

ADVECTION-DISPERSION EQUATION FOR POLLUTANT CONCENTRATION



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TRIBHUVAN UNIVERSITY
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BY
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DEDICATION

To

My Mother
Basandhari Paudel

Along with
My Wife
Sujata Paudel

My Son
Krishal Paudel



STUDENT'S DECLARATION

This thesis entitled “**Advection-Dispersion Equation for Pollutant Concentration**”, which has been submitted to the Central Department of Mathematics, Institute of Science and Technology (IOST), Tribhuvan University, Nepal for the partial fulfillment of the Master of Philosophy (M. Phil.) Degree in Mathematics, is a genuine work that I carried out under my supervisor **Dr. Jeevan Kafle** and that no sources other than those listed in the Bibliography have been used in this work. Moreover, this work has not been published or submitted elsewhere for the requirement of any degree programme.

.....

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RECOMMENDATION

This is to recommend that Mr. **Keshav Paudel** has prepared this thesis entitled “**Advection-Dispersion Equation for Pollutant Concentration** ” for the partial fulfillment of the Master of Philosophy (M. Phil.) in Mathematics under my supervision. To my knowledge, this work has not been submitted for any other degree. He has fulfilled all the requirements laid down by the Central Department of Mathematics, Institute of Science and Technology (IOST), Tribhuvan University (TU), Kirtipur for the submission of the thesis for the partial fulfillment of M. Phil. Degree in Mathematics.

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Date: 28 April 2022



LETTER OF APPROVAL

We certify that the Research Evaluation Committee of the Central Department of Mathematics, Tribhuvan University, Kirtipur approved this research work entitled “**Advection-Dispersion Equation for Pollutant Concentration**” carried out by Mr. **Keshav Paudel** in the scope and generality as a thesis in the partial fulfillment for the requirement of the M. Phil. degree in Mathematics.

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.....

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ABSTRACT

The advection-dispersion-reaction equation is used to describe the dispersion process. Here, we solve one-dimensional steady advection-dispersion equation numerically by using finite difference method. Also, we formulate the model to minimize the cost of wastewater treatment.

Analytical solution to unsteady advection-dispersion equation using Laplace transformation technique is derived to describe the pollutant concentration $C(x, t)$. We have obtained analytic unsteady solution by taking the water velocity u in the x -direction as a linear function of x and dispersion coefficient D as zero in case of concentration of pollutant in one region. Numerical studies show variation of C with time t . If the added pollutant rate along the river q is very small amount, the variation of C along the river at different times coincide to each other. In case of concentration of pollutant in two regions, analytical solutions are determined by taking dispersion coefficient D as non-zero.

A coupled system of advection-dispersion equations based water pollution model is presented that incorporates different parameters. We have proposed analytical solution for mathematical model. One dimensional model is used to observe the concentrations by taking dimension along the length of river. By considering the removal of pollutant by aeration, event of steady states is investigated. In this model, coupled advection-dispersion equations are solved by taking dispersion coefficient as zero and non-zero, respectively.

Keywords: Pollutant, Concentration, Laplace transformation, Dispersion, Analytical solution, Dissolved oxygen.

LIST OF ACRONYMS AND ABBREVIATIONS

PDE	Partial Differential Equation
ODE	Ordinary Differential Equation
FDM	Finite Difference Method
COD	Chemical Oxygen Demand
WHO	World Health Organization
DO	Dissolved Oxygen
ADE	Advection-Dispersion Equation

LIST OF SYMBOLS

A	Cross-section area of the river(m^2).
C	Pollutant concentration ($kg.m^{-3}$).
X	Dissolved oxygen concentration ($kg.m^{-3}$).
D	Dispersion coefficient of pollutant in the x -direction ($m^2.day^{-1}$).
D_x	Dispersion coefficient of dissolved oxygen in the x -direction ($m^2.day^{-1}$).
R	Substance decay rate (s^{-1}).
S	Saturated oxygen concentration ($kg.m^{-3}$).
Q	Rate of change of substance concentration due to a source ($kg.m^{-3}s$).
L	Polluted length of river (m).
c	Chemical oxygen demand concentration ($kg.m^{-3}$).
k	Half-saturated oxygen demand concentration for pollutant decay ($kg.m^{-3}$).
k_1	Degradation rate coefficient for pollutant (day^{-1}).
k_2	Degradation rate coefficient for dissolved oxygen (day^{-1}).
s	Laplace transform variable.
p	Rate of pollution ($kg.m^{-3}$) at the origin.
q	Added pollutant rate along the river ($kg.m^{-1}.day^{-1}$).
α	Mass transfer of oxygen from air to water ($m^2.day^{-1}$).
t	Time (day).
u	Water velocity in the x -direction ($m.day^{-1}$).
x	Position (m).

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Chapter 1

Introduction

1.1 Background

Water pollution takes place due to presence of unfavourable substances into water that reduces standard of water which is dangerous to human health [1]. Drinking water is required to be safe for public health. Being a common solvent, water is a main source of diseases. The data of world health organization (WHO) shows that 80% diseases and 3.1% deaths happen because of low water quality [6]. Advection means pollutant transfer by bulk water flow downstream. Diffusion is extend over of gas through water.

In the present situation, water pollution in rivers has been subject of study for researchers functioning in environment science, geology, mathematics etc. If important hydraulic and chemical processes are studied together, analytical solutions of mathematical models describing pollutant are hardly possible [32]. Now a days mathematical models have been used widely for acceptable tools of water quality management. The purpose of this thesis is to promote analytical solution of advection-dispersion equation. The poor water quality in Bagmati River in Kathmandu, Nepal was the motivation for this study. The rapid urban growth, low levels of awareness, industrial waste, continuous dumping of solid wastes, insufficient waste-water treatment facilities, domestic sewage are the main reasons for pollution in the Bagmati River [29].

1.2 Literature Review

Marusic [16] proposed a model which describes pollutant dispersion in river-type systems. He was success to describe changes in pollutant concentration with time. Pochai, et al. [23] presented a model to control the level of pollutant optimally in waste-water.

This model would help to minimize initial water treatment cost. They solved convection-diffusion equation with constant coefficients by finite element method. It was used to optimize water treatment costs. Miller, et al. [17] used two mathematical models. The first was the hydrodynamic model. This model provides the water flow's elevation and velocity field. The dispersion model was the second. They used central different technique to simulate the model. This model gives the pollutant concentration field.

According to the study of Johari, et al. [8], advection-diffusion equation was used to forecast concentration of pollutant transport. Salkuyeh [26] presented convection-diffusion equation with exact solution. Pochai, et al. [23] and Tabuenca, et al. [30] presented water pollution models in one and two-dimensional respectively. Pochai [22, 21] studied hydrodynamic model in uniform reservoir and river respectively.

Pimpunchat et al. [19] presented a mathematical model for river pollution. They had used coupled advection-dispersion equation to investigate the effect of aeration. Carslaw and Jaeger [2] have derived analytical solutions to advection-dispersion equation with Laplace transforms and with Greens functions. Paudel et al.[18] presented analytical solution to advection-dispersion equation of pollutant concentration. They used Laplace transformation technique to solve this equation. Van Genuchten and Alves [31] introduced analytical solutions for a physical system in a semi-infinite domain with zero initial concentration. Savovic and Djordjevich [27] derived numerical solution to advection-diffusion equation in semi-infinite media. Kumar et al. [12] have presented analytical solutions to one-dimensional advectiondiffusion equation.

Unsteady advection-dispersion equations were presented with analytical solutions by Wadi et al. [32] and Pimpunchat [15]. These equations give a one-dimensional description of pollutant concentration. Pimpunchat et al.[20] presented two coupled well-known advection dispersion equations for the pollutant and dissolved oxygen concentrations, respectively. There are many technique to solve partial differential equations [9, 10, 24, 25]. Laplace transformation is also a very useful technique.

1.3 Objective

In the present situation, water pollution in rivers has been subject of study for researchers functioning in environment science, geology, mathematics etc. If important hydraulic and chemical processes are studied together, analytical solutions of mathematical models describing pollutant are hardly possible. Now a days mathematical models have been used widely for acceptable tools of water quality management. The objective of this thesis is

to promote analytical solution of advection-dispersion equation using Laplace transforms method and other techniques. We aim to solve coupled advection-dispersion equations using efficient way.

1.4 Structure of Thesis

According to the following plan, we shall proceed discussing the rest of the thesis. In Chapter 2, we solve one-dimensional steady advection-dispersion-reaction equation numerically by using finite difference method (FDM). Also, we analyze chemical oxygen demand (COD) concentration with the variation of different parameter values. In chapter 3, unsteady advection-dispersion equation is solved analytically. In this chapter advection-dispersion equation describes pollutant concentration $C(x, t)$. We use Laplace transformation technique to solve the equation. We have proposed analytical solution to coupled pair of nonlinear advection-dispersion equations for concentrations of pollutant and dissolved oxygen in chapter 4. Chapter 5 concludes the thesis with summary.

Chapter 2

Advection-Dispersion Equation for Chemical Oxygen Demand Concentration

2.1 Introduction

Water pollution takes place due to unfavourable substances come into water that reduces the standard of water which is dangerous to human health [1]. Drinking water is required to be safe for public health. Being a common solvent, water is a main source of diseases. The data of world health organization (WHO) shows that 80% diseases and 3.1% deaths happen because of low water quality [6]. Advection means pollutant transfer by bulk water flow downstream. Chemical oxygen demand (COD) is a measure of amount of oxygen consumed by reactions [28].

Here, we solve one-dimensional steady advection-diffusion-reaction equation numerically. To solve this equation we use Finite difference method (FDM). Also, we analyze chemical oxygen demand (COD) concentration with the variation of different parameter values. Figure 2.1 shows the Guheshwori Wastewater Treatment Plant, Kathmandu, Nepal [5].

2.2 Mathematical Model

The unsteady advection-dispersion equation [3] is given by the following relation

$$D \frac{\partial^2 c}{\partial x^2} = u \frac{\partial c}{\partial x} + \frac{\partial c}{\partial t}, \quad t \in [0; T] \quad (2.1)$$



Figure 2.1: Guheshwori Wastewater Treatment Plant

Here, we consider a steady state condition. The advection-dispersion-reaction equation [3, 4, 23] which describes the concentration of COD is

$$D \frac{d^2c}{dx^2} = u \frac{dc}{dx} + Rc - Q \quad (2.2)$$

where $c(x)$ is the COD concentration in $x \in [a; b]$ (kg/m^3). Here, u represents the flow velocity in the x - direction (m/s). The dispersion coefficient is D (m^2/s). The substance degradation rate is R (s^{-1}). Q is the rate of change in substance concentration due to a source (kg/m^3s). At $x = a$ and $x = b$, the boundary conditions are $c = c_0$ and $\frac{dc}{dx} = t_0$, respectively.

Consider (2.2) in the form

$$\frac{d^2c}{dx^2} = p(x) \frac{dc}{dx} + q(x)c + r(x) \quad (2.3)$$

where $p(x) = \frac{u}{D}$, $q(x) = \frac{R}{D}$, and $r(x) = \frac{-Q}{D}$.

2.2.1 Numerical Method

At $x_i \forall i = 1, 2, \dots, N$, where N is the number of nodes, the equation (2.3) becomes

$$\frac{d^2c}{dx_i^2} = p(x_i) \frac{dc}{dx_i} + q(x_i)c + r(x_i) \quad (2.4)$$

where x_i be the interior mesh points. Using the central difference method,

$$\left(\frac{c_{i+1} - 2c_i + c_{i-1}}{h^2} \right) - p(x_i) \left(\frac{c_{i+1} - c_{i-1}}{2h} \right) - q(x_i)c_i = r(x_i) \quad (2.5)$$

$$\Rightarrow -\left(1 + \frac{h}{2}p(x_i)c_{i-1}\right) - (-2 - h^2q(x_i))c_i - \left(1 - \frac{h}{2}p(x_i)c_{i+1}\right) = -h^2r(x_i) \quad (2.6)$$

$$\Rightarrow c_{i+1} = \frac{2(2 + qh^2)}{2 - ph}c_i - \frac{2 + ph}{2 - ph}c_{i-1} + \frac{2h^2r}{2 - ph} \quad (2.7)$$

The resulting equation system is written in tri-diagonal $N \times N$ -matrix form as

$$[K]\{c\} = \{G\} \quad (2.8)$$

where

$$\{c\} = \begin{Bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_N \end{Bmatrix} \quad \text{and} \quad \{G\} = \begin{Bmatrix} -h^2r(x_1) + (1 + \frac{h}{2}p(x_1)c_0) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -h^2r(x_N) + ((1 + \frac{h}{2}p(x_N))(2ht_0)) \end{Bmatrix}$$

The values of $[K]$ are k_{ij} for each $1 \leq i, j \leq N$. Then, from (2.8)

$$\{\tilde{c}\} = [B]\{G\} \quad (2.9)$$

where B be the inverse matrix of K and we obtain

$$\tilde{c}_i = \sum_{j=1}^N b_{ij}g_j, \text{ where } 1 \leq i \leq N \quad (2.10)$$

2.2.2 Optimal Control of Cost

Let r_α be the COD concentration that is eliminated at inflow sites and x_β be the observation nodes. Consequently, $g_\alpha - r_\alpha$ is the pollutant concentration following partial purification [4, 11, 23]. Then

$$\tilde{c}_\beta = b_{\beta 1}g_1 + \dots + b_{\beta \alpha}(g_\alpha - r_\alpha) + \dots + b_{\beta N}g_N \quad (2.11)$$

Let C_{ST} be the standard COD concentration. The level of water quality \tilde{c}_β must be equal to or lower than the standard level. The constraints are

$$\tilde{c}_\beta = \sum_{i=1}^m b_{\beta i}g_i + \sum_{j=1}^n b_{\beta \alpha_j}(g_{\alpha_j} - r_{\alpha_j}) \leq C_{ST} \quad (2.12)$$

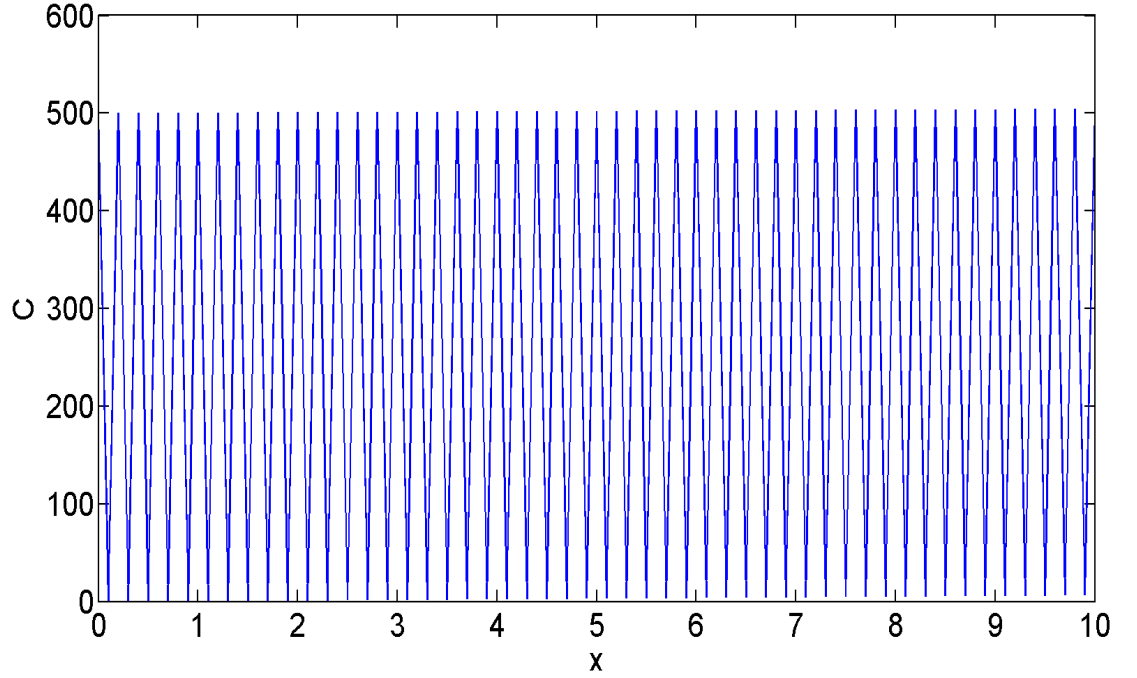


Figure 2.2: Numerical Solution of ADE by FTCS(2.7)

where n is the number of input points and m is the number of observation points. The system's cost of waste-water treatment is the objective function J , so

$$\text{Min}J(x) = \sum_{j=1}^m w_j r_{\alpha_j} \quad (2.13)$$

where w_j is the unit cost of waste-water treatment. By simplex method, the optimal control problem can be solved.

2.3 Result and Discussion

Here, we solve one-dimensional steady advection-diffusion-reaction equation numerically by using Finite Difference Method (FDM). Also, we analyze chemical oxygen demand (COD) concentration with the variation of different parameter values. Assume that there are different plants which discharge waste-water into the stream at certain distances so that the COD concentrations of the wastewater are also different which is shown in the figure 2.2 described by the equation (2.7).

2.4 Conclusion

We solve one-dimensional steady advection-dispersion-reaction equation. We use Finite Difference Method (FDM) to solve this equation numerically. Also, we examine chemical oxygen demand (COD) concentration in the variation of different parameters. Furthermore, we formulate to minimize the cost of waste-water treatment.

Chapter 3

Advection-Dispersion Equation for Pollutant Concentration

3.1 Introduction

In the present situation water pollution in rivers has been subject of study for researchers functioning in environment science, geology, mathematics etc. If important hydraulic and chemical processes are studied together, analytical solutions of mathematical models describing pollutant are hardly possible [32]. Now a days mathematical models have been used widely for acceptable tools of water quality management. The purpose of this thesis is to promote analytical solution of advection-dispersion equation. It is a particular case of research presented by Pimpunchat et al. [20]. The poor water quality in Bagmati River in Kathmandu, Nepal was the motivation for this study. The rapid urban growth, low levels of awareness, continuous dumping of solid wastes, domestic sewage, industrial waste, insufficient waste-water treatment facilities are the main reasons for pollution in the Bagmati River [29].

3.2 Concentration of Pollutant in One Region

3.2.1 Mathematical Model

Here pollutant concentration is described by advection-dispersion equation [18, 19, 32]:

$$\frac{\partial(AC)}{\partial t} = D \frac{\partial^2(AC)}{\partial x^2} - \frac{\partial(uAC)}{\partial x} - k_1 \frac{X}{X+k} AC + qH(x); \quad 0 \leq x < L, t > 0 \quad (3.1)$$

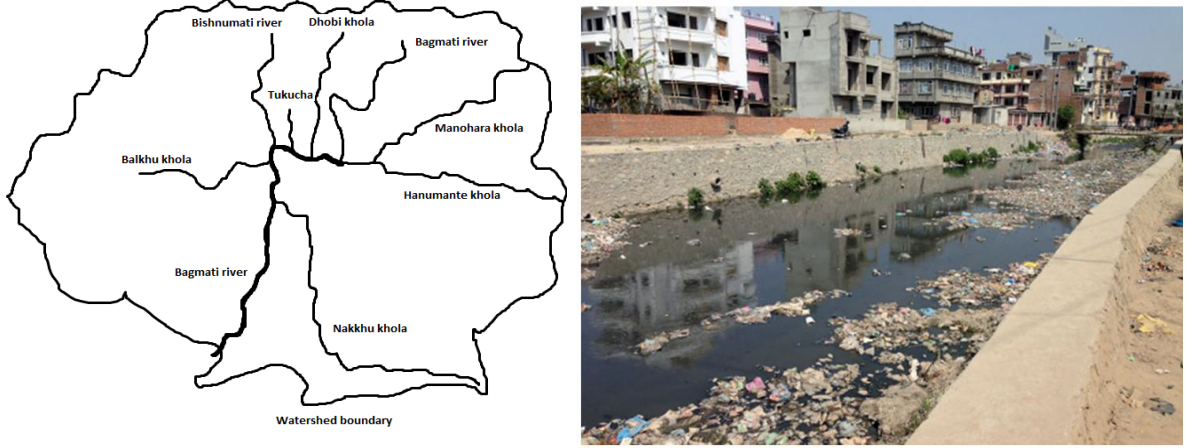


Figure 3.1: Map of Bagmati River and its tributaries (Left) and Polluted Bagmati River (Right) [29]

the Heaviside function is denoted by $H(x)$ and defined by

$$H(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

Here, u is the water velocity in the direction of x , C is the pollutant concentration, k_1 is the pollutant degradation rate coefficient, D is the pollutant dispersion coefficient in the direction of x , q is the added pollutant rate along the river, X is the concentration of dissolved oxygen in the river, k is the half-saturated oxygen demand concentration for pollutant decay, and A is the cross-sectional area of river. We take the parameters A , q , D , k_1 as constants [14].

In the general case when $k \neq 0$, it is not possible to apply Laplace transform [32]. We apply Laplace transformation by taking $k = 0$. In this model we take zero dispersion D (i.e. $D = 0$) [20]. Using these above conditions, the model equation (3.1) becomes:

$$\frac{\partial(AC)}{\partial t} = -\frac{\partial(uAC)}{\partial x} - k_1AC + q; \quad 0 \leq x < L, t > 0. \quad (3.3)$$

Let us consider $u(x) = 1 + ax$ [7, 18] for water velocity, where a is non-zero real constant has the dimension of inverse of space variable.

Equation (3.3) can be written as

$$\frac{\partial(C)}{\partial t} = -u\frac{\partial(C)}{\partial x} - Ca - k_1C + \frac{q}{A}; \quad 0 \leq x < L, t > 0. \quad (3.4)$$

These are the initial and boundary conditions:

$$C(x, 0) = 0; \quad x \geq 0 \quad (3.5)$$

$$C(0, t) = p; \quad t > 0 \quad (3.6)$$

where the pollutant concentration is $C(x, t)$ when dispersion coefficient $D = 0$. The initial rate of pollution is supposed to be zero. At the origin p is the rate of pollution.

3.2.2 Analytical Solution

The definition of the Laplace transformation method is an equation (3.7). To obtain the analytical solution, it is used. One can define the Laplace transformation as: If any function defined in $a \leq x \leq b$ and $t > 0$ is $f(x, t)$, then its Laplace transform with respect to t is denoted by:

$$L\{f(x, t)\} = F(x, s) = \int_0^{\infty} e^{-st} f(x, t) dt, \quad s \succ 0 \quad (3.7)$$

where s is the transform variable [13]. The inverse Laplace transformation is defined by:

$$L^{-1}\{F(x, s)\} = f(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(x, s) e^{-st} ds, \quad c \succ 0 \quad (3.8)$$

Applying Laplace transformation to equation (3.4) gives:

$$\begin{aligned} s\tilde{C}(x, s) - C(x, 0) &= -u \frac{\partial \tilde{C}}{\partial x} - \tilde{C}(x, s)a - k_1 \tilde{C}(x, s) + \frac{q}{As} \\ \Rightarrow s\tilde{C}(x, s) - C(x, 0) &= -u \frac{\partial \tilde{C}}{\partial x} - (a + k_1)\tilde{C}(x, s) + \frac{q}{As} \end{aligned}$$

Using (3.5), we get,

$$\begin{aligned} s\tilde{C}(x, s) &= -u \frac{\partial \tilde{C}}{\partial x} - (a + k_1)\tilde{C}(x, s) + \frac{q}{As} \\ \Rightarrow \frac{\partial \tilde{C}}{\partial x} + \left(\frac{s + a + k_1}{1 + ax} \right) \tilde{C} &= \frac{q}{As(1 + ax)} \end{aligned}$$

Integrating factor ($I.F.$) = $e^{\int \left(\frac{s+a+k_1}{1+ax} \right) dx} = (1 + ax)^{\left(\frac{s+a+k_1}{a} \right)}$ Thus

$$\begin{aligned} \tilde{C}(1 + ax)^{\left(\frac{s+a+k_1}{a} \right)} &= \frac{q}{As} \int (1 + ax)^{\left(\frac{s+k_1}{a} \right)} dx \\ \Rightarrow \tilde{C}(1 + ax)^{\left(\frac{s+a+k_1}{a} \right)} &= \frac{aq}{As(s + a + k_1)} (1 + ax)^{\left(\frac{s+a+k_1}{a} \right)} + C_1 \end{aligned}$$

where C_1 is the constant of integration.

Therefore

$$\begin{aligned} \tilde{C}(x, s) &= \frac{aq}{As(s + a + k_1)} \\ &\quad + C_1(1 + ax)^{-\left(\frac{s+a+k_1}{a} \right)} \end{aligned} \quad (3.9)$$

Taking laplace transform of (3.6)

$$\tilde{C}(0, s) = \frac{p}{s} \quad (3.10)$$

Using (3.10) on (3.9), we get,

$$\frac{p}{s} = \frac{aq}{As(s+a+k_1)} + C_1$$

Therefore

$$C_1 = \frac{p}{s} - \frac{aq}{As(s+a+k_1)}$$

Thus

$$\begin{aligned} \tilde{C}(x, s) &= \frac{aq}{As(s+a+k_1)} \\ &\quad + \frac{p}{s}(1+ax)^{-\left(\frac{s+a+k_1}{a}\right)} \\ &\quad - \frac{aq}{As(s+a+k_1)}(1+ax)^{-\left(\frac{s+a+k_1}{a}\right)} \end{aligned} \quad (3.11)$$

The inverse of Laplace transform is

$$\begin{aligned} C(x, t) &= \frac{aq}{A} \left[\frac{1}{k_1+a} - \frac{1}{k_1+a} e^{-(k_1+a)t} \right] \\ &\quad + p(1+ax)^{-\left(\frac{k_1+a}{a}\right)} H \left(t + \frac{\log(1+ax)}{a} \right) \\ &\quad - \frac{aq}{A} \left[\left(\frac{1}{k_1+a} - \frac{1}{k_1+a} e^{-(k_1+a)t} \right) (1+ax)^{-\left(\frac{k_1+a}{a}\right)} H \left(t + \frac{\log(1+ax)}{a} \right) \right] \end{aligned} \quad (3.12)$$

where $H \left(t + \frac{\log(1+ax)}{a} \right)$ is Heaviside function.

$$\begin{aligned} C(x, t) &= \frac{aq}{A(k_1+a)} \\ &\quad - \frac{aq}{A(k_1+a)} e^{-(k_1+a)t} \\ &\quad + p(1+ax)^{-\left(\frac{k_1+a}{a}\right)} \\ &\quad - \frac{aq}{A(k_1+a)} (1+ax)^{-\left(\frac{k_1+a}{a}\right)} \\ &\quad + \frac{aq}{A(k_1+a)} e^{-(k_1+a)t} (1+ax)^{-\left(\frac{k_1+a}{a}\right)} \end{aligned} \quad (3.13)$$

which can be written as

$$C(x^*, t) = q^* - q^* e^{-k_1^* t} + px^* - q^* x^* + q^* e^{-k_1^* t} x^*$$

where

$$k_1^* = (k_1 + a), \quad q^* = \frac{aq}{Ak_1^*}, \quad x^* = (1+ax)^{-\frac{k_1^*}{a}}$$

$$\therefore C(x^*, t) = px^* + q^*(1-x^*)(1-e^{-k_1^* t}) \quad (3.14)$$

3.2.3 Steady State Case

Equation (3.13) becomes:

$$\begin{aligned}
 C(x) = & \frac{aq}{A(k_1 + a)} \\
 & + p(1 + ax)^{-\left(\frac{k_1+a}{a}\right)} \\
 & - \frac{aq}{A(k_1 + a)}(1 + ax)^{-\left(\frac{k_1+a}{a}\right)}
 \end{aligned} \tag{3.15}$$

If we take $p = 0$ equation (3.15) becomes similar to the result derived by Wadi et al. [32].

3.3 Concentration of Pollutant in Two Regions

3.3.1 Mathematical Model

Pollutant concentration in one-dimension can be described by advection-dispersion equations [19]:

$$\frac{\partial(AC_1)}{\partial t} = D \frac{\partial^2(AC_1)}{\partial x^2} - \frac{\partial(uAC_1)}{\partial x} - k_1 \frac{X}{X+k} AC_1; \quad x \leq 0, t > 0 \tag{3.16}$$

$$\frac{\partial(AC_2)}{\partial t} = D \frac{\partial^2(AC_2)}{\partial x^2} - \frac{\partial(uAC_2)}{\partial x} - k_1 \frac{X}{X+k} AC_2 + qH(x); \quad 0 \leq x < L, t > 0 \tag{3.17}$$

Here C_1 and C_2 are pollutant concentrations in two regions respectively.

If we take $k \neq 0$, it is not possible to apply Laplace transform [32]. We apply Laplace transformation by taking $k = 0$.

Initial and boundary conditions together with equations (3.16) and (3.17) are:

$$C_1(x, 0) = 0, C_2(x, 0) = 0; \quad x \geq 0 \tag{3.18}$$

$$C_1(0, t) = C_2(0, t); \quad t > 0 \tag{3.19}$$

$$\frac{d}{dx} C_1(0, t) = \frac{d}{dx} C_2(0, t); \quad t > 0 \tag{3.20}$$

3.3.2 Analytical Solution

The definition of the Laplace transformation method is an equation (3.7). To solve the advection-dispersion equation, it is used. Applying this technique to equations (3.16) and

(3.17):

$$D \frac{d^2 \tilde{C}_1(x, s)}{dx^2} - u \frac{d \tilde{C}_1(x, s)}{dx} + \tilde{C}_1(x, 0) - (k_1 + s) \tilde{C}_1(x, s) = 0; \quad x \leq 0, s > 0 \quad (3.21)$$

$$D \frac{d^2 \tilde{C}_2(x, s)}{dx^2} - u \frac{d \tilde{C}_2(x, s)}{dx} + \tilde{C}_2(x, 0) - (k_1 + s) \tilde{C}_2(x, s) + \frac{q}{As} = 0; \quad x \geq 0, s > 0 \quad (3.22)$$

The boundary conditions (3.19) and (3.20) become:

$$\tilde{C}_1(0, s) = \tilde{C}_2(0, s), \quad (3.23)$$

$$\frac{d}{dx} \tilde{C}_1(0, s) = \frac{d}{dx} \tilde{C}_2(0, s). \quad (3.24)$$

Applying (3.18), solutions of equations (3.21) and (3.22) are:

$$\begin{aligned} \tilde{C}_1(x, s) = & \frac{D}{(k_1 + s)} \alpha_1 e^{(\delta + \sqrt{(\alpha+s)/D})x} \\ & + \frac{D}{(k_1 + s)} \alpha_2 e^{(\delta - \sqrt{(\alpha+s)/D})x} \end{aligned} \quad (3.25)$$

$$\begin{aligned} \tilde{C}_2(x, s) = & \frac{q}{As(k_1 + s)} + \frac{D}{(k_1 + s)} \alpha_3 e^{(\delta + \sqrt{(\alpha+s)/D})x} \\ & + \frac{D}{(k_1 + s)} \alpha_4 e^{(\delta - \sqrt{(\alpha+s)/D})x} \end{aligned} \quad (3.26)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are constants. To determine the constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ we use the boundary conditions (3.23) and (3.24). Hence, equations (3.25) and (3.26) are given by:

$$\tilde{C}_1(x, s) = \frac{q}{As(k_1 + s)} \left(\frac{\sqrt{(\alpha + s)/D} - \delta}{2\sqrt{\alpha + s}/D} \right) e^{(\delta + \sqrt{(\alpha+s)/D})x}, \quad x \leq 0, \quad (3.27)$$

$$\tilde{C}_2(x, s) = \frac{q}{As(k_1 + s)} - \frac{q}{As(k_1 + s)} \left(\frac{\sqrt{(\alpha + s)/D} + \delta}{2\sqrt{\alpha + s}/D} \right) e^{(\delta - \sqrt{(\alpha+s)/D})x}, \quad x \geq 0, \quad (3.28)$$

where $\alpha = D\beta^2$, $\delta = \frac{u}{2D}$, $\beta = \sqrt{\delta^2 + \frac{k_1}{D}}$. Using inverse Laplace transformation in equations (3.27) and (3.28),

we get,

$$\begin{aligned}
C_1(x, t) &= \frac{q}{4k_1A} e^{(\delta-\beta)x} \operatorname{erfc} \left(\frac{-x}{2\sqrt{Dt}} + \sqrt{\alpha t} \right) \\
&\quad + \frac{q}{4k_1A} e^{(\delta+\beta)x} \operatorname{erfc} \left(\frac{-x}{2\sqrt{Dt}} - \sqrt{\alpha t} \right) \\
&\quad - \frac{q}{2k_1A} e^{-k_1t} \operatorname{erfc} \left(\frac{-x}{2\sqrt{Dt}} + \sqrt{(\alpha - k_1)t} \right) \\
&\quad - \frac{q}{4k_1A} \frac{\delta}{\beta} e^{(\delta+\beta)x} \operatorname{erfc} \left(\frac{-x}{2\sqrt{Dt}} - \sqrt{\alpha t} \right) \\
&\quad - \frac{q}{4k_1A} \frac{\delta}{\beta} e^{(\delta-\beta)x} \operatorname{erfc} \left(\frac{-x}{2\sqrt{Dt}} + \sqrt{\alpha t} \right), \quad x \leq 0,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
C_2(x, t) &= \frac{q}{k_1A} - \frac{q}{k_1A} e^{(-k_1t)} \\
&\quad - \frac{q}{4k_1A} e^{(\delta+\beta)x} \operatorname{erfc} \left(\frac{x}{2\sqrt{Dt}} + \sqrt{\alpha t} \right) \\
&\quad - \frac{q}{4k_1A} e^{(\delta-\beta)x} \operatorname{erfc} \left(\frac{x}{2\sqrt{Dt}} - \sqrt{\alpha t} \right) \\
&\quad + \frac{q}{2k_1A} e^{(-k_1t)} \operatorname{erfc} \left(\frac{x}{2\sqrt{Dt}} - \sqrt{(\alpha - k_1)t} \right) \\
&\quad - \frac{q}{4k_1A} \frac{\delta}{\beta} e^{(\delta-\beta)x} \operatorname{erfc} \left(\frac{x}{2\sqrt{Dt}} - \sqrt{\alpha t} \right) \\
&\quad + \frac{q}{4k_1A} \frac{\delta}{\beta} e^{(\delta+\beta)x} \operatorname{erfc} \left(\frac{x}{2\sqrt{Dt}} + \sqrt{\alpha t} \right), \quad x \geq 0,
\end{aligned} \tag{3.30}$$

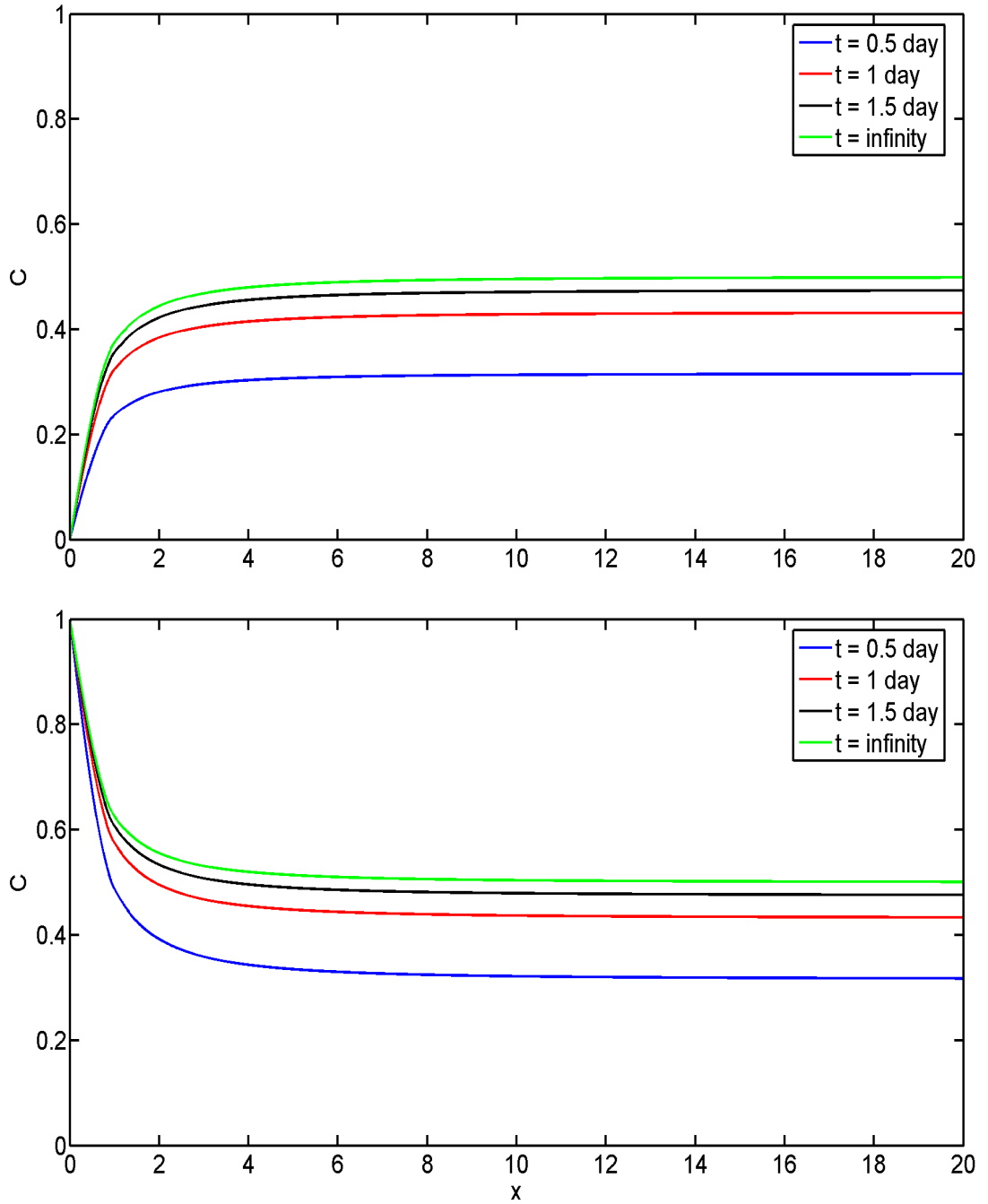


Figure 3.2: Analytical solution with dispersion for C at different time described by the equation (3.13) top: $p=0$, bottom: $p=1$.

3.4 Result and Discussion

Here, analytical solution to unsteady advection-dispersion equation is derived. Advection-dispersion equation describes pollutant concentration $C(x, t)$. Here we have used Laplace transformation technique to solve the equation. We have expressed variation of C with

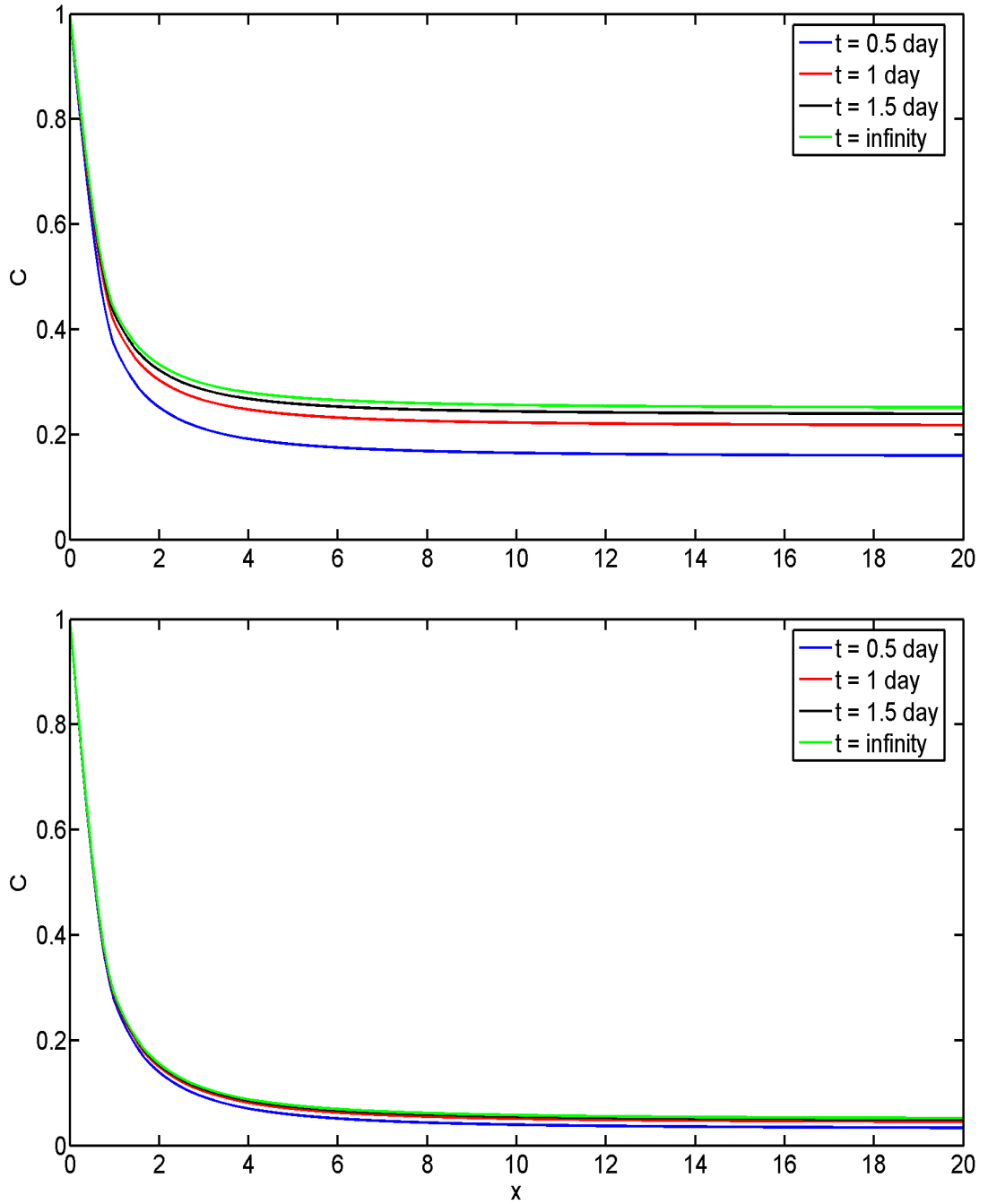


Figure 3.3: Analytical solution with dispersion for C at different time described by the equation (3.13) top: $q=0.5$, bottom: $q=0.1$.

time and space. Pollutant concentration is given by equation (3.13). Figure 3.2 on the top shows variation of C described by equation (3.13). We set parameters A, a, q, k to be 1 and $p = 0$ [20]. From figure 3.2 on the top, for $x \geq 0$ as t increases C increases. Pollutant concentration C extends to its maximum value when $t \rightarrow \infty$. Generally when x increases concentration of pollutant increases. From figure 3.2 on the bottom, C decreases when x

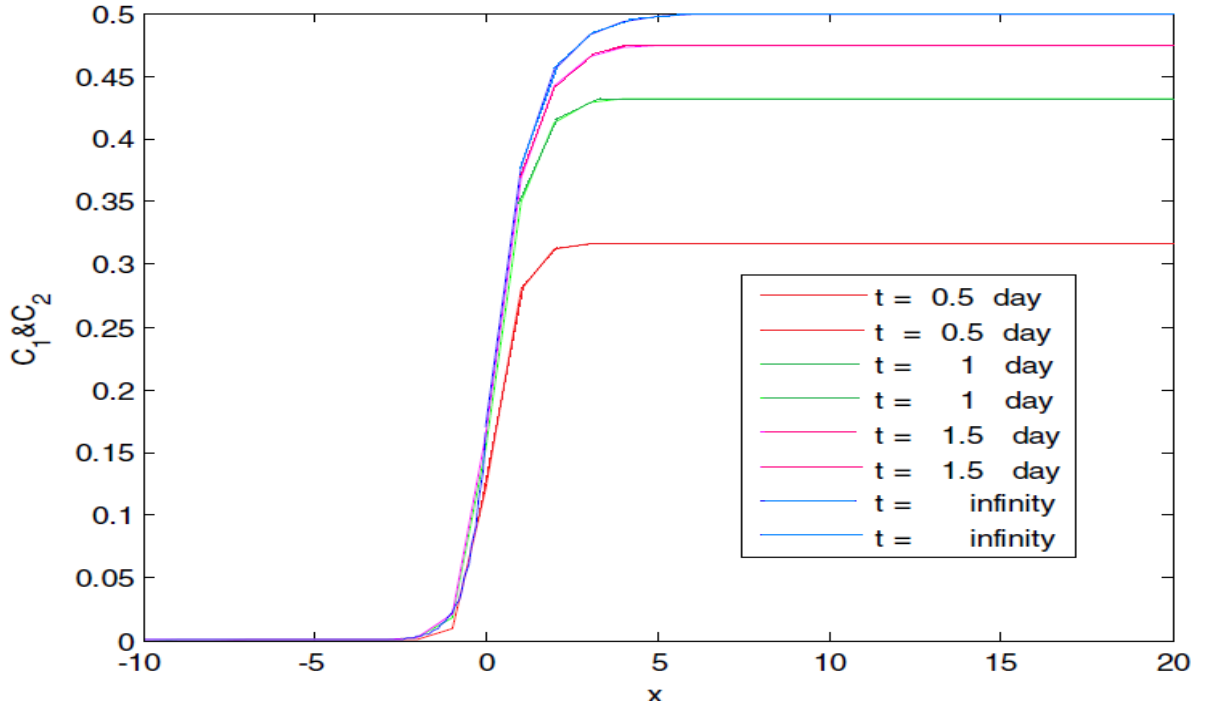


Figure 3.4: Concentrations C_1 and C_2 given by equations (3.29) and (3.30).

increases at any time.

The top of figure 3.3 displays the fluctuation of C along the river from source up to sink at different times $t = 0.5, 1.0, 1.5$ (days) and $t \rightarrow \infty$ and added pollutant rate along the river $q = 0.5$ and figure 3.3 on the bottom shows the variation of C along the river with $q = 0.1$. From figure 3.3, if the added pollutant rate along the river q decreases, the variations of C at different times come to near at each other. If q is in very small amount the variation of C along the river at different times coincide to each other.

Figure 3.4 shows variation of C_1 and C_2 . Here we consider a case $D \neq 0$. The figure 3.4 describes equations (3.29) and (3.30). We set parameters A, q, a, k to be 1 and $p = 0$ [20]. When $x \leq 0$, t has a negligibly little impact on C_1 . When $x \geq 0$, C_2 increases as t increases. When $t \rightarrow \infty$, concentrations reach to its maximum.

3.5 Conclusion

Analytical solution to unsteady advection-dispersion equation is derived. Advection-dispersion equation describes pollutant concentration $C(x, t)$. Here we use Laplace transformation technique to solve the equation. We have obtained analytic unsteady solution by taking the water velocity u in the x -direction as a linear function of x and dispersion

coefficient D as zero in case of concentration of pollutant in one region. Numerical studies shows variation of C with time t . If the added pollutant rate along the river q is in very small amount, the variation of C along the river at different times coincide to each other. In case of concentration of pollutant in two regions, analytical solutions are constructed by taking dispersion coefficient D as non-zero.

Chapter 4

Advection-Dispersion Equation for Concentrations of Pollutant and Dissolved Oxygen

4.1 Introduction

A coupled system of advection-dispersion equations based water pollution model is presented that incorporates different parameters. The major concerns of this research are to observe concentrations of pollutant and dissolved oxygen in the river. One dimensional model is used to observe concentrations. In this model coupled advection-dispersion equations are solved by taking dispersion coefficient as zero and non-zero.

4.2 Mathematical Model

We consider coupled non-linear equations for the concentrations of pollutant $C(x, t)$ and dissolved oxygen $X(x, t)$. When oxygen reacts with pollutant, coupling situation appears. To observe concentrations, one dimensional model is used. We take dimension along the length of river. So the concentrations $C(x, t)$ and $X(x, t)$ satisfy advection-dispersion equations. The coupled equations [15, 20] are expressed in one dimension as

$$\frac{\partial(AC)}{\partial t} = D \frac{\partial^2(AC)}{\partial x^2} - \frac{\partial(uAC)}{\partial x} - k_1 \frac{X}{X+k} AC + qH(x); \quad 0 \leq x < L, t > 0 \quad (4.1)$$

$$\frac{\partial(AX)}{\partial t} = D_x \frac{\partial^2(AX)}{\partial x^2} - \frac{\partial(uAX)}{\partial x} - k_2 \frac{X}{X+k} AC + \alpha(S - X); \quad 0 \leq x < L, t > 0 \quad (4.2)$$

where $H(x)$ is the Heaviside function defined by

$$H(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ 0 & \text{otherwise.} \end{cases} \quad (4.3)$$

Here, u is the water velocity in the direction of x , D is the pollutant's dispersion coefficient in the same direction, D_x is the dissolved oxygen's dispersion coefficient in the same direction, S is the saturation oxygen concentration, k_1 is the pollutant's degradation rate coefficient, k_2 is the dissolved oxygen's degradation rate coefficient, The mass transfer of oxygen from air to water is represented by α , the additional pollutant rate along the river is represented by q , the half-saturated oxygen demand concentration for pollutant decay is represented by k , and the cross-section of the river is represented by A .

We take the parameters A , u , q , α and S as constants [14, 32].

4.3 Steady State Analysis

Here we consider various special cases given in figure 4.1. The model we used is steady-state model.

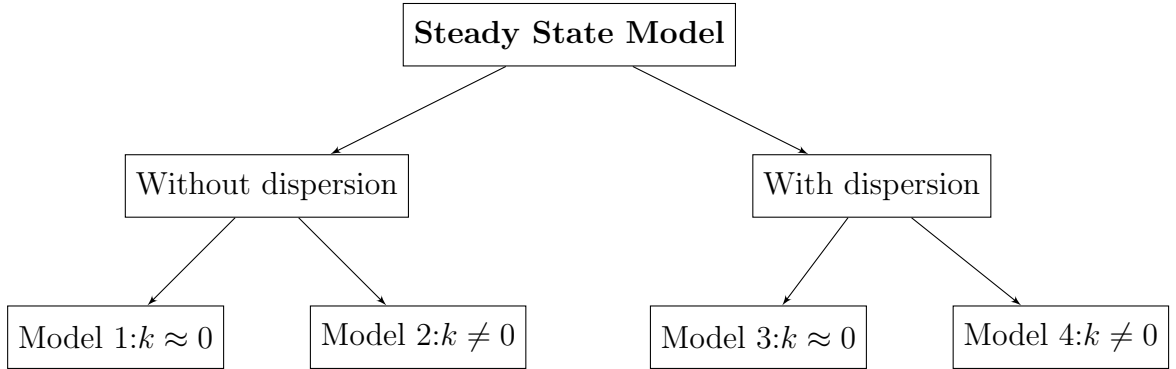


Figure 4.1: Special Cases of Model [20]

Model 1. We have no dispersion ($D = 0, D_x = 0$) in this model and k is negligible (*i.e* $k \approx 0$)[18].

$$\frac{d(uAC(x))}{dx} = -k_1AC(x) + q; \quad (x > 0, t > 0) \quad (4.4)$$

$$\frac{d(uAX(x))}{dx} = -k_2AC(x) + \alpha(S - X(x)); \quad (x > 0, t > 0) \quad (4.5)$$

We consider k negligible ($k \approx 0$). The boundary conditions are $C(0) = 0$ and $X(0) = S$.

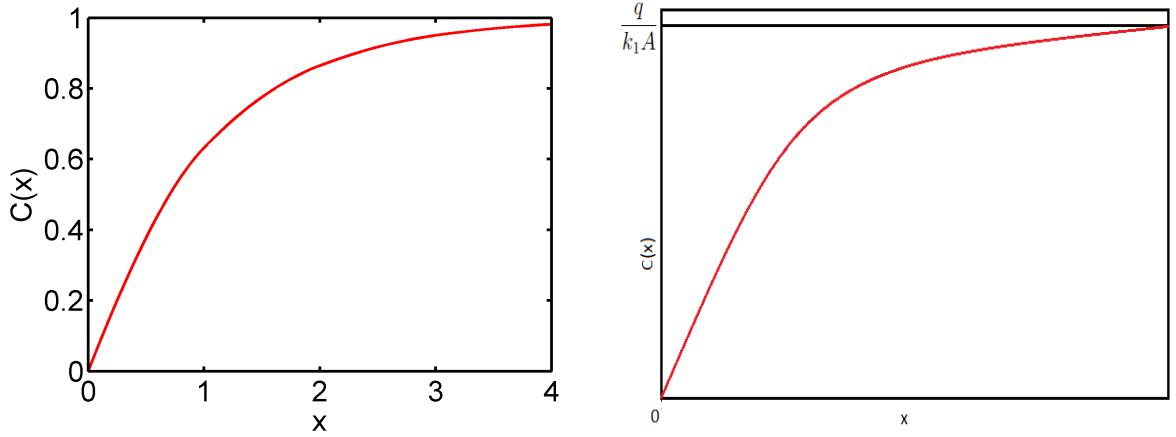


Figure 4.2: Solution for C without dispersion and $k \approx 0$.

From equation (4.4)

$$uA \frac{dC(x)}{dx} + k_1 A C(x) = q$$

$$\Rightarrow \frac{dC}{dx} + \left(\frac{k_1}{u} \right) C = \frac{q}{uA}$$

Integrating Factor

$$I.F. = \exp \left(\int \frac{k_1}{u} dx \right) = \exp \left(\frac{k_1 x}{uA} \right)$$

The solution is

$$C \exp \left(\frac{k_1 x}{u} \right) = \frac{q}{uA} \int \exp \left(\frac{k_1 x}{u} \right) dx + c_1$$

$$\Rightarrow C \exp \left(\frac{k_1 x}{u} \right) = \frac{qu}{uk_1 A} \exp \left(\frac{k_1 x}{u} \right) + c_1$$

$$\Rightarrow C \exp \left(\frac{k_1 x}{u} \right) = \frac{q}{k_1 A} \exp \left(\frac{k_1 x}{u} \right) + c_1$$

Using $C(0) = 0$,

$$c_1 = \frac{-q}{k_1 A}$$

The pollutant concentration is

$$C(x) = \frac{q}{k_1 A} \left\{ 1 - \exp \left(\frac{-k_1 x}{u} \right) \right\} \quad (4.6)$$

and so the limit is given by

$$C(x) = \lim_{x \rightarrow \infty} \frac{q}{k_1 A} \quad (4.7)$$

Figure 4.2 shows the variation of C in the range $0 \leq x \leq 4$ described by equation (4.6).

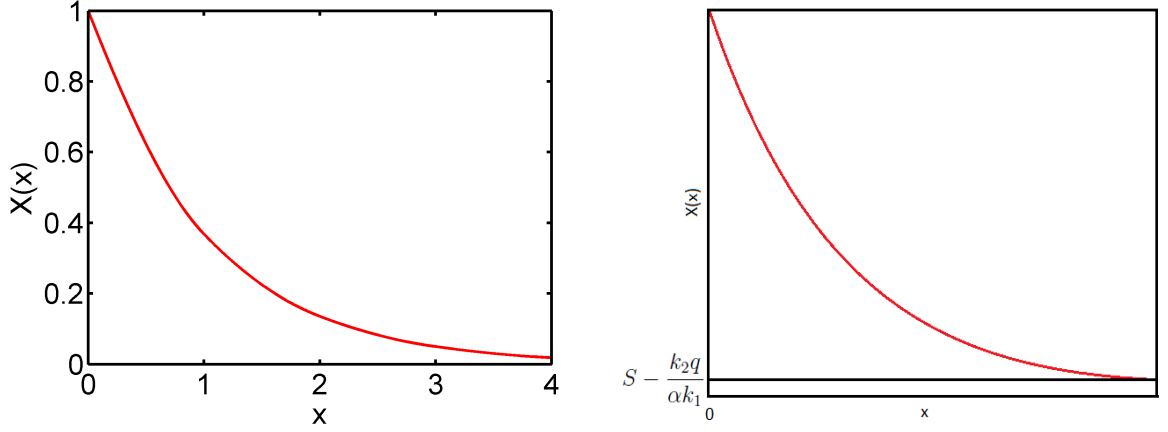


Figure 4.3: Solution for X without dispersion and $k \approx 0$

We consider parameters A, u, q, k_1 to be 1 [32] to test our model. Figure 4.2 shows that C increases as x increases. It reaches to maximum as $x \rightarrow \infty$. Generally concentration of pollutant increases as x increases. From equation (4.5)

$$uA \frac{dX}{dx} = -k_2AC + \alpha(S - X)$$

$$\Rightarrow \frac{dX}{dx} + \left(\frac{\alpha}{uA}\right) X = \left(\frac{\alpha S}{uA} - \frac{k_2C}{u}\right)$$

Integrating Factor

$$I.F. = \exp\left(\int \frac{\alpha}{uA} dx\right) = \exp\left(\frac{\alpha x}{uA}\right)$$

The solution is

$$X \exp\left(\frac{\alpha x}{uA}\right) = \int \left(\frac{\alpha S}{uA} - \frac{k_2C}{u}\right) \exp\left(\frac{\alpha x}{uA}\right) dx + c_2$$

$$\Rightarrow X \exp\left(\frac{\alpha x}{uA}\right) = \int \left[\frac{\alpha S}{uA} - \frac{k_2}{u} \frac{q}{k_1 A} \left\{1 - \exp\left(\frac{-k_1 x}{u}\right)\right\}\right] \exp\left(\frac{\alpha x}{uA}\right) dx + c_2$$

$$\Rightarrow X \exp\left(\frac{\alpha x}{uA}\right) = \int \left(\frac{\alpha S}{uA} - \frac{k_2 q}{u k_1 A}\right) \exp\left(\frac{\alpha x}{uA}\right) dx + \int \frac{k_2 q}{u k_1 A} \exp\left(\frac{\alpha - k_1 A}{uA} x\right) dx + c_2$$

$$\Rightarrow X \exp\left(\frac{\alpha x}{uA}\right) = \left(\frac{\alpha S}{uA} - \frac{k_2 q}{u k_1 A}\right) \frac{uA}{\alpha} \exp\left(\frac{\alpha x}{uA}\right) + \frac{k_2 q}{u k_1 A} \left(\frac{uA}{\alpha - k_1 A}\right) \exp\left(\frac{\alpha - k_1 A}{uA} x\right) + c_2$$

$$\Rightarrow X = S - \frac{k_2 q}{k_1 \alpha} + \frac{k_2 q}{k_1 (\alpha - k_1 A)} \exp\left(\frac{-k_1 x}{u}\right) + c_2 \exp\left(\frac{-\alpha x}{uA}\right)$$

Using $X(0) = S$,

$$c_2 = \frac{k_2 q}{k_1 \alpha} - \frac{k_2 q}{k_1 (\alpha - k_1 A)} = -\frac{k_2 q A}{\alpha (\alpha - k_1 A)}$$

$$\Rightarrow X = S - \frac{k_2 q}{\alpha k_1} + \frac{k_2 q}{k_1 (\alpha - k_1 A)} \exp\left(\frac{-k_1 x}{u}\right) - \frac{k_2 q A}{\alpha (\alpha - k_1 A)} \exp\left(\frac{-\alpha x}{uA}\right)$$

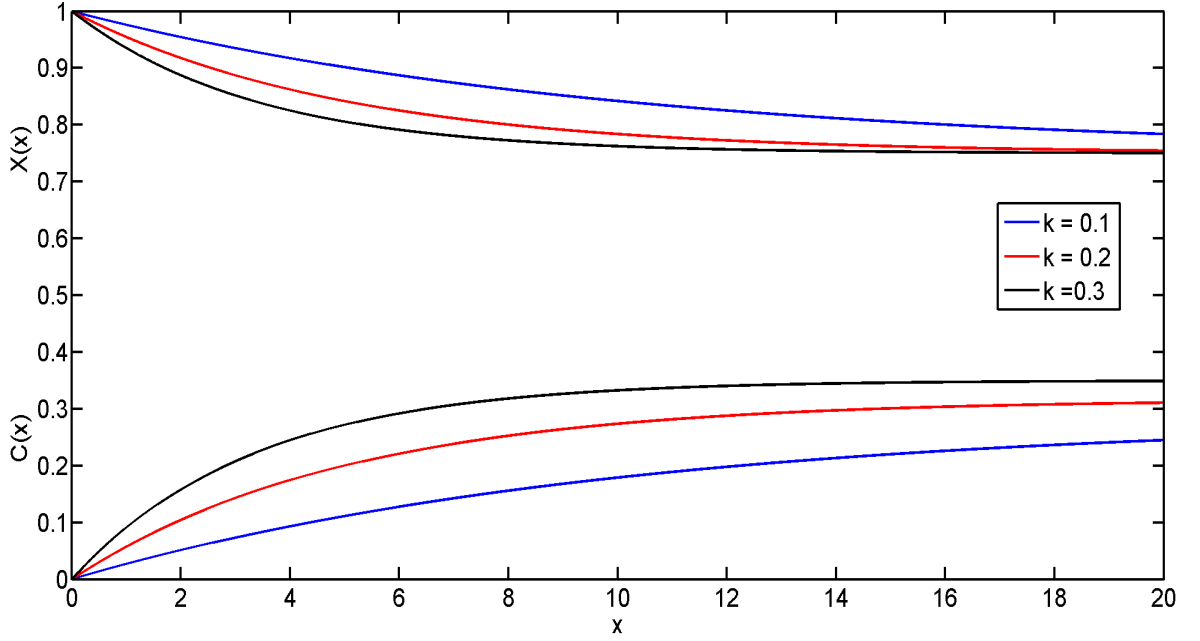


Figure 4.4: Solution for C and X with dispersion and $k \neq 0$.

The dissolved oxygen concentration is

$$\begin{aligned}
 X(x) = S - \frac{k_2 q}{\alpha k_1} &+ \frac{k_2 q}{k_1(\alpha - k_1 A)} \exp\left(\frac{-k_1 x}{u}\right) \\
 &- \frac{k_2 q A}{\alpha(\alpha - k_1 A)} \exp\left(\frac{-\alpha x}{u A}\right)
 \end{aligned} \quad (4.8)$$

and downstream

$$\lim_{x \rightarrow \infty} X(x) = S - \frac{k_2 q}{\alpha k_1} \quad (4.9)$$

This is shown in figure (4.3). The figure (4.3) shows that oxygen level decreases due to reaction with pollutants. Now the solutions for concentrations of pollutant and dissolved oxygen are

$$\lim_{x \rightarrow \infty} (C(x), X(x)) = \left(\frac{q}{k_1 A}, S - \frac{k_2 q}{\alpha k_1} \right) \quad (4.10)$$

Model 2. We have no dispersion ($D = 0, D_x = 0$) and k is non-zero (*i.e.* $k \neq 0$) [20].

$$\frac{d(uAC(x))}{dx} = -k_1 \frac{X(x)}{X(x) + k} AC(x) + q; \quad (x > 0, t > 0) \quad (4.11)$$

$$\frac{d(uAX(x))}{dx} = -k_2 \frac{X(x)}{X(x) + k} AC(x) + \alpha(S - X(x)); \quad (x > 0, t > 0) \quad (4.12)$$

Boundary conditions are $C(0) = 0$ and $X(0) = S$ which are same. The solutions are

$$\lim_{x \rightarrow \infty} (C(x), X(x)) = \left(\frac{q}{k_1 A} + \frac{\alpha k q}{k_2 \left(\frac{\alpha k_1 S}{k_2} - q \right)}, S - \frac{k_2 q}{\alpha k_1} \right) \quad (4.13)$$

The solutions depend upon k and q whereas in the model 1 these solutions depend on q only. If $q \geq \frac{\alpha k_1 S}{k_2}$, the solution does not exist. Figure 4.4 shows the variation of C and X in the range $0 \leq x \leq 20$ as k varies. To test our model we suppose $A, S, u, k_1, k_2, \alpha$ to be 1 and q to be $1/4$.

Model 3. We have dispersion terms $D \neq 0, D_x \neq 0$ and $k \approx 0$.

$$D \frac{d^2(AC)}{dx^2} - \frac{d(uAC)}{dx} - k_1 \frac{X}{X+k} AC + qH(x) = 0; \quad (x > L, t > 0) \quad (4.14)$$

$$D_x \frac{d^2(AX)}{dx^2} - \frac{d(uAX)}{dx} - k_2 \frac{X}{X+k} AC + \alpha(S-X) = 0; \quad (x > L, t > 0) \quad (4.15)$$

In this model ($k \approx 0$), the equations (4.14) and (4.15) become

$$D \frac{d^2(AC)}{dx^2} - \frac{d(uAC)}{dx} - k_1 AC + q = 0 \quad (4.16)$$

$$D_x \frac{d^2(AX)}{dx^2} - \frac{d(uAX)}{dx} - k_2 AC + \alpha(S-X) = 0 \quad (4.17)$$

From equation (4.16)

$$\begin{aligned} DA \frac{d^2 C}{dx^2} - uA \frac{dC}{dx} - k_1 AC + q &= 0 \\ \Rightarrow \frac{d^2 C}{dx^2} - \frac{u}{D} \frac{dC}{dx} - \frac{k_1}{D} C &= -\frac{q}{DA} \\ \Rightarrow \left(\tilde{D}^2 - \frac{u}{D} \tilde{D} - \frac{k_1}{D} \right) C &= -\frac{q}{DA} \end{aligned}$$

where $\tilde{D} = \frac{d}{dx}$, which is the second order differential equation.

Its Auxiliary Equation is

$$\begin{aligned} m^2 - \frac{u}{D} m - \frac{k_1}{D} &= 0 \\ \Rightarrow m &= \frac{\frac{u}{D} \pm \sqrt{\frac{u^2}{D^2} - 4 \left(-\frac{k_1}{D} \right)}}{2} \\ \Rightarrow m &= \frac{u}{2D} \pm \frac{\sqrt{u^2 + 4Dk_1}}{2D} \\ \Rightarrow m &= \delta \pm \beta \end{aligned}$$

where

$$\delta = \frac{u}{2D}$$

and

$$\beta = \frac{\sqrt{u^2 + 4Dk_1}}{2D}.$$

Therefore, the Complementary Function (C.F.) is

$$C.F. = c_1 e^{(\delta-\beta)x} + c_2 e^{(\delta+\beta)x}$$

and the Particular Integral (P.I.) is

$$\begin{aligned} P.I. &= \frac{1}{\tilde{D}^2 - \frac{u}{D}\tilde{D} - \frac{k_1}{D}} \left(-\frac{q}{DA} \right) \\ &\Rightarrow P.I. = \frac{q}{k_1 A} \end{aligned}$$

The general solution is

$$\begin{aligned} C(x) &= C.F. + P.I. \\ &\Rightarrow C(x) = c_1 e^{(\delta-\beta)x} + c_2 e^{(\delta+\beta)x} + \frac{q}{k_1 A} \end{aligned}$$

So the pollutant concentration is

$$C(x) = \begin{cases} \frac{q}{k_1 A} \left[1 - \left(\frac{\delta+\beta}{2\beta} \right) e^{(\delta-\beta)x} \right] & \text{if } x \geq 0 \\ \frac{q}{k_1 A} \left(\frac{\beta-\delta}{2\beta} \right) e^{(\delta+\beta)x} & \text{if } x < 0 \end{cases} \quad (4.18)$$

We use the conditions $C(\infty) < \infty$ and $C(-\infty) < \infty$ [19].

From equation (4.17)

$$\begin{aligned} AD_x \frac{d^2(X)}{dx^2} - uA \frac{d(X)}{dx} - k_2 AC + (\alpha S - \alpha X) &= 0 \\ \Rightarrow \frac{d^2 X}{dx^2} - \frac{u}{D_x} \frac{dX}{dx} - \frac{k_2}{D_x} C - \frac{\alpha}{AD_x} X + \frac{\alpha S}{AD_x} &= 0 \\ \Rightarrow \frac{d^2 X}{dx^2} - \frac{u}{D_x} \frac{dX}{dx} - \frac{\alpha}{AD_x} X &= \frac{k_2 C}{D_x} - \frac{\alpha S}{AD_x} \\ \Rightarrow \left(\tilde{D}^2 - \frac{u}{D_x} \tilde{D} - \frac{\alpha}{AD_x} \right) X &= \left(\frac{k_2 AC - \alpha S}{AD_x} \right) \end{aligned}$$

where $\tilde{D} = \frac{d}{dx}$, which is the second order differential equation.

Its Auxiliary Equation is

$$\begin{aligned} m^2 - \frac{u}{D_x} m - \frac{\alpha}{AD_x} &= 0 \\ \Rightarrow m &= \frac{\frac{u}{D_x} \pm \sqrt{\frac{u^2}{D_x^2} - 4 \left(-\frac{\alpha}{AD_x} \right)}}{2} \\ \Rightarrow m &= \frac{u}{2D_x} \pm \frac{\sqrt{u^2 + \frac{4\alpha D_x}{A}}}{2D_x} \end{aligned}$$

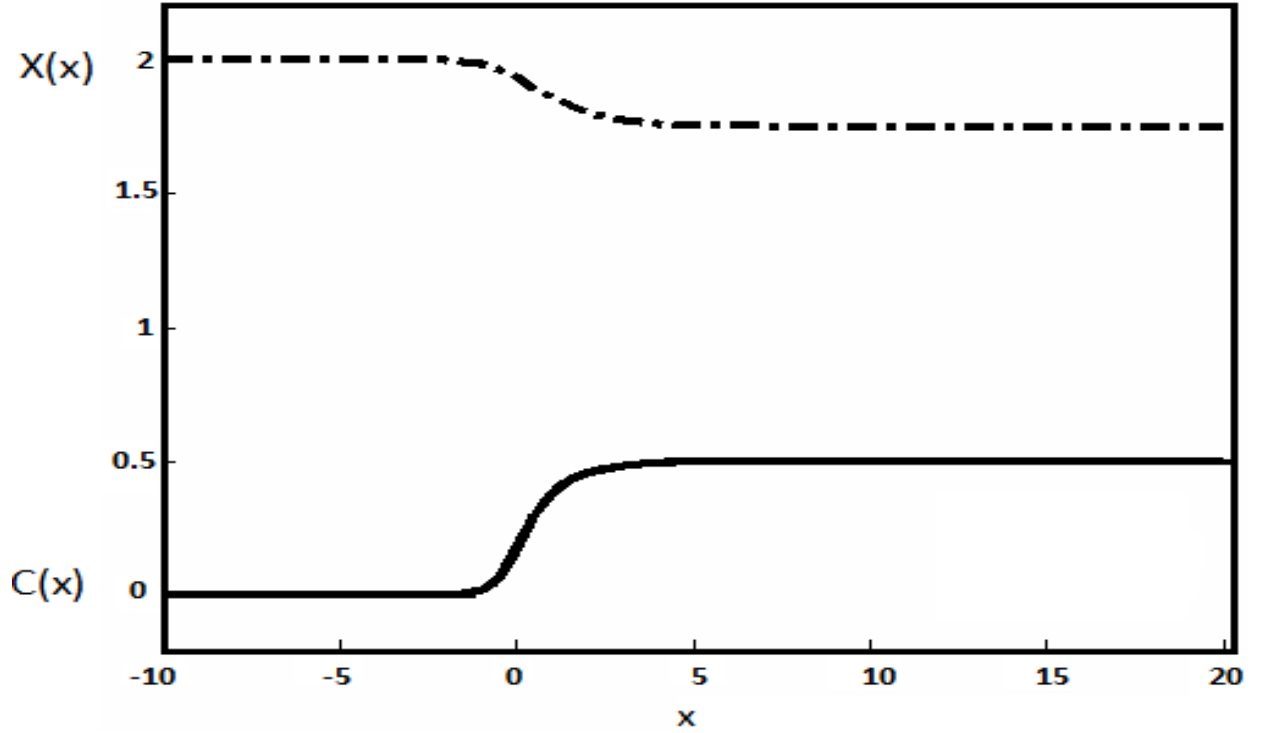


Figure 4.5: Solution for C and X with dispersion and $k \approx 0$.

$$\Rightarrow m = \gamma \pm \eta$$

where

$$\gamma = \frac{u}{2D_x}$$

and

$$\eta = \frac{\sqrt{u^2 + \frac{4\alpha D_x}{A}}}{2D_x}$$

Therefore the Complementary Function (C.F.) is

$$C.F. = c_3 e^{(\gamma-\eta)x} + c_4 e^{(\gamma+\eta)x}$$

and the Particular Integral (P.I.) is

$$P.I. = \frac{1}{\tilde{D}^2 - \frac{u}{D_x} \tilde{D} - \frac{\alpha}{AD_x}} \left(\frac{k_2 AC - \alpha S}{AD_x} \right)$$

where

$$C(x) = \begin{cases} \frac{q}{k_1 A} \left[1 - \left(\frac{\delta+\beta}{2\beta} \right) e^{(\delta-\beta)x} \right] & \text{if } x \geq 0 \\ \frac{q}{k_1 A} \left(\frac{\beta-\delta}{2\beta} \right) e^{(\delta+\beta)x} & \text{if } x < 0 \end{cases}$$

The general solution is

$$X(x) = C.F. + P.I.$$

So the dissolved oxygen concentration is

$$X(x) = \begin{cases} S - \frac{k_2q}{k_1\alpha} + \frac{k_2q}{k_1} \left[\left(\frac{\delta+\eta}{2\eta\alpha} - \frac{\delta+\beta}{4\beta\eta A^*} + \frac{\delta-\beta}{4\beta\eta B^*} \right) e^{(\gamma-\eta)x} - \frac{\delta+\beta}{2\beta A^*} x e^{(\delta-\beta)x} \right] & \text{if } x \geq 0 \\ S + \frac{k_2q}{k_1} \left[\left(\frac{\delta-\eta}{2\eta\alpha} - \frac{\delta+\beta}{4\beta\eta A^*} + \frac{\delta-\beta}{4\beta\eta B^*} \right) e^{(\gamma+\eta)x} - \frac{\delta-\beta}{2\beta B^*} x e^{(\delta+\beta)x} \right] & \text{if } x < 0 \end{cases} \quad (4.19)$$

$$A^* = 2AD_x(\delta - \beta) - uA,$$

$$B^* = 2AD_x(\delta + \beta) - uA$$

Here we used initial conditions $X(\infty) < \infty$ and $X(-\infty) = S$ [20]. The parameters A, u, q, D, D_x and k_2 are supposed to be 1 and α, S and k_1 to be 2. The solution for C and X is given in the figure 4.5.

Model 4. We have dispersion terms $D \neq 0, D_x \neq 0$ and $k \neq 0$ [19].

$$D \frac{d^2(AC)}{dx^2} - \frac{d(uAC)}{dx} - k_1 \frac{X}{X+k} AC + qH(x) = 0; \quad 0 \leq x < L, t > 0 \quad (4.20)$$

$$D_x \frac{d^2(AX)}{dx^2} - \frac{d(uAX)}{dx} - k_2 \frac{X}{X+k} AX + \alpha(S - X) = 0; \quad 0 \leq x < L, t > 0 \quad (4.21)$$

$C(-\infty) = 0$ and $X(-\infty) = S$ are the boundary conditions.

$$\lim_{x \rightarrow \infty} (C(x), X(x)) = \left[\frac{q}{k_1 A} \left(1 + \frac{k}{X(\infty)} \right), S - \frac{k_2 q}{k_1 \alpha} \right] \quad (4.22)$$

4.4 Conclusion

A coupled system of advection-dispersion equations based water pollution model is presented that incorporates different parameters. We have proposed simple analytical solution for mathematical model. One dimensional model is used to observe the concentrations by taking dimension along the length of river. By considering the removal of pollutant by aeration, event of steady states is investigated. In this model, coupled advection-dispersion equations are solved by taking dispersion coefficient as zero and non-zero respectively.

Chapter 5

Summary

We solve one-dimensional steady advection-dispersion-reaction equation. We use finite difference method (FDM) to solve this equation numerically. Also, we examine chemical oxygen demand (COD) concentration in the variation of different parameters. Furthermore, we formulate to minimize the cost of waste-water treatment.

Analytical solution to unsteady advection-dispersion equation is derived. Advection-dispersion equation describes pollutant concentration $C(x, t)$. Here we use Laplace transformation technique to solve the equation. We have obtained analytic unsteady solution by taking the water velocity u in the x -direction as a linear function of x and dispersion coefficient D as zero in case of concentration of pollutant in one region. Numerical studies shows variation of C with time t . If the added pollutant rate along the river q is in very small amount, the variation of C along the river at different times coincide to each other. In case of concentration of pollutant in two regions, analytical solutions are constructed by taking dispersion coefficient D as non-zero.

A coupled system of advection-dispersion equations based water pollution model is presented that incorporates different parameters. We have proposed simple analytical solution for mathematical model. One dimensional model is used to observe the concentrations by taking dimension along the length of river. By considering the removal of pollutant by aeration, event of steady states is investigated. In this model coupled advection-dispersion equations are solved by taking dispersion coefficient as zero and non-zero respectively.

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PUBLICATIONS

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Analytical Solution for Advection-Dispersion Equation of the Pollutant Concentration using Laplace Transformation

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Abstract: We present simple analytical solution for the unsteady advection-dispersion equation describing the pollutant concentration $C(x, t)$ in one dimension. In this model the water velocity in the x -direction is taken as a linear function of x and dispersion coefficient D as zero. In this paper by taking $k = 0$, k is the half saturated oxygen demand concentration for pollutant decay, we can apply the Laplace transformation and obtain the solution. The variation of $C(x, t)$ with different times t upto $t \rightarrow \infty$ (the steady state case) is taken into account advection-dispersion equation in our study.

Keywords: Pollutant, Concentration, Laplace transformation, Dispersion, Analytical solution

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Nomenclature

A	Cross-section area of the river (m^2).
C	Pollutant concentrations ($kg.m^{-3}$).
D	Dispersion coefficient of pollutant in the x -direction ($m^2.day^{-1}$).
k	Half-saturated oxygen demand concentration for pollutant decay ($kg.m^{-3}$).
k_1	Degradation rate coefficient for pollutant (day^{-1}).
L	Polluted length of river (m).
s	Laplace transform variable.
p	Rate of pollution ($kg.m^{-3}$) at the origin .
q	Added pollutant rate along the river ($kg.m^{-1}.day^{-1}$).
t	Time (day).
u	Water velocity in the x -direction ($m.day^{-1}$).
x	Position (m).

1 Introduction

Pollution of rivers has become a matter of concern for scientists working in environmental engineering, hydrology, chemical engineering, geology, soil physics, and mathematics. If important hydraulic and chemical processes are examined together, analytical solutions of the mathematical models representing pollutant transport are hardly possible [21]. To forecast water quality and to give reliable tools for water quality management in affected areas, mathematical models have been used widely. The purpose of this study is to promote analytical solution of one-dimensional unsteady advection-dispersion equation using Laplace transforms method. It is a particular case of the research made by Pimpunchat et al. [11] which was carried at Tha Chin River in Thailand. The poor water quality of the Bagmati River in Kathmandu, Nepal was the motivation for this study. The rapid urban growth, continuous dumping of solid wastes, domestic sewage, industrial waste, insufficient waste-water treatment facilities, low levels of awareness are the main reasons for pollution in the Bagmati River (Fig. 1) [18].

Pimpunchat et al. [12] presented a mathematical model for river pollution comprising a coupled pair of nonlinear equations and had investigated the effect of aeration on the degradation of pollutant. In some simplified cases they had obtained analytic steady-state solutions. Carslaw and Jaeger [1] have derived



Figure 1: Polluted Bagmati River in Kathmandu, Nepal [18]

analytical solutions for one-dimensional transport in composite media with Laplace transforms and with Green's functions. Marusic [10] presented a mathematical model for analyzing the hydrodynamics and pollutant dispersion in river-type systems. He was able to report changes in the pollutant concentration in the river with time. According to the study of Johari et al. [3], the results of one dimensional advection diffusion equation had successfully been used to predict the transportation of water pollution concentration by manipulating the velocity and diffusion parameters. Salkuyeh [16] put forward a method to find the exact solution of the system of ordinary differential equations obtained when we discretize the convection-diffusion equation with regard to the space variable. Pochai et al. [13] and Tabuenca et al. [19] presented the finite element method for solving the water pollution models in one and two dimensional water areas respectively. Van Genuchten and Alves [20] introduced analytical solutions for a physical system in a semi-infinite domain with zero initial concentration. Savovic and Djordjevic [17] presented numerical solution for the one dimensional advection-diffusion partial differential equation with variable coefficients in semi-infinite media. Kumar et al. [6] have derived analytical solutions for one-dimensional advection-diffusion equation in a longitudinal finite initially solute free domain with variable coefficients. Wadi et al. [21] presented simple analytical solutions for the unsteady advection-dispersion equations describing the pollutant concentration in one dimension. The solutions were obtained by using Laplace transformation technique. According to the study of Manitcharoen and Pimpunchat [9], the unsteady state solutions of pollutant concentration by considering advection-dispersion equations in one dimension were proposed by using the Laplace transform technique and the explicit finite difference technique, for analytical and numerical solutions, respectively. In recent years many techniques have been developed to find the solution of partial differential equations [4, 5, 14, 15]. One of such techniques is Laplace transform, which is a very useful technique.

2 Mathematical Model

The water pollution or the concentration of the pollutant $C(x, t)$ is assumed to vary with time t (days) along the length of the river $L(m)$ (the polluted length of the river) where it is assumed that the rate of pollutant addition along the river $q(kg/m \text{ day})$ is constant. An unsteady flow of water pollutant concentration in one dimension can be described by advection-dispersion equation [12]:

$$\frac{\partial(AC)}{\partial t} = D \frac{\partial^2(AC)}{\partial x^2} - \frac{\partial(uAC)}{\partial x} - k_1 \frac{X}{X+k} AC + qH(x); \quad 0 \leq x < L, t > 0 \quad (1)$$

where $H(x)$ is the Heaviside function defined by

$$H(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Here, u is the water velocity in x - direction, C is the concentration of pollutant, D is the dispersion coefficient of pollutant in x - direction, k_1 is the degradation rate coefficient of pollutant, q is the added pollutant rate along the river, k is the half saturated oxygen demand concentration for pollutant decay, X is the concentration of the dissolved oxygen within the river and A is the cross-section of area of river. We assume the stream reach is considered to be homogeneous system. So we take the parameters A, q, D, k_1 as constants over time and space [8].

In the general case when $k \neq 0$, it will be impossible to use Laplace transform to suggest an exact solution [21]. We apply Laplace transformation by taking $k = 0$. In this model we take zero dispersion D (i.e. $D = 0$) [11]. Using these above conditions, the model equation (1) becomes:

$$\frac{\partial(AC)}{\partial t} = -\frac{\partial(uAC)}{\partial x} - k_1AC + q; \quad 0 \leq x < L, t > 0 \quad (3)$$

Let us consider $u(x, t) = 1 + ax$ [2] for water velocity, where a is non-zero real constant has the dimension of inverse of space variable. Equation (3) can be written as

$$\frac{\partial(C)}{\partial t} = -u\frac{\partial(C)}{\partial x} - Ca - k_1C + \frac{q}{A}; \quad 0 \leq x < L, t > 0 \quad (4)$$

This equation is solved under the initial and boundary conditions as:

$$C(x, 0) = 0; \quad x \geq 0 \quad (5)$$

$$C(0, t) = p; \quad t > 0 \quad (6)$$

where $C(x, t)$ is the pollutant concentration for the case when dispersion coefficient $D = 0$, the initial rate of pollution along the river is supposed zero and p is the rate of pollution at the origin.

2.1 The analytical solution

Laplace transformation technique is defined by equation (7), and is used to get the analytical solution. The Laplace transformation may be defined as: If $f(x, t)$ is any function defined in $a \leq x \leq b$ and $t > 0$, then its Laplace transform with respect to t is denoted by:

$$L\{f(x, t)\} = F(x, s) = \int_0^\infty e^{-st} f(x, t) dt, \quad s > 0 \quad (7)$$

where s is called the transform variable [7]. The inverse Laplace transformation is denoted by $L^{-1}\{F(x, s)\} = f(x, t)$ and defined by the complex variable:

$$L^{-1}\{F(x, s)\} = f(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(x, s) e^{-st} f(x, t) ds, \quad c > 0 \quad (8)$$

Applying Laplace transformation to equation (4) gives:

$$s\tilde{C}(x, s) - C(x, 0) = -u\frac{\partial\tilde{C}}{\partial x} - \tilde{C}(x, s)a - k_1\tilde{C}(x, s) + \frac{q}{As}$$

$$s\tilde{C}(x, s) - C(x, 0) = -u\frac{\partial\tilde{C}}{\partial x} - (a + k_1)\tilde{C}(x, s) + \frac{q}{As}$$

Using (5), we get,

$$s\tilde{C}(x, s) = -u \frac{\partial \tilde{C}}{\partial x} - (a + k_1)\tilde{C}(x, s) + \frac{q}{As}$$

$$\frac{\partial \tilde{C}}{\partial x} + \left(\frac{s + a + k_1}{1 + ax} \right) \tilde{C} = \frac{q}{As(1 + ax)}$$

Integrating factor (I.F.) = $e^{\int \left(\frac{s+a+k_1}{1+ax} \right) dx} = (1 + ax)^{\left(\frac{s+a+k_1}{a} \right)}$

Thus

$$\tilde{C}(1 + ax)^{\left(\frac{s+a+k_1}{a} \right)} = \frac{q}{As} \int (1 + ax)^{\left(\frac{s+k_1}{a} \right)} dx$$

$$\tilde{C}(1 + ax)^{\left(\frac{s+a+k_1}{a} \right)} = \frac{aq}{As(s + a + k_1)} (1 + ax)^{\left(\frac{s+a+k_1}{a} \right)} + C_1$$

where C_1 is the constant of integration.

Therefore

$$\tilde{C}(x, s) = \frac{aq}{As(s + a + k_1)} + C_1(1 + ax)^{-\left(\frac{s+a+k_1}{a} \right)} \quad (9)$$

Taking laplace transform of (6)

$$\tilde{C}(0, s) = \frac{p}{s} \quad (10)$$

Using (10) on (9), we get,

$$\frac{p}{s} = \frac{aq}{As(s + a + k_1)} + C_1$$

Therefore

$$C_1 = \frac{p}{s} - \frac{aq}{As(s + a + k_1)}$$

Thus

$$\tilde{C}(x, s) = \frac{aq}{As(s + a + k_1)} + \frac{p}{s}(1 + ax)^{-\left(\frac{s+a+k_1}{a} \right)}$$

$$- \frac{aq}{As(s + a + k_1)} (1 + ax)^{-\left(\frac{s+a+k_1}{a} \right)} \quad (11)$$

The inverse of Laplace transform is

$$C(x, t) = \frac{aq}{A} \left[\frac{1}{k_1 + a} - \frac{1}{k_1 + a} e^{-(k_1+a)t} \right] + p(1 + ax)^{-\left(\frac{k_1+a}{a} \right)} H \left(t + \frac{\log(1 + ax)}{a} \right)$$

$$- \frac{aq}{A} \left[\left(\frac{1}{k_1 + a} - \frac{1}{k_1 + a} e^{-(k_1+a)t} \right) (1 + ax)^{-\left(\frac{k_1+a}{a} \right)} H \left(t + \frac{\log(1 + ax)}{a} \right) \right]$$

where

$$H \left(t + \frac{\log(1 + ax)}{a} \right)$$

is Heaviside function. This is used to capture the fact that pollutant is discharged for $\left(t + \frac{\log(1+ax)}{a} \right) > 0$ only.

$$C(x, t) = \frac{aq}{A(k_1 + a)} - \frac{aq}{A(k_1 + a)} e^{-(k_1+a)t}$$

$$+ p(1 + ax)^{-\left(\frac{k_1+a}{a} \right)} - \frac{aq}{A(k_1 + a)} (1 + ax)^{-\left(\frac{k_1+a}{a} \right)}$$

$$+ \frac{aq}{A(k_1 + a)} e^{-(k_1+a)t} (1 + ax)^{-\left(\frac{k_1+a}{a} \right)} \quad (12)$$

which can be written as

$$C(x^*, t) = q^* - q^* e^{-k_1^* t} + px^* - q^* x^* + q^* e^{-k_1^* t} x^*$$

where

$$k_1^* = (k_1 + a), \quad q^* = \frac{aq}{Ak_1^*}, \quad x^* = (1 + ax)^{-\frac{k_1^*}{a}}$$

$$\therefore C(x^*, t) = px^* + q^*(1 - x^*)(1 - e^{-k_1^*t}) \quad (13)$$

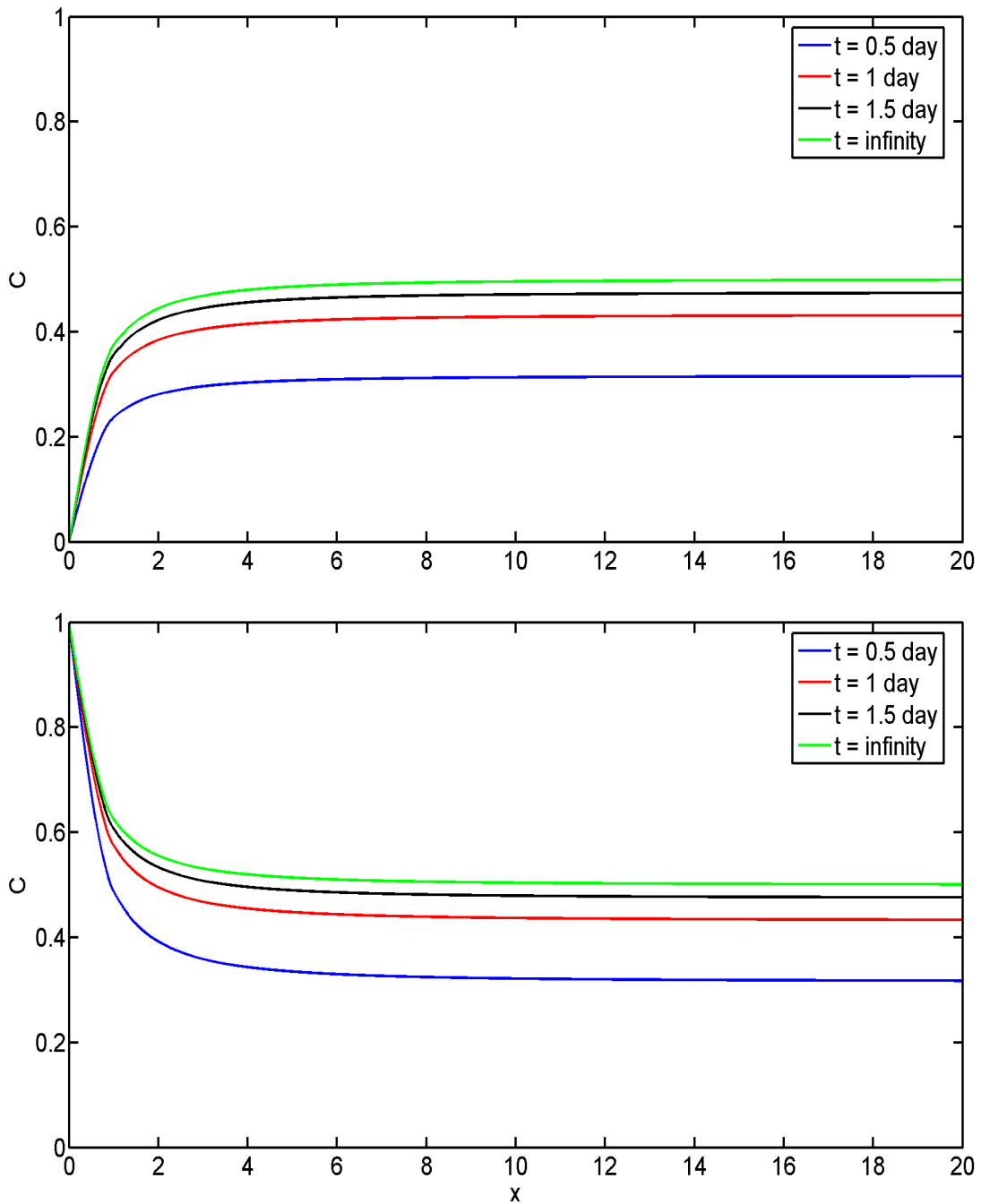


Figure 2: Analytical solution with dispersion for C at different time described by the equation (12) **top:** $p=0$, **bottom:** $p=1$.

2.1.1 Steady state case

Equation (12) for the steady state when $t \rightarrow \infty$ gives:

$$C(x) = \frac{aq}{A(k_1 + a)} + p(1 + ax)^{-\left(\frac{k_1+a}{a}\right)} - \frac{aq}{A(k_1 + a)}(1 + ax)^{-\left(\frac{k_1+a}{a}\right)} \quad (14)$$

For the special case when $p = 0$ the equation (14) will be the same as that obtained by Wadi et al. [21].

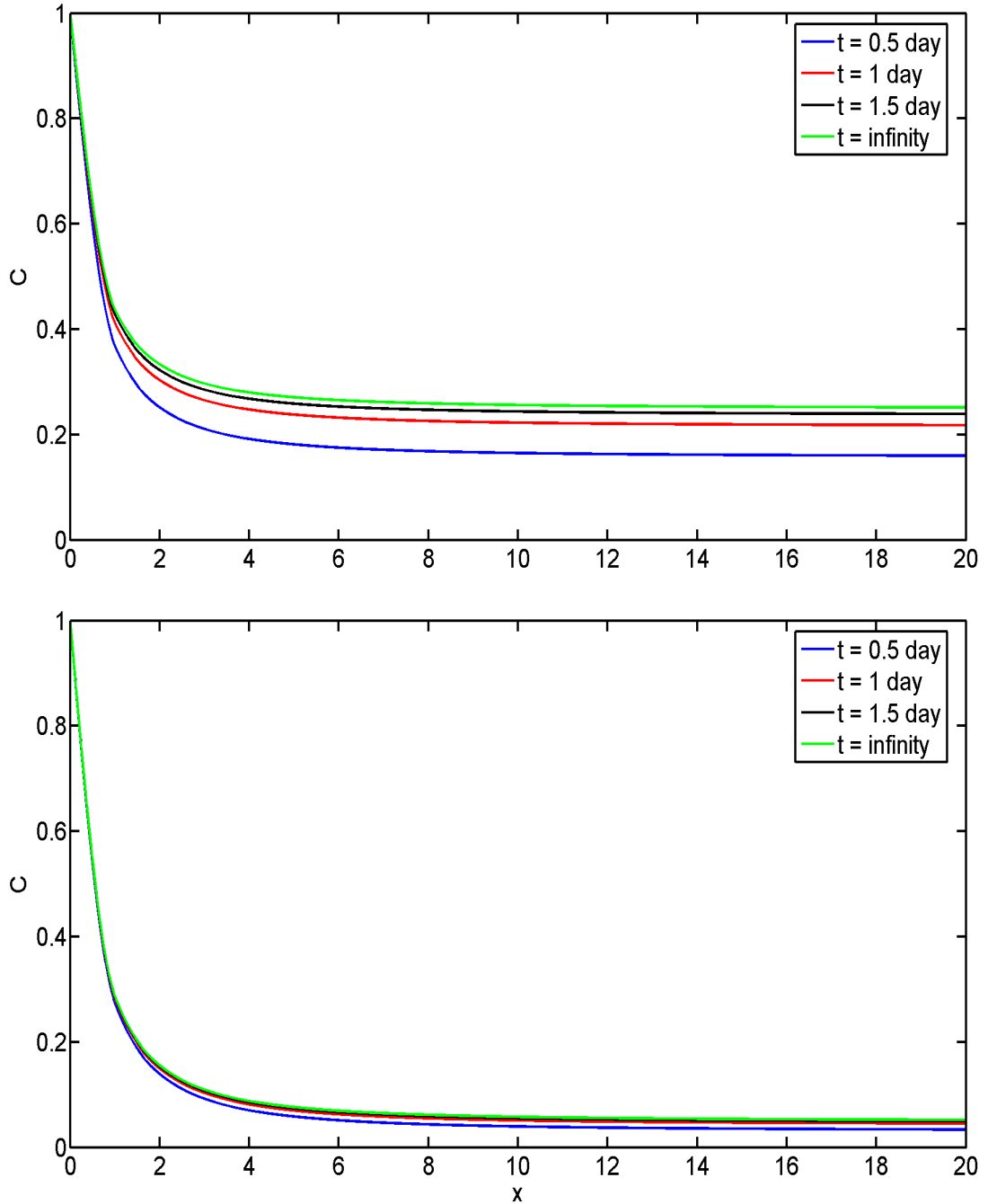


Figure 3: Analytical solution with dispersion for C at different time described by the equation (12) **top:** $q=0.5$, **bottom:** $q=0.1$.

3 Result and Discussion

We present simple analytical solution for the unsteady advection dispersion equation describing the pollutant concentration $C(x, t)$ in one dimension using Laplace transformation method for which zero dispersion D is taken. From solution we obtained variation of C with time and space. The time is given in days and the values of the concentration of pollutant are measured in ($kg.m^{-3}$). In general, the pollutant concentration is given by equation (12).

Figure 2 on the top shows the variation of C in the range $0 \leq x \leq 20$ with the time t described by equation (12) for $t = 0.5, 1.0, 1.5$ (days) and $t \rightarrow \infty$. To test our model we set the parameters A, q, a, k to be 1 and $p = 0$ [11]. From figure 2 on the top, for $x \geq 0$ as t increases C increases and reaches its maximum value as $t \rightarrow \infty$. In general, the concentration of pollutant increases as x increases. From figure 2 on the bottom, we see that: as x increases the value of C decreases for any time t , it reaches a constant value near the sink. The effect of the time t is very small near the upstream and dominant near the downstream. As t increases the value of C increases at any cross-section of the river. This result is in good agreement with that reported by Wadi et al. [21].

Figure 3 on the top shows the variation of C along the river from source up to sink at different times $t = 0.5, 1.0, 1.5$ (days) and $t \rightarrow \infty$ and added pollutant rate along the river $q = 0.5$ and figure 3 on the bottom shows the variation of C along the river with $q = 0.1$. From figure 3, if the added pollutant rate along the river q decreases, the variations of C at different times come to near at each other. If q is in very small amount the variation of C along the river at different times coincide to each other.

4 Conclusion

Simple analytical solution for the unsteady advection-dispersion equation describing the pollutant concentration $C(x, t)$ in one dimension is derived by using Laplace transformation technique. We have obtained analytic unsteady solution by taking the water velocity u in the x -direction as a linear function of x and dispersion coefficient D as zero. Numerical studies show that the variation of C with time t is caused only by pure convection and rate of pollutant addition along the river and if the added pollutant rate along the river q is in very small amount, the variation of C along the river at different times coincide to each other.

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Advection-Dispersion Equation for Concentrations of Pollutant and Dissolved Oxygen

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Abstract: A coupled system of advection-dispersion equations based water pollution model is presented that incorporates different parameters. The major concerns of the research are to observe the concentrations of pollutant and dissolved oxygen in the river. One dimensional model is used to observe the concentrations by taking the dimension along the length of the river. By considering the removal of pollutant by aeration, the event of steady states is investigated, since the processes of pollution and aeration are sustained. In this model coupled advection-dispersion equations are solved by taking dispersion coefficient as zero and non-zero respectively.

Keywords: Pollutant, Concentration, Dispersion, Analytical Solution, Dissolved Oxygen.

Nomenclature

A	Cross-section area of the river(m^2).
C	Pollutant concentration ($kg.m^{-3}$).
X	Dissolved oxygen concentration ($kg.m^{-3}$).
D	Dispersion coefficient of pollutant in the x -direction ($m^2.day^{-1}$).
D_x	Dispersion coefficient of dissolved oxygen in the x -direction ($m^2.day^{-1}$).
S	Saturated oxygen concentration ($kg.m^{-3}$).
k	Half-saturated oxygen demand concentration for pollutant decay ($kg.m^{-3}$).
k_1	Degradation rate coefficient for pollutant (day^{-1}).
k_2	Degradation rate coefficient for dissolved oxygen (day^{-1}).
L	Polluted length of river (m).
q	Added pollutant rate along the river ($kg.m^{-1}.day^{-1}$).
α	Mass transfer of oxygen from air to water ($m^2.day^{-1}$).
t	Time (day).
u	Water velocity in the x -direction ($m.day^{-1}$).
x	Position (m).

1 Introduction

Water pollution is one of the main environmental issues that we are facing, as more than 70% of the Earth's surface is water-covered. Water pollution takes place due to unfavourable substances come into water that replaces quality of water and which is dangerous to human health [1]. Drinking water is required to be safe for public health. Being a common solvent, water is a main source of diseases. The data of world health organization (WHO) shows that 80% diseases and 3.1% deaths happen because of low water quality [2]. The poor water quality of the Bagmati River in Kathmandu 1, Nepal was the motivation for this study. The rapid urban growth, low levels of awareness, industrial waste, continuous dumping of solid wastes, insufficient waste-water treatment facilities, domestic sewage are the main reasons for pollution in the Bagmati River in Kathmandu, Nepal.

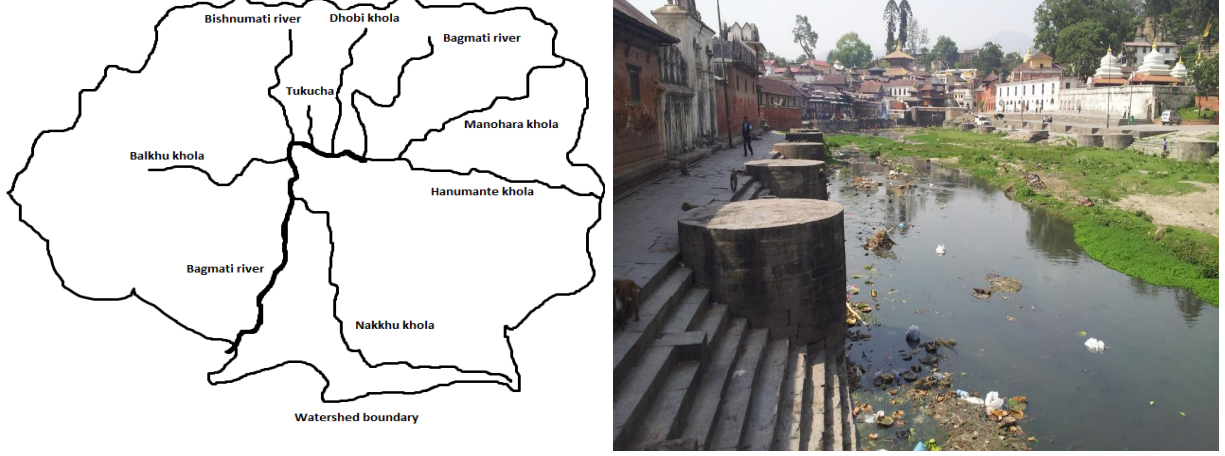


Figure 1: Map of Bagmati River and its tributaries (left) and Polluted Bagmati River [21] (Right).

28 Pimpunchat et al. [12] presented a mathematical model for river pollution. They had used coupled
 29 advection-dispersion equation to investigate the effect of aeration.

30 Paudel et al.[10] presented analytical solution to advection-dispersion equation of pollutant concentration.
 31 They used Laplace transformation technique to solve this equation. Pochai, et al. [13] and Tabuenca, et
 32 al. [18] presented water pollution models in one and two-dimensional respectively. Van Genuchten and
 33 Alves [19] introduced analytical solutions for a physical system in a semi-infinite domain with zero initial
 34 concentration. Savovic and Djordjevich [17] derived numerical solution to advection-diffusion equation
 35 in semi-infinite media. Kumar et al. [6] have presented analytical solutions to one-dimensional advec-
 36 tion-diffusion equation.

37 Wadi et al. [20] presented analytical solutions to unsteady advection-dispersion equations. These equa-
 38 tions describes pollutant concentration in one dimension. The solutions were obtained by using Laplace
 39 transformation technique. Marusic [9] proposed a model which describes pollutant dispersion in river-type
 40 systems. He was success to describe changes in pollutant concentration with time. According to the study
 41 of Johari, et al. [3], advection-diffusion equation was used to forecast concentration of pollutant transport.
 42 Salkuyeh [16] presented convection-diffusion equation with exact solution.

43 According to the study of Manitcharoen and Pimpunchat [8], the unsteady state solutions of advection-
 44 dispersion equations for the pollutant concentration were proposed. The Laplace transformation technique
 45 had been used for analytical and numerical solutions, respectively. In recent years many techniques have
 46 been developed to find the solution of partial differential equations [4, 5, 14, 15].

47 A coupled system of advection-dispersion equations based water pollution model is presented that incor-
 48 porates different parameters. One dimensional model is used to observe the concentrations by taking
 49 dimension along the length of river. Then, analytical solutions are constructed.

50

51 2 Mathematical Model

52 We consider coupled equations for pollutant $C(x, t)$ and dissolved oxygen $X(x, t)$ concentrations. When
 53 oxygen reacts with pollutant, coupling situation appears. To observe concentrations, one dimensional
 54 model is used. We take dimension along the length of river. So the concentrations $C(x, t)$ and $X(x, t)$
 55 satisfy advection-dispersion equations. The coupled equations [8, 11] are expressed in one dimension as

$$\frac{\partial(AC)}{\partial t} = D \frac{\partial^2(AC)}{\partial x^2} - \frac{\partial(uAC)}{\partial x} - k_1 \frac{X}{X+k} AC + qH(x); \quad 0 \leq x < L, t > 0 \quad (1)$$

56

57

$$\frac{\partial(AX)}{\partial t} = D_x \frac{\partial^2(AX)}{\partial x^2} - \frac{\partial(uAX)}{\partial x} - k_2 \frac{X}{X+k} AC + \alpha(S-X); \quad 0 \leq x < L, t > 0 \quad (2)$$

58

59 where $H(x)$ is the Heaviside function defined by

$$H(x) = \begin{cases} 1 & \text{if } 0 < x < L \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

60

61 Here, u is the water velocity in x - direction, D is the dispersion coefficient of pollutant in x - direction, D_x
 62 is the dispersion coefficient of dissolved oxygen in x - direction, S is the saturated oxygen concentration, k_1
 63 is the degradation rate coefficient of pollutant, k_2 is the degradation rate coefficient of dissolved oxygen, α
 64 is the mass transfer of oxygen from air to water, q is the added pollutant rate along the river, k is the half
 65 saturated oxygen demand concentration for pollutant decay and A is the cross-section of area of river.

66 We take the parameters A , u , q , α and S as constants [7, 20].

67

68 3 Steady State Analysis

69 Here we consider various special cases given in figure 2. The model we used is steady-state model.

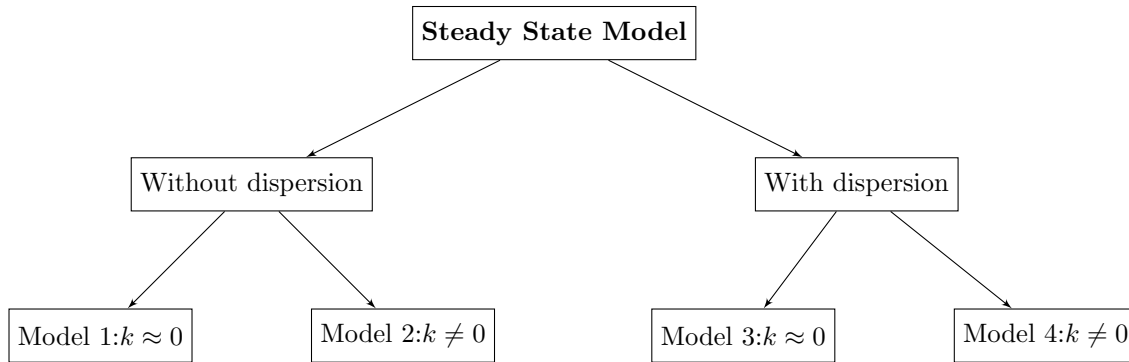


Figure 2: Special Cases of Model [11]

70

71 **Model 1.** We have no dispersion ($D = 0, D_x = 0$) in this model and k is negligible (*i.e* $k \approx 0$)[10].

72

$$\frac{d(uAC(x))}{dx} = -k_1 AC(x) + q; \quad (x > 0, t > 0) \quad (4)$$

73

74

$$\frac{d(uAX(x))}{dx} = -k_2 AC(x) + \alpha(S - X(x)); \quad (x > 0, t > 0) \quad (5)$$

We consider k negligible ($k \approx 0$). The boundary conditions are $C(0) = 0$ and $X(0) = S$.
 From equation (4)

$$uA \frac{dC(x)}{dx} + k_1 AC(x) = q$$

$$\Rightarrow \frac{dC}{dx} + \left(\frac{k_1}{u}\right) C = \frac{q}{uA}$$

Integrating Factor

$$I.F. = \exp\left(\int \frac{k_1}{u} dx\right) = \exp\left(\frac{k_1 x}{uA}\right)$$

The solution is

$$C \exp\left(\frac{k_1 x}{u}\right) = \frac{q}{uA} \int \exp\left(\frac{k_1 x}{u}\right) dx + c_1$$

$$\Rightarrow C \exp\left(\frac{k_1 x}{u}\right) = \frac{qu}{uk_1A} \exp\left(\frac{k_1 x}{u}\right) + c_1$$

$$\Rightarrow C \exp\left(\frac{k_1 x}{u}\right) = \frac{q}{k_1A} \exp\left(\frac{k_1 x}{u}\right) + c_1$$

Using $C(0) = 0$,

$$c_1 = \frac{-q}{k_1A}$$

75 The pollutant concentration is

$$C(x) = \frac{q}{k_1A} \left\{ 1 - \exp\left(\frac{-k_1 x}{u}\right) \right\} \quad (6)$$

76

77 and so the limit is given by

$$C(x) = \lim_{x \rightarrow \infty} \frac{q}{k_1A} \quad (7)$$

Figure 3 shows the variation of C in the range $0 \leq x \leq 4$ described by equation (6). We consider parameters A, u, q, k_1 to be 1 [20] to test our model. Figure 3 shows that C increases as x increases. It reaches to maximum as $x \rightarrow \infty$. Generally concentration of pollutant increases as x increases.

From equation (5)

$$uA \frac{dX}{dx} = -k_2AC + \alpha(S - X)$$

$$\Rightarrow \frac{dX}{dx} + \left(\frac{\alpha}{uA}\right) X = \left(\frac{\alpha S}{uA} - \frac{k_2C}{u}\right)$$

Integrating Factor

$$I.F. = \exp\left(\int \frac{\alpha}{uA} dx\right) = \exp\left(\frac{\alpha x}{uA}\right)$$

The solution is

$$X \exp\left(\frac{\alpha x}{uA}\right) = \int \left(\frac{\alpha S}{uA} - \frac{k_2C}{u}\right) \exp\left(\frac{\alpha x}{uA}\right) dx + c_2$$

$$\Rightarrow X \exp\left(\frac{\alpha x}{uA}\right) = \int \left[\frac{\alpha S}{uA} - \frac{k_2}{u} \frac{q}{k_1A} \left\{ 1 - \exp\left(\frac{-k_1 x}{u}\right) \right\}\right] \exp\left(\frac{\alpha x}{uA}\right) dx + c_2$$

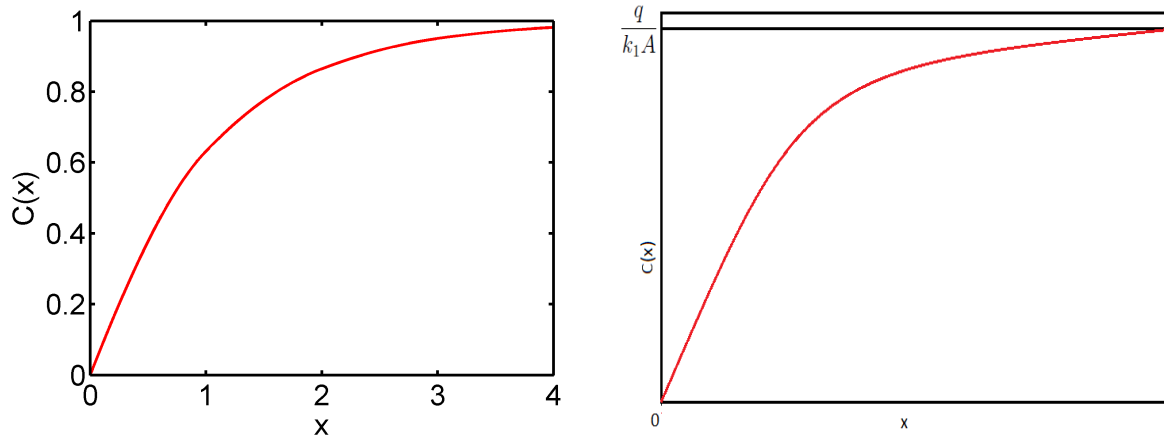


Figure 3: Solution for C without dispersion and $k \approx 0$.

$$\begin{aligned} \Rightarrow X \exp\left(\frac{\alpha x}{uA}\right) &= \int \left(\frac{\alpha S}{uA} - \frac{k_2 q}{uk_1 A}\right) \exp\left(\frac{\alpha x}{uA}\right) dx + \int \frac{k_2 q}{uk_1 A} \exp\left(\frac{\alpha - k_1 A}{uA} x\right) dx + c_2 \\ \Rightarrow X \exp\left(\frac{\alpha x}{uA}\right) &= \left(\frac{\alpha S}{uA} - \frac{k_2 q}{uk_1 A}\right) \frac{uA}{\alpha} \exp\left(\frac{\alpha x}{uA}\right) + \frac{k_2 q}{uk_1 A} \left(\frac{uA}{\alpha - k_1 A}\right) \exp\left(\frac{\alpha - k_1 A}{uA} x\right) + c_2 \\ \Rightarrow X &= S - \frac{k_2 q}{k_1 \alpha} + \frac{k_2 q}{k_1(\alpha - k_1 A)} \exp\left(\frac{-k_1 x}{u}\right) + c_2 \exp\left(\frac{-\alpha x}{uA}\right) \end{aligned}$$

Using $X(0) = S$,

$$c_2 = \frac{k_2 q}{k_1 \alpha} - \frac{k_2 q}{k_1(\alpha - k_1 A)} = -\frac{k_2 q A}{\alpha(\alpha - k_1 A)}$$

$$\Rightarrow X = S - \frac{k_2 q}{\alpha k_1} + \frac{k_2 q}{k_1(\alpha - k_1 A)} \exp\left(\frac{-k_1 x}{u}\right) - \frac{k_2 q A}{\alpha(\alpha - k_1 A)} \exp\left(\frac{-\alpha x}{uA}\right)$$

78

79 The dissolved oxygen concentration is

$$\begin{aligned} X(x) &= S - \frac{k_2 q}{\alpha k_1} \\ &\quad + \frac{k_2 q}{k_1(\alpha - k_1 A)} \exp\left(\frac{-k_1 x}{u}\right) \\ &\quad - \frac{k_2 q A}{\alpha(\alpha - k_1 A)} \exp\left(\frac{-\alpha x}{uA}\right) \end{aligned} \tag{8}$$

80

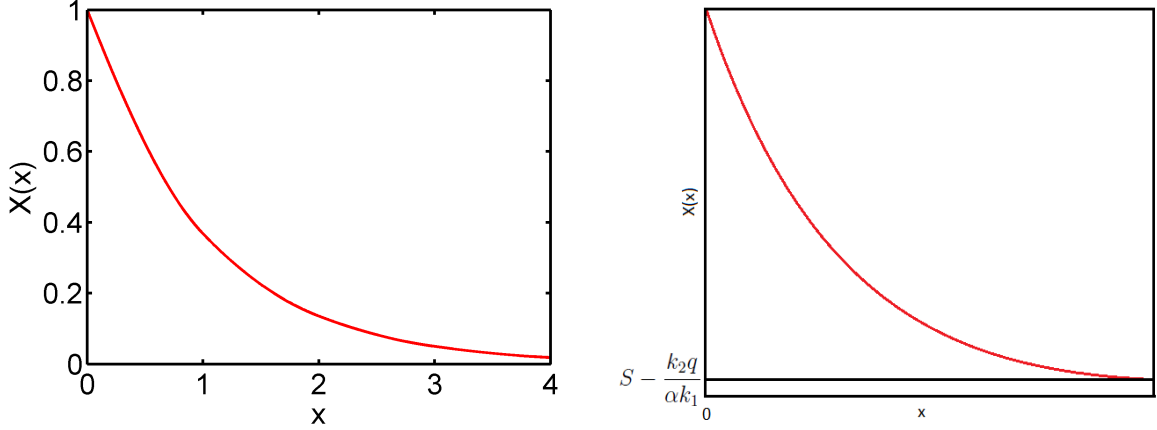
81 and downstream

82

$$\lim_{x \rightarrow \infty} X(x) = S - \frac{k_2 q}{\alpha k_1} \tag{9}$$

83

84 This is shown in figure (4). The figure (4) shows that oxygen level decreases due to reaction with pollutants.


 Figure 4: Solution for X without dispersion and $k \approx 0$

85

86 Now the solutions for concentrations of pollutant and dissolved oxygen are

87

$$\lim_{x \rightarrow \infty} (C(x), X(x)) = \left(\frac{q}{k_1 A}, S - \frac{k_2 q}{\alpha k_1} \right) \quad (10)$$

88

 89 **Model 2.** We have no dispersion ($D = 0, D_x = 0$) and k is non-zero (*i.e.* $k \neq 0$) [11].

90

$$\frac{d(uAC(x))}{dx} = -k_1 \frac{X(x)}{X(x) + k} AC(x) + q; \quad (x > 0, t > 0) \quad (11)$$

91

92

$$\frac{d(uAX(x))}{dx} = -k_2 \frac{X(x)}{X(x) + k} AC(x) + \alpha(S - X(x)); \quad (x > 0, t > 0) \quad (12)$$

93

 94 Boundary conditions are $C(0) = 0$ and $X(0) = S$ which are same. The solutions are

$$\lim_{x \rightarrow \infty} (C(x), X(x)) = \left(\frac{q}{k_1 A} + \frac{\alpha k q}{k_2 \left(\frac{\alpha k_1 S}{k_2} - q \right)}, S - \frac{k_2 q}{\alpha k_1} \right) \quad (13)$$

95

 96 The solutions depend upon k and q whereas in the model 1 these solutions depend on q only. If $q \geq \frac{\alpha k_1 S}{k_2}$,

 97 the solution does not exist. Figure 5 shows the variation of C and X in the range $0 \leq x \leq 20$ as k varies.

 98 To test our model we suppose $A, S, u, k_1, k_2, \alpha$ to be 1 and q to be 1/4.

 99 **Model 3.** We have dispersion terms $D \neq 0, D_x \neq 0$ and $k \approx 0$.

100

$$D \frac{d^2(AC)}{dx^2} - \frac{d(uAC)}{dx} - k_1 \frac{X}{X + k} AC + qH(x) = 0; \quad (x > L, t > 0) \quad (14)$$

101

$$D_x \frac{d^2(AX)}{dx^2} - \frac{d(uAX)}{dx} - k_2 \frac{X}{X + k} AC + \alpha(S - X) = 0; \quad (x > L, t > 0) \quad (15)$$

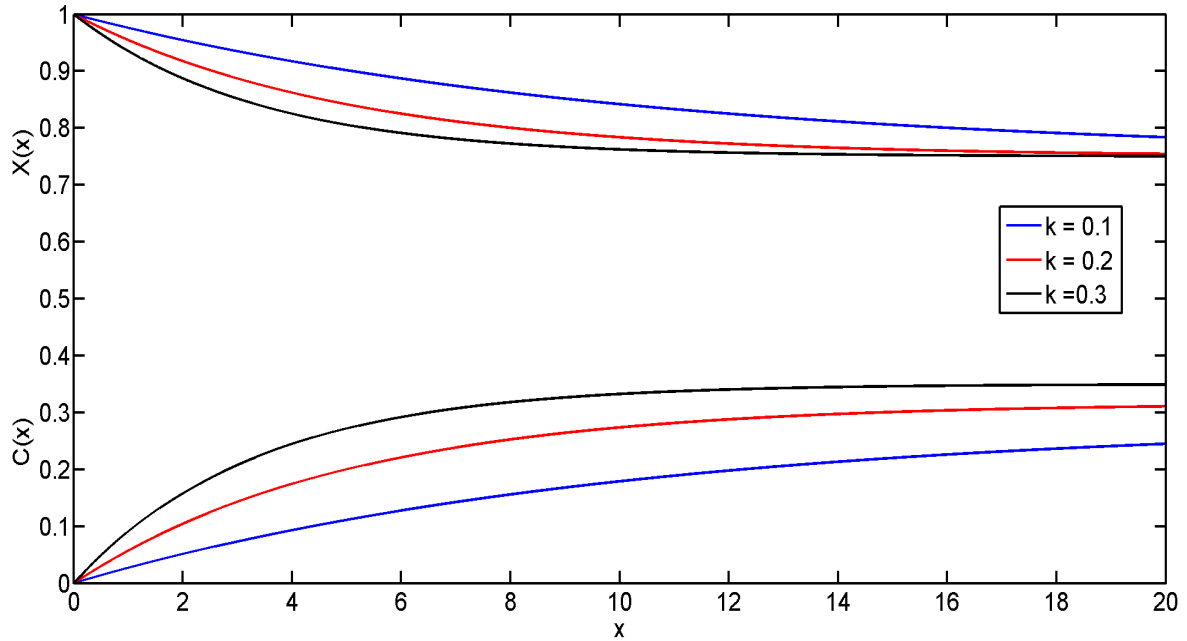


Figure 5: Solution for C and X with dispersion and $k \neq 0$.

102 In this model ($k \approx 0$), the equations 14 and 15 become

$$D \frac{d^2(AC)}{dx^2} - \frac{d(uAC)}{dx} - k_1 AC + q = 0 \quad (16)$$

103

$$D_x \frac{d^2(AX)}{dx^2} - \frac{d(uAX)}{dx} - k_2 AC + \alpha(S - X) = 0 \quad (17)$$

From equation 16

$$DA \frac{d^2 C}{dx^2} - uA \frac{dC}{dx} - k_1 AC + q = 0$$

$$\Rightarrow \frac{d^2 C}{dx^2} - \frac{u}{D} \frac{dC}{dx} - \frac{k_1}{D} C = -\frac{q}{DA}$$

$$\Rightarrow \left(\tilde{D}^2 - \frac{u}{D} \tilde{D} - \frac{k_1}{D} \right) C = -\frac{q}{DA}$$

where $\tilde{D} = \frac{d}{dx}$, which is the second order differential equation.

Its Auxiliary Equation is

$$m^2 - \frac{u}{D} m - \frac{k_1}{D} = 0$$

$$\Rightarrow m = \frac{\frac{u}{D} \pm \sqrt{\frac{u^2}{D^2} - 4 \left(-\frac{k_1}{D}\right)}}{2}$$

$$\Rightarrow m = \frac{u}{2D} \pm \frac{\sqrt{u^2 + 4Dk_1}}{2D}$$

$$\Rightarrow m = \delta \pm \beta$$

where

$$\delta = \frac{u}{2D}$$

and

$$\beta = \frac{\sqrt{u^2 + 4Dk_1}}{2D}.$$

Therefore the Complementary Function (C.F.) is

$$C.F. = c_1 e^{(\delta-\beta)x} + c_2 e^{(\delta+\beta)x}$$

and the Particular Integral (P.I.) is

$$\begin{aligned} P.I. &= \frac{1}{\tilde{D}^2 - \frac{u}{D}\tilde{D} - \frac{k_1}{D}} \left(-\frac{q}{DA} \right) \\ &\Rightarrow P.I. = \frac{q}{k_1 A} \end{aligned}$$

The general solution is

$$\begin{aligned} C(x) &= C.F. + P.I. \\ &\Rightarrow C(x) = c_1 e^{(\delta-\beta)x} + c_2 e^{(\delta+\beta)x} + \frac{q}{k_1 A} \end{aligned}$$

104 So the pollutant concentration is

$$C(x) = \begin{cases} \frac{q}{k_1 A} \left[1 - \left(\frac{\delta+\beta}{2\beta} \right) e^{(\delta-\beta)x} \right] & \text{if } x \geq 0 \\ \frac{q}{k_1 A} \left(\frac{\beta-\delta}{2\beta} \right) e^{(\delta+\beta)x} & \text{if } x < 0 \end{cases} \quad (18)$$

We use the conditions $C(\infty) < \infty$ and $C(-\infty) < \infty$ [12].

From equation 17

$$\begin{aligned} AD_x \frac{d^2(X)}{dx^2} - uA \frac{d(X)}{dx} - k_2 AC + (\alpha S - \alpha X) &= 0 \\ \Rightarrow \frac{d^2 X}{dx^2} - \frac{u}{D_x} \frac{dX}{dx} - \frac{k_2}{D_x} C - \frac{\alpha}{AD_x} X + \frac{\alpha S}{AD_x} &= 0 \\ \Rightarrow \frac{d^2 X}{dx^2} - \frac{u}{D_x} \frac{dX}{dx} - \frac{\alpha}{AD_x} X &= \frac{k_2 C}{D_x} - \frac{\alpha S}{AD_x} \\ \Rightarrow \left(\tilde{D}^2 - \frac{u}{D_x} \tilde{D} - \frac{\alpha}{AD_x} \right) X &= \left(\frac{k_2 AC - \alpha S}{AD_x} \right) \end{aligned}$$

where $\tilde{D} = \frac{d}{dx}$, which is the second order differential equation.

Its Auxiliary Equation is

$$\begin{aligned} m^2 - \frac{u}{D_x} m - \frac{\alpha}{AD_x} &= 0 \\ \Rightarrow m &= \frac{\frac{u}{D_x} \pm \sqrt{\frac{u^2}{D_x^2} - 4 \left(-\frac{\alpha}{AD_x} \right)}}{2} \end{aligned}$$

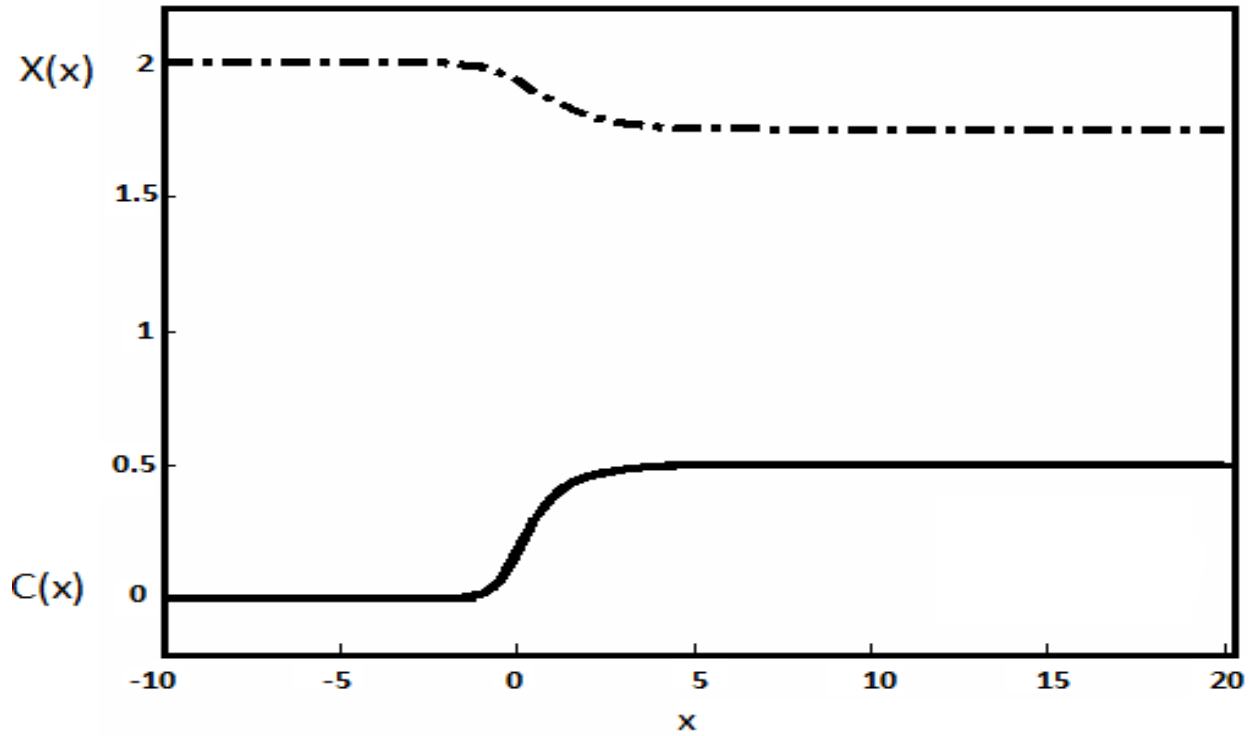


Figure 6: Solution for C and X with dispersion and $k \approx 0$.

$$\Rightarrow m = \frac{u}{2D_x} \pm \frac{\sqrt{u^2 + \frac{4\alpha D_x}{A}}}{2D_x}$$

$$\Rightarrow m = \gamma \pm \eta$$

where

$$\gamma = \frac{u}{2D_x}$$

and

$$\eta = \frac{\sqrt{u^2 + \frac{4\alpha D_x}{A}}}{2D_x}$$

Therefore the Complementary Function (C.F.) is

$$C.F. = c_3 e^{(\gamma-\eta)x} + c_4 e^{(\gamma+\eta)x}$$

and the Particular Integral (P.I.) is

$$P.I. = \frac{1}{\tilde{D}^2 - \frac{u}{D_x} \tilde{D} - \frac{\alpha}{AD_x}} \left(\frac{k_2 AC - \alpha S}{AD_x} \right)$$

where

$$C(x) = \begin{cases} \frac{q}{k_1 A} \left[1 - \left(\frac{\delta+\beta}{2\beta} \right) e^{(\delta-\beta)x} \right] & \text{if } x \geq 0 \\ \frac{q}{k_1 A} \left(\frac{\beta-\delta}{2\beta} \right) e^{(\delta+\beta)x} & \text{if } x < 0 \end{cases}$$

The general solution is

$$X(x) = C.F. + P.I.$$

So the dissolved oxygen concentration is

$$X(x) = \begin{cases} S - \frac{k_2 q}{k_1 \alpha} + \frac{k_2 q}{k_1} \left[\left(\frac{\delta + \eta}{2\eta \alpha} - \frac{\delta + \beta}{4\beta \eta A^*} + \frac{\delta - \beta}{4\beta \eta B^*} \right) e^{(\gamma - \eta)x} - \frac{\delta + \beta}{2\beta A^*} x e^{(\delta - \beta)x} \right] & \text{if } x \geq 0 \\ S + \frac{k_2 q}{k_1} \left[\left(\frac{\delta - \eta}{2\eta \alpha} - \frac{\delta + \beta}{4\beta \eta A^*} + \frac{\delta - \beta}{4\beta \eta B^*} \right) e^{(\gamma + \eta)x} - \frac{\delta - \beta}{2\beta B^*} x e^{(\delta + \beta)x} \right] & \text{if } x < 0 \end{cases} \quad (19)$$

$$A^* = 2AD_x(\delta - \beta) - uA,$$

$$B^* = 2AD_x(\delta + \beta) - uA$$

Here we used initial conditions $X(\infty) < \infty$ and $X(-\infty) = S$ [11]. The parameters A, u, q, D, D_x and k_2 are supposed to be 1 and α, S and k_1 to be 2. The solution for C and X is given in the figure 6.

Model 4. We have dispersion terms $D \neq 0, D_x \neq 0$ and $k \neq 0$ [12].

$$D \frac{d^2(AC)}{dx^2} - \frac{d(uAC)}{dx} - k_1 \frac{X}{X+k} AC + qH(x) = 0; \quad 0 \leq x < L, t > 0 \quad (20)$$

$$D_x \frac{d^2(AX)}{dx^2} - \frac{d(uAX)}{dx} - k_2 \frac{X}{X+k} AX + \alpha(S - X) = 0; \quad 0 \leq x < L, t > 0 \quad (21)$$

Boundary conditions are $C(-\infty) = 0$ and $X(-\infty) = S$.

$$\lim_{x \rightarrow \infty} (C(x), X(x)) = \left[\frac{q}{k_1 A} \left(1 + \frac{k}{X(\infty)} \right), S - \frac{k_2 q}{k_1 \alpha} \right] \quad (22)$$

4 Conclusion

A coupled system of advection-dispersion equations based water pollution model is presented that incorporates different parameters. We have proposed analytical solution for mathematical model. One dimensional model is used to observe the concentrations by taking dimension along the length of river. By considering the removal of pollutant by aeration, event of steady states is investigated. In this model coupled advection-dispersion equations are solved by taking dispersion coefficient as zero and non-zero respectively.

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PRESENTATIONS

1. K. Paudel, P. S. Bhandari, and J. Kafle. Analytical solution to advection-dispersion equation of the pollutant concentration using laplace transformation. *International Conference on Mathematics and its Application*, April 9-11, 2021.
2. K. Paudel and J. Kafle. Mathematical modeling of water pollution assessment using finite difference method. *Seminar Cum Workshop on Mathematics & its Application*, August 29-30, 2020.

**ANALYTICAL SOLUTION FOR ADVECTION-DISPERSION EQUATION
OF THE POLLUTANT CONCENTRATION USING LAPLACE
TRANSFORMATION**

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Abstract: We present simple analytical solution for the unsteady advection-dispersion equation describing the pollutant concentration $C(x; t)$ in one dimension. In this model the water velocity in the x -direction is taken as a linear function of x and dispersion coefficient D as zero. In this paper by taking $k = 0$, k is the half saturated oxygen demand concentration for pollutant decay, we can apply the Laplace transformation and obtain the solution. The variation of $C(x; t)$ with different time t upto $t \rightarrow \infty$ (the steady state case) is taken into account in our study.

Keywords: Pollutant, Concentration, Laplace transformation, Dispersion

Minimization of Water Treatment Cost Using Finite Difference Method

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Abstract: Water pollution assessment problems arise frequently in environmental science. The convection-diffusion equation is used to describe the diffusion process in environmental science, such as the pollutant transport in the atmosphere, oceans, lakes, rivers or groundwater. Here, a finite difference method (FDM) for solving the one-dimensional steady advection-diffusion-reaction equation is proposed; it is then used to optimize water treatment costs.

Keyword: Finite difference method, Steady state, Water pollution control, Optimization, Convection-diffusion equation.

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