILISTIC METRIC SPACE**, International Conference on Analysis and its Applications 2021 organized by Nepal Mathematical Society (NMS), Nepal in collaboration with the Central Department of Mathematics, Tribhuvan University(TU); Department of Mathematics, School of Science, Kathmandu University (KU) Nepal; South Asian University (SAU), New Delhi, India; Association of Nepalese Mathematicians in America (ANMA) USA; Department of Mathematics Valmeeki Campus, Nepal Sanskrit University (NSU), Nepal.

4. March 15 - March 17, 2019: **ON CONTRACTION CONDITION AND FIXED POINT RESULTS IN PROBABILISTIC METRIC SPACE**, International Workshop on Fixed Point Theory and its Application organized by the Department of Mathematics Jamia Millia Islamia, New Delhi, India.

5. January 12 - January 15, 2019: **ON g-CONTRACTION CONDITION IN PROBABILISTIC METRIC SPACE**, 7th National Conference on Mathematics And its Application organized by Nepal Mathematical Society, Butwal, Nepal.

6. November 16 - November 18, 2018: **ON SOME CONTRACTION CONDITIONS IN PROBABILISTIC METRIC SPACE**, 23rd international Conference of International Academy of Physical Sciences organized by Nepal Academy of Science and Technology, Kathmandu, Nepal.

.

Award and Fellowship Achievement:

1. Ph. D. fellowship 76/77 under S and T-3 granted by University Grant Commission Nepal.

2. NMS Fellowship Award -2020 By Nepal Mathematical Society Sponsored by Nick Simons Foundation Nepal.

3. Best Paper Presentation Award for young mathematicians at the 3rd international workshop on Applied Non-Linear Analysis(IWANA-2022) held at Bangsaen Heritage Hotel, Bangsaen Beach, Chonburi, Thailand.

References

[1] M. Aamri and D.E. Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, Journal of Mathematical Analysis and Applications **270** (2002), no. 1, 181–188.

[2] M. Abbas and B. E. Rhoades, *Common fixed point theorems for occasionally weakly compatible mappings satisfying a generalized contractive condition*, Mathematical Communications **13** (2008), no. 2, 295–301.

[3] A. Abian, *Fixed point theorems of the mappings of partially ordered sets*, Rendiconti del Circolo Matematico di Palermo **20** (1971), no. 2, 139–142.

[4] S. Abian and A. B. Brown, *A theorem on partially ordered sets, with applications to fixed point theorems*, Canadian Journal of Mathematics **13** (1961), 78–82.

[5] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed point theory for lipschitzian-type mappings with applications*, vol. 6, Springer, 2009.

[6] MA Al-Thagafi and N. Shahzad, *Generalized i-nonexpansive selfmaps and invariant approximations*, Acta Mathematica Sinica, English Series **24** (2008), no. 5, 867–876.

[7] J. Ali, M. Imdad, and D. Bahuguna, *Common fixed point theorems in menger spaces with common property (ea)*, Computers & Mathematics with Applications **60** (2010), no. 12, 3152–3159.

[8] A. Aliouche and V. Popa, *Common fixed point theorems for occasionally weakly compatible mappings via implicit relations*, Filomat **22** (2008), no. 2, 99–107.

[9] H. Amann, *Order structures and fixed points*, Ruhr-universitat, Bochum, mimeographed lecture notes (1977).

[10] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundamenta Mathematicae **3** (1922), no. 1, 133–181.

[11] A. Bhatt, H. Chandra, and DR Sahu, *Common fixed point theorems for occasionally weakly compatible mappings under relaxed conditions*, Nonlinear Analysis: Theory, Methods & Applications **73** (2010), no. 1, 176–182.

[12] R. M. T. Bianchini, *Su un problema di s. reich riguardonte la teoria dei punti fissi*, Bollettino dell'Unione Matematica Italiana **5** (1972), 103–108.

[13] R. K. Bishta and V. Rakocevic, *On a probabilistic version of meir-keeler type fixed point theorem for a family of discontinuous operators*, Applied General Topology **22** (2021), no. 2, 435–446.

[14] A. Björner, *Order-reversing maps and unique fixed points in complete lattices*, Algebra Universalis **12** (1981), no. 1, 402–403.

[15] M. Black, *Vagueness*, Philosophy of science **4** (1937), 427–455.

[16] P. Bohl, *Über die bewegung eines mechnischen systems in der nähe einer gleichgewichtslage.*, Walter de Gruyter, Berlin/New York Berlin, New York, 1904.

[17] H. F. Bohnenblust and S. Karlin, *On a theorem of ville*, Contributions to the Theory of Games **1** (1950), 155–160.

[18] H. Bouhadjera, A. Djoudi, and B. Fisher, *A unique common fixed point theorem for occasionally weakly compatible maps*, Bulletin of the Allahabad Mathematical Society **24** (2009), no. 1, 1–6.

[19] N. Bourbaki, *Sur le théorème de zorn*, Archiv der Mathematik **2** (1949), no. 6, 434–437.

[20] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proceedings of the American Mathematical society **20** (1969), 458–464.

[21] L. Brillouin, *Science and information theory*, Academic Press, 2nd edition, New york, 1956.

[22] A. Brondsted, *Common fixed points and partial orders*, Proceedings of the American Mathematical Society **77** (1979), no. 3, 365–368.

[23] L. E. J. Brouwer, *Über abbildung von mannigfaltigkeiten*, Mathematische annalen **71** (1911), no. 1, 97–115.

[24] F. E. Browder, *On the convergence of successive approximations for non linear funcctional equations*, Nederlandse Akademie van Wetenschappen. Proceedings series A **71** (1968), 27–35.

[25] A. Bucur, *About applications of fixed point theory*, Scientific Bulletin **22** (2017), no. 1, 13–17.

[26] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Transactions of the American Mathematical Society **215** (1976), 241–251.

[27] S. Carl and S. Heikkilä, *Fixed point theory in ordered sets and applications: from differential and integral equations to game theory*, Springer Science & Business Media, 2010.

[28] A. L. Cauchy, *A. l., lecons de calcul differentiel et de calcul intgral*, vol. 2, Mallet-Bachelier, Paris, 1884.

[29] H. Chandra and A. Bhatt, *Fixed point theorems for occasionally weakly compatible maps in probabilistic semi-metric space*, International Journal of Mathematical Analysis **3** (2009), no. 12, 563–570.

[30] S. S. Chang, *A common fixed point theorem for commuting mappings*, Proceedings of the American Mathematical society **83** (1981), 645–652.

[31] S. S. Chang, Y. J. Cho, and S. M. Kang, *Non linear operator theory in probabilistic metric space*, Nova Science Publishers, Inc., Huntington, New York, 2001.

[32] S. S. Chang, B. S. Lee, Y. J. Cho, Y. Q. Chen, S. M. Kang, and J. S. Jung, *Generalized contraction mapping principle and differential equations in probabilistic metric spaces*, Proceedings of the American mathematical society **124** (1996), no. 8, 2367–2376.

[33] S. K. Chatterjea, *Fixed point theorems*, Comptes rendus de l'Acadmie bulgare des Sciences **25** (1972), 727–730.

[34] A. K. Chaudhary, *A study on contraction conditions in probabilistic metric space*, 2018, pp. 1–53.

[35] ———, *Fixed point results for single self maps in probabilistic metric space*, International Journal of Mathematics And its Applications **7** (2019), no. 4, 23–30.

[36] ———, *On probabilistic metric space*, Proceedings of Nepal Mathematical Society for 7th national conference on mathematics and its applications-2019 **5** (2019), 128–134.

[37] A. K. Chaudhary and K. Jha, *Contraction conditions in probabilistic metric spaces*, American Journal of Mathematics and Statistics **9** (2019), no. 5, 199–202.

[38] A. K. Chaudhary, K. Jha, and K. B. Manandhar, *A common fixed point theorem in menger space with compatible mapping of type (k)*, Advances in Mathematics: Scientific Journal **11** (2022), no. 10, 883–892.

[39] A. K. Chaudhary, K. Jha, K. B. Manandhar, and D. Mihet, *Some common fixed point theorems for occasionally weakly compatible mappings in menger space*, Journal of Computer Science and Applied Mathematics **3** (2022), no. 2, 65–76.

[40] A. K. Chaudhary, K. Jha, K. B. Manandhar, and P. P. Murthy, *A common fixed point theorem in menger space with compatible mapping of type (p)*, International Journal of Mathematical Sciences and Engineering Applications **15** (2021), no. 2, 59–70.

[41] A. K. Chaudhary, K. Jha, K. B. Manandhar, and H. K. Pathak, *A common fixed point theorem in menger space with weakly compatible mapping of type (p)*, Advances in Mathematics: Scientific Journal **11** (2022), no. 11, 1019–1031.

[42] S. Chauhan, S. Kumar, and B. D. Pant, *Common fixed point theorems for occasionally weakly compatible mappings in menger spaces*, Journal of Advvances Research Pure Math **3** (2011), no. 4, 17–23.

[43] Y. C. Cheh, *Common fixed point of continuous mapping in metric spaces*, Publications de l'Institut Mathmatique **27** (1980), no. 41, 21–25.

[44] Y. J. Cho, *Survey on metric fixed point theory and applications in in advances in real and complex analysis with applications book*, Birkhauser, Singapore, 2017.

[45] Y. J. Cho, P. P. Murthy, and M. Stojakovic, *Compatible mappings of type(a) and common fixed points in menger space*, Communications of the Korean Mathematical Society **7** (1992), no. 2, 325–339.

[46] Y. J. Cho, B. K. Sharma, and D. R. Sahu, *Common fixed point theorems in menger spaces with common property (ea)*, Mathematica Japonica **42** (1995), no. 1, 91–98.

[47] L. J. B. Ciric, *Generalized contractions and fixed-point theorems*, Publications de lInstitut Mathmatique **12** (1971), no. 26, 19–26.

[48] ———, *On a fixed point of generalized contractions on probabilistic metric spaces*, Publications de l'Institut Mathmatique **18** (1975), no. 32, 71–78.

[49] Lj. B. Ćirić, *A generalization of banachs contraction principle*, Proceedings of the American Mathematical society **45** (1974), no. 2, 267–273.

[50] Lj. B. Ćirić, B. Samet, and C. Vetro, *Common fixed point theorems for families of occasionally weakly compatible mappings*, Mathematical and Computer Modelling **53** (2011), no. 5-6, 631–636.

[51] G. Constantin and I. Istratescu, *Elements of probabilistic analysis with applications*, Springer, 1989.

[52] S. Dalal, *Common fixed point theorem for occasionally weakly compatible mappings in menger space*, Advances in Fixed Point Theory **3** (2013), no. 2, 451–463.

[53] J. Danes, *Two fixed point theorems in topological and metric spaces*, Bulletin of the Australian Mathematical Society **14** (1976), 21–25.

[54] K. M. Das and V. Naik, *Common fixed point theorems for commuting maps on metric spaces*, Proceedings of the American Mathematical society **77** (1979), 269–373.

[55] L. de Broglie, *Une remarue sur l interaction entre la matiere et le champ electromagnetique*, Comptes rendus de l'Acadmie des Sciences **200** (1935), 361–363.

[56] R. Demarr, *Common fixed points for isotone mappings*, Colloquium Mathematicum, vol. 13, Institute of Mathematics Polish Academy of Sciences, 1964, pp. 45–48.

[57] R. O. Duda and P. E. Hart, *Pattern classification and scene analysis*, Wiley, New york, 1973.

[58] J. Dugundji and A. Granas, *A. granas et m. lassonde*, Fixed Point Theory **1**, no. 61.

[59] A. S. Eddington, *Fundamental theory*, Oxford University Press, New york and London, 1953.

[60] M. Edelstein, *An extension of banachs contraction principle*, Proceedings of the American Mathematical society **12** (1961), 7–10.

[61] Ky. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proceedings of the National Academy of Sciences **38** (1952), no. 2, 121–126.

[62] G. Fechner, *Elements of psychophysics*, LEIPZIG, 1860.

[63] B. Fischer and M. S. khan, *Some fixed point theorems for commuting mappings*, Mathematische Nachrichten **106** (1982), 323–326.

[64] A. Francal, *On the possible connection between quantum mechanics and gravitation in quantum mechanics*, A half century later (1977), 19–38.

[65] M. Frechet, *Sur quelques points du calcul functionnel*, Rendiconti del Circolo Matematico di Palermo **22** (1906), 1–74.

[66] M. Geraghty, *On contractive mappings*, Proceedings of the American Mathematical society **40** (1973), 604–608.

[67] I. L. Glicksberg, *A further generalization of the kakutani fixed point theorem, with application to nash equilibrium points*, Proceedings of the American Mathematical Society **3** (1952), no. 1, 170–174.

[68] K. Goebel, *A coincidence theorem*, Bulletin L'Acadmie Polonaise des Science, Srie des Sciences Mathmatiques **16** (1968), 7–10.

[69] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, no. 28, Cambridge university press, 1990.

[70] I. Golet, *On contractions in probabilistic metric spaces*, Radovi Matematicki **13** (2004), no. 1, 87–92.

[71] S. Gupta, S. Kumar, and G. K. Soni, *Common fixed point results in menger spaces*, International Research Journal of Engineering and Technology **5** (2018), no. 3, 1–4.

[72] V. Gupta, M. S. Khan, B. Singh, and S. Kumar, *Existence and uniqueness of fixed point for meir-keeler type contractive condition in menger spaces*, Computational and Mathematical Methods **3** (2021), no. 2, 1–9.

[73] L. F. Guseman, *Fixed point theorems for mappings with a contractive iterate at a point*, Proceedings of the American Mathematical Society **26** (1970), no. 4, 615–618.

[74] J. Hadamard, *Sur quelques applications de l'indice de kronecker*, Bussiere, 1910.

[75] O. Hadzic and E. Pap, *Fixed point theory in probabilistic metric spaces*, Kluwer Academic Publishers, Dordrecht, 2003.

[76] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of reich*, Canadian Mathematical Bulletin **16** (1973), no. 2, 201–206.

[77] F. Hausdorff, *Grunduge der mengelehre*, Leipzig : Veit und Comp., 1914.

[78] T. L. Hicks, *Fixed point theory in probabilistic metric spaces*, Review of Research faculty of Novi Sad **13** (1983), 63–72.

[79] H. Hjelmslev, *Die nuterliche geometrie*, Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg **2** (1923), 1–36.

[80] M. Imdad and J. Ali, *Jungcks common fixed point theorem and ea property*, Acta Mathematica Sinica, English Series **24** (2008), no. 1, 87–94.

[81] K. Iseki, *On common fixed points of mappings*, Bulletin of the Australian Mathematical Society **10** (1974), 365–370.

[82] S. Ishikawa, *Fixed points by a new iteration method*, Proceedings of the American Mathematical Society **44** (1974), no. 1, 147–150.

[83] V. I. Istratescu, *Fixed point theory*, vol. 2, D. Reidel Publishing Company, Dordrecht., 1981.

[84] V. I. Istratescue, *Introducere in teoria spatiilor metrice probabilistie cu aplicati*, Bucharest: Editions Editura, Technica, 1974.

[85] ———, *Some remarks nonarchimedean on probabilistic metric spaces*, Glas. Mat. **11** (1976), no. 31, 155–161.

[86] V.I. Istratescue and I. Sacuiu, *Fixed point theorems for contraction mappings on probabilistic metric spaces*, Revue Roumaine de Mathematiques Pures et Appliquees **18** (1973), 1375–1380.

[87] J. Jachymski, *Equivalence of some contractivity properties over metrical structures*, Proceedings of the American Mathematical society **125** (1997), 2327–2335.

[88] M. F. Janowitz, *An order theoretic model for cluster analysis*, Siam Journal on Applied Mathematics **34** (1978), 55–72.

[89] I. H. Jebril, S. K. Datta, S. Sarkar, and N. G. Biswas, *Common fixed point theorems in probabilistic metric spaces using lukasiecz t-norm and product t-norm*, Journal of Statistics Applications and Probability **10** (2021), no. 3, 635–639.

[90] K. Jha, *Some applications of banach's contractin principle*, Nepal Journal of Science and Technology **4** (2002), 135–140.

[91] K. Jha, M. Imdad, and U. Rajopadhyaya, *Fixed point theorems for occasionally weakly compatible mappings in semi-metric space*, Annals of Pure and Applied Mathematics **5** (2014), no. 2, 153–157.

[92] K. Jha, R. P. Pant, and K. B. Manandhar, *A common fixed point theorem for reciprocal continuous compatible mappings in metric space*, Annals of Pure and Applied Mathematics **5** (2014), no. 2, 120–124.

[93] K. Jha, R. P. Pant, and S. L. Singh, *On the existence of common fixed point for compatible mappings*, Punjab University Journal of Mathematics **37** (2005), 39–47.

[94] K. Jha, V. Popa, and K. B. Manandhar, *Common fixed points for compatible mappings of type (k) in metric space*, International Journal of Mathematical Sciences and Engineering Applications **8** (2014), 383–391.

[95] G. Jungck, *Commuting mappings and fixed points*, American Mathematical Monthly **83** (1976), 261–263.

[96] ———, *Compatible mappings and common fixed points*, International Journal of Mathematical Sciences **9** (1986), 771–779.

[97] ———, *Common fixed points for commuting and compatible maps on compacta*, Proceedings of the American Mathematical society **103** (1988), 977–983.

[98] ———, *Fixed points for compatible maps on the unit interval*, Proceedings of the American Mathematical society **115** (1992), 495–499.

[99] ———, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far eastern journal of mathematics and science **4** (1996), 199–215.

[100] G. Jungck, K. B. Moon, S. Park, and B. E. Rhoades, *On generalizations of the meir-keeler tpe contraction maps: Correction*, Journal of Mathematical Analysis and Applications **180** (1993), 221–222.

[101] G. Jungck, P. P. Murthy, and Y. J. Cho, *Compatible mappings of type (a) and common fixed points*, Mathematica Japonica **38** (1993), 381–390.

[102] G. Jungck and B. E. Rhoades, *Some fixed point theorems for compatible mappings*, International Journal of Mathematics and Mathematical Sciences **16** (1993), 417–428.

[103] ______, *Fixed points for set valued functions without continuity*, Indian Journal of pure and applied mathematics **29** (1998), no. 3, 227–238.

[104] ______, *Fixed point theorems for occasionally weakly compatible mappings*, Fixed point theory **7** (2006), no. 2, 287–296.

[105] ______, *Fixed point theorems for occasionally weakly compatible mappings*, Fixed point theory **7** (2006), no. 2, 287–296.

[106] S. Kakutani, *A generalization of brouwer fixed point theorem*, Duke Math. J. **8** (1941), 457–459.

[107] R. Kannan, *Some results on fixed points*, Bulletin of the Calcutta Mathematical Society **60** (1968), 71–76.

[108] M. A. Khamsi and W. A. Kirk, *An introduction to metric spaces and fixed point theory*, John Wiley & Sons, 2011.

[109] M. S. Khan, *Two fixed point theorems in topological and metric spaces*, Bulletin L'Acadmie Polonaise des Science, Srie des Sciences Mathmatiques **29** (1981), 499–507.

[110] F. Khojasteh, M. Abbas, and S. Costache, *Two new types of fixed point theorems in complete metric spaces*, Abstract and Applied Analysis, vol. 2014, 2014, p. 5 pages.

[111] W. Kirk and N. Shahzad, *Fixed point theory in distance spaces*, Springer, 2014.

[112] W. A. Kirk and B. Sims, *Handbook of metric fixed point theory*, Australian Mathematical Society GAZETTE **29** (2002), no. 2.

[113] E. P. Klement, R. Mesiar, and E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[114] B. Knaster, *Un theoreme sur les functions d'ensembles*, Annales de la Socie'te Polonaise de Mathematique **6** (1928), 133–134.

[115] J. K. Kohli and S. Vashistha, *Common fixed point theorems in probabilistic metric spaces*, Acta Mathematica Hungarica **115** (2007), no. 1-2, 37–47.

[116] I. Kubiaczyk and S. Sharma, *Some common fixed point theorems in menger space under strict contractive conditions.*, Southeast Asian Bulletin of Mathematics **32** (2008), no. 1, 117–124.

[117] S. Kumar and B. D. Pant, *A common fixed point theorem in probabilistic metric space using implicit relation*, Filomat **22** (2008), no. 2, 43–52.

[118] D. Kurepa, *Fixpoints of decreasing mappings of ordered sets*, Publications de l'Institut Mathmatique **18** (1975), no. 32, 111–116.

[119] J. Liouville, *Mémoire sur le développement des fonctions ou parties de fonctions en séries de sinus et de cosinus.*, Journal de Mathématiques Pures et Appliquées (1836), 14–32.

[120] R. Lipschitz, *Lehrbuch der analysis: bd. grundlagen der analysis*, vol. 1, M. Cohen & sohn (F. Cohen), 1877.

[121] Z Lu, *On common fixed points for converse commuting self-maps on a metric spaces*, Acta Analysis Functionalis Applicata **4** (2002), no. 3, 226–228.

[122] K. B. Manandhar, K. Jha, and Y. J. Cho, *common fixed point theorem in intuitionistic fuzzy metric spaces using compatible mappings of type (k)*, Bulletein of Society for Mathematics Services and Standards **3** (2014), 81–87.

[123] K. B. Manandhar, K. Jha, and G. Porru, *Common fixed point theorem of compatible mappings of type (k) in fuzzy metric spaces*, Electronic Journal of Mathematical Analysis and Applications **2** (2014), no. 2, 248–253.

[124] W. R. Mann, *Mean value methods in iteration*, Proceedings of the American Mathematical Society **4** (1953), no. 3, 506–510.

[125] S. Mantro, S. Kumar, S. S. Bhatia, and S. M. Kang, *Common fixed point theorems for contraction mapping using weak reciprocal continuity*, International Journal of Pure and Applied Mathematics **88** (2013), no. 1, 77–90.

[126] A. Meir and E. Keeler, *A theorem on contraction mapping*, Journal of Mathematical Analysis and Applications **28** (1969), 326–329.

[127] K. Menger, *Untersuchungen uber allgemeine metrik*, Mathematische Annalen **100** (1928), 75–163.

[128] ———, *Zur metrik der kurben; untersuchungen uber allgemeine metrik*, Mathematische Annalen **103** (1930), 466–501.

[129] ———, *Kurbentheorie*, Leipzig, Teubner, Reprinted by Chelsea, Bronx, NY 1967, 1932.

[130] ———, *Statistical metrics*, Proceedings of the National Academy of Sciences of the United States of America **28** (1942), 535–537.

[131] ———, *Probabilistic theories of relations*, Ibid **37** (1951), 178–180.

[132] ———, *Geometrie generale*, Memory of Science and Mathematics **124** (1954), 466–501.

[133] D. Mihet, *Inegalitata triunghiului si puncte fixed in pm-spatii*, Ph.D. thesis, Doctoral Thesis West of Timisora, in Engilish, 2001.

[134] ———, *A note on hicks type contractions on generalized menger spaces*, West University of Timisoara, Seminarul de Teoria Probabilitatilor si Aplicatii **133** (2002).

[135] ———, *Weak-hicks contractions*, Fixed Point Theory **6** (2005), no. 1, 71–78.

[136] D. Miheţ, *Fixed point theorems in probabilistic metric spaces*, Chaos, Solitons & Fractals **41** (2009), no. 2, 1014–1019.

[137] S. N. Mishra, *Common fixed points of compatible mappings in probabilistic metric space*, Mathematica Japonica **36** (1991), no. 2, 283–289.

[138] R. E. Moore, *Methods and applications of interval analysis*, SIAM, USA, 1979.

[139] O. Onicescu, *O. et systems aleatoires*, Bucharest: Editions de I Academie de la R.P., Romanie, 1964.

[140] J. R. Oppenheimer, *The flying trapeze: Three crisis of physicist*, Oxford University Press, New york and London, 1962.

[141] B. D. Pant, M. Abbas, and S. Chauhan, *Coincidences and common fixed points of weakly compatible mappings in menger spaces*, The Journal of Indian Mathematical Society. **80** (2013), no. 1-2, 127–139.

[142] B. D. Pant and S. Chauhan, *Common fixed point theorems for occasionally weakly compatible mappings using implicit relation*, The Journal of Indian Mathematical Society **77** (2010), no. 1, 13–21.

[143] ______, *Common fixed point theorem for occasionally weakly compatible mappings in menger space*, Surveys in Mathematics and its Applications **6** (2011), 1–7.

[144] B. D. Pant, B. M. L. Tivari, and S. L. Singh, *Common fixed point theorem for densifying mappings in probabilistic metric spaces*, Honam Mathematical Journal **5** (1983), no. 1, 151–154.

[145] R. P. Pant, *Common fixed points of non-commuting mappings*, Journal of Mathematical Analysis and Applications **188** (1994), 436–440.

[146] ______, *A common fixed point theorem under a new condition*, Indian Journal of Pure and Applied Mathematics **30** (1999), 147–152.

[147] R. P. Pant and R. K. Bisht, *Occasionally weakly compatible mappings and fixed points*, Bulletin of the Belgian Mathematical Society-Simon Stevin **19** (2012), no. 4, 655–661.

[148] R. P. Pant, A. Pant, M. Nikolić, and S. N. Ješić, *A characterization of completeness of menger pm-spaces*, Journal of Fixed Point Theory and Applications **21** (2019), no. 4, 1–9.

[149] E. Pap, O. Hadžić, and R. Mesiar, *A fixed point theorem in probabilistic metric spaces and an application*, Journal of Mathematical Analysis and Applications **202** (1996), no. 2, 433–449.

[150] S. Park, *On general contractive type conditions*, Journal of the Korean Mathematical Society **17** (1980), no. 1, 131–140.

[151] V. Parvaneh, G. G. Branach, and G. Gharab, *Some common fixed point theorems in complete metric spaces*, International Journal of Pure and Applied Mathematics **76** (2012), no. 1, 1–8.

[152] A. Pasini, *Some fixed point theorems of the mappings of partially ordered sets*, Rendiconti del Seminario Matematico della Universita di Padova **51** (1974), 167–177.

[153] V. Pata, *Fixed point theorems and applications*, Springer Nature, Switzerland, 2019.

[154] H. K. Pathak, Y. J. Cho, S. S. Chang, and S. M. Kang, *Compatible mappings of type (p) and fixed point theorem in metric spaces and probabilistic metric spaces*, Novi Sad Journal of Mathematics **26** (1996), no. 2, 87–109.

[155] H. K. Pathak, S. M. Kang, and J. H. Baek, *Weak compatible mappings of type (a) and common fixed points in menger spaces*, Communications of the Korean Mathematical Society **10** (1995), no. 1, 67–83.

[156] P.Cousot and R. Cousot, *Constructive versions of Tarski's fixed point theorems*, Pacific Journal of Mathematics **81** (1979), no. 1, 43–57.

[157] G. Peano, *Demonstration de lintgrabilite des equations diffrentielles ordinaires*, Mathematische Annalen **2** (1890), no. 37, 182–228.

[158] E. Picard, *Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives*, Journal de Mathématiques pures et appliquées **6** (1890), 145–210.

[159] H. Poincaré, *Sur les courbes définies par les équations différentielles*, Journal des Mathmatiques Pures et Appliques **4** (1886), no. 2, 151–217.

[160] H. Poincare, *Science and hypothesis*, The science Press, Lancaster Pa., 1905.

[161] ______, *The foundation of science*, The science Press, Lancaster Pa., 1913.

[162] R. P.Pant, V. Pant, and V. P. Pandey, *Generalization of meir-keeler type fixed point theorems*, Tamkang Journal of Mathematics **35** (2004), no. 3, 179–188.

[163] J. M. Prager, *Segmentation of static and dynamic schemes*, 1979.

[164] V. Radu, *Lectures on probabilistic analysis, surveys, lecture notes and monographs, series on probability, statistics and applied mathematics*, Universitatea din Timisoara, Timisoara **2** (1994).

[165] S. Reich, *Some remarks concerning contraction mappings*, Canadian Mathematical Bulletin **14** (1971), no. 1, 121–124.

[166] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Transactions of the American Mathematical Society **226** (1977), 257–290.

[167] ______, *Contractive definitions revisited*, Contemporary Mathematics **21** (1983), 189–205.

[168] ______, *Proving fixed point theorems using general principles*, Indian Journal of Pure and Applied Mathematics **8** (1996), 741–770.

[169] R.Kannan, *Some results on fixed points*, American Mathematical Monthly **76** (1969), 405–408.

[170] N. Rosen, *Statistical geometry and fundamental particles*, Physical Review **72** (1947), 298–303.

[171] ______, *Quantum geometry*, Annals of Physics **19** (1962), 165–172.

[172] M. R.Singh and Y. R. Singh, *Compatible mappings of type (e) and common fixed point theorems of meir-keeler type*, International Journal of Mathematical Sciences and Engineering Applications **1** (2007), no. 2, 299–315.

[173] I. A. Rus, *Some fixed point theorems in metric spaces*, Rendiconti dell'Istituto di Matematica dell'Universita di Trieste **3** (1971), 169–172.

[174] ______, *Generalized contractions and applications*, Cluj University Press, 2001.

[175] B. Sadovskii, *A fixed-point principle*, Functional Analysis and Its Applications **1** (1967), no. 2, 151–153.

[176] K. P. R. Sastry, G. A. Naidu, P. V. S. Prasad, and S. S. A. Sasatri, *Common fixed point theorems in menger spaces*, Global Journal of Mathematical Sciences: Theory and Practical **3** (2011), no. 1, 41–46.

[177] H. Scarf, *The approximation of fixed points of a continuous mapping*, Siam Journal on Applied Mathematics **15** (1967), no. 5, 1328–1343.

[178] J. Schauder, *Der fixpunktsatz in funktionalraümen*, Studia Mathematica **2** (1930), no. 1, 171–180.

[179] B. Schweizer, *Functions systems*, Mathematische Annalen **172** (1967), 1–16.

[180] B. Schweizer, H. Sherwood, and R. M. Tardiff, *Contractions on probabilistic metric spaces: examples and counterexamples.*, Stochastica **12** (1988), no. 1, 5–17.

[181] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific Journal of Mathematics **10** (1960), 314–334.

[182] ______, *Probabilistic metric space*, Dover Publications INC., New york, 1983.

[183] V. M. Sehgal, *Some fixed point theorems in functional analysis and probability*, Wayne State University, 1966.

[184] ______, *Some fixed and common fixed point theorems in metric spaces*, Canadian Mathematical Bulletin **17** (1974), no. 2, 257–259.

[185] V. M. Sehgal and A. T. Bharucha Reid, *Fixed point of contraction mapping in probabilistic metric spaces*, Mathematical Systems Theory **6** (1972), 97–102.

[186] A. N. Serstnev, *Random normed spaces : problems of completeness*, Kazan. Gos. Univ. Ucen. Zap. **122** (1962), 3–20.

[187] S. Sessa, *On a weakly commutativity condition in a fixed point considerations*, Publications de l'Institut Mathmatique **32** (1982), no. 46, 149–153.

[188] C. E. Shanon, *The mathematical theory of communication*, Bell Labs Technical Journal **27** (1948), 379–423.

[189] B. K. Sharma and N. K. Sahu, *Common fixed points of three continuous mappings*, Mathematics Student **59** (1991), 77–80.

[190] S. Sharma and K. Choubey, *Common fixed points theorems for weakly compatible mappings in menger spaces*, Journal of Korean Society of Mathematical Education Series B: Pure and Applied Mathematics **10** (2003), no. 4, 245–254.

[191] S. Sharma and B. Deshpandey, *Common fixed point theorems for weakly compatible mappings without continuity in menger spaces*, Journal of Korean Society of Mathematical Education Series B: Pure and Applied Mathematics **10** (2003), no. 2, 133–144.

[192] R. N. Shepard, *Multi dimentional scaling, tree fitting, and clustering*, Science **210** (1980), 390–398.

[193] H. Sherhood, *On the completion of probabilistic metric space*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **6** (1966), 62–64.

[194] A. H. Sidiqui, *Functional analysis with applications*, Tata Me Gray Publication Comp.Ltd., India, 1986.

[195] B. Singh and S. Jain, *Semi-compatibility and fixed point theorem in menger space*, Journal of the Chungcheong Mathematical Society **17** (2004), no. 1, 1–17.

[196] ———, *Common fixed point theorem in menger space through weak compatibility*, ournal of Mathematical Analysis and Applications **301** (2005), no. 2, 439–448.

[197] S. Singh, *A study on fixed point theory with applications*, Createspace Independent Publishing Platform, India, 2013.

[198] S. L. Singh and B. D. Pant, *Common fixed point theorems for commuting mappings in probabilistic metric spaces*, Honam Mathematical Journal **5** (1983), no. 1, 139–150.

[199] S. P. Singh, B. Watson, and P. Srivastava, *Fixed point theory and best approximation: the kkm-map principle*, vol. 424, Springer Science & Business Media, 2013.

[200] S.Kumar and A. Rani, *Some common fixed point theorems in menger space*, Journal of Applied mathematics **3** (2012), 235–245.

[201] D. Smart and D. Roger, *Fixed point theorems*, vol. 66, Cup Archive, 1980.

[202] V. Srinivas and S. Ravi, *Fixed point theorem using occasionally weakly compatible mappings in metric space*, Journal of Computer and Mathematical Sciences **10** (2019), no. 3, 461–466.

[203] S. S. Stevens, *Measurements,psycho physics, and utility,in measurement, definitions and theories*, C. west Church Man, and P. Ratoosh, eds, Wiley, New york, 1959.

[204] J. Sun and Y. Sun, *A general principle on ordered sets and its applications to fixed point theory*, Applicable Analysis **34** (1989), no. 1-2, 129–137.

[205] A. Tarski, *A lattice-theoretical fixpoint theorem and its applications.*, Pacific journal of Mathematics **5** (1955), no. 2, 285–309.

[206] K. Taş, M. Telci, and B. Fisher, *Common fixed point theorems for compatible mappings*, International Journal of Mathematics and Mathematical Sciences **19** (1996), no. 3, 451–456.

[207] A. Tychonoff, *Ein fixpunktsatz*, Mathematische Annalen **111** (1935), no. 1, 767–776.

[208] J. S. Vandergraft, *Newtons method for convex operators in partially ordered spaces*, Siam Journal on Numerical Analysis **4** (1967), no. 3, 406–432.

[209] C. Vetro, *Some fixed point theorems for occasionally weakly compatible mappings in probabilistic semi-metric spaces*, International Journal of Modern Mathematics **4** (2009), no. 3, 277–284.

[210] A. Wald, *On a statistical generalization of metric spaces*, Proceedings of the National Academy of Sciences of the United States of America **29** (1943), 196–197.

[211] ———, *On a statistical generalization of metric spaces*, Reports of a Mathematical Colloquium, Second Series **5** (1944), no. 6, 76–79.

[212] H. Weyl, *Symmetry*, Princeton University Press, NJ, 1952.

[213] J. S. W. Wong, *Common fixed points of commuting monotone mappings*, Canadian Journal of Mathematics **19** (1967), 617–620.

[214] Y. Yakimovsky, *Boundary and object detection in real world maps*, Journal of the Association for Computing Machinery **23** (1976), 599–618.