STRUCTURE AND PROPERTIES OF FATOU, JULIA, ESCAPING AND FAST ESCAPING SETS OF HOLOMORPHIC SEMIGROUPS

A THESIS SUBMITTED TO THE

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BY

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DECLARATION

Thesis entitled **" Structure and Properties of Fatou, Julia, Escaping and Fast Escaping Sets of Holomorphic Semigroups "** which is being submitted to the Central Department of Mathematics, Institute of Science and Technology (IOST), Tribhuvan University, Nepal for the award of the degree of Doctor of Philosophy (Ph.D.), is a research work carried out by me under the supervision of Associate Prof. Dr. Ajaya Singh, Central Department of Mathematics, Tribhuvan University and co-supervised by Prof. Dr. Prakash Muni Bajracharya.

This research is original and has not been submitted earlier in part or full in this or any other form to any university or institute, here or elsewhere, for the award of any degree.

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Bishnu Hari Subedi

RECOMMENDATION

This is to recommend that **Mr. Bishnu Hari Subedi** has carried out research entitled **" Structure and Properties of Fatou, Julia, Escaping and Fast Escaping Sets of Holomorphic Semigroups "** for the award of Doctor of Philosophy (Ph.D.) in **Mathematics** under our supervision. To our knowledge, this work has not been submitted for any other degree.

He has fulfilled all the requirements laid down by the Institute of Science and Technology (IOST), Tribhuvan University, Kirtipur for the submission of the thesis for the award of Ph.D. degree.

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LETTER OF APPROVAL

On the recommendation of Associate Prof. Dr. Ajaya Singh and Prof. Dr. Prakash Muni Bajracharya, this Ph.D. thesis submitted by **Mr. Bishnu Hari Subedi** entitled **"Structure and Properties of Fatou, Julia, Escaping and Fast Escaping Sets Holomorphic Semigroups"** forwarded by Central Department Research Committee (CDRC) to the Dean, IOST, T.U.

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ABSTRACT

We study the dynamical behavior of a semigroup generated by holomorphic functions in the complex plane. In particular, we concentrate on semigroup dynamics, where semigroups are generated by transcendental entire functions. It is a study of the behavior of the compositions of a finite set of holomorphic functions in the complex plane. We study Fatou, Julia, escaping and fast escaping sets of such semigroups. The principal aim of this thesis is to investigate the structure and the properties of these sets in the more general settings of holomorphic semigroups. In this thesis, we see to what extent, the structure and the properties of the Fatou, Julia, escaping and fast escaping sets of classical holomorphic dynamics are preserved and generalized to semigroup dynamics, and what new phenomena can occur. A holomorphic semigroup is not abelian in general; however, a cyclic semigroup is abelian, so differences in the dynamics can occur in the structure and the properties of these sets. If a semigroup is abelian, such types of differences will narrow down, and most of the structure and the properties of these sets of classical holomorphic dynamics are preserved and generalized. In this thesis, we generalize the notion of abelian semigroups to nearly abelian semigroups, and we investigate the identical structure and the properties of these sets in such semigroups. On the basis of the algebraic notion of different indices such as finite index, cofinite index and Rees index, we also investigate subsemigroups whose Fatou, Julia and escaping sets coincide with their corresponding parent semigroup. In the holomorphic semigroup setting, there may be empty Fatou sets and empty escaping sets; hence we also investigate certain holomorphic semigroups whose Fatou sets and escaping sets are non-empty on the basis of (partial)fundamental sets and Carlemen sets. Finally, we define fast escaping sets of transcendental semigroups, and we discuss some fundamental structure and properties of these sets.

LIST OF SYMBOLS

$\mathbb{Z}$	set of integers
$\mathbb{N}$	set of positive integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers or real line
$\mathbb{R}-\mathbb{Q}$	set of irrational numbers
$\mathbb{C}$	set of complex numbers or complex plane
$z=x+iy$	a complex number
$\Re(z)$	real part of z
$\Im(z)$	imaginary part of z
$\mathbb{D}$	unit disk $\{z\in\mathbb{C}:\lvert z\rvert<1\}$
$\mathbb{H}^u$	upper half plane $\{z\in\mathbb{C}:\Im(z)>0\}$
$\mathbb{H}_l$	lower half plane $\{z\in\mathbb{C}:\Im(z)<0\}$
$l\mathbb{H}$	left half plane $\{z\in\mathbb{C}:\Re(z)<0\}$
$\mathbb{H}r$	right half plane $\{z\in\mathbb{C}:\Re(z)>0\}$
$\mathbb{C}_\infty$	set of extended complex numbers or Riemann sphere $\mathbb{C}\cup\{\infty\}$
S^1	unit circle
f^n	nth iterates of a function f
$O^+(z)$	forward orbit of a point z under f
$O^-(z)$	backward orbit of z under f

λ	lambda
$\mathscr{F}$	family of holomorphic functions
$F(f)$	Fatou set of f
$J(f)$	Julia set of f
$C(f)$	set of critical points of f
$CV(f)$	set of critical values of f
Γ	big gamma
$AV(f)$	set of asymptotic values of f
$SV(f)$	set of singular values of f
$\mathscr{S}$	Speicer class
$\mathscr{B}$	Eremenko-Lyubich class
$P(f)$	post singular set of f
$\rho(f)$	order of f
$\lambda(f)$	lower order of f
$\sigma(f)$	type of f
$M(r,f)$	maximum modulus of f
$m(r,f)$	minimum modulus of f
$M^n(r,f)$	nth iterate of $M(r,f)$
$m^n(r,f)$	nth iterate of $m(r,f)$
γ	small gamma
$I(f)$	set of escaping points (escaping set) of f
$\lvert . \rvert$	absolute value
$\emptyset$	empty set
∂A	boundary of a set A
$\overline{A}$	closure of a set A
$A^{'}$	derived set of a set A
$Int.A$	interior of a set A
$Ext.A$	exterior of a set A

A^C	complement of a set A
$A - B$	set difference of a set A by a set B
$A(f)$	set of fast escaping points (fast escaping set) of f
$A_R^L(f)$	Lth level of $A(f)$
$A_R(f)$	0th level of $A(f)$
S	(holomorphic) semigroup
T	(holomorphic) subsemigroup
I	ideal of (holomorphic) semigroup S
$\langle A \rangle$	semigroup generated by a set A
$K(S)$	kernel of S
$F(S)$	Fatou set of S
$J(S)$	Julia set of S
$I(S)$	escaping set of S
$F_1(S)$	completely invariant Fatou set of S
$J_1(S)$	completely invariant Julia set of S
$I_1(S)$	completely invariant escaping set of S
exp	exponential
$\forall$	for all
$P(S)$	post singular set of S
S_U	stabilizer of a set U of S
$R(S)$	regular set of S
$L(S)$	limit set of S
Φ	family of conformal maps $\phi : z \to az + b$
$\Phi(S)$	set of commutators of S
S'	conjugate of S
$A(S)$	fast escaping set of S
$A_R^L(S)$	Lth level of $A(S)$
$A_R(S)$	0th level of $A(S)$

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Chapter 1

INTRODUCTION

1.1 Introduction

We study the dynamical behavior of a semigroup of holomorphic functions in the complex plane. This is a natural generalization of the study of the dynamics of a given holomorphic function. The main idea in the dynamics of a given holomorphic function is to study the iteration of such a function, and in the semigroup case we have finite set of holomorphic functions and each iterative step we have several functions to compose with. The dynamical study of a holomorphic semigroup was initiated by Hinkkanen and Martin in a series of papers [46, 47, 48]. The main goal of these works was to extend the classical theory of the dynamics associated to the iteration of a rational function of a complex variable, to the more general setting of the dynamics associated to an arbitrary semigroup of rational functions. In the first step, they generalized the notions of Fatou and Julia sets to the context of semigroups and then investigated the structure of these sets and their basic properties. Poon [75] extended the study to the dynamics on semigroups generated by transcendental entire functions and discussed several properties of Fatou and Julia sets. Kumar and Kumar [61, 62, 63] initiated the study of an escaping set of a semigroup generated by transcendental entire functions. They extended several results of the escaping sets of classical transcendental dynamics to the more general setting of the dynamics of transcendental semigroups. In addition, there are few others who are working in this field. Among them Stankewitz [103, 104] and Haung [43] studied respectively on completely invariant Fatou and Julia sets for holomorphic semigroups, and the dynamics of semigroups of transcendental meromorphic functions. Some of them are jointly working in this field. The joint works of Stankewitz and Sumi [105, 106], Kriete and Sumi [57], Haung and Cheng [52], and Wang and Haung [127] also appeared in this field.

The dynamics of iteration of a holomorphic function has been investigated successfully over the past 100 years after its formal foundation laid by Pierre Joseph Louis Fatou (1878-1929) and Gaston Maurice Julia (1893-1978), and is still an active and exciting area of modern research. The holomorphic dynamical theory in its century of history has undergone many changes, from the classical study of the Fatou, Julia, Mandelbrot, escaping and fast escaping sets of a single rational or transcendental function, to the dynamical study of these sets in the holomorphic semigroup setting. The objects of interest in both classical and semigroup holomorphic dynamics are Fatou, Julia, escaping and fast escaping sets. It is noted that the Fatou, Julia, escaping and fast escaping sets of a single holomorphic function is a particular case of the Fatou, Julia, escaping and fast escaping sets, respectively, of a holomorphic semigroup. In this thesis, *classical holomorphic dynamics* refers the iteration theory of a single holomorphic function, and *holomorphic semigroup dynamics* refers the dynamical theory generated by the composition of various classes of holomorphic functions. In holomorphic semigroup dynamics, the algebraic structure of a semigroup is naturally attached to the dynamics, and hence the situation is little bit complicated. In holomorphic semigroup dynamics, some fundamental results of the classical holomorphic dynamics may not be preserved and generalized, and new phenomena may occur. In addition, the complication in the study of the dynamics of holomorphic semigroups makes the intricate structure and properties of Fatou, Julia, escaping and fast escaping sets less visible. Except in some explicit results, we can not describe the real structure of Fatou, Julia, escaping and fast escaping sets of a holomorphic semigroup. In general, there is no obvious visible structure of these sets in holomorphic semigroup dynamics.

This manuscript is organized as follows:

In this first chapter, we introduce briefly the Fatou, Julia, escaping and fast escaping sets of holomorphic semigroups with short historical background in less technical detail. We also state briefly the rationale and objectives of the research. In short, this chapter is a mirror of this thesis and has been written independently of other chapters.

In Chapter 2, we collect introductory and background materials which are required for the understanding of the results of this thesis. In this chapter, we collect a number of useful notations, necessary definitions and various useful and well-known results of the classical dynamics of a single holomorphic function.

In Chapter 3, we introduce different aspects of holomorphic semigroups.

In Chapter 4, we compare results of the classical holomorphic dynamics and the holomorphic semigroup dynamics, to see how far results of the classical holomorphic dynamics are preserved and generalized in the holomorphic semigroup dynamics and what new phenomena can occur.

In Chapter 5, we investigate the structure and properties of the Fatou, Julia and escaping sets in the more general setting of nearly abelian holomorphic semigroups. We focus on how far the results of abelian holomorphic semigroups are preserved and generalized for nearly abelian holomorphic semigroups.

In Chapter 6, we investigate subsemigroups of holomorphic semigroups whose dynamics coincide with the dynamics of their corresponding semigroups. That is, we investigate subsemigroups whose Fatou, Julia and escaping sets coincide with Fatou, Julia and escaping sets of their corresponding semigroups.

In Chapter 7, we investigate some concrete examples of transcendental semigroups which have non-empty Fatou and escaping sets.

In Chapter 8, we define fast escaping sets of transcendental semigroups and discuss some fundamental structure and properties of this set in semigroup settings.

At the end of the thesis, we give a brief Chapter-9 containing summary and conclusions of our research findings with interesting open problems and suggestions for further research.

Almost all results provided in this thesis have been published in the Nepali Mathematical Sciences Report [107, 110, 118, 122], Universal Journal of Applied Mathematics [114], Turkish Journal of Mathematics [115], Journal of Institute of Engineering-Tribhuvan University [117], Indian Journal of Mathematics [116], Nepal Journal of Science and Technology [119], Journal of Nepal Mathematical Society [120, 121] and arXiv pre-prints [108, 109, 111, 112, 113]. However, this thesis is not a compilation of these papers. The main reason for this is that the papers have been written as the research progressed, so the special research results have been followed by more general ones. However, this order is completely reversed in the thesis: the special results are proved as consequences of general ones. We have also tried to retain some of the flavor of the original research by commenting frequently on the history and motivation for the main ideas.

Mathematical statements in this thesis are classified as theorems, corollaries, propositions and lemmas. Theorems are main original results of this thesis. Corollaries are consequences of the theorems. Propositions are relevant results of other authors. Lemmas are technical results that are needed for the proof of the theorems. The square shape $\square$ denotes the end of a mathematical statement and typically found at the end of the proofs of the theorems (corollaries, lemmas) and end of the solution of examples.

We now have a few words about the arrangement of material of this thesis. The first number of the definition (or conjecture, lemma, proposition, theorem, example etc.) indicates the chapter number, the second number indicates the section of the concerned chapter and the third number indicates its serial number. The numbers within the bracket [ ] indicate alphabetical referenced serial number in the reference section. We have not collected any more materials (research papers, books, monographs etc.) in reference section which are not cited in this thesis.

1.2 Rationale

We study Fatou, Julia, escaping and fast escaping sets of a semigroup of holomorphic functions in the complex plane. This is a natural generalization of these sets of a given holomorphic function.

A *holomorphic semigroup* S is a semigroup of holomorphic functions defined on the complex plane $\mathbb{C}$ or extended complex plane $\mathbb{C}_\infty$ or certain subsets thereof with the semigroup operation being the functional composition. Let

$$\mathscr{F} = \{f_\alpha : f_\alpha \text{ is a holomorphic function for all } \alpha \in \Delta\},$$

where index set Δ is allowed to be infinite in general unless stated otherwise. When a semigroup S generated by $\mathscr{F}$, we write $S = \langle f_\alpha \rangle_{\alpha \in \Delta}$ or simply $S = \langle f_\alpha \rangle$. Holomorphic semigroup S is said to be a *rational semigroup* or a *transcendental semigroup* depending on whether $\mathscr{F}$ is a collection of rational functions or transcendental entire functions. In particular, S is said to be a *polynomial semigroup* if $\mathscr{F}$ is a collection of polynomials of degree at least 2. The holomorphic semigroup S is said to be *abelian* if $f_\alpha \circ f_\beta = f_\beta \circ f_\alpha$ for all generators f_α, f_β of S. A semigroup generated by finitely many holomorphic functions $f_i, (i = 1, 2, \ldots, n)$ is called *finitely generated holomorphic semigroup*, and we write it by $S = \langle f_1, f_2, \ldots, f_n \rangle$. If S is generated by a single holomorphic function f, then S is called a *cyclic holomorphic semigroup*, and we write it by $S = \langle f \rangle$. In this case, each $g \in S$ can be written as $g = f^n$, where f^n is the nth iterate of f with itself.

We say that $\mathscr{F}$ is a *normal family* if every sequence $(f_\alpha) \subseteq \mathscr{F}$ has a subsequence (f_{α_k}) which is uniformly convergent or divergent on all compact subsets of $\mathbb{C}$. If there is a neighborhood U of a point z such that $\mathscr{F}$ is a normal family in U, then we say that $\mathscr{F}$ is normal at z. We say that a function f is *iteratively divergent* at $z \in \mathbb{C}$ if $f^n(z) \to \infty$ as $n \to \infty$. A semigroup S is *iteratively divergent* at z if every $f \in S$ is iteratively divergent at z. The study of holomorphic semigroup dynamics begins with a description of the *Fatou set* $F(S)$ of a semigroup S which is defined as a maximal open set where S is a normal family, and its complement is the *Julia set* $J(S)$. Also, the *escaping set*

$$I(S) = \{z \in \mathbb{C} : S \text{ is iteratively divergent at } z\}$$

of a semigroup S has equally played an important role in transcendental semigroup dynamics. There is a dynamically significant subset of an escaping set which we called by *fast escaping set* of a transcendental semigroup. Let S be a transcendental semigroup. Let us define a set $A_R(S) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R, f) \text{ for all } f \in S \text{ and } n \in \mathbb{N}\}$ where $M(r, f) = \max_{|z|=r} |f(z)|$, and $M^n(r, f)$ denotes the nth iterates of $M(r, f)$. $R > 0$ can be taken any value such that $M(r, f) > r$ for $r \geq R$. The fast escaping set $A(S)$ of a transcendental semigroup S consists the set $A_R(S)$ and all its pre-images.

If $S = \langle f \rangle$, then the Fatou, Julia, escaping and fast escaping sets are respectively denoted by $F(f), J(f)$, $I(f)$ and $A(f)$. Therefore, this definition generalizes the Fatou, Julia, escaping and fast escaping sets of a single holomorphic function. A cyclic semigroup $S = \langle f \rangle$ is abelian, however, a non-cyclic semigroup may not be abelian. Therefore, fundamental contrasts in the dynamics are appeared by different algebraic structure of corresponding semigroups. In this sense, classical holomorphic dynamics is a dynamical study of cyclic semigroups whereas semigroup dynamics is a dynamical study of non-cyclic holomorphic semigroups. The theory of classical holomorphic dynamics is well developed and well understood since last 100 years in the work of Fatou [35, 36, 37, 38], Julia [53, 54, 55], Baker [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], Eremenko [31], Eremenko and Lyubich [32, 33, 34], Bergweiler [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27], Rippon and Stallard [85, 86, 87, 88, 89, 90, 91], Schleicher [93, 94], Rempe [78, 79, 80, 81, 82, 83, 84], Osborne [69, 70, 71, 72, 73, 74], Sixsmith [98, 99, 100, 101] and many others. In addition, nowadays, it has a large number of published results, own journals, a growing number of monographs [15, 51, 65, 68], a lot of summer /winter research schools, a lot of fully funded research projects and many other open research fronts. The naive reason for a number of interesting developments of the classical holomorphic dynamics is that it appears naturally in almost mathematical contexts, and the structure of its fundamental sets look naturally beautiful fractal images of the real universe or nature earth. In contrast, there are comparatively less research, less number of published research papers, no monographs, very less number of conferences / workshops / research schools, very few number of fully funded research projects, and comparatively very less visible structure of Fatou, Julia and escaping sets in holomorphic semigroup dynamics. Therefore, to enrich this field of study little bit more, we decided to study the structure and properties of Fatou, Julia, escaping and fast escaping sets of holomorphic semigroups.

1.3 Objectives

Structure and properties of Fatou and Julia sets of rational semigroups were studied by Hinkannen and Martin [46, 47, 48] and that of transcendental semigroups by Poon [75], Haung [43], and others. Escaping sets of transcendental semigroups were studied first by Kumar and Kumar [61, 62, 63]. In this thesis, we study further structure and properties of Fatou, Julia and escaping sets of holomorphic semigroups. We also study fast escaping sets of transcendental semigroups and discuss their basic structure and properties. In this context, we have the following objectives of this thesis.

1. To what extent, the structure and properties of Fatou, Julia, escaping and fast escaping sets of the classical holomorphic dynamics are generalized and preserved in the holomorphic semigroup dynamics.
2. What new phenomenna occur in the structure and properties of these sets of holomorphic semigroups.

To achieve these objectives, we have proved altogether sixty new assertions as lemmas, theorems and corollaries. We have also investigated seventeen new examples. Among them fourty two assertions and twelve examples are published in the international and national Journals of Mathematics [110, 114, 115, 116, 117, 118, 119, 120, 121, 122].

Chapter 2

PRELIMINARIES

2.1 Notations

We first give some notations used throughout the thesis. We denote the set of integers, the set of integers greater than zero, the set of rational numbers, the set of real numbers (the real line) and the set of irrational numbers respectively by $\mathbb{Z},\ \mathbb{N},\ \mathbb{Q},\ \mathbb{R}$ and $\mathbb{R}-\mathbb{Q}$. The complex plane and the extended complex plane (Riemann sphere) are denoted respectively by $\mathbb{C},\ \mathbb{C}_\infty = \mathbb{C}\cup\{\infty\}$. The disc in $\mathbb{C}$ of radius r with center at z_0 is denoted by $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$, the unit disc by $\mathbb{D} = D_1(0) = \{z \in \mathbb{C} : |z| < 1\}$, and the unit circle by $S^1 = \{z \in \mathbb{C} : |z| = 1\} = \partial\mathbb{D}$, where $|z|$ represents the modulus of the complex number z. We denote the left half plane, right half plane, lower half plane and upper half plane respectively by $l\mathbb{H} = \{z \in \mathbb{C} : \Re(z) < 0\}$, $\mathbb{H}r = \{z \in \mathbb{C} : \Re(z) > 0\}$, $\mathbb{H}_l = \{z \in \mathbb{C} : \Im(z) < 0\}$, and $\mathbb{H}^u = \{z \in \mathbb{C} : \Im(z) > 0\}$, where $\Re(z)$ and $\Im(z)$ respectively denote the real and imaginary parts of a complex number z, and the annulus by $A(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$ for $0 < r < R$. For any set $A \subset \mathbb{C}$ (or $\mathbb{C}_\infty$), the closure, the boundary, the interior, the exterior, the derived set and the compliment of A are denoted by $\overline{A},\ \partial A,\ Int.A,\ Ext.A,\ A'$ and A^c respectively.

2.2 Background on function theory

We frequently use terms open set, closed set, topological space, Hausdorff space, homeomorphism, connected set, bounded set, unbounded set, component of a set, compact set, perfect set, holomorphic function, meromorphic function, entire function, transcendental entire function, conformal map, singularities, uniform convergence, continuity, equicontinuity etc. related to classical complex analysis. For defnitions, examples and other related theorems, we refer [29, 40, 42, 49].

We frequently need to use maximum modulus and minimum modulus of a holomorphic function. We begin with the following definitions.

Definition 2.2.1 (Maximum modulus and minimum modulus). *Let f be holomorphic in $|z| < R < \infty$. The maximum modulus and minimum modulus of f are defined respectively by*

$$M(r, f) = \max_{|z|=r} |f(z)|$$

and

$$m(r, f) = \min_{|z|=r} |f(z)|$$

for all $r < R$.

According to the classical maximum modulus principle, the maximum and minimum modulus of a holomorphic function f are always attained in the boundary. It is noted that maximum modulus $M(r, f)$ describes the growth of an entire function f in the whole complex plane $\mathbb{C}$. The notion of maximum modulus and minimum modulus are used to define order and type of an entire function.

Definition 2.2.2 (Order and type of entire function). *The order $\rho(f)$ and lower order $\lambda(f)$ of an entire function f are defined respectively by*

$$\rho(f) = \limsup_{r\to\infty} \frac{\log \log M(r, f)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log \log m(r, f)}{\log r}.$$

Also, type of f of order ρ is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log M(r, f)}{r^{\rho}}$$

where $M(r, f)$ and $m(r, f)$ denote respectively the maximum modulus and minimum modulus of the function f. We say that the growth of f is minimal type if $\sigma = 0$, mean type if $0 < \sigma < \infty$, and maximal type if $\sigma = \infty$.

It is noted that order is an important characterization of an entire function, which compares the rate of growth of maximum modulus with the growth of the modulus of simple entire function e^z. For example, the exponential function $z \to e^z$, the sine function $z \to \sin z$ and the cosine function $z \to \cos z$ has order 1, the function $z \to \exp(z^d)$ has order d. Also, the iterated maximum modulus $M^n(r, f)$ gives an upper bound for $|f^n|$. That is, if $|z| \leq r$, then $|f^n(z)| \leq M^n(r, f)$. However, the point z satisfying $|z| \geq r$ can go faster to ∞. That is, for an entire function f, there exists $z \in \mathbb{C}$ such that $|f^n(z)| \geq M^n(r, f)$ for all $n \in \mathbb{N}$ and $r \geq R$ with $r > 0$. However, in the case of iterated minimum modulus $m^n(r, f)$, this fact may not hold. That is, for an entire function f, there exists $R > 0$ with $m(r, f) > r$ for all $r \geq R$, we can not always find $r > 0$ such that $m^n(r, f) \to \infty$ as $n \to \infty$.

2.3 Background on classical holomorphic dynamics

The subject holomorphic dynamics is an iteration theory of a holomorphic function. This subject formally begins with the description of the local behavior of a holomorphic function near its fixed points which enable us to establish some basic properties of Fatou-Julia theory.

Definition 2.3.1 (**nth iterate**). *Let f be a holomorphic function. For any $n \in \mathbb{N}$, the nth iterate of f is a function $f^n = f \circ f \circ \ldots \circ f$ which is defined by $f^0(z) =$ an identity, $f^1(z) = f(z)$, $f^2(z) = f(f(z))$, $\ldots, f^n(z) = f^{n-1}(f(z))$.*

It is noted that $f^n(z)$ is defined for all $z \in \mathbb{C}$ if f is an entire function, and it is defined for all $z \in \mathbb{C}$ except for a countable set of poles of f^n for each $n = 1, 2, \ldots, n$ if f is a meromorphic function. If f is a rational function, then f has a meromorphic extension to $\mathbb{C}_\infty$ and f^n is defined and meromorphic in $\mathbb{C}_\infty$. However, if f is transcendental, there is no way to define $f(\infty)$. The iterate of any function f form a group if f is invertible and semigroup, otherwise. Moreover, the the set of the form

$$O^+(z) = \{f^n(z)\}_{n=0}^{\infty}$$

is called a *forward orbit* of z under f. The finite orbit $\{f^n(z)\}_{n=0}^{p-1}$ is called a *cycle*. The *backward orbit* of z under f is the set

$$O^-(z) = \{f^{-n}(z)\}_{n=0}^{\infty} = \{w : f^n(w) = z\} = \bigcup_{n \geq 0} f^{-n}(z)$$

where $f^{-n}(z) = \{w : f^n(w) = z\}$. That is, backward orbit of z is a set of pre-images of z under f. The *large (grand) orbit* is the set $\{w : \text{ there exist } m, n \in \mathbb{N} : f^n(z) =$

$f^m(w)\}$. That is, it is a set of point $w \in \mathbb{C}$ whose orbits eventually intersects the orbit of z. Thus, z and w have same grand orbit if and only if $f^n(z) = f^m(w)$ for some choice of $m \geq 0$ and $n \geq 0$. A point $z \in \mathbb{C}$ is said to have *finite grand orbit* or it is an *exceptional* under f if its grand orbit is finite set.

Definition 2.3.2 (Periodic point and fixed point). *A point $z \in \mathbb{C}$ is called a periodic point of the function f if $f^n(z) = z$ for some $n \in \mathbb{N}$. The smallest n is called its period. In particular, if $f(z) = z$, then z is called a fixed point of f.*

It is noted that unlike the case of rational functions, a transcendental entire function may not have fixed points, for example, $z \to e^z + z$ has no fixed points. However, Bergweiler [17] proved the following result.

Proposition 2.3.1 (Number of fixed points of f^n). *Let f be a transcendental entire function. For $n \geq 2$, f^n has infinitely many (repelling) fixed points.*

Definition 2.3.3 (Pre-periodic point). *The point z is called a pre-periodic (or eventually periodic) if $f^{k+n}(z) = f^k(z)$ for some integer $k \geq 0,\ n > 0$ and strictly pre-periodic if it is pre-periodic but not periodic.*

Let z is a periodic point of period n with $(f^n)'(z) = \lambda$, where f' denotes the complex differentiation of f with respect to z and

$$(f^n)'(z) = \prod_{i=0}^{n-1} f'(f^i(z)) = \prod_{i=1}^{n-1} f'(z_i) \tag{2.3.1}$$

where $z_i = f^{i-1}(z)$. The complex number λ obtained in this way is called *multiplier*. It is noted that the multiplier at z is same as that any other point in the cycle of z.

Definition 2.3.4 (Attracting and super attracting periodic point). *Let f be a holomorphic function. The point z is called an attracting (or super-attracting) periodic point of f if $|\lambda| < 1\ \ (or\ \ |\lambda| = 0)$ where λ is a multiplier of f as defined in (2.3.1).*

Definition 2.3.5 (Repelling periodic point). *The point z is called a repelling periodic point or unstable point of a holomorphic function f if $|\lambda| > 1$ where λ is a multiplier of f.*

Definition 2.3.6 (Indifference periodic point). *The point z is called an indifferent periodic point or a neutral point of a holomorphic function f if $|\lambda| = 1$ where λ is a multiplier of f.*

It is noted that in the case of (supper) attracting periodic point, nearby points are attracted to the orbit under iteration by f; in the case of repelling periodic points, points close to the orbit move away; and in the case of indifference periodic points, nearby points stay near z but does not converge to z.

Definition 2.3.7 (Rationally indifference and irrationally indifference periodic point). *The periodic point z is said to be a parabolic or rationally indifferent if $\lambda^m = 1$ for some $m \in \mathbb{N}$. The periodic point z is said to be an irrationally indifferent if $\lambda = e^{2\pi i\theta}, \ \theta \in \mathbb{R} - \mathbb{Q}$, where λ is a multiplier of f.*

It is noted that in the case of rationally indifference periodic points, the nearby dynamics is completely known but in the case of irrationally indifference periodic points, there are certain values of θ, where nearby dynamics is still not known.

Definition 2.3.8 (Picard exceptional value). *Let f be a holomorphic function. A number $w \in \mathbb{C}$ is a Picard exceptional value or an omitted value of f if f does not assume w in $\mathbb{C}$.*

By classical Picard's theorem, f has at most one or two finite omitted values according to f is an entire or a meromorphic function. For examples: $f(z) = \lambda \tan z, (-1 < \lambda < 1)$ has two omitted values $\pm\lambda i$, and $f(z) = \lambda e^z$ has an omitted value 0.

Definition 2.3.9 (Conjugate function). *A holomorphic function f defined in a domain $U \subset \mathbb{C}$ is said to be conformally conjugate to another holomorphic function g defined in a domain $V \subset \mathbb{C}$ if there is a conformal map $\phi : U \to V$ such that $g = \phi \circ f \circ \phi^{-1}$.*

If we waive inversibility of ϕ, that is, simply $g \circ \phi = \phi \circ f$ holds, then we say f is *semi-conjugate* to g. An irrationally indifference periodic point of a holomorphic function f is a *Siegel point* or *Cremer point* if f is locally conjugated to a linear function or not.

There are two types of points for which the inverse of holomorphic functions are not well defined, namely critical values and asymptotic values and collectively they are known as singular values, and holomorphic dynamics in a large extent is almost determined by such values.

Definition 2.3.10 (Critical value, asymptotic value and singular value). *For any holomorphic function f,*

$$CV(f) = \{w \in \mathbb{C} : w = f(z) \text{ for some } z \text{ such that } f'(z) = 0\}$$

is the set of critical values of f. The set $AV(f)$ consisting of all $w \in \mathbb{C}$ such that there exists a curve $\Gamma : [0, \infty) \to \mathbb{C}$ so that $|\Gamma(t)| \to \infty$ and $f(\Gamma(t)) \to w$ as $t \to \infty$ is the set of asymptotic values of f, and $SV(f) = \overline{CV(f) \cup AV(f)}$ is the set of singular values of f.

It is noted that among the entire functions only transcendental entire functions may have asymptotic values. It is clear that polynomials can not have finite asymptotic values. If f is a rational or transcendental entire, then immediate basin of any cycle of attracting and parabolic periodic points contain singular values of f. There are certain holomorphic functions whose finite asymptotic values can also be critical values. For exapmle, the function $f(z) = z^2e^{-z^2}$ has $AV(f) = \{0\}$ and $CV(f) = \{0, 1/e\}$.

Definition 2.3.11 (Finite type and bounded type function). *Let f be a holomorphic function. If $SV(f)$ is finite, then f is said to be of finite type. If $SV(f)$ is bounded, then f is said to be of bounded type.*

Definition 2.3.12 (Speiser class and Eremenko-Lyubich class). *Let f be a holomorphic function. The sets*

$$\mathscr{S} = \{f : f \text{ is of finite type}\}$$

and

$$\mathscr{B} = \{f : f \text{ is of bounded type}\}$$

are respectively known as Speiser class and Eremenko-Lyubich class.

It is noted that $\mathscr{S} \subset \mathscr{B}$ and $\mathscr{B} - \mathscr{S} \neq \emptyset$. For example, $\dfrac{\sin z}{z} \in \mathscr{B} - \mathscr{S} \neq \emptyset$ (that is, this function has asymptotic value 0 and the set of critical values is an infinite set of real numbers, all of modulus not greater than 1). The most important class of functions in the class $\mathscr{S}$ are functions of the exponential family λe^z, $(\lambda \neq 0)$, and the cosine family $\cos(\alpha z + \beta)$, $(\alpha \neq 0)$.

Definition 2.3.13 (Post singular point and post singular set). *Let f be a holomorphic function. The post-singular point of f is the point on the orbit of a singular value. That is, if z is a singular value of f, then $f^n(z)$ is a post-singular point for $n \geq 0$. The set of all post-singular points is called post-singular set and it is denoted by*

$$P(f) = \bigcup_{n \geq 0} f^n(SV(f))$$

It is noted that for a transcendental entire function f, $P(f)$ contains at least two points. Bakar [6] proved that $P(f^k) = P(f)$ for all $k \in \mathbb{N}$. The Cremer point of any transcendental entire function f is contained in $P(f)$ ([51, Theorem 4.17]).

Definition 2.3.14 (Post singularly bounded (finite) function). *A holomorphic function f is called post-singularly bounded (finite) if its post-singular set is bounded (finite).*

Definition 2.3.15 (Hyperbolic holomorphic function). *A holomorphic function $f \in \mathscr{B}$ is said to be hyperbolic if every point in $SV(f)$ belongs to the basin of attracting periodic cycle of f.*

For example, the function $f(z) = \frac{\pi}{2}\sin z$ is hyperbolic. For this function, $SV(f) = \{\pm\frac{\pi}{2}\}$ which are super-attracting fixed points, so by definition, f is hyperbolic.

The chief objects of classical holomorphic dynamics are Fatou and Julia sets of a holomorphic function and their definitions depend on normal family of the sequence of nth iterates of a holomrphic function.

Definition 2.3.16 (Normal family). *A family $\mathscr{F}$ of holomorphic functions in a domain $D \subset \mathbb{C}_\infty$ is called a normal family in D if every sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{F}$ contains a subsequence $(f_{n_k})_{n_k\in\mathbb{N}}$ which converges uniformly on every compact subset of D either to a holomorphic function f or to ∞. The family $\mathscr{F}$ is called normal at a point $z \in \mathbb{C}$ if this point has a neighborhood where it is normal.*

The theory of normal family of holomorphic functions was investigated by P. Montel in 1907. It is noted that the set $\mathscr{F} = \{f^n : n \in \mathbb{N}\}$ forms a normal family in the basin of attraction of any attracting periodic points of the function f.

Two most important fundamental objects of holomorphic dynamics are Fatou and Julia sets which are defined as follows.

Definition 2.3.17 (Fatou set, Julia set and Fatou component). *Let f be a holomorphic function. The Fatou set $F(f)$ of the function f is defined as the maximal open set where the sequence $(f^n)_{n\in\mathbb{N}}$ forms a normal family, and its complement is the Julia set $J(f)$. A connected component of $F(f)$ is called Fatou component.*

For any holomorphic function f, there are different Fatou components of dynamical interest.

Definition 2.3.18 (Pre-periodic and periodic Fatou component). *Let $U \subset F(f)$ be a Fatou component of a holomorphic function f such that $f^n(U)$ is contained in some component of $F(f)$ which is usually denoted by U_n. Then U is called pre-periodic if $U_n = U_m$, for some integers $n > m \geq 0$. In particular, if $U_n = U_0 = U$ (that is, $f^n(U) \subset U$) for some smallest positive integer $n \geq 1$, then U is called a periodic Fatou component of period n.*

Let U is a periodic Fatou component of period n. The set $\{U_0, U_1 \ldots, U_{n-1}\}$ is called the *periodic cycle* of U. If $U_1 = f(U) \subset U$, then U is called an *invariant domain*.

Definition 2.3.19 (Wandering domain). *A component U of the Fatou set $F(f)$ of a holomorphic function f is a wandering domain if it is not pre-periodic.*

It is noted that if f is a rational or an entire function with a finite number of singular values, then f has no (simply or multiply connected) wandering domains. Periodic Fatou components of a holomorphic function can also further classified and analyzed.

Proposition 2.3.2 (**Classification of periodic Fatou components**). *For a holomorphic function f, a periodic Fatou component U of period p is one of the following types.*

1. *(Super) attracting domain if U contains a (super) attracting periodic point z_0 of period p such that $f^{np}(z) \to z_0$ as $n \to \infty$ for all $z \in U$.*

2. *Parabolic domain or Leau domain if ∂U contains an indifferent periodic point z_0 of period p such that $f^{np}(z) \to z_0$ as $n \to \infty$ for all $z \in U$.*

3. *Seigal disk if there is an analytic homeomorphism $\phi : U \to \mathbb{D}$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi\alpha} z$ for some $\alpha \in \mathbb{R} - \mathbb{Q}$.*

4. *Hermann ring if there is an analytic homeomorphism $\phi : U \to A$, where A is an annulus $\{z : 1 < |z| < r\},\ r > 1$ such that $(\phi \circ f^p \circ \phi^{-1})(z) = e^{2\pi\alpha} z$ for some $\alpha \in \mathbb{R} - \mathbb{Q}$.*

5. *Baker domain if $f^{np}(z) \to \infty$ as $n \to \infty$ for all $z \in U$.*

Definition 2.3.20 (**Forward, backward and completely invariant set**). *Let f a holomorphic function. A set $U \subset \mathbb{C}$ is called forward invariant under f if $f(z) \in U$ for all $z \in U$ (or $f(U) \subset U$), backward invariant under f if $f^{-1}(z) \in U$ for all $z \in U$ (or $f^{-1}(U) \subset U$) (here $f^{-1}(U)$ denotes the pre-image U under f), and completely invariant if it is both forward and backward invariant.*

It is noted that a rational function (and a meromorphic function) has at most two completely invariant domains ([15, Theorem 5.6.1]), and a transcendental entire function has at most one completely invariant domain ([7]). For examples: $f(z) = z^2$ has two completely invariant domains, namely $\mathbb{D}$ and $\{z \in \mathbb{C} : |z| > 1\}$, where $J(f) = S^1$ is also completely invariant. Likewise, $f(z) = \tan z$ has two completely invariant domains, namely, $\mathbb{H}^u$ and $\mathbb{H}_l$, where $J(f) = \mathbb{R} \cup \{\infty\}$ is also completely invariant. The function $f(z) = \lambda e^z,\ (0 < \lambda < 1/e)$ has a completely invariant domain, namely, the attracting domain (which is $F(f)$ itself) of the attracting fixed point.

From [15, 18, 51, 68], we can state the following structure and properties of the Fatou and Julia sets of any holomorphic function.

Proposition 2.3.3 (**Structure and Properties of the Fatou and Julia sets**). *Let f be a holomorphic function.*

1. *$F(f)$ is an open set, and $J(f)$ is a non-empty and a closed prefect set. In particular, $J(f)$ is uncountable. $J(f)$ is unbounded if f is a transcendental entire function.*

2. *$F(f) = \emptyset$ for every entire function of finite type for which all singular values are either pre-periodic or escape to ∞, and $F(f) \neq \emptyset$ for any entire function with attracting periodic points.*

3. *$F(f) = F(f^n)$, and $J(f) = J(f^n)$ for all $n \geq 2$.*

4. *$F(f)$ and $J(f)$ are completely invariant under f. Either $J(f) = \mathbb{C}$ or Int.$J(f) = \emptyset$.*

5. *$J(f)$ is the closure of the set repelling periodic points. In particular, repelling periodic points are dense in $J(f)$.*

6. *$J(f)$ is never totally disconnected if f is a transcendental entire function.*

7. *There is a single unbounded component of the Fatou set (which is immediate basin of ∞) if f is a polynomial.*

8. *$F(f)$ does not have wandering domains if f is a rational function.*

9. *$F(f)$ does not have Herman rings if f is a transcendental entire function.*

10. *$F(f)$ does not have wandering domains if f is an entire function of finite type.*

11. *$F(f)$ does not have Baker domains if f is a transcendental entire function of bounded type.*

12. *$F(f)$ contains all (super) attracting fixed points and cycles, all Siegel points and cycles. $J(f)$ contains all repelling fixed points and cycles, all rationally indifference fixed points and cycles, and all Cremer points and cycles.*

13. *Every unbounded component and a pre-periodic component of $F(f)$ is simply connected if f is a transcendental entire function, and for such a function, a multiply connected Fatou component is bounded and wandering.*

14. *Any component of $F(f)$ is simply connected if f is a bounded type transcendental function.*

15. *If $F(f)$ has an unbounded component, then all components are simply connected if f is a transcendental entire function.*

16. *Any component of $F(f)$ is simply connected if f is a transcendental entire function bounded on some curve tending to ∞.*

17. *All component of $F(f)$ are simply connected if f is an entire function and it has finite asymptotic values.*

18. *The number of components of $F(f)$ is either $0, 1, 2$ or ∞ if f is a rational function and either 0, 1 or ∞ if f is a transcendental entire function, and in this case, the number of multiply connected components is either 0 or ∞.*

19. *If f is a hyperbolic transcendental entire function, then $P(f)$ is a compact subset of $F(f)$.*

20. *Every component of $F(f)$ is bounded if and only if a hyperbolic transcendental entire function f has no finite asymptotic values and every component of $F(f)$ contains at most finitely many critical points.*

The order $\rho(f)$ of an entire function f plays crucial role in the structure of Fatou components. For a polynomial or an entire function of order 0, the attracting basin of any finite attracting periodic point is bounded. For example, the polynomial function $f(z) = z^2$ has bounded attracting basin $\mathbb{D}$ of the finite attracting periodic point 0. However, in the case of transcendental entire function f, there are bounded basin of attraction if $\rho(f) \leq 1/2$. The one of the most important open problem in the field of holomorphic dynamics is the conjecture of Baker concerned about the certain order of f and bounded components of the Fatou set.

Conjecture 2.3.1 (Baker's conjecture). *If a hoomorphic function f has order less than 1/2 or has order at most 1/2, minimal type, then all components of the Fatou set $F(f)$ are bounded.*

This conjecture arosed by Baker in his paper [11] in 1981 and remain open in general. Anderson and Hinkkanen [1, Theorem 1] and Stallard [102, Theorem 3A, Page 49] proved the following result.

Proposition 2.3.4 (A partial solution to the Baker conjecture). *Let f be a transcendental entire function of order less than 1/2. Then every pre-periodic or periodic components of $F(f)$ is bounded.*

From this result, we can say that a case has been settled down and there is open problem only for wandering domains and Baker domains of period greater than 1. By using the notion of log-regularity, Andersion and Hinkkanen [1, Theorem 2] proved the following result which is considered a most general answer to the Baker's conjecture 2.3.1.

Proposition 2.3.5 (A more strong partial solution to the Baker conjecture). *Let f is a transcendental entire function of order less than 1/2 such that for some positive constant c*

$$\frac{\phi'(x)}{\phi(x)} \geq \frac{1+c}{x} \tag{2.3.2}$$

for sufficiently large x, where $\phi(x) = \log M(e^x, f)$. Then every components of $F(f)$ is bounded.

It is noted that the function f is *log-regular* if condition (2.3.2) is satisfied by the function $\phi(x) = \log M(e^x, f)$. It is given in [45, Page 205] that the regularity condition (2.3.2) holds for all transcendental entire functions of finite order and positive lower order.

2.4 Escaping set and Eremenko's conjecture

The third fundamental object of the holomorphic dynamics, in particular of the transcendental dynamics is the *escaping set*.

Definition 2.4.1 (Escaping set). *Let f be a holomorphic function. The escaping set of f is defined by $I(f) = \{z : f^n(z) \to \infty$ as $n \to \infty\}$. A point $z \in I(f)$ is called an escaping point.*

The escaping set $I(f)$ has played significant role in polynomial dynamics, and in transcendental dynamics. The set $I(f)$ of a polynomial f (of degree $d \geq 2$) is an open subset of the Fatou set $F(f)$, and the Julia set $J(f)$ equals the boundary of $I(f)$. However, for a general transcendental entire function f, $I(f)$ is no longer open but as in polynomial, boundary of $I(f)$ is the Julia set $J(f)$. The set $I(f)$ was studied first time for a general transcendental entire function f by Eremenko [31]. He [31, Theorems 1, 2, 3, 4 (Corrollary)] proved the following results.

Proposition 2.4.1 (Structure and properties of the escaping set). *Let f be a transcendental entire function. Then*

1. $I(f) \neq \emptyset$,
2. $J(f) \cap I(f) \neq \emptyset$,
3. *The closure $\overline{I(f)}$ of $I(f)$ has no bounded components,*
4. $J(f) = \overline{I(f)}$ *when $f \in \mathscr{B}$.*

Furthermore,

5. $I(f) = I(f^n)$, *for $n \geq 2$,*
6. $I(f)$ *is completely invariant.*

In view of the third statement of Proposition 2.4.1, he posed a conjecture:

Conjecture 2.4.1 (Eremenko's conjecture). *Let f be a transcendental entire function. Then every component of $I(f)$ is unbounded.*

This conjecture is considered an important open problem of transcendental dynamics, and nowadays it is famous as *Eremenko's conjecture*. This conjecture was proved first by Rempe [78, Theorem 1.1] for certain class of transcendental entire functions.

Proposition 2.4.2 (A partial solution to Eremenko conjecture). *Let $f \in \mathscr{B}$ be an entire function and $P(f)$ is bounded. Then every component of $I(f)$ is unbounded.*

If U is a Fatou component such that $U \cap I(f) \neq \emptyset$, then by normality $U \subset I(f)$. We call such type of Fatou component by an *escaping Fatou component*. It is not always necessary that the boundary of an escaping Fatou component must lie in $I(f)$. For example, the Fatou function $f(z) = e^{-z} + z + 1$ has $\mathbb{H}r$ as an escaping Fatou component.

The conjecture 2.4.1 has been proved for a general transcendental entire function for the fast escaping set $A(f)$, which consists of points that tend to infinity as fast as possible under iteration. This set is a subset of $I(S)$, and it was introduced first time by Bergweiler and Hinkkanen [22], and defined in the following form by Rippon and Stallard [89].

Definition 2.4.2 (Fast escaping set). *For a transcendental entire function f, the fast escaping set is defined by*

$$A(f) = \{z \in \mathbb{C} : \exists L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| \geq M^n(R, f) \text{ for } n \in \mathbb{N}\}$$

where $M(r, f), r > 0$ represents the maximum modulus of f, and $M^n(r, f)$ denotes nth iterate of $M(r, f)$ with respect to r. $R > 0$ can be taken to be any number such that $M(r, f) > r$ for $r \geq R$.

Most of properties and structure of the fast escaping set $A(f)$ were proved by Rippon and Stallard on the basis of the properties of certan subsets of the set $A(f)$, which is defined as follows.

Definition 2.4.3 (Lth label of fast escaping set). *For any $L \in \mathbb{Z}$, the Lth label of the fast escaping set is defined by*

$$A_R^L(f) = \{z : |f^n(z)| \geq M^{n+L}(R, f) \text{ for } n \in \mathbb{N}, n + L \geq 0\}$$

where $L \in \mathbb{Z}$ and $R > 0$ be such that $M(r, f) > r$ for $r \geq R$.

It is noted that each of the level of $A(f)$ is a closed set, and since $M^{n+1}(R, f) > M^n(R, f)$ for all $n \geq 0$, we must have $A_R^L(f) \subset A_R^{L-1}(f)$ for all $L \in \mathbb{Z}$. This implies that

$$A(f) = \bigcup_{L \in \mathbb{N}} A_R^{-L}(f) \text{ and } A_R^{-L}(f) \subset A_R^{-(L+1)}(f),\ L \in \mathbb{N}.$$

The concept of level provides a new understanding of the structure of $A(f)$ as a countable union of closed sets. On the basis of Definition 2.4.3, Rippon and Stallard [89, Theorem 1.1] obtained the following strongest result in the direction of Eremenko's conjecture 2.4.1.

Proposition 2.4.3 (A more strong partial solution to Eremenko conjecture). *Let f be a transcendental entire function, and $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Then for each $L \in \mathbb{Z}$, every component of $A_R^L(f)$ is closed and unbounded. In particular, every component of $A(f)$ is closed and unbounded.*

Since $A(f) \subset I(f)$ and $A(f) \neq \emptyset$ (see [22]), so this result provides a partial answer to Eremenko's conjecture. That is, the set $I(f)$ must has at least one unbounded component. Like escaping sets, we can get from [89, 99] the following further similar structure and properties of fast escapimg sets.

Proposition 2.4.4 (Properties of fast escaping sets). *Let f be a transcendental entire function. Then the following hold.*

1. $A(f) \neq \emptyset$.
2. $A(f) = A(f^n)$ *for* $n \geq 2$.
3. $A(f)$ *is completely invariant.*
4. $A(f)$ *is independent of R.*
5. $J(f) \cap A(f) \neq \emptyset$.
6. $J(f) = \partial A(f)$.
7. $J(f) = \overline{A(f) \cap J(f)}$.
8. $A(f)$ *has no bounded components.*
9. $\overline{U} \subset A(f)$ *for any Fatou component U that meets $A(f)$.*
10. $\overline{U} \subset A(f)$ *for any multiply connected Fatou component U.*
11. *If U is a Fatou component and $U \cap A(f) \neq \emptyset$, then U is a wandering domain.*

12. $J(f) \subset \overline{A(f)}$, *and if* f *does not have wandering domain, then* $A(f) \subset J(f)$.

It is noted that almost properties of the fast escaping set $A(f)$ were proved by Rippon and Stallard on the basis of levels ([89, Theorems 1.1, 1.2, 2.2, 2.6, 4.4, 5.1]). Furthermore, statements 8 and 9 of Proposition 2.4.4 are much stronger properties of $A(f)$ than the set $I(f)$. Also, according to the statement (8) of Proposition 2.4.4, Eremenko's conjecture hold for the subset $A(f)$ of $I(f)$.

2.5 Connection between Baker and Eremenko conjecture

In recent years, active research in the field of escaping set has been devoted mostly to see the structure that has number of strong dynamical properties as well as able to establish a connection between the conjecture of Baker and the conjecture of Eremenko. The new research in this direction has become possible by introducing spider's web structure. This new set structure is defined as follows.

Definition 2.5.1 (Spider's web). *A set* E *is an (infinite) spider's web if* E *is connected, and there exists a sequence of bounded simply connected domains* G_n *with* $G_n \subset G_{n+1}$ *for* $n \in \mathbb{N},\ \partial G_n \subset E$ *for* $n \in \mathbb{N}$ *and* $\bigcup_{n\in\mathbb{N}} G_n = \mathbb{C}$.

Rippon and Stallard [89, Lemma 7.1] proved the following basic properties of spider's web structure which are useful for many purposes.

Proposition 2.5.1 (Properties of spider's web). *Let* f *be a transcendental entire function, and let* $R > 0$ *be such that* $M(r, f) > r$ *for* $r \geq R$ *and* $L \in \mathbb{Z}$.

1. *If G is a bounded components of* $A_R^L(f)^C$, *then* $\partial G \subset A_R^L(f)$, *and* f^n *is a proper map of G onto the bounded component of* $A_R^{n+L}(f)^C$ *for each* $n \in \mathbb{N}$.
2. *If* $A_R^L(f)^C$ *has bounded component, then* $A_R^L(f)$ *is a spider's web, and hence every component of* $A_R^L(f)^C$ *is bounded.*
3. $A_R(f)$ *is a spider's web if and only if* $A_R^L(f)$ *is a spider's web.*
4. *For* $R' > R$, *then* $A_R(f)$ *is a spider's web if and only if* $A_R'(f)$ *is a spiders web.*
5. *If* $I(f), J(f), I(f) \cap J(f)$ *contain spider's web, then each of set is a spider's web.*

It is noted that if $I(f)$ is a spider's web, then $I(f)$ is connected and unbounded, and so Eremenko's conjecture holds. In [85], Rippon and Stallard have proved that $A_R(f)$, $A(f)$ and $I(f)$ are spider's web for a transcendental entire function f whenever f has a multiply connected Faou component. With these strong dynamical properties, it is better to ask: Which function f that gives $A_R(f)$ a structure of spider's web? Several classes of functions that gives $A_R(f)$ a structure of spider's web are derived using the idea of following results of Rippon and Stallard [89, Theorem 1.9].

Proposition 2.5.2 (Functions that give spider's web structure for $I(f)$). *Let f be transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Then $A_R(f)$ is a spider's web if one of the following holds:*

1. *f has a multiply connected Fatou component.*
2. *f is of order less than 1/2.*
3. *f has finite order of Fabray gaps.*
4. *f has Hayman gaps.*
5. *f exihibits the pits effects.*

Osborne et al. [74, Theorem 1.1] recently investigated classes of transcendental entire functions stated in Proposition 2.5.2 are also enough to give

$$m^n(r, f) \to \infty \text{ as } n \to \infty \tag{2.5.1}$$

for any $r > 0$, where $m(r, f)$ is a minimum modulus of f. In general, it is not always necessary that $m^n(r, f) \to \infty$ as $n \to \infty$ for any $r > 0$. Therefore, $A(f)$ may not always be a subset of $I(f)$ if we replace $M(r, f)$ by $m(r, f)$ in Definition 2.4.2 of $A(f)$. Suppose f is a transcendental entire function satisfying the condition (2.5.1). Osborne et al. [74, Page 49] defined a set of the form

$$B_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq \widehat{m}^n(R, f), \text{ for } n \in \mathbb{N}\}$$

where

$$\widehat{m}(r, f) = \max_{0 \leq t \leq r} m(t, f), \quad \text{for } r \in [0, \infty).$$

Indeed, the set $B_R(f)$ consists of points whose modulus of iterates under the function f grows at least as fast as iterates of the maximal function: $\widehat{m}(r, f)$. Osborne et al. [74, Theorem 1.2] recently proved the following result.

Proposition 2.5.3. *If $m^n(r, f) \to \infty$ as $n \to \infty$ for any $r > 0$, then there exists $R > 0$ such that $\widehat{m}^n(R, f) \to \infty$ as $n \to \infty$, and for any such R, there exists $r \geq R$ such that $m^n(r, f) \geq \widehat{m}^n(R, f)$.*

The set $B_R(f)$ is well defined and independent of R as well as completely invariant under f. They also formed a set

$$B(f) = \bigcup_{l \geq 0} f^{-l}(B_R(f)).$$

of all pre-images of the set $B_R(f)$. This set is again well defined and independent of R as well as completely invariant under f if $m^n(r, f) \to \infty$ as $n \to \infty$ for all $r > 0$. As $M^n(r, f) \geq \widehat{m}^n(r, f)$ for any integer $n \geq 0$, there always have $A(f) \subset B(f)$, and so $B(f) \neq \emptyset$. The most concerned issue for the development of the set $B(f)$ is to find the class of functions for which $A(f) = B(f)$. This condition appeared as a milestone for both Baker and Eremenko's conjectures as in the following results due to Osborne et al. [74, Theorem 1.3].

Proposition 2.5.4 (A solution for both Baker and Eremenko conjecture). *Let f be a transcendental entire function such that $m^n(r, f) \to \infty$ as $n \to \infty$ for any $r > 0$. Then*

1. *$B(f) = A(f)$ if and only if there exists $r \geq R > 0$ such that $m^n(r, f) \geq M^n(R, f)$ for integer $n \geq 0$ and $M^n(R, f) \to \infty$ as $n \to \infty$.*
2. *If (1) holds, then $B(f)$ and $A(f)$ are spider's web and $F(f)$ has no unbounded components.*

The hypothesis $B(f) = A(f)$ is a very strong condition to hold both of Baker's and Eremenko's conjectures. If this condition is loosed, the essence of the above second condition may not hold. The subset $B(f)$ of the escaping set is a recently investigated tool that can be used to make spider's web structure of $A(f)$ and $I(f)$ together with to get Fatou set that has no unbounded components.

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Chapter 3

THE NOTION OF HOLOMORPHIC SEMIGROUPS

3.1 The general notion of semigroups

The purpose of this section is to introduce the general notion of semigroup structure and substructure. We only define here semigroups and groups, and for more detailed study of classical semigroup theory, we refer [44, 50].

Definition 3.1.1 (**Semigroup, subsemigroup, monoid and group**). *A semigroup is an ordered pair $(S, \circ)$, where S is a non-empty set and $\circ$ is a binary composition on S (by which we mean a map $\circ : S \times S \to S$) that satisfies the associative law*

$$a \circ (b \circ c) = (a \circ b) \circ c \tag{3.1.1}$$

for all $a, b, c \in S$. A non-empty subset $T \subset S$ is called a subsemigroup if $a \circ b \in T$ for all $a, b \in T$. A monoid is a semigroup S with an element $i \in S$ (which is known as identity) such that

$$i \circ a = a \circ i = a \tag{3.1.2}$$

for all $a \in S$. A monoid S is a group if for all $a \in S$, there exists $a^{-1} \in S$ such that $a \circ a^{-1} = a^{-1} \circ a = i$, where i is an identity element in S satisfying the relation (3.1.2).

Probably the simplest examples of semigroups are additive and multiplicative semigroups of various number sets such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. An important example of a semigroup is the set of all self-maps τ_X of a non-empty set X- the full transformation semigroup on a set X under the functional composition. Also, every group is a semigroup. Note that every semigroup S can be extended to a monoid S^i by adjoining an extra element i to S such that $i \circ f = f \circ i = f$ for all $f \in S$.

For any $A \subset S$, the intersection of all subsemigroups of S containing A is called a subsemigroup generated by A. It is denoted by $\langle A \rangle$, and it is a set of all elements of S that can be expressed as a finite composition (product) of elements in A. That is,

$$\langle A \rangle = \{a_1 \circ a_2 \circ \ldots \circ a_n : a_1, a_2, \ldots, a_n \in A \text{ and } n \in \mathbb{N}\}.$$

It is a least subsemigroup of S containing the set A. A subsemigroup T of S is said to be generated by a non-empty set A if $T = \langle A \rangle$. If $A = \{a\}$ for any $a \in S$, then $T = \langle a \rangle = \{a, a^2, a^3, \ldots, a^n, \ldots\}$ is called a *cyclic* subsemigroup of S generated by a. If $S = \langle a \rangle$ for some $a \in S$, then S is called a *cyclic* semigroup. It is noted that either $\langle a \rangle$ is algebraically similar to $\mathbb{N}$ or it is a cyclic group of order n for some $n \in \mathbb{N}$.

3.2 The notion of holomorphic semigroups

From Section 3.1, we can say that semigroups are very classical algebraic structures with a binary composition that satisfies associative law (3.1.1). It naturally arose from the general mapping of a set into itself. Hence, a set of holomorphic functions on $\mathbb{C}$ or $\mathbb{C}_\infty$ or certain subsets thereof naturally forms a semigroup.

Definition 3.2.1 (Holomorphic semigroup and subsemigroup). *A holomorphic semigroup S is a semigroup of holomorphic functions defined on $\mathbb{C}$ or $\mathbb{C}_\infty$ or certain subsets thereof with the semigroup operation being the functional composition. A non-empty subset $T \subseteq S$ is a* subsemigroup *of S if $f \circ g \in T$ for all $f, g \in S$.*

Let

$$\mathscr{F} = \{f_\alpha : f_\alpha \text{ is a holomorphic function for all } \alpha \in \Delta\}, \tag{3.2.1}$$

where Δ is an index set which is allowed to be infinite in general unless stated otherwise. In this sense, a holomorphic semigroup is a set S of holomorphic functions from $\mathscr{F}$ such that $f_{\alpha+\beta}(z) = f_\alpha(f_\beta(z))$ for all $z \in \mathbb{C}$ or $\mathbb{C}_\infty$ or certain subsets thereof and for all $\alpha, \beta, \alpha + \beta \in \Delta$. If $\Delta \subseteq \mathbb{N}$, then S is called a *discrete semigroup*.

We are more interested in special holomorphic semigroups whose each element can be expressed as a finite composition of certain holomorphic functions. More formally, such a semigroup is defined as follows:

Definition 3.2.2 (Holomorphic semigroup generated by holomorphic functions). *Let $\mathscr{F}$ be a family of holomorphic functions as defined in (3.2.1). A holomorphic semigroup S generated by $\mathscr{F}$ is a semigroup of all elements that can be expressed as a finite composition of elements in $\mathscr{F}$. We denote such a holomorphic semigroup by $S = \langle f_\alpha \rangle_{\alpha \in \Delta}$ or simply by $S = \langle f_\alpha \rangle$.*

Holomorphic semigroup S is said to be a *rational semigroup* or a *transcendental semigroup* depending on whether $\mathscr{F}$ is a collection of rational functions or transcendental entire functions. In particular, S is said to be a *polynomial semigroup* if $\mathscr{F}$ is a collection of polynomial functions. The transcendental semigroups or polynomial semigroups are also called *entire semigroups*. It is noted that if $f \in S$, then there is a finite number of functions f_α for $\alpha \in \Delta$ such that $f = f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_m}$ for some $m \in \mathbb{N}$, and $\alpha_1, \alpha_2, \ldots \alpha_m \in \{1, 2, \ldots, m\}$.

Definition 3.2.3 (Cancellative semigroup). *A holomorphic semigroup S is said to be right cancellative if $f \circ g = h \circ g \implies f = h$, left cancellative if $h \circ g = h \circ f \implies g = f$ for all $f, g, h \in S$, and cancellative if it is both right and left cancellative.*

Definition 3.2.4 (Abelian holomorphic semigroup). *A holomorphic semigroup $S = \langle f_\alpha \rangle$ is abelian if $f_\alpha \circ f_\beta = f_\beta \circ f_\alpha$ for all generators f_α, f_β of S.*

Definition 3.2.5 (Finitely generated holomorphic semigroup and cyclic semigroup). *A semigroup generated by finitely many holomorphic functions $f_j, (j = 1, 2, \ldots, n)$ is called a finitely generated holomorphic semigroup, and we write $S = \langle f_1, f_2, \ldots, f_n \rangle$. If S is generated by only one holomorphic function f, then S is called a cyclic holomorphic semigroup, and we write $S = \langle f \rangle$. In this case, each $g \in S$ can be written as $g = f^n$, where f^n is the nth iterate of f with itself.*

By Definition 3.2.2 of a holomorphic semigroup, we at once get the following result.

Proposition 3.2.1 ([115, Proposition 1.1])**.** *Let $S = \langle f_\alpha \rangle$ be a holomorphic semigroup. Then for every $f \in S$, f^m (for all $m \in \mathbb{N}$) can be written as $f^m = f_{\alpha_1} \circ f_{\alpha_2} \circ f_{\alpha_3} \circ \cdots \circ f_{\alpha_p}$, where $\alpha_i \in \{\alpha : \alpha \in \Delta\}$ for some $p \in \mathbb{N}$.*

Example 3.2.1 ([121, Example 1.1])**.** *Let S be a set consisting of all powers z which are either all powers of 2 or all powers of 3 or the product of all powers of 2 and 3. Then S forms a semigroup under the functional composition. It is a finitely generated polynomial semigroup generated by two polynomials $z \to z^2$ and $z \to z^3$. In this case, $S = \langle z^2, z^3 \rangle$.*

Example 3.2.2 ([121, Example 1.2])**.** *Let $\alpha \in \mathbb{C}$ such that $\Re(\alpha) \geq 0$. For any $k \in \mathbb{N}$, the function $f_k(z) = e^{-\alpha k} z$ for all $z \in \mathbb{C}$ is holomorphic in $\mathbb{C}$, and so $S = \{f_k : k \in \mathbb{N}\}$ is a holomorphic (entire) semigroup. Also, each $f \in S$ can be written as $f(z) = f_1^l(z)$ for some $l \in \mathbb{N}$. Therefore, $S = \langle f_1 \rangle$.*

Example 3.2.3 ([121, Example 1.3])**.** *If we choose Δ a set of positive rational numbers $\mathbb{Q}^+$ in Example 3.2.2, then every function $f_r(z) = e^{-\alpha r}z$ for all $z \in \mathbb{C}$ and $r \in \mathbb{Q}^+$ for some $\alpha \in \mathbb{C}$ such that $\Re(\alpha) \geq 0$ can be written as a finite composition of $f_t(z) = e^{-\alpha t}z$ with $t \in \mathbb{Q}^+$. Hence, $S = \langle f_t \rangle_{t \in \mathbb{Q}^+}$ is a holomorphic semigroup generated by the set $\{f_t : t \in \mathbb{Q}^+\}$.*

It is noted that these are examples of abelian holomorphic semigroups. The following holomorphic semigroups are non-abelian.

Example 3.2.4 ([121, Example 1.4])**.** $S_1 = \langle e^z, e^{-z} \rangle$ *and* $S_2 = \langle \sin z, \cos z \rangle$.

There are certain subsets of semigroups with a stronger closure property rather than that of subsemigroups.

Definition 3.2.6 (Left ideal, right ideal and two sided ideal). *Let I be a non-empty subset of a holomorphic semigroup S. We say that I is a left ideal (or right ideal) of S if $f \circ h \in I$ (or $h \circ f \in I$) for all $f \in S$ and $h \in I$, that is, $SI \subset I$ (or $IS \subset I$). We say that I is a two sided ideal (or simply ideal) if it is both left and right ideal.*

It is noted that if S is an abelian semigroup, then the notions of left ideal, right ideal and two sided ideal coincide. Evidently, every ideal (left, right or two sided) is a subsemigroup, but not every subsemigroup is an ideal.

Ideals can be constructed naturally from the semigroup S. For example, the sets

$$SK = \{f \circ g : f \in S, g \in K\} = \bigcup_{g \in K} S \circ g,$$

$$KS = \{g \circ f : f \in S, g \in K\} = \bigcup_{g \in K} g \circ S,$$

and

$$SKS = \{f \circ g \circ h : f, h \in S, g \in K\} = \bigcup_{g \in K} S \circ g \circ S$$

for any non-empty subset K of a holomorphic semigroup S, are respectively left, right and two sided ideals. Likewise, for any $g \in S$, the sets $S \circ g$, $g \circ S$ and $S \circ g \circ S$ are respectively left, right and two sided ideals. In general, g may not be in $g \circ S$ (or $S \circ g$ or $S \circ g \circ S$) for each $g \in S$. If it happens to be in $g \circ S$ (or $S \circ g$ or $S \circ g \circ S$), then $g = g \circ f$ (or $g = f \circ g$ or $g = f \circ g \circ h$) for some $f, h \in S$. In this case, $g \circ S$ (or $S \circ g$ or $S \circ g \circ S$) is a smallest right (or left or two sided) ideal containing g, which is a right (or left or two sided) ideal generated by g. Otherwise, $g \circ S$ (or $g \circ S$ or $S \circ g \circ S$) is said to be quasi-generated by g. It is obvious that the union of any non-empty family of left (or right or two sided) ideals of S is a left (or right or two sided) ideal of S.

On the basis of some topological structure of the complement of $g \circ S$ in S, we can define the following types of holomorphic semigroups. Recall that a space X is *compact* if every open covering of X contains a finite subcover.

Definition 3.2.7 (F-semigroup and C-semigroup). *Let S is a holomorphic semigroup and $g \in S$. Then we say*

1. *S is a F-(right) semigroup if $S - g \circ S$ is finite;*
2. *S is a C-(right) semigroup if $S - g \circ S$ is relatively compact (that is, $\overline{S - g \circ S}$ is compact in S).*

Analogously, we can define F-(left) semigroup and C-(left) semigroup of any holomorphic semigroup S. We say only F-semigroup and C-semigroup onward for such a holomrphic semigroup S on the assumption that left/right is clear from the context. For example, holomorphic semigroup of Example 3.2.2 is both F-semigroup and C-semigroup and that of Example 3.2.3 is a C-semigroup. A cyclic holomorphic semigroup is also both F-semigroup and C-semigroup.

There are certain type of left (or right) ideals which are connected to two sided ideals. That is, on the basis of such ideals of holomorphic semigroups, we can construct two sided ideals. This ideal structure is defined as follows.

Definition 3.2.8 (Minimal left (or right) ideal). *A left (or right) ideal M of a holomorphic semigroup S is said to be minimal if for every left (or right) ideal I of S such that $I \subseteq M$, then $M = I$.*

It is noted that minimal left (or right) ideal of S may be empty. Also, there are semigroups which have minimal left ideals but no minimal right ideals. If it is non-empty for a certain holomorphic semigroup S, then for every $f \in M$, there must be $M \circ f = M$ (or $f \circ M = M$) and $S \circ f = M$ (or $f \circ S = M$). That is, if semigroup S contains at least one minimal left (or right) ideal, then it has the kernel. Also, a minimal left (or right) ideal is always contained in every two sided ideal of S. We can also make a two sided ideal by the help of the minimal left (or right) ideals. For any holomorphic semigroup S, let us define

$$K(S) = \bigcup \{M : M \text{ is a minimal left (or right) ideal of } S\}.$$

The set $K(S)$ is called a *kernel* of S (in fact, kernel is a two sided ideal of the semigroup). This set is non-empty if and only if S has at least one minimal left (or right) ideal, and in such a case, it is itself a minimal left (or right) ideal. Therefore, as stated above, it is contained in every two sided ideal of S. For all $f \in S$, we have

$$K(S) \circ f = \bigcup \{M \circ f : M \text{ is a minimal left (or right) ideal of } S\} \subseteq K(S).$$

Hence, $K(S)$ is also a left (or right) ideal. From this discussion, we can conclude the following result.

Proposition 3.2.2 (Example of minimal two sided ideal). *For any holomorphic semigroup S, $K(S)$ is a minimal two sided ideal of S if it is non-empty.*

Proof. See for instance [16, Theorem 2.9]. □

We can define a special type of holomorphic semigroup S where $K(S)$ is non-empty. This type semigroup has some special features such as every left (or right) ideal of S includes minimal one and every left (or right) ideal of S contains a special element which is called an idempotent. Recall that an element $e \in S$ is called an *idempotent* if $e \circ e = e$.

Definition 3.2.9 (Abundant semigroup). *A holomorphic semigroup S is said to be abundant if every left (or right) ideal of S includes a minimal one, and every minimal left (or right) ideal contains an idempotent element.*

It is obvious that $K(S) \neq \emptyset$ if S is abundant. There are topologically significant examples of abundant semigroups which can be defined as follows.

Definition 3.2.10 (Compact right topological holomorphic semigroup). *Let S be a holomorphic semigroup and $g \in S$.*

1. *We define a right translation map $F_g : S \to S$ with respect to g by $F_g(h) = h \circ g$ for all $h \in S$.*
2. *We define a compact holomorphic right topological semigroup by the pair (S, τ), where τ is a topology on S such that the space (S, τ) is compact and Hausdorff, and right translation map F_g for every $g \in S$ is continuous with respect to τ.*

It is noted that a left translation map, and a compact holomorphic left topological semigroup are defined similarly. Also, note that in a compact right topological semigroup, we do not require that left translation maps are continuous. We say only translation map and compact holomorphic topological semigroup if left or right is clear from the context.

Example 3.2.5 ([121, Example 2.1]). *Let $S = \{f_t : t \in \mathbb{Q}\}$ be a holomorphic semigroup, where f_t is a function of Example 3.2.3 for all $t \in \mathbb{Q}$. The collection of all subsets of S forms a topology τ on S and hence it is also open cover of S. There are some finite number of elements in τ that can cover S. Hence, space (X, τ) is compact and Hausdorff. Therefore, this semigroup is compact holomorphic topological semigroup, and hence abundant.*

It is noted that in the holomorphic semigroup S of Example 3.2.5, there is an element $f_0(z) = e^{-\alpha 0}z$ such that $f_0 \circ f_0 = f_0$ which an idempotent by definition. Therefore, except the holomrphic semigroup in general, an abundant semigroup, and in particular, compact holomorphic topological semigroup has idempotents.

Theorem 3.2.1 ([121, Theorem 2.1]). *Let S be a compact holomorphic topological semigroup. Then there is an element $e \in S$ such that $e \circ e = e$.*

This theorem can be proved as a standard application of Zorn's lemma from set theory. It states that *if every chain $\mathscr{C}$ in a partially ordered set $(S, \leq)$ has upper bound in S, then $(S, \leq)$ has a maximal element*. It is noted that partial ordered set is a system consisting of non-empty set S and a relation denoted by $\leq$ satisfying the properties of anti-symmetry, reflexivity and transitivity. A chain $\mathscr{C}$ in a partial ordered set $(S, \leq)$ is a subset of S such that for every $x, y \in \mathscr{C}$, either $x \leq y$ or $x \geq y$. An element $m \in S$ is a maximal element of $(S, \leq)$ if $m \leq x$ for $x \in S$ implies $m = x$.

Sketch of the Proof of Theorem 3.2.1. The proof of this theorem follows from the following two facts:
Fact 1: S has a minimal close subsemigroup.
Let τ be a family of all closed subsemigroups of the semigroup S. Then $\tau \neq \emptyset$, and it is a topology of closed sets partially ordered by the reverse inclusion. Let $\mathscr{C} \subset \tau$ be a chain in $(\tau, \supseteq)$. S is compact and $\mathscr{C}$ has the finite intersection property. Therefore, $\cap_{T \in \mathscr{C}} T$ is non-empty and serves as a least upper bound of $\mathscr{C}$. Then by Zorn's lemma, τ has a maximal element M (say), where $M \subseteq \cap_{T \in \mathscr{C}} T$. In reality of this context, M is minimal closed subsemigroup of S.
Fact 2: If $e \in M$, then $M = \{e\}$ and e is an idempotent.
We can consider two cases of the proof of this fact 2.
Case 1: We prove $M = M \circ e = \{e\}$. Let $e \in M$. Then $M \circ e$ is a subsemigroup of M. The map $F_e : M \to M \circ e,\ F_e(h) \to h \circ e$ is a right translation map of a topological holomorphic semigroup S restricted to M. Then it is continuous. M is compact, so $M \circ e$ is also compact as a image of a compact set under continuous map F_e. By fact 1, M is minimal, and $M \circ e$ is non-empty. We must have $M = M \circ e = \{e\}$. This proves that e is an idempotent.

Case 2: We prove $N = \{f \in M : f \circ e = e\} = M$. By the construction of the set N, it is a subset of M and closed under the functional composition. This shows that N is a subsemigroup of M. By Case 1, $e \in M = M \circ e$ can be written as $e = f \circ e$ for some $f \in M$. This shows that $e \in N \neq \emptyset$. Finally, N can be written as the intersection of closed subsets of M and $F_e^{-1}\{\{e\}\}$. This proved N is a non-empty closed subsemigroup of M. Then as in Case 1, $N = \{e\}$. □

From Theorem 3.2.1, we may also conclude the following assertion.

Theorem 3.2.2 ([121, Theorem 2.2]). *Every compact holomorphic topological semigroup is abundant.*

Chapter 4

DYNAMICS OF HOLOMORPHIC SEMIGROUPS

In this chapter, we compare existing results of the classical holomorphic dynamics with some of the existing results of holomorphic semigroup dynamics. We also prove some results of holomorphic semigroup dynamics, and we see whether there is a connection or contrast with classical one. Also, we see how far the results generalize, and what new phenomena appear.

4.1 The Fatou, Julia and escaping sets

Let a family $\mathscr{F}$ of holomorphic functions is a *normal family* in some neighborhood of a point $z \in \mathbb{C}$. If $\mathscr{F}$ is a semigroup S such that it is normal family in a neighborhood U of a point $z \in \mathbb{C}$, then we say S is normal at z. We say that a holomorphic function f is *iteratively divergent* at $z \in \mathbb{C}$ if $f^n(z) \to \infty$ as $n \to \infty$. A semigroup S is *iteratively divergent* at z if every $f \in S$ is iteratively divergent at z. A semigroup S is said to be *iteratively bounded* at z if there is an element $f \in S$ which is not iteratively divergent at z.

Like in classical holomorphic dynamics (that is, based on the Fatou-Julia-Eremenko theory of a holomorphic function), the Fatou, Julia and escaping sets in the settings of a holomorphic semigroup are defined as follows:

Definition 4.1.1 (Fatou, Julia and escaping sets). *The Fatou set of a holomorphic semigroup S is defined by $F(S) = \{z \in \mathbb{C} : S \;\;\text{is normal at}\;\; z\}$, and the Julia set*

$J(S)$ of S is the complement of $F(S)$. Let S is a transcendental semigroup. Then the escaping set of S is defined by $I(S) = \{z \in \mathbb{C} : S$ is iteratively divergent at $z\}$ We call each point of the set $I(S)$ an escaping point.

It is obvious that $F(S)$ is the largest open subset (of $\mathbb{C}$ or $\mathbb{C}_\infty$) on which the semigroup S is normal. And its complement $J(S)$ is a closed set for any semigroup S. However, the escaping set $I(S)$ is neither an open nor a closed set (if it is non-empty) for any transcendental semigroup S. If $S = \langle f \rangle$, then the Fatou, Julia and escaping sets are respectively denoted by $F(f), J(f)$ and $I(f)$. Furthermore, let $f \in S$. Then $T = \langle f \rangle$ is a cyclic subsemigroup of S generated by f. From these facts, we can say that the Fatou, Julia and escaping sets of the subsemigroup $T = \langle f \rangle$ are respectively the Fatou, Julia and escaping sets of the holomorphic function f. Hence, Definition 4.1.1 generalizes the Fatou, Julia and escaping sets of a holomorphic function of Sections 2.3 and 2.4.

Any maximally connected subset U of the Fatou set $F(S)$ is called a *Fatou component*. As in classical holomorphic dynamics, a Fatou component U of a holomorphic semigroup S can be simply connected or multiply connected. Kumar and Kumar [60, Theorems 4.1, 4.2, 4.3, 4.5] extended some results of classical holomorphic dynamics related to simply and multiply connected Fatou components to holomorphic semigroup dynamics.

If S is a polynomial semigroup, then $I(S)$ is a Fatou component containing ∞, and so it is an open subset of Fatou set $F(S)$. If S is a transcendental semigroup, then escaping set $I(S)$ is neither an open nor a closed set (if it is non-empty). The following immediate result holds from Definition 4.1.1 of an escaping set.

Theorem 4.1.1. *Let S be a holomorphic semigroup. Then $z \in \mathbb{C}$ is an escaping point under S if and only if every sequence $(g_k)_{k \in \mathbb{N}}$ in S is iteratively divergent at z.*

Example 4.1.1. *For $k \in \mathbb{N}$, let us consider a semigroup $S = \langle \{f_k\} \rangle$, where $f_k(z) = e^{-1-\frac{1}{k}} e^z$. For $x > 1$, we have $f_k^n(x) \to \infty$ as $n \to \infty$. Furthermore, it is easy to see that $f^n(x) \to \infty$ as $n \to \infty$ for every $f \in S$. Therefore, by Theorem 4.1.1, $x > 1$ is in $I(S)$.*

Definition 4.1.1 of escaping set is different than that of the definition given by Kumar and Kumar [61, Definition 2.1]. According to our definition, the escaping set of Example 4.1.1 is non-empty. However, according to Kumar and Kumar, the escaping set of this example is empty because the sequence $(f_k)_{k=1}^{\infty} \subset S$ does not contain subsequences tending to infinity. We can slightly generalize Theorem 4.1.1 to the following assertion which can be an alternative definition of an escaping set.

Theorem 4.1.2 ([115, Lemma 3.3]). *Let a complex number $z \in \mathbb{C}$ is an escaping point of a holomorphic semigroup S. Then every non-convergent sequence in S has a subsequence which diverges to ∞ at z.*

Proof. Let $z \in \mathbb{C}$ be an escaping point of a holomorphic semigroup S. Then by Definition 4.1.1, $f^n(z) \to \infty$ as $n \to \infty$ for all $f \in S$. By Proposition 3.2.1, for each $n \in \mathbb{N}$, we have $f^n = f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_n}$, where f_{α_i} is a generator of S for all $i = 1, 2, \ldots n$. We can consider a sequence $(g_n)_{n \in \mathbb{N}}$ in S representing $g_1 = f,\ g_2 = f^2, \ldots, g_n = f^n, \ldots$ (say), where each $g_i = f_{\alpha_1} \circ f_{\alpha_2} \circ \cdots \circ f_{\alpha_j}$ for $i = 1, 2, \ldots, n, \ldots$ and $j = 1, 2, \ldots, i$ such that $g_n(z) \to \infty$ as $n \to \infty$ or there is a non-convergent sequence in S which contains $(g_n)_{n \in \mathbb{N}}$ as a subsequence such that $g_n(z) \to \infty$ as $n \to \infty$. $\square$

Example 4.1.2. *Let us consider a semigroup $S = \langle f, g \rangle$, where $f(z) = \lambda e^z$ and $g(z) = \mu e^z$ with $\lambda < \mu < e^{-1}$. For $x > 1$, we have $h_n(x) \to \infty$ as $n \to \infty$ for every $(h_n) \subset S$. Therefore, by Theorem 4.1.2, every $x > 1$ is in $I(S)$.*

It is noted that if for a sequence $(h_n) \subset S$ such that $h_n(z) \to \infty$ as $n \to \infty$, then we can not conclude always that there is an element $f \in S$ such that $h_n = f^k$ for some $k \in \mathbb{N}$ with $f^k(z) \to \infty$ as $k \to \infty$. For example, if we choose $h_n(z) = f \circ g \circ f \circ g^2 \circ \cdots \circ f \circ g^n$ in Example 4.1.2, then there does not exist an element $l \in S$ such that $l^k(z) \to \infty$ as $k \to \infty$.

The following immediate relations hold between the Fatou, Julia and escaping sets of a holomorphic semigroup S and its cyclic subsemigroup from Definition 4.1.1. Indeed, this is a connection between classical and semigroup holomorphic dynamics.

Theorem 4.1.3. *Let S be a holomorphic semigroup. Then*

1. *$F(S) \subset F(f)$ for all $f \in S$ and hence $F(S) \subset \bigcap_{f \in S} F(f)$.*
2. *$J(f) \subset J(S)$ for all $f \in S$.*
3. *$I(S) \subset I(f)$ for all $f \in S$ and hence $I(S) \subset \bigcap_{f \in S} I(f)$.*

It is noted that we deal Theorem 4.1.3 (3) in the case of transcendental semigroups even though it holds for polynomial semigroups.

In analogy to classical rational dynamics ([15, Theorem 4.2.4]), Hinkkanen and Martin ([46, Lemma 3.1 and Corollary 3.1]) proved the following assertion.

Proposition 4.1.1. *Let S be a rational semigroup. Then the Julia set $J(S)$ is perfect and $J(S) = \overline{\bigcup_{f \in S} J(f)}$.*

In analogy to classical transcendental dynamics ([51, Theorem 3.10]), Poon ([75, Theorems 4.1 and 4.2]) proved the following assertion.

Proposition 4.1.2. *Let S be a transcendental semigroup. Then the Julia set $J(S)$ is perfect and $J(S) = \overline{\bigcup_{f\in S} J(f)}$.*

From Theorem 4.1.3 ((1) and (3)), we can say that the Fatou and escaping sets of a holomorphic semigroup may be empty. For example, the Fatou set of the semigroup $S = \langle f, g\rangle$ generated by the functions $f(z) = e^{\lambda z}$ and $g(z) = e^{s\lambda z} + 2\pi i/\lambda$ for all $s \in \mathbb{N}$ is empty for $\lambda > 1/e$ (and non-empty for $0 < \lambda < 1/e$ (see [75, Example 2.1])). The escaping set of the semigroup $S = \langle f, g\rangle$ generated by functions $f(z) = e^z$ and $g(z) = e^{-z}$ is empty (that is, the particular (say) $h = g \circ f^k \in S$ is iteratively bounded at any $z \in I(f)$). It is difficult to generalize holomorphic semigroups that can have empty escaping sets. From the fact that we developed in Chapter 3.2, we can say that an abundant holomorphic semigroup has empty escaping set. It is not known that the semigroup $S = \langle e^z, e^{-z}\rangle$ is abundant or not. We know that the Fatou set may be empty but the escaping set is non-empty in classical holomorphic dynamics. This is a contrast feature of the escaping set in classical and semigroup holomorphic dynamics. From the same Theorem part (2), and Propositions 4.1.1 and 4.1.2, we can say that, in classical and semigroup holomorphic dynamics, the Julia set is a closed, non-empty, unbounded and a perfect set. There are several transcendental semigroups whose Fatou and escaping sets are non-empty. From the following examples of Kumar and Kumar [63, Examples 3.2 and 3.3] and [61, Examples 2.6 and 2.7], we get non-empty Fatou and escaping sets.

Example 4.1.3. *Let $S = \langle f, g\rangle$, where $f(z) = e^z + \lambda$ and $g(z) = e^z + \lambda + 2\pi i$ for all $\lambda \in \mathbb{C} - \{0\}$. Then $F(S) = F(f) \neq \emptyset$ and $I(S) = I(f) \neq \emptyset$.*

Example 4.1.4. *Let $S = \langle f, g\rangle$, where $f(z) = \lambda \sin z$ and $g(z) = \lambda \sin z + 2\pi$ for all $0 < |\lambda| < 1$. Then $F(S) = F(f) \neq \emptyset$ and $I(S) = I(f) \neq \emptyset$.*

Example 4.1.5. *Let $S = \langle f, g\rangle$, where $f(z) = e^{\lambda z}$ and $g(z) = e^{s\lambda z} + 2\pi i/\lambda$ for all $\lambda \in \mathbb{C} - \{0\}$ and $s \in \mathbb{N}$. Then $I(S) = I(f) \neq \emptyset$ and $F(S) = F(f) \neq \emptyset$ for $0 < \lambda < e^{-1}$.*

Example 4.1.6. *Let $S = \langle f, g\rangle$, where $f(z) = \lambda \sin z$ for all $\lambda \in \mathbb{C} - \{0\}$ and $g(z) = f^n + 2\pi$ for all $n \in \mathbb{N}$. Then $I(S) = I(f) \neq \emptyset$ and $F(S) = F(f) \neq \emptyset$.*

Kumar and Kumar [61, Theorem 3.4] generalized these examples to the following result.

Proposition 4.1.3. *Let $S = \langle f, g\rangle$ be a transcendental semigroup generated by a periodic function f with period p and another function g defined by $g = f^n + p,\ n \in \mathbb{N}$. Then $F(S) = F(f)$ and $I(S) = I(f)$.*

Next, we see series of results where classical transcendental dynamics (in particular, results related to escaping set) can be generalized to transcendental semigroup dynamics. If $I(S) \neq \emptyset$, then the statement $\partial I(f) = J(f)$ ([31, Statement 1, Page 339]) of classical holomorphic dynamics can be generalized to semigroup dynamics. The following results is due to Kumar and Kumar [61, Lemma 4.2 and Theorem 4.3] which yields a generalized answer in semigroup dynamics.

Proposition 4.1.4. *Let S be a transcendental semigroup such that $I(S) \neq \emptyset$. Then*

1. *$Int.(I(S)) \subset F(S)$ and $Ext.(I(S)) \subset F(S)$, where $Int.$ and $Ext.$ respectively denote the interior and exterior of $I(S)$.*
2. *$\partial I(S) = J(S)$, where $\partial I(S)$ denotes the boundary of $I(S)$.*

Proof. 1. We refer, for instance, [61, Lemma 4.2].

2. The facts $Int.(I(S)) \subset F(S)$ and $Ext.(I(S)) \subset F(S)$ yield $J(S) \subset \partial I(S)$. The fact $\partial I(S) \subset J(S)$ is obvious. □

From Proposition 4.1.4, the fact $J(S) \subset \overline{I(S)}$ follows trivially. If $I(S) \neq \emptyset$, then we prove the following result which is a generalization of Eremenko's result $I(f) \cap J(f) \neq \emptyset$ (Proposition 2.4.1(3)) of classical transcendental dynamics to holomorphic semigroup dynamics.

Theorem 4.1.4. *Let S be a transcendental semigroup such that $I(S) \neq \emptyset$. Then $I(S) \cap J(S) \neq \emptyset$*

Lemma 4.1.1. *Let f be a transcendental entire function and U be a multiply connected component of $F(f)$. Then $f^n(z) \to \infty$ locally uniformly on U.*

Proof. See, for instance, [6, Theorem 3.1]. □

Proof of Theorem 4.1.4. Case (1). Suppose $F(S)$ has a multiply connected component U. Then by Theorem 4.1.3 (1), U is also a multiply connected component of $F(f)$ for each $f \in S$. By Lemma 4.1.1, for each $f \in S$, $f^n(z) \to \infty$ locally uniformly on U and on ∂U. It follows by normality (and also by Theorem 4.1.2) that every non-convergent sequence in S has a subsequence which diverges to ∞ locally uniformly on U and ∂U. This proves that $f^n(z) \to \infty$ for all $z \in U$, and $z \in \partial U$ for all $f \in S$. Again, by Theorem 4.1.3 (3) , $U \subset I(S)$. As $\partial U \subset J(f)$ for all $f \in S$, so, by Theorem 4.1.3 (2), $\partial U \subset J(S)$. This proves that $I(S) \cap J(S) \neq \emptyset$.
Case (2). Suppose that every components of $F(S)$ are simply connected. Let U be an arbitrary simply connected Fatou component of $F(S)$. Then by Theorem 4.1.3 (1),

U is also a simply connected component of $F(f)$ for all $f \in S$. Then by Proposition 2.4.1(2), we have $I(f) \cap J(f) \neq \emptyset$ for each $f \in S$. Again, by Theorem 4.1.3 (3) and Propositions 4.1.2 and 4.1.4 (2), we must conclude $I(S) \cap J(S) \neq \emptyset$. □

Theorem 4.1.5. *Let S be a transcendental semigroup such that $I(S) \neq \emptyset$. Then $\overline{I(S)}$ has no bounded components.*

Lemma 4.1.2. *A multiply connected component of the Fatou set $F(S)$ of a transcendental semigroup S lies in escaping set $I(S)$.*

Proof. Let U be a multiply connected component of $F(S)$. Then it is a multiply connected component of $F(f)$ for all $f \in S$. Then by Lemma 4.1.1, $f^n \to \infty$ locally uniformally on U for all $f \in S$. This proves $U \subset I(f)$ for all $f \in S$. Hence $U \subset I(S)$. □

Proof of Theorem 4.1.5. By Theorem 4.1.3(3), we can write $I(S) \subset I(f)$ for all $f \in S$. Therefore, $\overline{I(S)} \subset \overline{I(f)}$ for all $f \in S$. By Proposition 2.4.1(3), $\overline{I(f)}$ has no bounded components. We prove that $\overline{I(S)}$ also has no bounded components. Suppose for the contrary that A be a bounded component of $\overline{I(S)}$. Then it is bounded component of $\overline{I(f)}$ for each $f \in S$. In such a case, there is a domain B (possibly homeomorphic to an annulus) which separates A from ∞. Therefore, $B \cap I(f) = \emptyset$ for all $f \in S$. This shows that $B \subset F(f)$ for all $f \in S$. Hence, $B \subset F(S)$. Let C be the bounded component of $\mathbb{C} - B$, then $C \cap J(f) \neq \emptyset$ for all $f \in S$. Therefore, $C \cap J(S) \neq \emptyset$. This proves that A is contained in a multiply connected component of $F(S)$. Then by Lemma 4.1.2, $A \subset I(S)$, a contradiction. □

There are certain classes of transcendental semigroups whose escaping sets satisfy Eremenko's Conjecture 2.4.1. However, in general, as in classical transcendental dynamics, it is an open problem of transcendental semigroup dynamics. In each of above examples 4.1.3, 4.1.4, 4.1.5 and 4.1.6, we have $I(S) = I(h) = I(f)$ for all $h \in S$, where every components of $I(f)$ is unbounded and hence Conjecture 2.4.1 holds for such transcendental semigroups. More generally, there are certain classes of transcendental semigroups, whose escaping sets satisfy Conjecture 2.4.1. First, we need to define the following types of holomorphic semigroups.

Definition 4.1.2 (Bounded (or finite) type semigroup). *A holomorphic semigroup S is said to be bounded type (or finite type) if each of its generators is of bounded type (or finite type).*

There is a more specific case of Theorem 4.1.5 which was proved by Kumar and Kumar ([61, Theorem 4.7] and [63, Theorem 5.9]).

Proposition 4.1.5. *Let S be a finitely generated bounded type transcendental semigroup. Then $F(S) = F(f)$ for all $f \in S$ and $\overline{I(S)}$ has no bounded components.*

It is noted that for a finite type transcendental semigroup S, $F(S) = F(f)$ for all $f \in S$ was proved first by Poon [75, Theorem 5.1]. There is a specialty of Fatou and Julia sets if a transcendental semigroup S is of bounded type. This specialty was proved by Huang and Cheng [52, Theorem 3], Huang [43, Theorem 2] and Kumar and Kumar [61, Theorem 4.5].

Proposition 4.1.6. *Let $S = \langle f_1, f_2, \ldots, f_n \rangle$ be a bounded (or finite) type transcendental semigroup. Then $I(S) \subset J(S)$ and $J(S) = \overline{I(S)}$.*

The following result proved by Huang and Cheng [52, Theorem 3] and Huang [43, Theorem 2].

Proposition 4.1.7. *Let $S = \langle f_1, f_2, \ldots, f_n \rangle$ be a bounded (or finite) type transcendental semigroup. Then for all $z \in F(S)$, there does not exist any sequence $(g_k)_{k \in \mathbb{N}}$ in S such that $g_k(z) \to \infty$ as $k \in \infty$.*

As a consequence of Proposition 4.1.7, Kumar and Kumar [63, Theorem 5.10] proved the following result.

Proposition 4.1.8. *Let $S = \langle f_1, f_2, \ldots, f_n \rangle$ be a bounded (or finite) type transcendental semigroup. Then every component of $F(S)$ is simply connected.*

Proposition 4.1.8 is a generalization of the following fact proved by Eremenko and Lyubich [34, Proposition 3] of classical transcendental dynamics.

Proposition 4.1.9. *If f is a bounded type transcendental entire function. Then all components of $F(f)$ are simply connected.*

We can generalize the definitions of post singularly bounded (or finite) transcendental entire function to post singularly bounded semigroup as shown below.

Definition 4.1.3 (Post singularly bounded (or finite) transcendental semigroup). *A transcendental semigroup S is said to be post-singularly bounded (or post-singularly finite) if each $g \in S$ is post-singularly bounded (or finite). Post singular set of a post singularly bounded semigroup S is defined by*

$$P(S) = \overline{\bigcup_{f \in S, n \geq 1} f^n(SV(f))}$$

It is noted that if a transcendental semigroup S is post singularly bounded (or finite), then each $f \in S$ is post singularly bounded (or finite). However, the converse may not true. The fact $P(f^k) = P(f)$ for all $k \in \mathbb{N}$ shows that f^k is post singularly bounded (or finite) if f is post singularly bounded (or finite). The following result was proved by Kumar and Kumar [61, Theorems 3.8 and 3.14] which generalizes the Proposition 2.4.2 to a particular class of transcendental semigroup. In other words, the Conjecture 2.4.1 holds in a particular type of transcendental semigroup.

Proposition 4.1.10. *Let $f \in \mathscr{B}$ (or $f \in \mathscr{S}$) periodic with period p and post singularly bounded (or finite). Let $g = f^n + p,\ n \in \mathbb{N}$. Then $S = \langle f, g \rangle$ is post singularly bounded (or finite) and all components of $I(S)$ are unbounded.*

Recall that a transcendental entire function f is *hyperbolic* if the post singular set $P(f)$ is a compact subset of the Fatou set $F(f)$. It is noted that Proposition 2.4.2 applies, in particular, to hyperbolic transcendental entire functions. We can generalize the definition of the hyperbolic transcendental entire function to the hyperbolic semigroup as shown below.

Definition 4.1.4 (Hyperbolic semigroup). *A transcendental semigroup S is said to be hyperbolic if each $g \in S$ is hyperbolic (that is, if $P(S)$ is a compact subset of $F(S)$).*

It is noted that if a transcendental semigroup S is hyperbolic, then each $f \in S$ is hyperbolic. However, the converse may not true. The fact $P(f^k) = P(f)$ for all $k \in \mathbb{N}$ shows that f^k is hyperbolic if f is hyperbolic. The following result was proved by Kumar and Kumar [61, Theorem 3.16] where the Conjecture 2.4.1 holds.

Proposition 4.1.11. *Let $f \in \mathscr{B}$ periodic with period p and hyperbolic. Let $g = f^n + p,\ n \in \mathbb{N}$. Then $S = \langle f, g \rangle$ is hyperbolic and all components of $I(S)$ are unbounded.*

Example 4.1.7. *$f(z) = e^{\lambda z}$ is hyperbolic entire function for each $\lambda \in (0, \frac{1}{e})$. The semigroup $S = \langle f, g \rangle$ where $g = f^m + p$ and $p = \frac{2\pi i}{\lambda}$, is a hyperbolic transcendental semigroup.*

We generalized Proposition 4.1.11 to finitely generated hyperbolic semigroup with some modifications in [110, Theorem 3.1]. This theorem will be the good source of non-empty escaping set of transcendental semigroup as well as we get Conjecture 2.4.1 in transcendental semigroup dynamics.

Theorem 4.1.6 ([110, Theorem 3.1])**.** *Let $S = \langle f_1, f_2, \ldots, f_n \rangle$ is an abelian bounded (or finite) type transcendental semigroup in which each f_i is hyperbolic for $i = 1, 2, \ldots, n$. Then the semigroup S is hyperbolic, and all components of $I(S)$ are unbounded.*

Lemma 4.1.3. *Let f and g be transcendental entire functions. Then*

$$SV(f \circ g) \subset SV(f) \cup f(SV(g))$$

Proof. See, for instance, [22, Lemma 2]. □

Lemma 4.1.4 ([110, Lemma 3.2])**.** *Let f and g be permutable transcendental entire functions. Then $f^m(SV(g)) \subset SV(g)$ and $g^m(SV(f)) \subset SV(f)$ for all $m \in \mathbb{N}$.*

Proof. First, we prove that $f(SV(g)) \subset SV(g)$. Then we use induction to prove

$$f^m(SV(g)) \subset SV(g).$$

Let $w \in f(SV(g))$. Then $w = f(z)$ for some $z \in SV(g)$. In this case, z is either a critical value or an asymptotic value of g.

First, let us suppose that z is a critical value of g. Then $z = g(u)$ with $g^{'}(u) = 0$. f and g are permutable functions, so $w = f(z) = f(g(u)) = (f \circ g)(u) = (g \circ f)(u)$. Also, $(f \circ g)^{'}(u) = f^{'}(g(u))g^{'}(u) = 0$. This shows that u is a critical point of $f \circ g = g \circ f$ and w is a critical value of $f \circ g = g \circ f$. By permutability of f and g, we can write $f^{'}(g(u))g^{'}(u) = g^{'}(f(u))f^{'}(u) = 0$ for any critical point u of $f \circ g$. We have $g^{'}(u) = 0$. Therefore, either $f^{'}(u) = 0 \Rightarrow u$ is a critical point of f or $g^{'}(f(u)) = 0 \Rightarrow f(u)$ is a critical point of g. This shows that $w = g(f(u))$ is a critical value of g. Hence, $w \in SV(g)$.

Next, suppose that z is an asymptotic value of the function g. We have to prove that $w = f(z)$ is also asymptotic value of g. Then there exists a curve $\gamma : [0, \infty) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $g(\gamma(t)) \to z$. Therefore, $f(g(\gamma(t))) \to f(z) = w$ as $t \to \infty$ along γ. We have $f \circ g = g \circ f$, so $f(g(\gamma(t))) \to f(z) = w \Rightarrow g(f(\gamma(t))) \to f(z) = w$ as $t \to \infty$ along γ. This shows w is an asymptotic value of g. This proves our assertion.

Assume that $f^k(SV(g)) \subset SV(g)$ for some $k \in \mathbb{N}$ with $k \leq m$. Then

$$f^{k+1}(SV(g)) = f(f^k(SV(g))) \subset f(SV(g)) \subset SV(g)$$

Therefore, by induction, for all $m \in \mathbb{N}$, we must have $f^m(SV(g)) \subset SV(g)$. The next part $g^m(SV(f)) \subset SV(f)$ can be proved similarly as above. □

Lemma 4.1.5 ([110, Lemma 3.3])**.** *Let f and g are two permutable hyperbolic transcendental entire functions. Then $f \circ g$ is also hyperbolic.*

Proof. We have to prove that $P(f \circ g)$ is a compact subset of $F(f \circ g)$. From [58, Lemma 3.2], we can write $F(f \circ g) \subset F(f) \cap F(g)$. This shows that $F(f \circ g)$ is a subset of $F(f)$ and $F(g)$. This lemma will be proved if we prove $P(f \circ g)$ is a compact subset of $F(f) \cup F(g)$. By Definition 2.3.13, we can write

$$\begin{aligned}
P(f \circ g) &= \overline{\bigcup_{m \geq 0} (f \circ g)^m (SV(f \circ g))} \\
&= \overline{\bigcup_{m \geq 0} f^m (g^m (SV(f \circ g)))} \qquad \text{(by using permutabilty of } f \text{ and } g\text{)} \\
&\subset \overline{\bigcup_{m \geq 0} f^m (g^m (SV(f) \cup f(SV(g)))} \qquad \text{(by Lemma 4.1.3)} \\
&= \overline{\bigcup_{m \geq 0} f^m (g^m (SV(f))) \cup g^m (f^{m+1} (SV(g)))} \\
&\subset \overline{\bigcup_{m \geq 0} f^m (SV(f)))} \cup \overline{\bigcup_{m \geq 0} g^m (SV(g)))} \qquad \text{(by Lemma 4.1.4)} \\
&= P(f) \cup P(g).
\end{aligned}$$

f and g are hyperbolic, so $P(f)$ and $P(g)$ are compact subset of $F(f)$ and $F(g)$. Therefore, the set $P(f) \cup P(g)$ must be compact subset of $F(f) \cup F(g)$. □

Proof of Theorem 4.1.6. Any $f \in S$ can be written as $f = f_{i_1} \circ f_{i_2} \circ f_{i_3} \circ \cdots \circ f_{i_m}$. By permutability of each f_i, we can rearrange f_{i_j} and ultimately represented by

$$f = f_1^{t_1} \circ f_2^{t_2} \circ \ldots \circ f_n^{t_n}$$

where each $t_k \geq 0$ is an integer for $k = 1, 2, \ldots, n$. Lemma 4.1.5 can be applied repeatably to show that each of $f_1^{t_1}, f_2^{t_2}, \ldots, f_n^{t_n}$ is hyperbolic. Again, by repeated application of same lemma, we can say that f is itself hyperbolic and so the semigroup S is hyperbolic. Next part follows from [108, Theorem 3.3] by the assumption of this theorem. □

The one of the most important result of classical holomorphic dynamics is either $J(f) = \mathbb{C}$ or $\mathbb{C}_\infty$ or $J(f)$ has empty interior for any holomorphic function f on $\mathbb{C}$ or $\mathbb{C}_\infty$ (see [15, Theorem 4.2.3] for rational function and [18, Lemma 3] for general holomorphic function). There are lot of examples of transcendental entire functions and rational functions whose Julia set is the entire complex plane or extended complex plane. For example,

1. $J(\lambda z e^z) = \mathbb{C}$ for a suitable value of λ,

2. $J(e^z) = \mathbb{C}$,

3. $J(\lambda z e^z/z) = \mathbb{C}_\infty$ for a suitable value of λ,

4. $J(\lambda \tan z) = \mathbb{C}_\infty$ for a suitable value of λ,

5. $J((z-2)^2/z^2) = \mathbb{C}_\infty$,

6. $J((z^2+1)^2/4z(z^2-1)) = \mathbb{C}_\infty$.

However, the analogous result is not hold in semigroup dynamics. Hinkkanen and Martin [46, Example-1] provided the following example that shows that Julia set of a rational semigroup S may have non-empty interior even if $J(S) \neq \mathbb{C}_\infty$.

Example 4.1.8. *Entire semigroup $S = \langle z^2, z^2/a \rangle$, where $a \in \mathbb{C}, |a| > 1$ has Fatou set $F(S) = \{z : |z| < 1 \text{ or } |z| > |a|\}$ and Julia set $J(S) = \{z : 1 \leq |z| \leq |a|\}$.*

Let U be a component of Fatou set $F(f)$. Then $f(U)$ is contained in some component V of $F(f)$. It is noted that if f is a rational function, then $V = f(U)$. If f is a transcendental function, then it is possible that $V \neq f(U)$. Let us recall the following result of Bergweiler [18] of classical holomrphic dynamics.

Proposition 4.1.12. *If f is entire, then $V - f(U)$ contains at most one point which is an asymptotic value of f.*

The following example of Huang [43, Example 2] shows that Proposition 4.1.12) can not be preserved for general semigroup dynamics. This is a contrast between classical holomorphic dynamics and semigroup dynamics.

Example 4.1.9. *Let $S = \langle z^n, az^n \rangle$, where $n > 2$ and $|a| > 1$. The Fatou set $F(S)$ contains following components*

$$U = \left\{ \sqrt[n]{\frac{1}{|a|}} < |z| < \sqrt[n]{\sqrt[n-1]{\frac{1}{|a|}}} \right\} \text{ and } V = \{|z| > 1\}.$$

For a function $f(z) = az^n$ in semigroup S, $f(U) \subset V$ and $V - f(U)$ is an unbounded domain.

Definition 4.1.5 (Backward orbit and exceptional set). *Let S be a holomorphic semigroup. We define the backward orbit of any $z \in \mathbb{C}$ (or $\mathbb{C}_\infty$) by*

$$O^-(z) = \{w \in \mathbb{C}_\infty : \text{there exists } f \in S \text{ such that } f(w) = z\}$$

and the Fatou exceptional set of S is defined by $E(S) = \{z \in \mathbb{C}_\infty : O^-(z)$ is finite$\}$. Any $z \in E(S)$ is called exceptional value.

It is noted that if S finitely generated rational semigroup, then $E(S) \subset F(S)$, otherwise we can not assert it. For example ([43, example 1]), semigroup $S = \langle f_m \rangle$, where $f_m(z) = a^m z^n, m \in \mathbb{N}, n \geq 2$ and $|a| > 1$, is an infinitely generated polynomial semigroup. Then, $E(S) = \{0, \infty\}$. It is easy to see that 0 is a limit point of $J(f_m) = \{z : |z| = |a|^{\frac{-m}{n-1}}\}$, and hence $0 \in J(S)$. In the case of finitely generated rational semigroup S, we always have $E(S) \subset F(S) \subset F(f)$ for any $f \in S$. Hence $E(S)$ contains at most two points. However, if S finitely generated transcendental semigroup, then we can not assert $E(S) \subset F(S)$ in general because for a transcendental function, it is difficult to determine whether Fatou exceptional value belongs Fatou set or Julia set. For example, 0 is the Fatou exceptional value of $f(z) = e^{\lambda z}$. It is known in classical holomorphic dynamics that $0 \in J(f)$ if $\lambda > 1/e$ and $0 \in F(f)$ if $\lambda < 1/e$. Poon and Yang [77] gave the following characterization whether a Fatou exceptional value belongs to the Fatou set or Julia set.

Proposition 4.1.13. *Let f is transcendental entire function. If $F(f)$ has no unbounded component, then Fatou exceptional value always belongs to Julia set.*

In the case of finitely generated transcendental semigroup S, if $E(f) \subset F(f)$ for all $f \in S$, then we can say $E(S) \subset F(S) \subset F(f)$ for any $f \in S$. Hence $E(S)$ contains at most one point. This fact is a generalization of classical holomorphic dynamics to semigroup dynamics and so it is a nice connection between these two types of dynamics. Huang [43, Proposition 1] proved the following result which also shows a connection between classical holomorphic dynamics and semigroup dynamics.

Proposition 4.1.14. *Let S be a holomorphic semigroup. If $z \notin E(S)$, then $J(S) \subseteq \overline{O^-(z)}$.*

If $z \in J(S)$ and $z \notin E(S)$, then $J(S) = \overline{O^-(z)}$ for any holomorphic semigroup S. This says that $\overline{O^-(z)}$ clusters at each point of $J(S)$. This result proved for rational semigroup by Hinkkanen and Martin [46, Lemma 3.2].

4.2 Invariant feature of Fatou, Julia and escaping sets

The main contrast between classical and semigroup holomorphic dynamics will appear in the invariant feature of the Fatou, Julia and escaping sets. Note that invariant feature is considered a very basic and fundamental structure of these sets.

Definition 4.2.1 (Forward, backward and completely invariant set). *Let S be a holomorphic semigroup. A set $U \subset \mathbb{C}$ is said to be S-forward invariant if $f(U) \subset U$ for all $f \in S$, it is said to be S-backward invariant if $f^{-1}(U) = \{z \in \mathbb{C} : f(z) \in U\} \subset U$ for all $f \in S$, and it is said to be S-completely invariant if it is both S-forward and S-backward invariant.*

Proposition 4.2.1. *Let S be a holomorphic semigroup. Then the Fatou set $F(S)$ is S-forward invariant and the Julia set $J(S)$ is S-backward invariant.*

This Proposition 4.2.1 was proved by Hinkkanen and Martin [46, Theorem 2.1] if S is a rational semigroup and it was proved by Poon [75, Theorem 2.1] if S is a transcendental semigroup.

Kumar and Kumar [61, Theorem 4.1] proved the following result that shows escaping set $I(S)$ is also S-forward invariant. Here, we provide another proof based on our Definition 4.1.1 of an escaping set.

Theorem 4.2.1 ([122, Theorem 3.1])**.** *The escaping set $I(S)$ of a transcendental semigroup S is S-forward invariant.*

Proof. Let $z \in I(S)$. Then by Definition 4.1.1, semigroup S is iteratively divergent at z. Therefore, for any $g \in S$, the subsemigroup $S \circ g = \{f \circ g : f \in S\}$ is also iteratively divergent at z. That is, $(f \circ g)^n(z) \to \infty$ as $n \to \infty$ for all $f \in S$. By Proposition 3.2.1, for all $n \in \mathbb{N}$, we have $(f \circ g)^n = h_{n_i} \circ g$ for some $h_{n_i} \in S$ where n_i depends on n. Therefore, $(f \circ g)^n(z) \to \infty$ as $n \to \infty \Rightarrow (h_{n_i} \circ g)(z) \to \infty$. Then by Theorem 4.1.2, there is a subsequence $(h_{n_{i_j}} \circ g)_{n_{i_j} \in \mathbb{N}}$ of the sequence $((h_{n_i} \circ g)$ such that $(h_{n_{i_j}} \circ g)(z) = h_{n_{i_j}}(g(z)) \to \infty$ as $n_{i_j} \to \infty$. This proves that S diverges at $g(z)$, so $g(z) \in I(S)$ for all $g \in S$. Hence $I(S)$ is S-forward invariant. □

Hinkkanen and Martin [46, Example1] provided the following example that show that the Fatou set $F(S)$ need not be backward invariant and the Julia set $J(S)$ need not be forward invariant.

Example 4.2.1. *For a rational semigroup $S = \langle z^2, z^2/a \rangle$, where $a \in \mathbb{C}, |a| > 1$, the Fatou set $F(S) = \{z : |z| < 1$ or $|z| > |a|\}$ is not S-backward invariant and Julia set $J(S) = \{z : 1 \leq |z| \leq |a|\}$ is not S-forward invariant.*

It is noted that the escaping sets of Examples 4.1.1 and 4.1.2 are not S-backward invariant. Fatou [35] and Julia [53] independently proved that the Fatou and Julia sets of a rational function are completely invariant. Note that all the three sets $F(f)$, $J(f)$ and $I(f)$ of a transcendental entire function are also completely invariant. This is a fundamental contrast between classical holomorphic dynamics and holomorphic semigroup dynamics.

We prove under certain conditions, the Fatou set of a holomorphic semigroup is S-backward invariant, and the Julia set is S-forward invariant.

Theorem 4.2.2 ([122, Theorem 3.2])**.** *Let S be an abelian holomorphic semigroup. Then the Fatou set $F(S)$ is S-backward invariant and Julia set $J(S)$ is S-forward invariant.*

Proof. We prove that if $g(z) \in F(S)$, then $z \in F(S)$ for all $g \in S$. This follows that $g^{-1}(F(S)) \subset F(S)$ for all $g \in S$. Suppose, $g(z) \in F(S)$. Let U be a neighborhood of $g(z)$ such that $\overline{U} \subset F(S)$. Then there is a subsequence (f^{n_j}) such that $f^{n_j}(g(z)) \to f(g(z))$ uniformly on U, where f is holomorphic function or constant ∞. S is abelian, so we have $g(f^{n_j}(z)) \to g(f(z))$ uniformly on U. This shows that $g \circ f^{n_j} \to g \circ f$ uniformly on U. This proves that $z \in F(S)$ for all $g \in S$. □

From Theorems 4.2.1 and 4.2.2, we can say that the sets $F(S)$ and $J(S)$ are S-completely invariant if S is an abelian holomorphic semigroup. The following example of Hinkkanen and Martin [46, Example 2] is best for Theorem 4.2.2.

Example 4.2.2. *The semigroup $S = \langle T_n(z) : n = 0, 1, 2, \ldots \rangle$ generated by Tchebyshev polynomials $T_n(z)$ defined by $T_0(z) = 1, T_1(z) = z$ and $T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$ is abelian. Therefore, by Theorem 4.2.2, Fatou set $F(S)$ is S-backward invariant and Julia set $J(S)$ is S-forward invariant.*

In [62, Theorem 2.1], Kumar et al. provided the following condition for backward invariance of $I(S)$. Here, we give another proof based on Definition 4.1.1 of escaping set.

Theorem 4.2.3 ([122, Theorem 3.3])**.** *The escaping set $I(S)$ of transcendental semigroup S is S-backward invariant if S is abelian.*

Proof. We prove that if $g(z) \in I(S)$, then $z \in I(S)$ for all $g \in S$. This follows that $g^{-1}(I(S)) \subset I(S)$ for all $g \in S$. This will be proved if we are able to prove its contrapositive statement: if $z \notin I(S)$, then $g(z) \notin I(S)$ for all $g \in S$. Let $z \notin I(S)$. Then there is some $f \in S$ which is iteratively bounded at z. That is, $f^n(z) \nrightarrow \infty$ as $n \to \infty$. In this case, there exists a sequence $(f_k)_{k \in \mathbb{N}}$ in S containing f which is iteratively bounded at z and all subsequences of this sequence containing f are also iteratively bounded at z. Now, for any $g \in S$, $(f_k \circ g)_{k \in \mathbb{N}}$ is a sequence in S. Then S is abelian and g is a transcendental entire function, so by the continuity of g at $z \in \mathbb{C}$, we can write $(f_k \circ g)(z) = (g \circ f_k)(z)$ for all $k \in \mathbb{N}$. From which it follows that the sequence $(f_k \circ g)_{k \in \mathbb{N}}$ is iteratively bounded at z. Therefore, all subsequence of this sequences are iteratively bounded at z. From the fact $(f_k \circ g)(z) = (g \circ f_k)(z)$ for all $k \in \mathbb{N}$, we can say that all subsequences of the sequence $(f_k \circ g)_{k \in \mathbb{N}}$ are iteratively bounded at $g(z)$. That is, $g(z) \notin I(S)$ for all $g \in S$. Therefore, $g^{-1}(I(S)) \subset I(S)$ for all $g \in S$. This proves that $I(S)$ is backward invariant. □

From the result of Proposition 4.2.1 and Theorems 4.2.1, 4.2.2 and 4.2.3, we can conclude that the Fatou, Julia, and escaping sets are S-completely invariant if S is an abelian transcendental semigroup. For example, the following

1. $\langle z + \gamma \sin z,\ z + \gamma \sin z + 2k\pi \rangle$,
2. $\langle z + \gamma \sin z,\ -z - \gamma \sin z + 2k\pi \rangle$,
3. $\langle z + \gamma e^z,\ z + \gamma e^z + 2k\pi i \rangle$,
4. $\langle z - \sin z,\ z - \sin z + 2\pi \rangle$,
5. $\langle e^{\gamma z},\ e^{\gamma z + \frac{2\pi i}{\gamma}} \rangle$, where $0 < \gamma < e^{-1}$,

are abelian transcendental semigroups, so their Fatou, Julia and escaping sets are S-completely invariant.

Theorems 4.2.2 and 4.2.3 give a kind of connection between classical and semigroup holomorphic dynamics. We got a completely invariant structure of the Fatou, Julia and escaping sets in both classical and semigroup holomorphic dynamics because of their associated abelian semigroups.

If S is a finitely generated rational semigroup, then Sumi [123, Lemma 1.1.4 (2)] proved the following result.

Proposition 4.2.2. *If $S = \langle f_1, f_2, \dots, f_n \rangle$ is a finitely generated rational semigroup, then $F(S) = \bigcap_{i=1}^{n} f_i^{-1}(F(S))$ and $J(S) = \bigcup_{i=1}^{n} f_i^{-1}(J(S))$*

In the case of finitely generated transcendental semigroup, we prove following result which is analogous to Proposition 4.2.2.

Theorem 4.2.4 ([122, Theorem 3.4])**.** *If $S = \langle f_1, f_2, \dots, f_n \rangle$ is a finitely generated transcendental semigroup, then $F(S) = \bigcap_{i=1}^{n} f_i^{-1}(F(S))$ and $J(S) = \bigcup_{i=1}^{n} f_i^{-1}(J(S))$*

Proof. The Fatou set $F(S)$ is S-forward invariant in general (Theorem 4.2.2). So, $f_i(F(S)) \subset F(S)$ and it follows $F(S) \subset \bigcap_{i=1}^{n} f_i^{-1}(F(S))$ for all i.

Next, let $z_0 \in \bigcap_{i=1}^{n} f_i^{-1}(F(S))$. Then (say) $w_i = f_i(z_0) \in F(S)$ for all i. The semigroup S is normal at w_i for all i. In other words, every $g \in S$ is equicontinuous at w_i for all i. That is, for any $\epsilon > 0$, there is $\delta > 0$ such that $d(g(w), g(w_i)) <$

ϵ, whenever $d(w, w_i) < \delta$ for all $w \in F(S)$ and $i = 1, 2, 3, \ldots, n$. Where d represents Euclidean metric on $\mathbb{C}$. For such δ, there is $\eta > 0$ such that $d(f_i(z), f_i(z_0)) < \delta$, whenever $d(z, z_0) < \eta$ for all $z \in \bigcap_{i=1}^{n} f_i^{-1}(F(S))$ and $i = 1, 2, 3, \ldots, n$. Thus, ultimately, we conclude that $d(g(f_i(z)), g(f_i(z_0))) < \epsilon$ whenever $d(z, z_0) < \eta$ for all $z \in \bigcap_{i=1}^{n} f_i^{-1}(F(S))$ and $i = 1, 2, 3, \ldots, n$. We can also write $S = \bigcup_{i=1}^{n}(S \circ f_i)$. So, S is equicontinuous at z_0. That is, $z_0 \in F(S)$. Hence, $F(S) = \bigcap_{i=1}^{n} f_i^{-1}(F(S))$. Second part of the theorem easily follows as

$$J(S) = \mathbb{C} - F(S) = \mathbb{C} - \bigcap_{i=1}^{n} f_i^{-1}(F(S)) = \bigcup_{i=1}^{n}(\mathbb{C} - f_i^{-1}(F(S)) = \bigcup_{i=1}^{n} f_i^{-1}(J(S)).$$

□

The relation $J(S) = \bigcup_{i=1}^{n} f_i^{-1}(J(S))$ for the Julia set is called *backward self similarity*. With this property, dynamics of a holomorphic semigroup can be regarded as *backward iterated function system*-BIFS associated to the set $\{f_1, f_2, \ldots, f_n\}$ of generators of the semigroup S. The Julia set $J(S)$ is considered as an attractor or invariant for BIFS. Indeed, $J(S)$ is a unique attractor, which is usually a fractal.

Theorems 4.2.1 and 4.2.3 can be used directly to express the escaping set $I(S)$ as a finite intersection of all pre-images of itself under the generators of the semigroup S if it is finitely generated. The following result is due to Kumar et al. [62, Theorem 2.6]. Here, we give an alternative proof.

Theorem 4.2.5 ([122, Theorem 3.5]). *If $S = \langle f_1, f_2, \ldots, f_n \rangle$ is a finitely generated transcendental semigroup, then $I(S) = \bigcap_{i=1}^{n} f_i^{-1}(I(S))$ if S is abelian semigroup.*

Proof. $I(S)$ is completely invariant under the assumption of the theorem (see for instance Theorems 4.2.1, 4.2.3). So, we have $f_i(I(S)) \subset I(S)$ and $f_i^{-1}(I(S)) \subset I(S)$ for all $1 \leq i \leq n$. From which we get respectively $I(S) \subset \bigcap_{i=1}^{n} f_i^{-1}(I(S))$ and $I(S) \supset \bigcap_{i=1}^{n} f_i^{-1}(I(S))$ for all $1 \leq i \leq n$. Thus, we get $I(S) = \bigcap_{i=1}^{n} f_i^{-1}(I(S))$. □

There are certain holomorphic semigroups whose dynamical behavior coincide with classical holomorphic dynamics. That is, there are certain holomorphic semigroups whose Fatou, Julia and escaping sets are same as Fatou, Julia and escaping sets of each of its generators.

Theorem 4.2.6 ([122, Theorem 4.1]). *Let S be an abelian rational semigroup. Then $J(S) = J(f)$ and $F(S) = F(f)$ for all $f \in S$ of degree at least two.*

We recall the following results of Fatou [38] and Julia [54] concerning commuting rational functions.

Proposition 4.2.3. *Let f and g be two rational functions of degree at least two such that $f \circ g = g \circ f$. Then $J(f) = J(g)$.*

Proof of Theorem 4.2.6. S is abelian, so we have $f_i \circ f_j = f_j \circ f_i$ for all generators f_i and f_j with $i \neq j$. Then by Proposition 4.2.3, $J(f_i) = J(f_j)$ for all i and j with $i \neq j$. Also, every $f \in S$ permutes with each generator f_i for all i, so again by the same Proposition 4.2.3, $J(f) = J(f_i)$ for all i. This fact together with the fact of Proposition 4.1.1, we can conclude that $J(S) = J(f)$ for all $f \in S$. □

The analogous result in transcendental semigroup dynamics may not hold in general because of the essence of Proposition 4.2.3 is still unanswered for permutable transcendental entire functions. Julia sets for two permutable entire functions were studied in [77, 97, 125], where we found certain conditions from which we can get the essence of Proposition 4.2.3. If we expose extra conditions in the statement, then result analogous to Theorem 4.2.6 holds in the case of a transcendental semigroup. One of the analogous result was proved by Poon [75, Theorem 5.1]. Here, we only give sketch of the proof similar to the proof of Theorem 4.2.6.

Proposition 4.2.4. *Let f and g be two transcendental entire functions of finite type. Then $f \circ g$ is of finite type. Moreover, if f and g are permutable, then $J(f) = J(g)$.*

Proposition 4.2.5. *Let S be an abelian transcendental semigroup in which each generator is of finite type. Then $J(S) = J(f)$ for all $f \in S$.*

Proof of Proposition 4.2.5. Since semigroup S is abelian, then by Proposition 4.2.4, $J(f_i) = J(f_j)$ for all generators f_i and f_j with $i \neq j$. Each $f = f_{i_1} \circ f_{i_2} \circ \ldots f_{i_m}$ is of finite type and $J(f) = J(f_i)$ for all i. This fact together with the fact of the Proposition 4.2.3, we can conclude that $J(S) = J(f)$ for all $f \in S$. □

We expect that the condition mentioned in the Proposition 4.2.5 will also be enough to hold $I(S) = I(f)$ for all $f \in S$.

Theorem 4.2.7 ([122, Theorem 4.2])**.** *Let S is an abelian transcendental semigroup in which each generator is of finite type (or bounded type). Then $I(S) = I(f)$ for all $f \in S$.*

Lemma 4.2.1. *Let f and g are transcendental entire functions of finite (or bounded) type. Then $f \circ g$ is of finite (or bounded) type.*

Proof. See, for instance, [75, Lemma 5.1]. □

Lemma 4.2.2 ([122, Lemma 4.1])**.** *If f and g are permutable transcendental entire functions of finite type (or bounded type), then $I(f) = I(g)$.*

Proof. As given in the statement of this lemma, Poon [75, Lemma 5.2] showed that $F(f) = F(g)$ (Proposition 4.2.4). Eremenko and Lyubich [34] proved that if transcendental function $f \in \mathscr{B}$, then $I(f) \subset J(f)$ and $J(f) = \overline{I(f)}$. For any function of finite type (or bounded type), we must have $\overline{I(f)} = \overline{I(g)}$. This Lemma will be proved if we show $J(f) - I(f) = J(g) - I(g)$. Let $z \in J(f) - I(f)$. Then z is a non-escaping point of $J(f)$, and so the sequence (f^n) has a bounded subsequence at z. $J(f) = J(g)$ implies that the sequence (g^n) has also a bounded subsequence at z. Therefore, $z \in J(g) - I(g)$. Hence, $J(f) - I(f) \subset J(g) - I(g)$. By similar fashion, we can show that $J(g) - I(g) \subset J(f) - I(g)$. Hence, we got our claim. □

Proof of Theorem 4.2.7. S is an abelian semigroup, so we have $f_i \circ f_j = f_j \circ f_i$ for all generator f_i and f_j with $i \neq j$. By Lemma 4.2.2, we have $I(f_i) = I(f_j)$. Any $f \in S$ can be written as $f = f_{i_1} \circ f_{i_2} \circ f_{i_3} \circ \cdots \circ f_{i_m}$. By permutability of each f_i, we can rearrange f_{i_j}, and ultimately represented by

$$f = f_1^{t_1} \circ f_2^{t_2} \circ \ldots \circ f_n^{t_n},$$

where each $t_k \geq 0$ is an integer for $k = 1, 2, \ldots, n$. Lemma 4.2.1 can be applied repeatably to show each of $f_1^{t_1}, f_2^{t_2}, \ldots, f_n^{t_n}$ is of finite (or bounded) type and so f is itself finite (or bounded) type. Each f_i permutes with f, and hence again by Lemma 4.2.2, $I(f_i) = I(f)$ for all $f \in S$. Therefore, $I(S) = I(f)$ for all $f \in S$. □

4.3 Completely invariant Fatou and Julia sets

From Proposition 4.2.1, we can say that the sets $F(S)$ and $J(S)$ need not be S-completely invariant. In this section, we generalize the completely invariant notion of the Fatou and Julia sets of a single transcendental entire function to the completely invariant notion of these sets in transcendental semigroup dynamics. In rational semigroups, and in particular, in polynomial semigroups, there are few studies over such completely invariant Fatou and Julia sets (see, for instance, [103, 104, 105] for more detail) but there is nothing study over transcendental semigroups. In this section, we give little bit attention on completely invariant Julia and Fatou sets of transcendental semigroups.

Definition 4.3.1 (Completely invariant Julia and Fatou sets). *Let S be a transcendental semigroup. We define a completely invariant Julia set of S by*

$$J_1(S) = \bigcap \{G : G \text{ is a closed, completely invariant set under each } f \in S\}$$

The completely invariant Fatou set $F_1(S)$ is defined as the complement of $J_1(S)$ in $\mathbb{C}$.

It is noted that in transcendental semigroup S, $J_1(S)$ exists, is closed, and completely invariant under each $f \in S$, and it contains the Julia set of each element of S. The corresponding Fatou set $F_1(S)$ is open, completely invariant, and contained in the Fatou set of each element of S.

The sets $J_1(S)$ and $F_1(S)$ of Definition 4.3.1 may and may not be the sets $J(S)$ and $F(S)$ respectively. The following examples can help to compere the sets $J_1(S)$ & $J(S)$ and $F_1(S)$ & $F(S)$.

Example 4.3.1. *Let $S = \langle f, g \rangle$ be a transcendental semigroup generated by $f(z) = \lambda \sin z$ and $g(z) = \lambda \sin z + 2\pi$, where $0 < |\lambda| < 1$. Then $J_1(S) = J(f) = J(g)$. It is also verified that $J(S) = J(f) = J(g)$. In this case, $J_1(S) = J(S)$, and so, $F_1(S) = F(S)$.*

There are other examples (see for instance [63, Example 3.2], and [75, Example 2.1]) of transcendental semigroups similar to Example 4.3.1.

Example 4.3.2. *Let $S = \langle f, g \rangle$ be a transcendental semigroup generated by $f(z) = \lambda e^z, (0 < \lambda < e^{-1})$ and $g(z) = \lambda e^z, (\lambda > 1/e)$. Then, by Devaney [30], $J(f)$ is a Cantor set (bouquet), and $J(g) = \mathbb{C}$. In this case, $F(g) = \emptyset$. Therefore, we have $F(S) = \emptyset$ and $J(S) = \mathbb{C}$. It is easy to verify that $J_1(S) = J(S) = \mathbb{C}$ and $F_1(S) = F(S) = \emptyset$.*

Example 4.3.3. *Let $S = \langle f, g \rangle$ be a transcendental semigroup generated by $f(z) = \lambda \sin z$, where $\lambda \in \mathbb{C}$ is chosen in such a way that there are two attracting cycles and $|\Re(\lambda)| \geq \pi/2$, and $g(z) = \mu e^z$, where $\mu \in (0, 1/e)$. Then, by Osborne [74, Example 6.4], $J(f)$ is a spider's web, and by Devaney [30], $J(g)$ is a Cantor bouquet. In this case, it is easy to verify that $J_1(S) = J(S) = \mathbb{C}$ and $F_1(S) = F(S) = \emptyset$.*

It is noted that the Cantor's bouquet, and the spider's web are structurally different sets. The Cantor's bouquet is closed, and has uncountably many components with a single unbounded complement, whereas the spider's web is connected with infinitely many complementary components, each of which is bounded. Hence, the Julia set $J(S)$ that contains both $J(f)$ and $J(g)$ of Example 4.3.3 must be the entire complex plane $\mathbb{C}$. In all of these three examples, we have $J_1(S) = J(S)$ and $F_1(S) = F(S)$. However, in the first Example 4.3.1, we have $J(f) = J(g)$, but in the next two Examples 4.3.2 and 4.3.3, we have $J(f) \neq J(g)$. The following example of entire (polynomial) semigroup [103, Example-2], we have $J(f) \neq J(g)$ as well as $F_1(S) \neq F(S)$ and $J_1(S) \neq J(S)$.

Example 4.3.4. *Let $S = \langle z^2, z^2/a \rangle$, where $a \in \mathbb{C}, |a| > 1$. Then the Julia set $J(S) = \{z : 1 \leq |z| \leq |a|\}$ which not forward invariant. Therefore, $J_1(S) \neq J(S)$. In this case,*

$J_1(S) = \mathbb{C}_\infty$. Note that $J(f) = \{z : |z| = 1\}$ and $J(g) = \{z : |z| = |a|\}$. The Fatou set $F(S) = \{z : |z| < 1$ or $|z| > |a|\}$ is not backward invariant, and so $F_1(S) \neq F(S)$. In this case, it is obvious that $F_1(S) = \emptyset$.

One of the main result in classical complex dynamics is that if Julia set has non-empty interior, then Julia set explodes, and it becomes whole complex plane $\mathbb{C}$. This result can be generalized to completely invariant Julia set $J_1(S)$. For this, we workout some constructions for the comparison of sets $J_1(S)$ and $J(S)$. Let $S = \langle f_1, f_2, \ldots f_n \rangle$ be a finitely generated transcendental semigroup. It is noted that $J(h) \subset J_1(S)$ for all $h \in S$ and so $\bigcup_{h \in S} J(h) \subset J_1(S)$. Let us define the following countable collections of sets:

$$\mathscr{E}_0 = \{J(h)\}$$

for all $h \in S$.

$$\mathscr{E}_1 = \bigcup_{h \in S} h^{-1}(\mathscr{E}_0) \cup \bigcup_{h \in S} h(\mathscr{E}_0)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\mathscr{E}_{n+1} = \bigcup_{h \in S} h^{-1}(\mathscr{E}_n) \cup \bigcup_{h \in S} h(\mathscr{E}_n)$$

and

$$\mathscr{E} = \bigcup_{n=0}^{\infty} \mathscr{E}_n$$

where $h^{-1}(\mathscr{E}_i) = \{h^{-1}(E) : E \in \mathscr{E}_i\}$ and $h(\mathscr{E}_i) = \{h(E) : E \in \mathscr{E}_i\}$ for any collection of sets $\mathscr{E}_i, (i = 1, 2, \ldots)$, and a function $h \in S$. The following result will be an alternative description of the set $J_1(S)$ of a transcendental semigroup S.

Theorem 4.3.1. *For a transcendental semigroup $\langle f_1, f_2, \ldots f_n \rangle$, we have $J_1(S) = \overline{\bigcup_{E \in \mathscr{E}} E}$.*

Proof. By Definition 4.3.1, $J_1(S)$ is closed, completely invariant under each $h \in S$, and contains $J(h)$ for all $h \in S$. Therefore, we can write

$$J_1(S) \supset \overline{\bigcup_{E \in \mathscr{E}} E}$$

The set $\overline{\bigcup_{E \in \mathscr{E}} E}$ is closed, and contains $J(h)$ for all $h \in S$. It remains to show that it is also completely invariant under each $h \in S$. h is a continuous closed map, so under each $h \in S$, $h(\overline{\bigcup_{E \in \mathscr{E}} E})$ and $h^{-1}(\overline{\bigcup_{E \in \mathscr{E}} E})$ are closed sets. It proves our claim. □

Corollary 4.3.1. *The set $J_1(S)$ is a perfect set.*

Proof. $J(h) \subset J_1(S)$ for all $h \in S$ and $J(h)$ is perfect, unbounded, and contains an infinite number of points for each $h \in S$. This corollary will be proved if we show $J_1(S)$ has no isolated points. Suppose $\alpha \in J_1(S)$ is an isolated point. Then it an isolated point of some $E \in \mathscr{E}$. Choose a neighborhood U of α so that $U - \{\alpha\} \subset F_1(S)$ where $F_1(S)$ is completely invariant Fatou set of S. $h^{-1}(F_1(S)) \subset F_1(S)$ and $h(F_1(S)) \subset F_1(S)$ for all $h \in S$, so each $h \in S$ omits $J_1(S)$ on $U - \{\alpha\}$, which implies that every element in S is normal on U. Which is a contradiction. □

Theorem 4.3.2. *Let S be a transcendental semigroup. If $J_1(S)$ has non-empty interior, then $J_1(S) = \mathbb{C}$.*

Proof. Let $Int.J_1(S) \neq \emptyset$, where $Int.J_1(S)$ denotes the interior of $J_1(S)$. Then there exists a disk $D = \{|z - z_0| < r\} \subset J_1(S)$ such that it intersects $J(h)$ for some $h \in S$. Then by [51, Theorem 3.9], for each finite value a, there is sequence $z_k \to z_0 \in J(h)$ and a sequence of positive integers $n_k \to \infty$ such that $f^{n_k}(z_k) = a, (k = 1, 2, 3, \dots)$ except at most for a finite value. Then by backward invariance of $J(h)$, $z_k \in J(h)$ and by forward invariance of $J(h)$, $a \in J(h)$. It shows that every finite value is in $J(h)$, except at most a single value. Since $h \in S$ is arbitrary, so we must have $J_1(S) = \mathbb{C}$. □

Corollary 4.3.2. *If $J_1(S) \neq \mathbb{C}$, then $F_1(S)$ is unbounded.*

Proof. If $F_1(S)$ is bounded, then $J_1(S)$ has interior points. By Theorem 4.3.2, $J_1(S) = \mathbb{C}$, which is a contradiction. □

Theorem 4.3.3. *Let S be a transcendental semigroup which contains functions f and g such that $J(f) \neq J(g)$. Then $J_1(S) = \mathbb{C}$.*

Lemma 4.3.1. *Let $S = \langle f, g \rangle$ be a transcendental semigroup such that $J(f) \neq J(g)$. Then $J_1(S) = \mathbb{C}$.*

Proof. Let U be a completely invariant component of $F(f)$. Then by [51, Theorem 4.36], U is unbounded, simply connected, and $\partial U = J(f)$. Likewise, a completely invariant component V of $F(g)$ is unbounded, simply connected, and $\partial V = J(g)$. $J(f) \neq J(g)$ implies that $\partial U \neq \partial V$. By [51, Theorem 3.8], $J(f)$ and $J(g)$ are unbounded, so $U \cap J(g) \neq \emptyset$ and $V \cap J(f) \neq \emptyset$. The fact $\partial U \neq \partial V$ implies that $J(f)$ must intersect interior of V, and $J(g)$ must intersect interior of U.

Let $z \in J(f) \cap Int.V$, where $Int.V$ is an interior of V. Then by the forward invariance of $J_1(S)$ and $Int.V$ under the function g, its nth iterates, that is, $g^n(z) \in J_1(S)$ and $g^n(z) \in Int.V$ for all $n \in \mathbb{N}$. Likewise, we can write $f^n(z) \in J_1(S)$

intersects open sets $Int.U$ and $Int.V$. Therefore, $J_1(S)$ intersects $Int.U \cap Int.V$. $J_1(S)$ is perfect and completely invariant set, so it contains all limits of the sequences (f^n) and (g^n). This prove that $Int.J_1(S) \neq \emptyset$ and hence by Theorem 4.3.2, $J_1(S) = \mathbb{C}$. □

Proof of Theorem 4.3.3. The proof follows by using Lemma 4.3.1 to the Theorem 4.3.2. □

$F_1(S)$ may also have a completely invariant component as in classical transcendental dynamics.

Theorem 4.3.4. *If S is a transcendental semigroup, then $F_1(S)$ has at most one completely invariant component.*

Proof. For a transcendental entire function f, $F(f)$ has at most one completely invariant component ([51, Theorem 4.38]). By Theorem 4.1.3(1), $F_1(S) \subset F(h)$ for all $h \in S$. This follows the assertion. □

$F_1(S)$ can have either 0 or infinitely many multiply connected components as in classical transcendental dynamics.

Theorem 4.3.5. *Let $F_1(S)$ be a completely invariant Fatou set of transcendental semigroup S. Then number of multiply connected components of $F_1(S)$ is either 0 or ∞.*

Proof. For a transcendental entire function f, the number of multiply connected components of $F(f)$ are either 0 or ∞ ([51, Theorem 4.43]). By Definition 4.3.1, $F_1(S) \subset F(h)$ for all $h \in S$. This follows the assertion. □

4.4 Completely invariant escaping sets

From Theorem 4.2.1, we can say that the set $I(S)$ is not neccessirly S-completely invariant. In this section, we generalize the completely invariant notion of the escaping set of a single transcendental entire function to the completely invariant notion of this set in transcendental semigroup dynamics. By Theorem 4.2.3, the escaping set $I(S)$ of an abelian transcendental semigroup S is S-backward invariant. This is a condition for a completely invariant escaping set of a transcendental semigroup. It is a generalization of completely invariant property of classical escaping sets of a function to more general settings of semigroups. In this section, we generalize the classical completely invariant notion of escaping sets of a function to the completely invariant notion of escaping sets of transcendental semigroups.

Definition 4.4.1 (Completely invariant escaping set). *Let S be a transcendental semigroup. We define a completely invariant escaping set of S by*

$$I_1(S) = \bigcap_{i \in \mathbb{N}} \{G_i : G_i \text{ is a completely invariant set under each } f \in S \text{ and each } G_i$$
$$\text{contains points } z \in \mathbb{C} \text{ such that } f^n(z) \to \infty \text{ as } n \to \infty \text{ for every } f \in S\}$$

By Definition 4.4.1, the set $I_1(S)$ is completely invariant under each element of S, and this set is same as the escaping set $I(S)$ if and only if S is an abelian semigroup. There are non-trivial transcendental semigroups from which one can get completely invariant escaping sets. The following assertion will be a good source of several examples.

Theorem 4.4.1. *Suppose that $S = \langle f, g \rangle$ and $I(f) = I(g)$. Then $I_1(S) = I(S)$.*

Proof. Since $I(f)$ is completely invariant under f and $I(g)$ is completely invariant under g. If $I(f) = I(g)$, then $I(h) = I(f) = I(g) = I(S)$ for all $h \in S$. In this case, $I_1(S) = I(S)$. □

For an example, semigroup $S = \langle f, g \rangle$ generated by functions $f(z) = e^{\lambda z},\ \lambda \in \mathbb{C} \setminus \{0\}$ and $g(z) = f^k + \frac{2\pi i}{\lambda},\ k \in \mathbb{N}$ is completely invariant. Here, we can find that $I(h) = I(f) = I(g) = I(S)$ for all $h \in S$. Another example of same kind is a semigroup $S = \langle f, g \rangle$ generated by the functions $f(z) = \lambda \sin z,\ \lambda \in \mathbb{C} \setminus \{0\}$ and $g(z) = f^k + 2\pi,\ k \in \mathbb{N}$. Note that in both of examples, the semigroup S is not abelian. From this discussion, we can conclude that escaping set $I(S)$ may be completely invariant even if semigroup S is not abelian. If S is an abelian transcendental semigroup, then $I(S)$ is nothing other than the set $I_1(S)$. There are transcendental semigroups, where escaping sets and completely invariant escaping sets might be empty.

Example 4.4.1. *Suppose that $S = \langle f, g \rangle$, where $f(z) = e^z$ and $g(z) = e^{-z}$. Then both $I_1(S)$ and $I(S)$ are empty sets. For $z \in I(f)$, then $g(f^n(z)) = 1/e^{f^n(z)} = 1/f(f^n(z)) = 1/f^{n+1}(z) \to 0$ as $n \to \infty$.*

Theorem 4.4.2. *Let S be a transcendental semigroup. Then $I_1(S) \subset I(S)$.*

Proof. Let $z \in I_1(S)$. Then by Definition 4.4.1, $z \in G_i$ for all i and $f^n(z) \to \infty$ as $n \to \infty$ for every $f \in S$. This proves that $z \in I(S)$. □

Let S be a transcendental semigroup such that $I_1(S) \neq \emptyset$. Then by Theorem 4.4.2, we can write $I_1(S) \subset I(h)$ for all $h \in S$. $I(h)$ is completely invariant for every $h \in S$, so their intersection $\bigcap_{h\in S} I(h)$ is also completely invariant. Define

$$E_0 = \bigcap_{h\in S} I(h)$$

$$E_1 = \bigcup_{h\in S} h^{-1}(E_0) \cup \bigcup_{h\in S} h(E_0)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$E_{n+1} = \bigcup_{h\in S} h^{-1}(E_n) \cup \bigcup_{h\in S} h(E_n)$$

and

$$E = \bigcap_{n\in\mathbb{N}\cup\{0\}} E_n \tag{4.4.1}$$

Theorem 4.4.3. *Let S be a transcendental semigroup. The set $E = \bigcap_{n\in\mathbb{N}\cup\{0\}} E_n$ is non-empty.*

Proof. We show that $I_1(S) \subset E_n$ for every $n \in \mathbb{N}\cup\{0\}$ by induction. $I_1(S) \subset E_0$ is obvious. By the completely invariant property of $I_1(S)$ under each $h \in S$, $I_1(S)$ is subset of each sets $h^{-1}(E_0)$ and $h(E_0)$ for all $h \in S$. This shows $I_1(S) \subset E_1$. Let us suppose $I_1(S) \subset E_n$. $E_{n+1} = h^{-1}(E_n) \cup h(E_n)$ for all $h \in S$, so, by the similar fashion as above, $I_1(S)$ is subset of each of the sets $h^{-1}(E_n)$ and $h(E_n)$ for all $h \in S$. This shows that $I_1(S) \subset E_{n+1}$ for each $n \in \mathbb{N}\cup\{0\}$. This proves that $E \neq \emptyset$. □

The following result will be a convenient description of a completely invariant escaping set of a transcendental semigroup.

Theorem 4.4.4. *Let S be a transcendental semigroup. Then $E = I_1(S)$, where E is a set as defined in (4.4.1).*

First, we prove the following lemma.

Lemma 4.4.1. *The closure $\overline{E}$ of any $E \subset \mathbb{C}$ is completely invariant under a transcendental entire function f if and only if the set E itself is completely invariant under the same function f.*

Proof. Let $\overline{E}$ is completely invariant under f. Then $f(\overline{E}) \subset \overline{E}$ and $f^{-1}(\overline{E}) \subset \overline{E}$. Let $z \in \overline{E}$, then $f(z) \in f(\overline{E})$, and so $f(z) \in \overline{E}$. Also, $z \in \overline{E} \implies$ there exists sequence $(z_n)_{n\in\mathbb{N}}$ in E such that $z_n \to z$ as $n \to \infty$. f is a continuous function f, so $f(z_n) \to f(z)$ as $n \to \infty$. As $f(z) \in \overline{E}$, we must have $f(z_n) \in E$. Note that $f(z_n) \in f(E)$ as $z_n \in E$. Thus, we must have $f(E) \subset E$.

Next, let $z \in \overline{E}$, then $f^{-1}(z) \in f^{-1}(\overline{E}) \subset \overline{E}$. So there exists $f^{-1}(z_n) \in E$ such that $f^{-1}(z_n) \to f^{-1}(z)$ as $n \to \infty$. However, it is obvious that $f^{-1}(z_n) \in f^{-1}(E)$. Thus we must have $f^{-1}(E) \subset E$.

The converse part of this lemma follows from [15, Theorem 3.2.3]. □

It is noted that under the assumption of Lemma 4.4.1, not only the closure of completely invariant set is completely invariant but also its complement, interior and boundary are also completely invariant (see, for instance, [15, Theorem 3.2.3]).

Proof of Theorem 4.4.4. By Definition 4.4.1, $I_1(S)$ is completely invariant for every $h \in S$, and is contained in $I(h)$ for all $h \in S$. Hence, by Theorem 4.4.3, $I_1(S)$ is contained in E_n for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $I_1(S) \subset E$.

On the other hand, set E is contained in $I(h)$, where each $I(h)$ is completely invariant. We need to show that E is completely invariant for all $f \in S$, and for every $z \in E, \ \ f^n(z) \to \infty$ as $n \to \infty$ for all $f \in S$.

Any $f \in S$ is continuous in $\mathbb{C}$, and $E \subset \mathbb{C}$. Therefore, by the usual topological argument, $f(\overline{E}) \subset \overline{f(E)} \Rightarrow f^{-1}(\overline{f(E)})$ is closed in $\mathbb{C}$ for all $f \in S \Rightarrow f$ is a continuous closed map. This shows that $f(\overline{E})$ and $f^{-1}(\overline{E})$ are both closed sets in $\mathbb{C}$. As each $f \in S$ is continuous closed map and $f(\overline{E_n}) \subset \overline{E_n}$ and $f^{-1}(\overline{E_n}) \subset \overline{E_n}$ for all n, then $f(\overline{E}) \subset \overline{E}$ and $f^{-1}\overline{(E)} \subset \overline{E}$. By Lemma 4.4.1, it proves that E is completely invariant under each $f \in S$.

Finally, any $z \in E \Rightarrow z \in E_n$ for all n. Again, E_n is a union of all images and pre-images of E_{n-1} under each $f \in S$. By this way, the point z belongs to the image or pre-image of E_0 under each $f \in S$. E_0 is contained in $I(f)$ for all $f \in S$, so, $f^n(z) \to \infty$ for all $f \in S$. Hence, $E \subset I_1(S)$. □

Chapter 5

NEARLY ABELIAN HOLOMORPHIC SEMIGROUPS

In this chapter, we extend some results of abelian transcendental semigroups to more general setting of nearly abelian transcendental semigroups. The principal feature of nearly abelian rational semigroups was investigated by Hinkannen and Martin [46, Theorem 4.1]. In such a case, they found that the Julia set $J(S)$ of a rational semigroup S is same as the Julia set $J(f)$ of each $f \in S$.

5.1 Nearly abelian transcendental semigroups

We defined abelian holomorphic semigroups, and we discussed some results associated with it. There is also a slightly larger family of transcendental semigroups that can fulfill this criteria. We call these semigroups by nearly abelian and it is considered the more general form than that of abelian semigroups.

Definition 5.1.1 (Nearly abelian transcendental semigroup). *We say that a transcendental semigroup S is nearly abelian if there is a family $\Phi = \{\phi_i\}$ of conformal maps of the form $az + b$ for some non-zero a such that*

1. $\phi_i(F(S)) = F(S)$ *for all* $\phi_i \in \Phi$*, and*
2. *for all* $f, g \in S$*, there is a* $\phi \in \Phi$ *such that* $f \circ g = \phi \circ g \circ f$*.*

It is noted that a particular example of a nearly abelian semigroup is an abelian semigroup. An abelian semigroup follows trivially from a nearly abelian semigroup if we choose ϕ an identity function.

The Definition 5.1.1 of nearly abelian transcendental semigroup looks more restrictive on affine maps of the form $\phi(z) = az + b,\ a \neq 0$, and this type of function can play the role of semiconjugacy to certain class of transcendental entire functions. Recall that a function f is (semi) conjugate to another function g if there is a continuous function ϕ such that $\phi \circ f = g \circ \phi$. For example, the transcendental entire function $f_1(z) = \lambda \cos z,\ \lambda \in \mathbb{C}$ is semi-conjugate to another transcendental entire function $f_2(z) = -\lambda \cos z$ because there is a function $\phi(z) = -z$ such that $\phi \circ f_1 = f_2 \circ \phi$. If there is a transcendental semigroup generated by such type of semi-conjugate functions, then semigroup will more likely to be nearly abelian.

Theorem 5.1.1 (**An example of nearly abelian transcendental semigroup**). *Let $S = \langle f_1, f_2, \ldots f_n, \ldots \rangle$ be a transcendental semigroup, and let ϕ be an entire function of the form $z \to az + b$ for some non zero a with $a,\ b \in \mathbb{C}$ such that $f_i \circ \phi = f_i \circ \phi$ for all i, and $f_j = \phi \circ f_i$ with $i \neq j$. Then semigroup S is nearly abelian.*

To prove this Theorem 5.1.1, we need the following lemma.

Lemma 5.1.1. *Let $S = \langle f_1, f_2, \ldots f_n, \ldots \rangle$ be a transcendental semigroup and let ϕ be an entire function of the form $z \to az + b$ for some non zero a such that $a,\ b \in \mathbb{C}$. If $\phi \circ f_i = f_j \circ \phi$ for all f_i and f_j with $i \neq j$, then $\phi(F(S)) = F(S)$ and $\phi(J(S)) = J(S)$.*

Proof. First, we prove that if $\phi \circ f_i = f_j \circ \phi$ for all i and j with $i \neq j$, then $\phi \circ f = g \circ \phi$ for all $f,\ g \in S$. By definition, any $f,\ g \in S$ can be written as $f = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}$, and $g = f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_n}$, where $i_k, j_k \in \{1, 2 \ldots\}$, and $k = 1, 2, \ldots, n$. Now, $\phi \circ f = \phi \circ f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n} = f_{j_1} \circ \phi \circ \circ f_{i_2} \circ \ldots \circ f_{i_n} = \ldots = f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_n} \circ \phi = g \circ \phi$. This proves our claim.

Let $w \in \phi(F(S))$. Then there is $z_0 \in F(S)$ such that $w = \phi(z_0)$. Let $U \subset F(S)$ is a neighborhood of z_0 such that $|f(z) - f(z_0)| < \epsilon/2$ for all $z \in U$ and $f \in S$. This shows that $f(U)$ has diameter less than ϵ for all $f \in S$. The function ϕ has the bounded first derivative $a \neq 0$, so it is a Lipschitz with Lipschitz constant $k = \sup |\phi'(z)| = a$. Now, for any $g \in S$, the diameter of $g(\phi(U)) = \phi(f(U))$ is less than $k\epsilon$. Hence $w = \phi(z_0) \in F(S)$. This shows that $\phi(F(S)) \subset F(S)$.

Next, let $w \in \phi(J(S))$. Then $w = \phi(z_0)$ for some $z_0 \in J(S)$. Let z_0 be a repelling fixed point for some $f \in S$, but not a critical point of ϕ. Then $\phi \circ f = g \circ \phi$ gives g has a fixed point at $\phi(z_0)$ with same multiplier as that of f at z_0. Thus, ϕ maps repelling fixed points of any $f \in S$ to repelling fixed points of another $g \in S$. By [75, Theorem

4.1 and 4.2], Julia set of a transcendental semigroup is perfect, and

$$J(S) = \overline{\bigcup_{f \in S} J(f))},$$

where repelling periodic points are dense in $J(f)$ for each $f \in S$. Therefore, by above discussion, it then follows that $\phi(J(S)) \subset J(S)$.

Finally, the fact $\phi(\mathbb{C}) = \mathbb{C}$ is obvious. Using this fact in $F(S) = \mathbb{C} - J(S)$ and $J(S) = \mathbb{C} - F(S)$, we get

$$\phi(F(S)) = \mathbb{C} - \phi(J(S)) \text{ and } \phi(J(S)) = \mathbb{C} - \phi(F(S)). \tag{5.1.1}$$

Again, using facts $\phi(J(S)) \subset J(S)$ and $\phi(F(S)) \subset F(S)$ in 5.1.1, we will get required reverse inclusions $F(S) \subset \phi(F(S))$, and $J(S) \subset \phi(J(S))$. □

It is noted that this Lemma 5.1.1 tells us that the first condition $\phi_i(F(S)) = F(S)$ of the nearly abelian semigroup holds obviously if a semigroup is generated by (semi) conjugate functions. This is a way that one can replace the first condition of Definition 5.1.1 of the nearly abelian transcendental semigroup.

Proof of Theorem 5.1.1. The first part for nearly abelian semigroup follows from Lemma 5.1.1. The second part follows from the following simple calculations. The hypothesis $\phi \circ f_i = f_j \circ \phi$ for all f_i and f_j with $i \neq j$ gives $f \circ \phi = \phi \circ g$ for all $f,\ g \in S$. From the hypothesis $\phi \circ f_i = f_j$ for all $f \neq j$, we can get $\phi \circ f = g$ for all $f, g \in S$. Therefore, $\phi \circ g \circ f = f \circ \phi \circ f = f \circ g$ for all $f,\ g \in S$. □

There are general and particular examples of transcendental entire functions that can fulfill the essence of Theorem 5.1.1, and so the semigroup generated by these functions is nearly abelian.

Example 5.1.1. *Let ϕ be an entire function of the form $z \to -z + c$ for some $c \in \mathbb{C}$. Let f be a transcendental entire function with $f \circ \phi = f$, and a function g is defined by $g = \phi \circ f$. Then functions f and g are conjugates, and the semigroup $S = \langle f, g \rangle$ is nearly abelian.*

Solution. Let $f,\ g$ and ϕ be as in the statement of the question. It is clear that ϕ^2 = identity. Then $g \circ \phi = \phi \circ f \circ \phi = \phi \circ f$. This proves that functions f and g are conjugates. The condition $\phi(F(S)) = F(S)$ for all $\phi \in \Phi$ of the definition of nearly abelian semigroup follows from Lemma 5.1.1. The second condition follows from the Theorem 5.1.1. More explicitly, it follows from the following calculation.

$f \circ g = f \circ \phi \circ f = f \circ f = f^2 = \phi^2 \circ f^2 = \phi \circ \phi \circ f \circ f = \phi \circ g \circ f$. Therefore, the semigroup $S = \langle f, g \rangle$ generated by these two functions f and g is nearly abelian. From the fact $g = \phi \circ f$, we can say that ϕ is not an identity. □

Example 5.1.2. *Let $f(z) = e^{z^2} + \lambda$, and $g = \phi \circ f$ where $\phi(z) = -z$. Then the semigroup $S = \langle f, g \rangle$ is nearly abelian. Likewise, functions $f(z) = \lambda \cos z$ and $g = \phi \circ f$, where $\phi(z) = -z$, generate the nearly abelian semigroup.*

Solution. The given functions in the question fulfills all conditions such as $f \circ \phi = f$, ϕ^2 = identity as well as $\phi \circ f = g \circ \phi$ of Theorem 5.1.1 and Example 5.1.1. Therefore, the semigroup $S = \langle f, g \rangle$ is nearly abelian. Note that $\phi \circ f = -f \neq f$, so ϕ is not an identity. □

It is noted that Example 5.1.1 is just for a nice general example of Theorem 5.1.1, and it says there is a nearly abelian transcendental semigroup. Unfortunately, this example does not generate many more examples of transcendental entire functions that can generate transcendental semigroup. Basically, it generates even functions or translates of even functions. For example: If we set $h(z) = f(z + c/2)$, then $h(z) = f(z + c/2) = (f \circ \phi)(z) = f(c - z - c/2) = f(c/2 - z) = h(-z)$. That is, h is an even function.

Hinkkanen and Martin [46, Theorem 4.1] proved that the Julia set of the nearly abelian rational semigroup is same as the Julia set of each of its element. Indeed, this is a generalization of the result of abelian rational semigroup that we proved in Theorem 4.2.6. It will be difficult to say the same in general if we take abelian transcendental semigroup. That is, if we have abelian transcendental semigroup S, it would not always $J(S) = J(f)$ for all $f \in S$. It would be sometime in certain case, and one of the case was proved by Poon [75, Theorem 5.1].

Proposition 5.1.1. *Let $S = \langle f_1, f_2, \dots f_n \rangle$ is an abelian finite type transcendental semigroup. Then $F(S) = F(f)$ for all $f \in S$.*

Indeed, this result looks like an extension work of the following results of Singh and Wang [97, Theorems 2, 3] of classical transcendental dynamics.

Proposition 5.1.2. *Let f and g are two permutable transcendental entire functions. If both f and g have no wandering domains, then $J(f) = J(f \circ g) = J(g)$.*

Proposition 5.1.3. *Let f and g are two permutable transcendental entire functions. If both f and g are of bounded type, then $J(f) = J(f \circ g) = J(g)$.*

Our particular interest is how far the result of Poon [75, Theorem 5.1] can be generalized to nearly abelian transcendental semigroups.

Theorem 5.1.2. *Let S be a nearly abelian semigroup generated by transcendental entire functions of finite type. Then for each $g \in S$, we have $I(S) = I(g)$, $J(S) = J(g)$ and $F(S) = F(g)$.*

First, we state and prove the following results.

Lemma 5.1.2. *Let f and g be two transcendental entire functions of finite type such that $f \circ g = \phi \circ g \circ f$, where $\phi(z) = az + b$, and $|a| = 1$. Then $F(f) = F(g)$.*

Proof. f and g are transcendental entire functions of finite type such that $f \circ g = \phi \circ g \circ f$. Then by [51, Theorems 4.29 and 4.32], f and g do not have Baker domains and wandering domains. That is, there are subsequences (f^{n_i}) and (g^{n_i}) which do not diverge to ∞ for all z in their respective Fatou sets. In order to prove this lemma, we only need to show that $F(f) \subset F(g)$, that is, $g(F(f)) \subset F(g)$. By symmetry, the reverse inclusion $F(g) \subset F(f)$ holds similarly.

Consider a point $z_0 \in F(f)$ and a neighborhood U of z_0 such that $\overline{U} \subseteq F(f)$. Then $g(U)$ is a neighborhood of $g(z_0)$. Consider a sequence (f^n) of iterates of f on $g(U)$. By assumption, there is a subsequence (f^{n_i}) of (f^n) converges to a holomorphic function $h : U \to \mathbb{C}$. In such a case, f^{n_i} converges to $g \circ h$ on $g(U)$, that is, $g \circ f^{n_i} \to g \circ h = \xi$ uniformly on U. By assumption, $f^{n_i} \circ g = \phi \circ g \circ f^{n_i} \to \phi \circ \xi = \varphi$ uniformly on U. This proves that $\{f^n \circ g : n \in \mathbb{N}\}$ is a normal family on U, and so $\{f^n : n \in \mathbb{N}\}$ is a normal family on $g(U)$. Since $F(f)$ is a maximal open set where $\{f^n : n \in \mathbb{N}\}$ is normal. Therefore, we have $g(U) \subseteq F(f)$. This proves that $g(z_0) \in F(f)$, and hence $g(F(f)) \subset F(g)$. □

Lemma 5.1.3. *Let f and g be two transcendental entire functions of finite type such that $f \circ g = \phi \circ g \circ f$, where $\phi(z) = az + b$, and $|a| = 1$. Then $I(f) = I(g)$.*

Proof. By Lemma 5.1.2, we have $J(f) = J(g)$. By Lemma 4.2.2, the rest of the proof of this lemma follows. □

Proof of Theorem 5.1.2. We prove $I(S) = I(f)$ for all $f \in S$ and the remaining equalities follows from Proposition 4.1.4. The semigroup S is nearly abelian, so for all $f, g \in S$, there is $\phi \in \Phi$ such that $f \circ g = \phi \circ g \circ f$. Also, $f = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}$, and $g = f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_n}$, where $i_k, j_k \in \{1, 2 \ldots\}$, $k = 1, 2, \ldots, n$, and each f_{i_k}, f_{j_k} is of finite type. By [75, Lemma 5.1], every f and g in S is of finite type. By Lemma 5.1.3, we have $I(f) = I(g)$ for all $f, g \in S$. By Theorem 4.1.3(3), $I(S) = \bigcap_{f \in S} I(f)$. Therefore, this fact together with the fact $I(f) = I(g)$ for all $f, g \in S$ of Lemma 5.1.3, we can conclude the assertion of Theorem 5.1.2. □

5.2 Conjugate semigroups of a holomorphic semigroup

The notion of nearly abelian holomorphic semigroup can be useful for defining conjugate semigroups of a holomorphic semigroup. The family of $\Phi = \{\phi_i\}$ of conformal maps of the form $z \to az + b$ for some non-zero a used in Definition 5.1.1 has given a spacial name.

Definition 5.2.1 (Commutator). *Let S be a nearly abelian transcendental semigroup. The set of the form $\Phi(S) = \{\phi : \text{ there are } \ f, g \in S \ \text{ such that } \ f \circ g = \phi \circ g \circ f\}$ is called the set of commutators of S.*

We write $\phi = [f, g]$ if $f \circ g = \phi \circ g \circ f$. Note that $[f, g]^{-1} = [g, f]$ and for any $f \in S,\ [f, f]$ = identity.

Definition 5.2.2. *Let $S = \langle f_1, f_2, f_3, \ldots, f_n \rangle$ be a nearly abelian transcendental semigroup, and $\Phi(S)$ be a set of its commutators. Let us define a set*

$$S^{'} = \langle \phi \circ f_1 \circ \phi^{-1},\ \phi \circ f_2 \circ \phi^{-1}, \ldots,\ \phi \circ f_n \circ \phi^{-1} \rangle \tag{5.2.1}$$

where $\phi \in \Phi(S)$ such that $\phi = [f_i, f_j]$ and $\phi^{-1} = [f_j, f_i]$. If we let $g_i = \phi \circ f_i \circ \phi^{-1}$, then we say function f_i is conjugate to g_i by a conformal map $\phi : \mathbb{C} \to \mathbb{C}$ of the form $az + b$ for some non-zero a. The semigroup $S^{'}$ is then called a conjugate semigroup of the semigroup S.

Theorem 5.2.1 ([118, Theorem 2.1])**.** *Let $S^{'}$ be a conjugate semigroup of a holomorphic semigroup S. Then $S^{'}$ is nearly abelian if and only if S is nearly abelian.*

Proof. Let S be a nearly abelian transcendental semigroup. Then $f_i \circ f_j = \phi \circ f_j \circ f_i$ for $\phi \in \Phi(S)$ and $f_i,\ f_j \in S$. Now for any $\phi \circ f_i \circ \phi^{-1},\ \phi \circ f_j \circ \phi^{-1} \in S^{'}$, we have

$$\begin{aligned}
(\phi \circ f_i \circ \phi^{-1}) \circ (\phi \circ f_j \circ \phi^{-1}) =& \phi \circ f_i \circ f_j \circ \phi^{-1} \\
=& \phi \circ \xi \circ f_j \circ f_i \circ \phi^{-1} \ \text{ for some } \ \xi \in \Phi(S) \\
=& \xi \circ \phi \circ f_j \circ f_i \circ \phi^{-1} \\
=& \xi \circ (\phi \circ f_j \circ \phi^{-1}) \circ (\phi \circ f_i \circ \phi^{-1})
\end{aligned}$$

This shows that the conjugate semigroup $S^{'}$ of a nearly abelian semigroup S is a nearly abelian.

Conversely, suppose that the semigroup $S^{'}$ is nearly abelian. Then $g_i \circ g_j = \phi \circ g_j \circ g_i$ for $\phi \in \Phi(S)$ and $g_i, g_j \in S^{'}$, where $g_i = \phi \circ f_i \circ \phi^{-1}$ and $g_j = \phi \circ f_j \circ \phi^{-1}$. From which get $f_i = \phi^{-1} \circ g_i \circ \phi$ and $f_j = \phi^{-1} \circ g_j \circ \phi$. Now, for any $f_i,\ f_j \in S$, we have

$$\begin{aligned} f_i \circ f_j =& (\phi^{-1} \circ g_i \circ \phi) \circ (\phi^{-1} \circ g_j \circ \phi) \\ =& \phi^{-1} \circ g_i \circ g_j \circ \phi \\ =& \phi^{-1} \circ \phi \circ g_j \circ g_i \circ \phi \\ =& g_j \circ g_i \circ \phi \\ =& \phi \circ f_j \circ \phi^{-1} \circ \phi \circ f_i \circ \phi^{-1} \circ \phi \\ =& \phi \circ f_j \circ f_i \end{aligned}$$

This shows that semigroup S is nearly abelian if its conjugate semigroup $S^{'}$ is nearly abelian. □

The image of the Fatou, Julia and escaping sets of a nearly abelian semigroup under commutator $\phi \in \Phi(S)$ is respectively the Fatou, Julia and escaping sets of its conjugate semigroup.

Theorem 5.2.2 ([118, Theorem 1.1])**.** *Let S be a nearly abelian transcendental semigroup, and $\Phi = \{\phi_i\}$ be a set of commutators of of the form $z \to az + b$ for some non-zero a. Let $S^{'}$ be a conjugate semigroup of S. Then there exists $\phi \in \Phi$ such that $\phi(I(S)) = I(S^{'}),\ \phi(J(S)) = J(S^{'})$ and $\phi(F(S)) = F(S^{'})$.*

To proof this theorem, we need the following lemma.

Lemma 5.2.1 ([118, Lemma 2.1])**.** *Let f and g be two transcendental entire functions, and ϕ be an entire function of the form $z \to az + b$, where $a \neq 0$ such that $\phi \circ f = g \circ \phi$. Then $\phi(I(f)) = I(g),\ \phi(J(f)) = J(g)$ and $\phi(F(f)) = F(g)$.*

Proof. Let $w \in \phi(I(f))$, then there is $z \in I(f)$ such that $w = \phi(z)$. The condition $z \in I(f) \implies f^n(z) \to \infty$ as $n \to \infty$. Now $g^n(w) = g^n(\phi(z)) = (g^n \circ \phi)(z) = (g^{n-1} \circ g \circ \phi)(z) = (g^{n-1} \circ \phi \circ f)(z) = (g^{n-2} \circ \phi \circ f^2)(z) = \ldots = (\phi \circ f^n)(z) = \phi(f^n(z))$. From $\phi(z) = az + b$, $(a \neq 0)$ and $f^n(z) \to \infty$ as $n \to \infty$, we must have $g^n(w) \to \infty$ as $n \to \infty$. This shows that $\phi(I(f)) \subset I(g)$. For opposite inclusion, we note that if $z \in I(g)$ then we must have $\phi(z) \in I(g)$. Because $\phi(z) \in \phi(I(g)) \subset I(g)$ (similar argument as above). As above $\phi(f^n(z)) = g^n(\phi(z)) \to \infty$ as $n \to \infty$. This shows that $z \in \phi(I(f))$, and so $I(g) \subset \phi(I(f))$. This proves that $\phi(I(f)) = I(g)$. Remaining equality obtained from the facts $\partial I(f) = J(f)$, and $F(f) = \mathbb{C} \setminus J(f)$. □

Proof of Theorem 5.2.2. Let $\phi \circ f_i \circ \phi^{-1} = g_i$ for all $i = 1, 2, \cdots, n$. From which we get $\phi \circ f_i = g_i \circ \phi$ for all $i = 1, 2, \cdots, n$. Any $f \in S$ and $g \in S^{'}$ can be written respectively as $f = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}$, and $g = g_{i_1} \circ g_{i_2} \circ \ldots \circ f g_{i_n}$. From which we get $\phi \circ f = \phi \circ f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n} = g_{i_1} \circ \phi \circ f_{i_2} \circ \ldots \circ f_{i_n} = g_{i_1} \circ g_{i_2} \circ \phi \circ \ldots \circ f_{i_n} = \ldots = g_{i_1} \circ g_{i_2} \circ \ldots \circ g_{i_n} \circ \phi = g \circ \phi$ for all $f \in S$ and $g \in S^{'}$. $S = \langle f_1, f_2, f_3, \ldots, f_n \rangle$ is a nearly abelian transcendental semigroup, so from Theorem 5.1.2, we have $I(S) = I(f),\ J(S) = J(f)$ and $F(S) = F(f)$ for all $f \in S$. Now $I(S) = I(f) \Longrightarrow \phi(I(S)) = \phi(I(f))$. By Lemma 5.2.1, $\phi(I(f)) = I(g)$. By Theorem 5.2.1, semigroup $S^{'}$ is nearly abelian, so again by Theorem 5.1.2, we have $I(S^{'}) = I(g)$. Thus we get $\phi(I(S)) = I(S^{'})$. Next two equality are obtained by the similar fashion. □

Analogous to [46, Theorem 4.3], every function of the nearly abelian transcendental semigroup S can be written as the composition of an element of commutator $\Phi(S)$, and the composition of the certain powers of its generators.

Theorem 5.2.3 ([118, Theorem 1.2]). *Let $S = \langle f_1, f_2, f_3, \ldots, f_n \rangle$ be a nearly abelian cancellative holomorphic semigroup. Then every element $f \in S$ can be written as $f = \phi \circ f_1^{t_1} \circ f_2^{t_2} \circ f_3^{t_3} \circ \cdots f_m^{t_m}$ where $\phi \in \Phi(S)$ if $\Phi(S)$ is a group or semigroup. Otherwise, $\phi \in G$, where $G = \langle \Phi(S) \rangle$ is a group generated by $\Phi(S)$, and t_i are non-negative integers.*

First, we need the following lemma.

Lemma 5.2.2 ([118, Lemma 3.1]). *Let S be a nearly abelian transcendental semigroup. Then for any $f \in S$, and for any $\phi \in \Phi(S)$, there is a conformal map ξ such that $f \circ \phi = \xi \circ f$. Moreover, there are $\xi_1, \xi_2 \in \Phi(S)$ such that $\xi = \xi_1 \circ \xi_2$.*

Proof. Given $\phi \in \Phi(S)$, there are $g, h \in S$ such that $g \circ h = \phi \circ h \circ g$. For any $f \in S$, we have

$$f \circ g \circ h = f \circ \phi \circ h \circ g. \tag{5.2.2}$$

Furthermore, there are $\xi_1, \xi_2 \in \Phi(S)$ such that

$$f \circ g \circ h = \xi_1 \circ g \circ f \circ h = \xi_1 \circ \xi_2 \circ f \circ h \circ g. \tag{5.2.3}$$

From the equations 5.2.2 and 5.2.3, we get $f \circ \phi = \xi_1 \circ \xi_2 \circ f = \xi \circ f$, where $\xi = \xi_1 \circ \xi_2$. □

Example 5.2.1 ([118, Example 3.1]). *For $f(z) = \lambda \cos z \in S$, there are $\phi(z) = \pi - z, \xi(z) = -z \in \Phi(S)$ such that $f \circ \phi = \xi \circ f$. The linear map $\phi(z) = \pi - z$ can be written as the composition of rotation through the angle π anticlockwise about the* *origin and the translation by* π*. The map* $\xi(z) = -z \in \Phi(S)$ *can be written as the composition of identity map and the rotation through the angle* π *anticlockwise about the origin.*

However, it is not always possible to find an element $\phi \in \Phi(S)$ such that $f \circ \phi = \xi \circ f$. For example if $f(z) = e^{z^2} + \lambda$ and $\xi(z) = -z$, then $\xi \circ f = -f \neq f = f \circ \phi$ for any suitable choice of $\phi \in \Phi(S)$.

Proof of Theorem 5.2.3. The proof of this theorem follows from the inductive application of Lemma 5.2.2 to each element $f = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}$ of S. □

Chapter 6

FATOU, JULIA AND ESCAPING SETS IN HOLOMORPHIC (SUB)SEMIGROUP DYNAMICS

In this chapter, we investigate under what conditions the Fatou, Julia and escaping sets of a holomorphic semigroup are respectively equal to the Fatou, Julia, and escaping sets of its proper subsemigroups.

6.1 Finite indexed and cofinite indexed subsemigroups

There are various notions of how large a substructure is inside of an algebraic object in order that the two structures share certain properties. One such a notion is *index*, and it plays an important role in general group theory and semigroup theory. It is used to measure the difference between a group (semigroup) and a subgroup (subsemigroup). It occurs in many important theorems of the group theory and semigroup theory. The notions of finite index, cofinite index and Rees index of a subsemigroup have been used to gauge the size of subsemigroup. If the subsemigroup T is big enough in semigroup S, then S and T share many properties.

It is possible that the Fatou, Julia, or escaping set of a holomorphic semigroup may be equal, respectively, to the Fatou, Julia, or escaping set of a proper subsemigroup.

Definition 6.1.1 (Finite index and cofinite index). *A subsemigroup T of a holomorphic* *semigroup S is said to be of finite index if there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1, where $S^1 = S \cup \{Identity\}$, such that*

$$S = (f_1 \circ T) \cup (f_2 \circ T) \cup \ldots \cup (f_n \circ T) \tag{6.1.1}$$

The smallest n that satisfies (6.1.1) is called the index of T in S. Similarly, a subsemigroup T of a holomorphic semigroup S is said to be of cofinite index if there exists finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that for any $f \in S$, there is $i \in \{1, 2, \ldots, n\}$ such that

$$f_i \circ f \in T \tag{6.1.2}$$

The smallest n that satisfies (6.1.2) is called the cofinite index of T in S.

It is noted that the size of a subsemigroup T of a semigroup S is measured in terms of index. If a subsemigroup T has a finite index or cofinite index in the semigroup S, then we say T is a finite indexed subsemigroup or a cofinite indexed subsemigroup respectively.

In semigroup theory, the cofinite index is also known as *Grigorochuk index*, and this index was introduced by Grigorochuk [41] in 1988. Maltcev and Ruskuc [64, Theorem 3.1] proved that for every element f of a finitely generated semigroup S, and every proper cofinite indexed subsemigroup T, one has $f \circ T \neq S$. It is noted that if the semigroup is a group, the notion of finite index and cofinite index coincide. The subsemigroup T of a finitely generated semigroup S consisting of all words of finite length (compositions of a finite number of holomorphic functions) has a finite index and a cofinite index in S.

From Definition 6.1.1, the finite index and cofinite index of subsemigroups of the following examples will be clear.

Example 6.1.1 ([115, Example 2.1]). *A subsemigroup $T = \langle \sin\sin z, \cos\cos z, \sin\cos z, \cos\sin z \rangle$ of the transcendental semigroup $S = \langle \sin z, \cos z \rangle$ has finite index 3 and cofinite index 2.*

Example 6.1.2 ([115, Example 2.2]). *A subset $T = \{$words (compositions) begining with $f\}$ of a holomorphic semigroup $S = \langle f, g \rangle$ is clearly a subsemigroup of S. Then T has an infinite index but cofinite index 1 in S.*

It is noted that in Example 6.1.2, S is finitely generated but T is not. Since any generating set of T must contain $\{f \circ g^n : n \geq 1\}$. The only cofinite subsemigroup of T is T itself. So T has cofinite index 1 in S.

Example 6.1.3 ([115, Example 2.3]). *Let $S = \langle f \rangle$ where f is a holomorphic function. Then the subsemigroup $T = \langle f^n : n \in \mathbb{N} \rangle$. has finite index n in S and cofinite index 1 in S.*

It is noted that in Example 6.1.3, the subsemigroup T has n different translates in S, which are $T, f \circ T, \ldots, f^{n-1} \circ T$. Here, the only cofinite subsemigroup of T is T itself. If we choose the subsemigroup of S to be S itself, then there are infinitely many translates of S, namely, $h \circ S = h \circ \langle f \rangle$ for all $h \in S$. So, S has an infinite index in itself. Again, it has cofinite index 1 in itself.

Using Theorem 4.1.3, we can prove the following assertion:

Lemma 6.1.1 ([115, Lemma 2.1]). *For any subsemigroup T of a holomorphic semigroup S, we have $F(S) \subset F(T), J(S) \supset J(T)$.*

Proof. We prove $F(S) \subset F(T)$. The next inclusion follows taking the complements. By Theorem 4.1.3, $F(S) \subset \cap_{f \in S} F(f)$, and $F(T) \subset \cap_{g \in T} F(g)$ for any subsemigroup T of the semigroup S. Since any $g \in T$ is also in S, so by same Theorem 4.1.3, we also have $F(S) \subset F(g)$ for all $g \in T$, and hence $F(S) \subset \cap_{g \in T} F(g)$. Now for any $z \in F(S)$, we have $z \in \cap_{g \in T} F(g)$ for all $g \in T$. This implies $z \in F(g)$ for all $g \in T$. This proves $z \in F(T)$ and hence $F(S) \subset F(T)$. □

Hinkannen and Martin [46, Theorem 2.4] proved that if a subsemigroup T has a finite index or a cofinite index in the rational semigroup S, then $F(S) = F(T)$ and $J(S) = J(T)$. In the following theorem, we prove the same result in the case of a general holomorphic semigroup. It is noted that by a general holomorphic semigroup, we mean either a rational semigroup or a transcendental semigroup.

Theorem 6.1.1 ([115, Theorem 2.1]). *If a subsemigroup T has a finite index or a cofinite index in the holomorphic semigroup S, then $F(S) = F(T)$ and $J(S) = J(T)$.*

Proof. By Lemma 6.1.1, $F(S) \subset F(T)$ for any holomorphic semigroup S. If S is a rational semigroup, the result follows from [46, Theorem 2.4]. We prove the reverse inclusion, if S is a transcendental semigroup.

Let the subsemigroup T of a semigroup S has finite index n. Then by Definition 6.1.1, there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that

$$S = f_1 \circ T \cup f_2 \circ T \cup \ldots \cup f_n \circ T$$

Then for any $g \in S$, there is an $h \in T$ such that $g = f_i \circ h$. Choose a sequence $(g_j)_{j \in \mathbb{N}}$ in S. Then each g_j is of the form $g_j = f_i \circ h_j$, where $h_j \in T$ and $1 \leq i \leq n$.

Here, we may assume the same i for all j. Hence, without loss of generality, we may choose a subsequence (g_{j_k}) of (g_j) such that $g_{j_k} = f_i \circ h_{j_k}$ for particular f_i, where (h_{j_k}) is a subsequence of (h_j) in T. Since on $F(T)$, the sequence (h_{j_k}) has a convergent subsequence so do the sequences (g_{j_k}) and (g_j) in $F(S)$. This proves $F(T) \subset F(S)$.

Let the subsemigroup T of a semigroup S have cofinite index n. Then by Definition 6.1.1, there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that for every $f \in S$, there is $i \in \{1, 2, \ldots, n\}$ such that $f_i \circ f \in T$. Let us choose a sequence $(g_j)_{j \in \mathbb{N}}$ in S. Then, for each j, there is an i with $1 \leq i \leq n$ such that $f_i \circ g_j = h_j \in T$. Let $z \in F(T)$. Then the sequence (h_j) has a convergent subsequence in T, and hence so does the sequence (g_j) in $F(S)$. This proves $F(T) \subset F(S)$. □

Next, we see a special subsemigroup of a holomorphic semigroup that yields a cofinite index.

Definition 6.1.2 (Stablizer, wandering component and stable domains). *For a holomorphic semigroup S, let U be a component of the Fatou set $F(S)$ and U_f be a component of the Fatou set containing $f(U)$ for some $f \in S$. The set of the form*

$$S_U = \{f \in S : U_f = U\}$$

is called the stabilizer of U on S. If S_U is non-empty, we say that a component U satisfying $U_f = U$ is a stable basin for S. The component U of $F(S)$ is said to be wandering if the set $\{U_f : f \in S\}$ contains infinitely many elements. That is, U is a wandering domain if there is sequence $(f_i)_{i \in \mathbb{N}}$ of elements of S such that $U_{f_i} \neq U_{f_j}$ for $i \neq j$.

It is noted that for any rational function f, we always have $U_f = U$, and hence U_S is non-empty for a rational semigroup S. However, if f is transcendental, it is possible that $U_f \neq U$. Therefore, S_U may be empty for a transcendental semigroup S. Bergweiler and Rohde [22] proved that $U_f - U$ contains at most one point which is an asymptotic value of f if f is an entire function.

Lemma 6.1.2 ([115, Lemma 2.2])**.** *Let S be a holomorphic semigroup. Then the stabilizer S_U (if it is non-empty) is a subsemigroup of S and $F(S) \subset F(S_U),\ J(S) \supset J(S_U)$.*

Proof. Let $f, g \in S_U$. Then by Definition 6.1.2, $U_f = U$ and $U_g = U$ where U_f and U_g are components of the Fatou set containing $f(U)$ and $g(U)$ respectively. Then $f(U) \subseteq U_f = U$ and $g(U) \subseteq U_g = U \Longrightarrow (f \circ g)(U) = f(g(U)) \subseteq f(U_g) = f(U) \subseteq U_f = U$ Since $(f \circ g)(U) \subseteq U_{f \circ g}$, so either $U_{f \circ g} \subseteq U$ or $U \subseteq U_{f \circ g}$. The only possibility in this case is $U_{f \circ g} = U$. Hence $f \circ g \in S_U$, which proves that S_U

is a subsemigroup of S. The proofs of $F(S) \subset F(S_U),\ J(S) \supset J(S_U)$ follow from Lemma 6.1.1. □

There may be a connection between having no wandering domains and the stable basins of cofinite index. We have established the connection in the following theorem for a general holomorphic semigroup S.

Theorem 6.1.2 ([115, Theorem 2.2]). *Let S be a holomorphic semigroup with no wandering domains. Let U be any component of Fatou set. Then the forward orbit $\{U_f : f \in S\}$ of U under S contains a stabilizer of U of cofinite index.*

Proof. If S is a rational semigroup, see, for instance, the proof of [46, Theorem 6.1]. If S is a transcendental semigroup, we sketch our proof in the following way. We are given that U is a non-wandering component of the Fatou set $F(S)$. So U has a finite forward orbit $U_1, U_2, \dots, U_n$ (say) with $U_1 = U$.
Case (i): If for every $i = 1, 2, \dots n$, there is $f_i \in S$ such that $f_i(U_i) \subseteq U_1$, then by Lemma 6.1.2, the stabilizer $S_{U_1} = \{f \in S : U_{1f} = U_1\}$ is a subsemigroup of S. For any $f \in S$ there is f_i for each $i = 1, 2, \dots, n$ such that $U_{1_{f_i \circ f}} = U_1$. This shows that $f_i \circ f \in S_{U_1}$. Therefore U_1 is a required stable basin such that the stabilizer S_{U_1} has a cofinite index in S.
Case (ii): If, for every $j = 2, \dots n$, there is $f_j \in S$ such that $f_j(U_j) \subseteq V$, where $V = U_j$ such that $j \geq 2$, then the number of components of forward orbits of V is strictly less than that of U. In this way, we can find a component $W = U_i$ for some $i \leq n$ whose forward orbit has fewest components. For every component W_g of the forward orbit of W, there is $f \in S$ such that $f(W_g) \subseteq W$. That is, $W_{g \circ f} = W$, and it follows that W is a required stable basin such that the stabilizer S_W has a cofinite index. □

We prove the following result which is an extension of [46, Theorem 2.4] to transcendental semigroup dynamics.

Theorem 6.1.3 ([115, Theorem 1.1]). *If a subsemigroup T has finite index or cofinite index in an abelian transcendental semigroup S, then $I(S) = I(T), J(S) = J(T)$ and $F(S) = F(T)$.*

We now prove a result analogous to Lemma 6.1.1 in the case of an escaping set of a transcendental semigroup.

Lemma 6.1.3 ([115, Lemma 3.1]). *For any subsemigroup T of a transcendental semigroup S, we have $I(S) \subset I(T)$.*

Proof. By Theorem 4.1.1, $I(S) \subset \cap_{f \in S} I(f)$ and $I(T) \subset \cap_{g \in T} I(g)$ for any subsemigroup T of S. Since $T \subset S$, the same theorem implies that $I(S) \subset I(g)$ for all

$g \in T$. Hence $I(S) \subset \cap_{g\in T} I(g)$. Now for any $z \in I(S)$, we have $z \in \cap_{g\in T} I(g)$ for all $g \in T$. This implies $z \in I(g)$ for all $g \in T$. By Definition 4.1.1, we have $g^n(z) \to \infty$ as $n \to \infty$ for all $g \in T$. This proves $z \in I(T)$ and hence $I(S) \subset I(T)$. □

Proof of Theorem 6.1.3. We prove $I(S) = I(T)$. The fact that $J(S) = J(T)$ follows from Lemma 4.1.4 (2). That $F(S) = F(T)$ is also obvious. By Lemma 6.1.3, we always have $I(S) \subset I(T)$ for any subsemigroup T of S. For proving this theorem, it is enough to show the reverse inclusion $I(T) \subset I(S)$.

Let a subsemigroup T of a semigroup S have finite index n. Then, by Definition 6.1.1, there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that

$$S = f_1 \circ T \cup f_2 \circ T \cup \ldots \cup f_n \circ T$$

Then, for any $g \in S$, there is $h \in T$ such that $g = f_i \circ h$. Choose a sequence $(g_j)_{j\in\mathbb{N}}$ in S. Then each g_j is of the form $g_j = f_i \circ h_j$, where $h_j \in T$, $1 \leq i \leq n$. Here, we may assume the same i for all j. Let $z \in I(T)$. Then by Lemma 4.1.2, every non-convergent sequence $(h_j)_{j\in\mathbb{N}}$ in T has a divergent subsequence $(h_{j_k})_{j_k\in\mathbb{N}}$ at the point z. That is, $h^n_{j_k}(z) \to \infty$ as $n \to \infty$ for all j_k. In this case, every sequence $(g_j)_{j\in\mathbb{N}}$ in S has a subsequence $(g_{j_k})_{k\in\mathbb{N}}$, where $g_{j_k} = f_i \circ h_{j_k}$ with $h^n_{j_k}(z) \to \infty$ as $n \to \infty$. Since S is an abelian transcendental semigroup, $g_{j_k} = f_i \circ h_{j_k} = h_{j_k} \circ f_i$. Thus, we may write $g^n_{j_k}(z) = h^n_{j_k}(f_i(z)) \to \infty$ as $n \to \infty$ This shows that $f_i(z) \in I(S)$. If f_i = identity for a particular i, we are done. If f_i is not identity, then it is an element of an abelian transcendental semigroup S, and in this case $I(S)$ is backward invariant by Theorem 4.2.3. So we must have $z \in I(S)$. Therefore, $I(T) \subset I(S)$.

Let a subsemigroup T of a semigroup S have cofinite index n. Then by Definition 6.1.1, there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that for every $f \in S$, there is $i \in \{1, 2, \ldots, n\}$ such that $f_i \circ f \in T$. Let us choose a sequence $(g_j)_{j\in\mathbb{N}}$ in S. Then for each j, there is a i with $1 \leq i \leq n$ such that $f_i \circ g_j = h_j \in T$. Let $z \in I(T)$. Then by Lemma 4.1.2, every non-convergent sequence $(h_j)_{j\in\mathbb{N}}$ in T has a divergent subsequence $(h_{j_k})_{j_k\in\mathbb{N}}$ at the point z. This follows that sequence $(f_i \circ g_j)$ has a divergent subsequence $(f_i \circ g_{j_k})$ (say) at z. Since S is abelian, we can write $(f_i \circ g_{j_k})(z) = (g_{j_k} \circ f_i)(z) = g_{j_k}(f_i(z)) = h_{j_k}(z)$ Now for any $z \in I(T)$, $h_{j_k} \in T$, we must have $h^n_{j_k}(z) = g^n_{j_k}(f_i(z)) \to \infty$ as$n \to \infty$. This implies that $f_i(z) \in I(S)$. If f_i = identity for a particular i, we are done. If f_i is not an identity, then it is an element of abelian transcendental semigroup S. So, as in the first part, we write $I(T) \subset I(S)$. □

6.2 Rees indexed subsemigroups

Let S be a holomorphic semigroup and $f \in S$. Then $S \circ f$ and $f \circ S$ are subsemigroups of S. It is noted that $S \circ f$ and $f \circ S$ may not be finitely generated even if the semigroup S is. For, if $S \circ f = \langle f_1, f_2, \dots, f_n \rangle$ where $f_i \in S$ for $i = 1, 2, \dots n$, then $f_i = g_i \circ f$, where $g_i \in S$. For any $g \in S$, we have $g^n \circ f \in S \circ f$ for all $n \geq 1$ but not every $g^n \circ f \in \langle f_1, f_2, \dots, f_n \rangle$. From this fact, we came to know that the notion of cofinite index fails to preserve the basic finiteness (finitely generated) condition of a subsemigroup. That is, if T is a subsemigroup of cofinite index in semigroup S, then S being finitely generated may not always imply that T is finitely generated. There is another notion of index which preserves the finiteness condition of a subsemigroup.

Definition 6.2.1 (**Rees index**). *Let S be a semigroup and T be a subsemigroup. The Rees index of T in S is defined as $|S - T| + 1$, where $|S - T|$ represents the cardinality of $S - T$. In this case, T is a large subsemigroup of S, and S is a small extension of T.*

The Rees index was first introduced by Jura [56] in the case where T is an ideal of the semigroup S. In such a case, the Rees index of T in S is the cardinality of factor semigroup S/T. From Definition 6.2.1, it is clear that the Rees index of T in S is the size of the complement $S - T$. For a subsemigroup to have finite Rees index in its parent semigroup is a fairly restrictive property, and it occurs naturally in semigroups (for instance, all ideals in the additive semigroup of positive integers are of finite Rees index). It is noted that Rees index does not generalize group index, and even the notion of finite Rees index does not generalize finite group index. That is, if G is an infinite group and H is a proper subgroup, the group index of H in G may be finite even though the Rees index is infinite. In fact, let G be an infinite group and H is a subgroup of G. Then H has finite Rees index in G if and only if $H = G$.

Next, we investigate how similar a semigroup S and its large subsemigroup T are. One basic similarity (proved first by Jura [56]) is the following result.

Proposition 6.2.1. *Let T be a large subsemigroup of a semigroup S. Then S is finitely generated if and only if T is finitely generated.*

Proof. See, for instance, [92, Theorem 1.1]. □

The abelian hypothesis can be deleted from Theorem 6.1.3 if we use the Rees index. So, we have the following generalization of Theorem 6.1.3.

Theorem 6.2.1 ([115, Theorem 2.3]). *Let T be a large subsemigroup of a finitely generated holomorphic semigroup S. Then $F(S) = F(T)$ and $J(S) = J(T)$.*

Proof. We prove $F(S) = F(T)$. The other equality follows by taking complements. By Lemma 6.1.1, it is clear that $F(S) \subset F(T)$. So, it is sufficient to prove that $F(T) \subset F(S)$. By Proposition 6.2.1, T is finitely generated. Let $X = \{f_1, f_2, \ldots, f_n\} \subset S$ be a generating set of T. Clearly, S is generated by the set $Y = X \cup (S - T)$ Every sequence (f_i) in $F(T)$ (where $f_i = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}$, and $i_n \in \{1, 2, \ldots, n\}$) has a convergent subsequence. Now each element g_m of a sequence (g_m) in S can be written as

$$g_m = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n} \circ h_{j_1} \circ h_{j_2} \circ \ldots \circ h_{j_k},$$

where $S - T = \{h_1, h_2, \ldots, h_k\} \subset S$ and $j_k \in \{1, 2, \ldots, k\}$. Since $S - T$ is finite, so a convergent sequence in $F(T)$ can be extended to a convergent sequence in $F(S)$. Thus, every sequence (g_m) in $F(S)$ has a convergent subsequence. Hence $F(T) \subset F(S)$. □

Theorem 6.2.2 ([115, Theorem 3.1]). *If a subsemigroup T of a finitely generated transcendental semigroup S has a finite Rees index, then $I(S) = I(T)$.*

Proof. If we prove $I(S) = I(T)$, then the equality $J(S) = J(T)$ will follow from Lemma 4.1.4 (2). The inclusion $I(S) \subset I(T)$ follows from Lemma 6.1.3. So we prove $I(T) \subset I(S)$.

By Theorem 6.2.1, T is finitely generated. Let $X = \{f_1, f_2, \ldots, f_n\} \subset S$ be a generating set of T. Clearly, S is generated by the set $Y = X \cup (S - T)$. By Lemma 4.1.2, every non-convergent sequence (f_i) in T (where $f_i = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}$, and $i_n \in \{1, 2, \ldots, n\}$) has a divergence subsequence (f_{n_k}) at each point of $I(T)$. Now each element g_m of the sequence (g_m) in S can be written as $g_m = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n} \circ h_{j_1} \circ h_{j_2} \circ \ldots \circ h_{j_k}$, where $S - T = \{h_1, h_2, \ldots, h_k\} \subset S$ is a finite set and $j_k \in \{1, 2, \ldots, k\}$. This shows that a divergent sequence in $I(T)$ can be extended to a divergent sequence in $I(S)$. So, every non-convergent sequence (g_m) in $I(S)$ has a divergent subsequence. Hence $I(T) \subset I(S)$. □

Chapter 7

SOME NON-CYCLIC HOLOMORPHIC SEMIGROUPS WITH NON-EMPTY FATOU AND ESCAPING SETS

It is known that for certain holomorphic semigroups, the Fatou sets and escaping sets might be empty. In this chapter, we obtain certain holomorphic semigroups of special interest, whose Fatou and escaping sets are non-empty.

7.1 (Partial) fundamental set of a holomorphic semigroup

From Theorem 4.1.1 (1) and (3), we can say that Fatou set and escaping set of holomorphic semigroup may be empty. The result [75, Theorem 5.1] is one of the case of non-empty Fatou set and that of [108, Theorem 3.3] is a case of the non-empty escaping set of a transcendental semigroup. We obtain another case of non-empty Fatou and escaping sets on the basis of the following definitions.

Definition 7.1.1 (**Discontinuous semigroup**). *A semigroup S is said to be discontinuous at a point $z \in \mathbb{C}$ if there is a neighborhood U of z such that $f(U) \cap U = \emptyset$ for all $f \in S$ or equivalently, translates of U by distinct elements of S (S-translates) are disjoint. The neighborhood U of z is also called a nice neighborhood of z.*

Remark 7.1.1. *Given a holomorphic semigroup S, there are two natural subsets associated with S.*

1. *The regular set $R(S)$ that consists of points $z \in \mathbb{C}$ at which S is discontinuous.*

2. *The limit set $L(S)$ that consists of points $z \in \mathbb{C}$ for which there is a point z_0, and a sequence (f_n) of distinct elements of S such that $f_n(z_0) \to z$ as $n \to \infty$.*

A set $X \subset \mathbb{C}$ is S-invariant or invariant under S if $f(X) = X$ for all $f \in S$. It is clear that both of the sets $R(S)$ and $L(S)$ are S-invariant. If U is a nice neighborhood, then $U \subset R(S)$. Thus $R(S)$ is an open set, whereas the set $L(S)$ a closed set, and $R(S) \cap L(S) = \emptyset$.

Definition 7.1.2 (Partial fundamental set and fundamental set). *A set U is called a partial fundamental set for the semigroup S if*

1. *$U \neq \emptyset$,*

2. *$U \subset R(S)$,*

3. *$f(U) \cap U = \emptyset$ for all $f \in S$.*

If in addition to (1), (2) and (3), U satisfies the property

4. *$\bigcup_{f \in S} f(U) = R(S)$,*

then U is called a fundamental set for S.

We say that $x, y \in \mathbb{C}$ are S-equivalent if there is an $f \in S$ such that $f(x) = y$. Condition (3) asserts that no two points of U are S-equivalent under semigroup S, and condition (4) asserts that every point of $R(S)$ is equivalent to some point of U. It is noted that if we replace (3) by $f^{-1}(U) \cap U = \emptyset$ for all $f \in S$, we say U is a backward partial fundamental set for S; if, in addition, U satisfies $\bigcup_{f \in S} f^{-1}(U) = R(S)$, then we say U is a backward fundamental set. Similar to the results of Hinkkanen and Martin [46, Lemma 2.2] in the case of a rational semigroup, we have proved the following in the case of transcendental semigroup S.

We prove the following assertion that shows that a partial fundamental set is in the Fatou set $F(S)$ and that a fundamental set is in the escaping set $I(S)$.

Theorem 7.1.1 ([115, Theorem 1.2])**.** *Let S be a holomorphic semigroup and U a partial fundamental set for S. Then $U \subset F(S)$. If, in addition, S is a transcendental semigroup and U is a fundamental set, then $U \subset I(S)$.*

Proof. Let S be a holomorphic semigroup. The set U is a non-empty open set, and $f(U) \cap U = \emptyset$ for all $f \in S$ by Definition 7.1.1. The statement $f(U) \cap U = \emptyset$ for all $f \in S$ implies that S omits U on U. U is an open set, so it contains more than two points. Then by Montel's theorem, S is normal on U. Therefore, $U \subset F(S)$.

Let S be a transcendental semigroup. To prove $U \subset I(S)$, we have to show that $f^n(z) \to \infty$ as $n \to \infty$ for all $f \in S$, and for all $z \in U$. The condition $f(U) \cap U = \emptyset$ for all $f \in S$ implies that $f^n(U) \cap U = \emptyset$, and $f \in S$ implies $f^n \in S$. Also, U is a fundamental set, so by Definition 7.1.2 (4), we have $\bigcup_{f \in S} f(U) = R(S)$. By Remark 7.1.1(2), there are no points in U which appear as the limit points under distinct $(f_m)_{m \in \mathbb{N}}$ in S. That is, (f_m) has a divergent subsequence (f_{m_k}) at each point of U. Thus, by Theorem 4.1.2 for any $z \in U$, $f^n(z) \to \infty$ as $n \to \infty$ for any $f \in (f_m)$. This shows that $U \subseteq I(S)$. □

Finally, we generalize Theorem 7.1.1 in the following form. We give a short sketch of the proof. For a more detailed proof, we refer to [76, Theorem 2.1].

Theorem 7.1.2 ([115, Theorem 4.1])**.** *Let U_1 and U_2 be two (partial) fundamental sets for transcendental semigroups S_1 and S_2 respectively. Suppose furthermore that $\mathbb{C} \setminus U_1 \subset U_2$ and $\mathbb{C} \setminus U_2 \subset U_1$. Then the semigroup $S = \langle S_1, S_2 \rangle$ is discontinuous, and $U = U_1 \cap U_2$ is a (partial) fundamental set for the semigroup S.*

Sketch of the proof. Let U_1, U_2 and S_1, S_2 be as given in the statement of the theorem. It is clear from Theorem 7.1.1 that $F(S_1) \neq \emptyset$, $F(S_2) \neq \emptyset$; also $I(S_1) \neq \emptyset$ and $I(S_2) \neq \emptyset$ if U_1 and U_2 are fundamental sets of S_1 and S_2 respectively. Note that $U \neq \emptyset$ by the assumption. Clearly, $f(U) \cap U = \emptyset$ for every $f \in S$. This proves S is discontinuous and that U is a (partial) fundamental set for S. □

It is noted that Theorems 7.1.1 and 7.1.2 hold if we have given (partial) backward fundamental set in the statements.

7.2 Carleman set and a non-empty Fatou set

In this section, we prove that there exist three transcendental entire functions that can have infinite number of domains which lie in the wandering (pre-periodic, or periodic) components of each of these functions and their compositions. This result is a generalization of the result of Kumar et al. [59]. By using this result, our aim is to prove the following assertion.

Theorem 7.2.1 ([117, Theorem 1]). *There is a non trivial transcendental semigroup S such that the Fatou set $F(S)$ has at least a simply connected component.*

It is noted that if the semigroup S is cyclic, then by [20, Theorem 1], the Fatou set $F(S)$ has both a simply and a multiply connected wandering domains. However, in the case of non-cyclic transcendental semigroups, the proof is not so easy. The reason behind is that the dynamics of individual transcendental entire functions differ largely from the dynamics of their compositions. To workout in this direction, we need a notion in approximation theory of entire functions. In our case, we can use the notion of Carleman set from which we obtain approximation of any holomorphic function by entire functions.

Definition 7.2.1 (Carleman Set). *Let F be a closed proper subset of $\mathbb{C}$ and $C(F) = \{f : F \to \mathbb{C} : f$ is continuous on F and analytic in the interior Int.F of $F\}$. Then f is called a Carleman set (for $\mathbb{C}$) if for any $g \in C(F)$, and any positive continuous function ϵ on F, there exists entire function h such that $|g(z) - h(z)| < \epsilon$ for all $z \in F$.*

The following important characterization of Carleman set was proved by Nersesjan in 1971 but we cite this from [39, Theorem 4, Page 157].

Proposition 7.2.1. *Let F be proper subset of $\mathbb{C}$. Then F is a Carleman set for $\mathbb{C}$ if and only if F satisfies:*

1. *$\mathbb{C}_\infty - F$ is connected;*
2. *$\mathbb{C}_\infty - F$ is locally connected at ∞;*
3. *for every compact subset K of $\mathbb{C}$, there is a neighborhood V of ∞ in $\mathbb{C}_\infty$ such that no component of Int.F intersects both K and V.*

It is noted that the space $\mathbb{C}_\infty - F$ is connected if and only if each component Z of open set $\mathbb{C} - F$ is unbounded. This fact together with Proposition 7.2.1 can be a nice tool for checking whether a set is a Carleman set for $\mathbb{C}$. The sets given in the following examples are Carlemen sets for $\mathbb{C}$.

Example 7.2.1. *The set $E = \{z \in \mathbb{C} : |z| = 1, \Re z > 0\} \cup \{z = x : x > 1\} \cup \bigcup_{n=3}^{\infty} \{z = re^{i\theta} : r > 1, \theta = \pi/n\}$ is a Carleman set by Proposition 7.2.1. This set is shown in figure 7.1.*

Example 7.2.2. *The set $E = G_0 \cup \bigcup_{k=1}^{\infty} (G_k \cup B_K \cup L_k \cup M_k)$, where*

$$G_0 = \{z \in \mathbb{C} : |z - 2| \leq 1\};$$

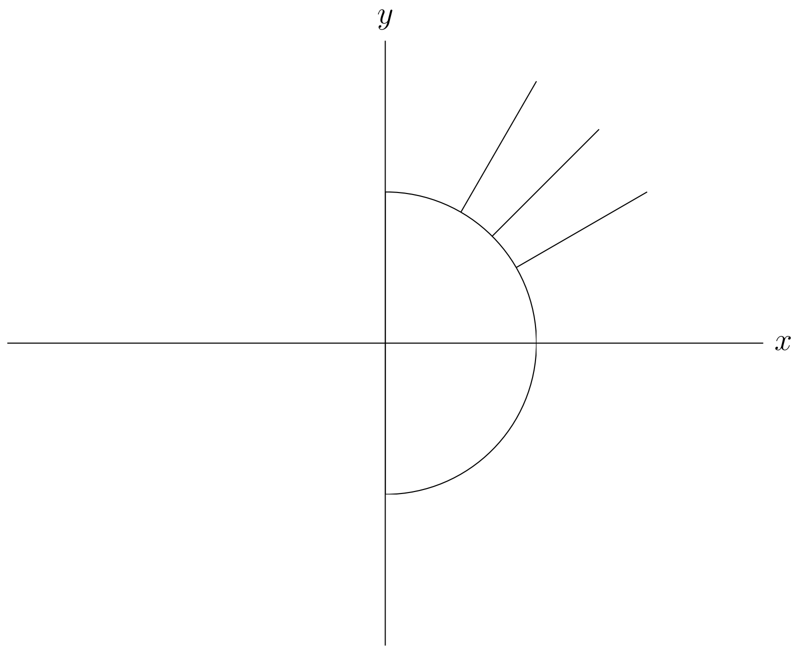

Figure 7.1: Carleman set E of Example 7.2.1

$$\begin{aligned} G_k &= \{z \in \mathbb{C} : |z-(4k+2)| \le 1\} \cup \{z \in \mathbb{C} : \Re(z) = 4k+2,\ \Im(z) \ge 1\} \\ &\cup \{z \in \mathbb{C} : \Re(z) = 4k+2,\ \Im(z) \le -1\},\ \ (k = 1,2,3,4,\ldots); \end{aligned}$$

$$\begin{aligned} B_k &= \{z \in \mathbb{C} : |z+(4k+2)| \le 1\} \cup \{z \in \mathbb{C} : \Re(z) = -(4k+2),\ \Im(z) \ge 1\} \cup \\ &\{z \in \mathbb{C} : \Re(z) = -(4k+2),\ \Im(z) \le -1\},\ \ (k = 1,2,3,4,\ldots); \end{aligned}$$

$$L_k = \{z \in \mathbb{C} : \Re(z) = 4k\},\ \ (k = 1,2,3,4,\ldots);$$

and

$$M_k = \{z \in \mathbb{C} : \Re(z) = -4k\},\ \ (k = 1,2,3,4,\ldots)$$

is a Carleman set by Proposition 7.2.1. This set is shown in figure 7.2.

From the help of the Carleman set of Example 7.2.2, Singh [96, Theorem 2] proved the following assertion.

Proposition 7.2.2. *There exist two transcendental entire functions f and g and a domain U such that U lies in the wandering component of the $F(f)$, $F(g)$, $F(f \circ g)$ and $F(g \circ f)$.*

In fact, Singh [96] also proved other results regarding the dynamics of two individual functions and their compositions (see for instance [96, Theorems 1, 3 and 4]) which

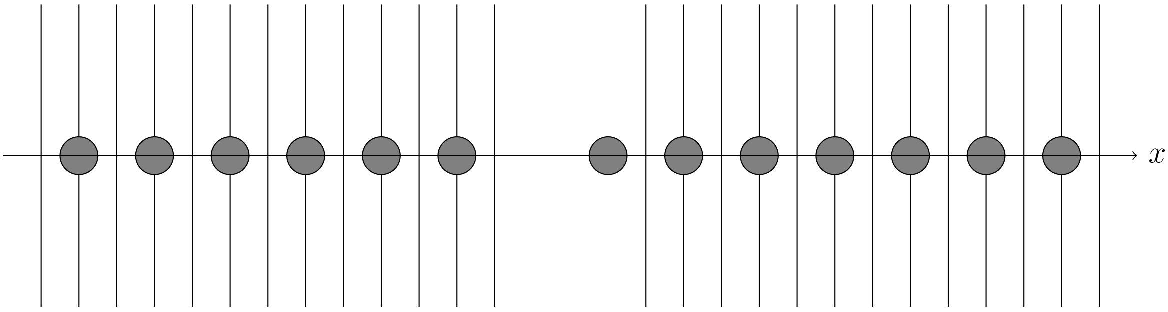

Figure 7.2: Carleman set E of Example 7.2.2

are also stricly based on the Carleman set of Example 7.2.2. Tomar [124] extended Proposition 7.2.2) to the following result.

Proposition 7.2.3. *There exist two transcendental entire functions f and g, and infinitely many domains in the angular region which lies in the wandering component of the $F(f)$, $F(g), F(f \circ g)$ and $F(g \circ f)$.*

It is noted that Singh [95, Theorems 3.2.1 - 3.2.6] studied different components of the Fatou set of a transcendental entire function in an angular region by using approximation theory of entire functions, in particular, by the help of Carleman set.

Kumar et al. [59, Theorem 2.1 to Theorem 2.15] extended these results, and for our purpose, we cite the following two assertions.

Proposition 7.2.4. *There exist two different transcendental entire functions f and g, and infinitely many domains which lie in different wandering component of the $F(f),\ F(g)$, $F(f \circ g)$ and $F(g \circ f)$.*

Proposition 7.2.5. *There exist two different transcendental entire functions f and g, and infinitely many domains which lie in different pre-periodic component of the $F(f),\ F(g)$, $F(f \circ g)$ and $F(g \circ f)$.*

Our particular interest of this study is that whether there are more than two transcendental entire functions that can have similarity between the dynamics of their compositions and dynamics of each of these individual functions. We investigate three transcendental entire functions such that each of individual function as well as their every composition consists of infinite number of domains which lie in the wandering (pre-periodic or periodic) component of the Fatou set. That is, we extend Propositions 7.2.4 and 7.2.5 to the following results:

Lemma 7.2.1 ([116, Theorem 1.1])**.** *There exist three different transcendental entire functions f, g and h, and infinitely many domains which lie in different wandering* *components of* $F(f), F(g), F(h), F(f\circ g), F(g\circ f), F(f\circ h), F(g\circ h), F(h\circ f), F(h\circ g), F(f\circ g\circ h), F(f\circ h\circ g), F(g\circ f\circ h), F(g\circ h\circ f), F(h\circ f\circ g)$ *and* $F(h\circ g\circ f)$.

Proof. Let

$$E = G_0 \cup \big(\bigcup_{k=1}^{\infty}(G_k \cup B_K \cup L_k \cup M_k)\big).$$

where G_0, G_k, B_k, L_k and M_k are sets as defined in Example 7.2.2. Then E is a Carleman set for $\mathbb{C}$. By the continuity of exponential function, for given $\epsilon > 0$, there exists $\delta > 0$, may depend on a given point w_0 such that

$$|w - w_0| < \delta \Longrightarrow |e^w - e^{w_0}| < \epsilon.$$

Let us choose $\epsilon = 1/2$, then there exist sufficiently small $\delta_k > 0,\ \delta_k^{'} > 0$ and $\delta_k^{''} > 0$ such that

$$|w - (\pi i + \log(4k+6))| < \delta_k \Longrightarrow |e^w + (4k+6)| < 1/2,\ (k = 1, 2, 3, \ldots);$$

$$|w - \log(4k-2)| < \delta_k^{'} \Longrightarrow |e^w - (4k+6)| < 1/2,\ (k = 1, 2, 3, \ldots);$$

and

$$|w - \log(4k-6)| < \delta_k^{''} \Longrightarrow |e^w - (4k-6)| < 1/2,\ (k = 3, 4, 5, \ldots).$$

In particular, let us choose sufficiently small $\delta_0 > 0, \lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$|w - \log 2| < \delta_0 \Longrightarrow |e^w - 2| < 1/2;$$

$$|w - (\pi i + \log 6)| < \lambda_1 \Longrightarrow |e^w + 6| < 1/2;$$

and

$$|w - (\pi i + \log 10)| < \lambda_2 \Longrightarrow |e^w + 10| < 1/2.$$

Next, let us define the following functions:

$$\alpha(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big); \\ \pi i + \log 6, & \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\ \pi i + \log(4k+6), & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{cases}$$

$$\beta(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \log(4k-2), & \forall z \in G_k,\ k = 2,3,4,\dots; \\ \pi i + \log 6, & \forall z \in G_1; \\ \pi i + \log(4k+6), & \forall z \in B_k,\ k = 1,2,3,\dots; \end{cases}$$

$$\gamma(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \log(4k-6), & \forall z \in G_k,\ k = 3,4,5\dots; \\ \pi i + \log 6, & \forall z \in G_2; \\ \pi i + \log 10, & \forall z \in G_1; \\ \pi i + \log(4k+6), & \forall z \in B_k,\ k = 1,2,3,\dots. \end{cases}$$

Let us define the following functions:

$$\epsilon_1(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \lambda_1, & \forall z \in G_k,\ k = 1,2,3,\dots; \\ \delta_k, & \forall z \in B_k,\ k = 1,2,3,\dots; \end{cases}$$

$$\epsilon_2(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \delta_k', & \forall z \in G_k,\ k = 2,3,4\dots; \\ \lambda_1, & \forall z \in G_1; \\ \delta_k, & \forall z \in B_k,\ k = 1,2,3,\dots; \end{cases}$$

and

$$\epsilon_3(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \delta_k'', & \forall z \in G_k,\ k = 3,4,5\dots; \\ \lambda_1, & \forall z \in G_2; \\ \lambda_2, & \forall z \in G_1; \\ \delta_k, & \forall z \in B_k,\ k = 1,2,3,\dots. \end{cases}$$

Clearly, the functions $\alpha(z),\ \beta(z)$ and $\gamma(z)$ are piecewise constant functions, so they are continuous on the set E and holomorphic in $Int.E.$ Also, since E is a Carleman set, so there exist entire functions $f_1(z),\ g_1(z)$ and $h_1(z)$ such that

$\forall z \in E,\ \ |f_1(z)-\alpha(z)| \le \epsilon_1(z),\ |g_1(z)-\beta(z)| \le \epsilon_2(z)$ and $|h_1(z)-\gamma(z)| \le \epsilon_3(z)$.

Consequently, we get transcendental entire functions $f(z)=e^{f_1(z)},\ g(z)=e^{g_1(z)}$ and $h(z)=e^{h_1(z)}$ which respectively satisfy the following:

$$\begin{aligned} &|f(z)-2| < 1/2, && \forall z \in G_0 \cup \left(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ &|f(z)+6| < 1/2, && \forall z \in G_k,\ k=1,2,3\ldots; \\ &|f(z)+(4k+6)| < 1/2, && \forall z \in B_k,\ k=1,2,3,\ldots; \end{aligned} \tag{7.2.1}$$

$$\begin{aligned} &|g(z)-2| < 1/2, && \forall z \in G_0 \cup \left(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ &|g(z)-(4k-2)| < 1/2, && \forall z \in G_k,\ k=2,3,4,\ldots; \\ &|g(z)+6| < 1/2, && \forall z \in G_1; \\ &|g(z)+(4k+6)| < 1/2, && \forall z \in B_k,\ k=1,2,3,\ldots; \end{aligned} \tag{7.2.2}$$

and

$$\begin{aligned} &|h(z)-2| < 1/2, && \forall z \in G_0 \cup \textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k); \\ &|h(z)-(4k-6)| < 1/2, && \forall z \in G_k, k=3,4,5\ldots; \\ &|h(z)+6| < 1/2, && \forall z \in G_2; \\ &|h(z)+10| < 1/2, && \forall z \in G_1; \\ &|h(z)+(4k+6)| < 1/2, && \forall z \in B_k, k=1,2,3\ldots. \end{aligned} \tag{7.2.3}$$

From (7.2.1), (7.2.2), and (7.2.3), we can say that each of the functions f, g and h maps $G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)$ into smaller disk $|z-2| < 1/2$ contained in G_0 and each of these function is a contracting mapping. Therefore, $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$ contains a fixed points z_1, z_2 and z_3 (say) such that

$$f^n(G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)) \longrightarrow z_1 \text{ as } n \longrightarrow \infty,$$

$$g^n(G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)) \longrightarrow z_2 \text{ as } n \longrightarrow \infty,$$

$$h^n(G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)) \longrightarrow z_3 \text{ as } n \longrightarrow \infty.$$

The fixed points z_1, z_2 and z_3 are respectively the attracting fixed points for the functions f, g and h, so $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$ lies in the attracting cycle, and hence $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$ is a subset of each of the Fatou set $F(f), F(g)$ and $F(h)$. $J(f) \neq \mathbb{C}$, $J(g) \neq \mathbb{C}$ and $J(h) \neq \mathbb{C}$, and so Julia set of each of the function f, g and h does not contain interior points, and hence Fatou set of each of these function contains all interior points. Fatou set of each of the function f, g and h contains Carleman set E.

Also, from (7.2.1), we can say that function f maps each G_k into smaller disk contained in B_1, and each B_k into smaller disk contained in B_{k+1}. In fact, G_k and B_k are contained in the wandering components of Fatou set $F(f)$. Again from (7.2.2), we can say that function g maps each of the domains G_k into the smaller disk contained in G_{k-1}, $(k = 2, 3, 4, \ldots)$, G_1 into smaller disk contained in B_1, and B_k, $(k = 1, 2, 3, \ldots)$ into the smaller disks contained in B_{k+1}. In fact, G_k and B_k are contained in the wandering components of the Fatou set $F(g)$. Likewise, from (7.2.3), we can say that domains G_k and B_k, $(k = 1, 2, 3 \ldots)$ are contained in the wandering components of $F(h)$.

Next, we examine the dynamical behavior of compositions of the functions f, g and h. The composite of any two and all of three of these functions satisfy the following.

Dynamical behavior of $f \circ g$:

$$\begin{aligned}
&|(f \circ g)(z) - 2| < 1/2, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big);\\
&|(f \circ g)(z) + 6| < 1/2, && \forall z \in G_k,\ k = 2, 3, 4 \ldots;\\
&|(f \circ g)(z) + 10| < 1/2, && \forall z \in G_1;\\
&|(f \circ g)(z) - (4k + 10)| < 1/2, && \forall z \in B_k,\ k = 1, 2, 3, \ldots.
\end{aligned} \tag{7.2.4}$$

The composition rule (7.2.4) shows that the domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ belong to $F(f \circ g)$, and in fact, each G_k and B_k is contained in the wandering components of $F(f \circ g)$.

Dynamical behavior of $g \circ f$:

$$\begin{aligned}
&|(g \circ f)(z) - 2| < 1/2, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big);\\
&|(g \circ f)(z) + 10| < 1/2, && \forall z \in G_k,\ k = 1, 2, 3, \ldots;\\
&|(g \circ f)(z) + (4k + 10)| < 1/2, && \forall z \in B_k,\ k = 1, 2, 3, \ldots.
\end{aligned} \tag{7.2.5}$$

From the composition rule (7.2.5), we can say that the domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ belong to $F(g \circ f)$, and in fact, each G_k and B_k belongs to the wandering component of $F(g \circ f)$.

Dynamical behavior of $f \circ h$:

$$\begin{aligned}
&|(f \circ h)(z) - 2| < 1/2, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big);\\
&|(f \circ h)(z) + 14| < 1/2, && \forall z \in G_1;\\
&|(f \circ h)(z) + 10| < 1/2, && \forall z \in G_2;\\
&|(f \circ h)(z) + 6| < 1/2, && \forall z \in G_k,\ k = 4, 5, 6, \ldots;\\
&|(f \circ h)(z) + (4k + 10)| < 1/2, && \forall z \in B_k,\ k = 1, 2, 3, \ldots.
\end{aligned} \tag{7.2.6}$$

As defined in above composition rule (7.2.6), the domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and $B_k,\ (k = 1, 2, 3, \ldots)$ belong to $F(f \circ h)$, and in fact, each G_k and B_k for all $k = 1, 2, 3, \ldots$ belongs to the wandering components of $F(f \circ h)$.

Dynamical behavior of $h \circ f$:

$$\begin{aligned}
&|(h \circ f)(z) - 2| < 1/2, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big);\\
&|(h \circ f)(z) + 10| < 1/2, && \forall z \in G_k,\ k = 1, 2, 3, \ldots;\\
&|(h \circ f)(z) + (4k + 10)| < 1/2, && \forall z \in B_k,\ k = 1, 2, 3, \ldots.
\end{aligned} \tag{7.2.7}$$

From the composition rule (7.2.7), we can say that the domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and B_k for all $k = 1, 2, 3, \ldots$ belong to $F(h \circ f)$ and , in fact, each G_k and B_k for all $k = 1, 2, 3, \ldots$ is contained in the wandering components of $F(h \circ f)$.

Dynamical behavior of $g \circ h$:

$$\begin{aligned}
&|(g \circ h)(z) - 2| < 1/2, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big);\\
&|(g \circ h)(z) + 14| < 1/2, && \forall z \in G_1;\\
&|(g \circ h)(z) + 10| < 1/2, && \forall z \in G_2;\\
&|(g \circ h)(z) + 6| < 1/2, && \forall z \in G_k;\ k = 3, 4, 5 \ldots;\\
&|(g \circ h)(z) + (4k + 10)| < 1/2, && \forall z \in B_k,\ k = 1, 2, 3, \ldots.
\end{aligned} \tag{7.2.8}$$

As defined in the above composition rule (7.2.8), the domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and $B_k,\ (k = 1, 2, 3, \ldots)$ belong to $F(g \circ h)$, and in fact, each G_k and B_k for all $k = 1, 2, 3, \ldots$ is contained in wandering components of $F(g \circ h)$.

Dynamical behavior of $h \circ g$:

$$
\begin{aligned}
&|(h\circ g)(z)-2|<1/2, && \forall z\in G_0\cup\big(\textstyle\bigcup_{k=1}^{\infty}(L_k\cup M_k)\big);\\
&|(h\circ g)(z)+10|<1/2, && \forall z\in G_k, k=1,2;\\
&|(h\circ g)(z)+6|<1/2, && \forall z\in G_3,;\\
&|(h\circ g)(z)-(4k-10)|<1/2, && \forall z\in G_k,\ k=4,5,6,\dots;\\
&|(h\circ g)(z)+(4k+10)|<1/2, && \forall z\in B_k,\ k=1,2,3,\dots.
\end{aligned}
\tag{7.2.9}
$$

As defined in the above composition rule (7.2.9), the domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and $B_k,\ (k=1,2,3,\dots)$ belong to $F(h\circ g)$, and in fact, each G_k and B_k for all $k=1,2,3,\dots$ is contained in wandering components of $F(h\circ g)$.

Dynamical behavior of $f \circ g \circ h$:

$$
\begin{aligned}
&|(f\circ g\circ h)(z)-2|<1/2, && \forall z\in G_0\cup\big(\textstyle\bigcup_{k=1}^{\infty}(L_k\cup M_k)\big);\\
&|(f\circ g\circ h)(z)+18|<1/2, && \forall z\in G_1;\\
&|(f\circ g\circ h)(z)+14|<1/2, && \forall z\in G_2;\\
&|(f\circ g\circ h)(z)+10|<1/2, && \forall z\in G_3;\\
&|(f\circ g\circ h)(z)+6|<1/2, && \forall z\in G_k,\ k=4,5,6,\dots;\\
&|(f\circ g\circ h)(z)+(4k+10)|<1/2, && \forall z\in B_k,\ k=1,2,3,\dots.
\end{aligned}
\tag{7.2.10}
$$

The composition rule (7.2.10) assigned above tells us that domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and $B_k,\ (k=1,2,3,\dots)$ lie in $F(f\circ g\circ h)$, and in fact, each G_k and B_k for all $k=1,2,3,\dots$ is contained in the wandering component of $F(f\circ g\circ h)$.

Dynamical behavior of $f \circ h \circ g$:

$$
\begin{aligned}
&|(f\circ h\circ g)(z)-2|<1/2, && \forall z\in G_0\cup\big(\textstyle\bigcup_{k=1}^{\infty}(L_k\cup M_k)\big);\\
&|(f\circ h\circ g)(z)+14|<1/2, && \forall z\in G_k,\ \text{for } k=1,2;\\
&|(f\circ h\circ g)(z)+10|<1/2, && \forall z\in G_3,\ \text{for } k=1,2;\\
&|(f\circ h\circ g)(z)+6|<1/2, && \forall z\in G_k,\ k=4,5,6,\dots;\\
&|(f\circ h\circ g)(z)+(4k+10)|<1/2, && \forall z\in B_k,\ k=1,2,3,\dots.
\end{aligned}
\tag{7.2.11}
$$

The composition rule (7.2.11) assigned above tells us that domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and $B_k,\ (k=1,2,3,\dots)$ lie in $F(f\circ g\circ h)$, and in fact, each G_k and B_k for all $k = 1, 2, 3, \ldots$ is contained in the wandering component of $F(f \circ h \circ g)$.

Dynamical behavior of $g \circ f \circ h$:

$$\begin{aligned}
&|(g \circ f \circ h)(z) - 2| < 1/2, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big); \\
&|(g \circ f \circ h)(z) + 18| < 1/2, && \forall z \in G_1; \\
&|(g \circ f \circ h)(z) + 14| < 1/2, && \forall z \in G_2; \\
&|(g \circ f \circ h)(z) + 10| < 1/2, && \forall z \in G_k,\ k = 3, 4, 5, \ldots; \\
&|(g \circ f \circ h)(z) + (4k + 10)| < 1/2, && \forall z \in B_k,\ k = 1, 2, 3, \ldots.
\end{aligned} \tag{7.2.12}$$

The composition rule (7.2.12) assigned above tells us that domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and $B_k,\ (k = 1, 2, 3, \ldots)$ lie in $F(g \circ f \circ h)$, and in fact, each G_k and B_k for all $k = 1, 2, 3, \ldots$ is contained in the wandering component of $F(g \circ f \circ h)$.

Dynamical behavior of $g \circ h \circ f$:

$$\begin{aligned}
&|(g \circ h \circ f)(z) - 2| < 1/2, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big); \\
&|(g \circ h \circ f)(z) + 14| < 1/2, && \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\
&|(g \circ h \circ f)(z) + (4k + 10)| < 1/2, && \forall z \in B_k,\ k = 1, 2, 3, \ldots.
\end{aligned} \tag{7.2.13}$$

The composition rule (7.2.13) assigned above tells us that domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and $B_k,\ (k = 1, 2, 3, \ldots)$ lie in $F(g \circ h \circ f)$, and in fact, each G_k and B_k for all $k = 1, 2, 3, \ldots$ is contained in the wandering component of $F(g \circ h \circ f)$.

Dynamical behavior of $h \circ f \circ g$:

$$\begin{aligned}
&|(h \circ f \circ g)(z) - 2| < 1/2, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big); \\
&|(h \circ f \circ g)(z) + 14| < 1/2, && \forall z \in G_1; \\
&|(h \circ f \circ g)(z) + 10| < 1/2, && \forall z \in G_k,\ k = 2, 3, 4, \ldots; \\
&|(h \circ f \circ g)(z) + (4k + 10)| < 1/2, && \forall z \in B_k,\ k = 1, 2, 3, \ldots.
\end{aligned} \tag{7.2.14}$$

The composition rule (7.2.14) assigned above tells us that domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and $B_k,\ (k = 1, 2, 3, \ldots)$ lie in $F(h \circ f \circ g)$, and in fact, each G_k and B_k for all $k = 1, 2, 3, \ldots$ is contained in the wandering component of $F(h \circ f \circ g)$.

Dynamical behavior of $h \circ g \circ f$:

$$\begin{aligned} &|(h \circ g \circ f)(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ &|(h \circ g \circ f)(z) + 14| < 1/2, && \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\ &|(h \circ g \circ f)(z) + (4k+10)| < 1/2, && \forall z \in B_k,\ k = 1, 2, 3, \ldots. \end{aligned} \tag{7.2.15}$$

The composition rule (7.2.15) assigned above tells us that domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and $B_k,\ (k = 1, 2, 3, \ldots)$ lie in $F(h \circ g \circ f)$, and in fact, each G_k and B_k for all $k = 1, 2, 3, \ldots$ is contained in the wandering component of $F(h \circ g \circ f)$.

From all of the above discussion, we found that the domains G_k and B_k for all $k = 1, 2, 3, \ldots$ are contained in the wandering domains of the functions $f,\ g,\ h$, and their compositions. □

Lemma 7.2.2 ([119, Theorem 1])**.** *There exist three different transcendental entire functions f, g and h, and infinitely many domains which lie in different pre-periodic components of $F(f), F(g), F(h), F(f \circ g), F(g \circ f), F(f \circ h), F(g \circ h), F(h \circ f), F(h \circ g), F(f \circ g \circ h), F(f \circ h \circ g), F(g \circ f \circ h), F(g \circ h \circ f), F(h \circ f \circ g)$ and $F(h \circ g \circ f)$.*

Proof. Let E, ϵ, δ, and δ_0 be as defined as in Lemma 7.2.1. For each $k = 2, 3, \ldots$, choose δ_k and $\delta_k^{'} > 0$ such that

$$|w - (\pi i + \log(4k-2))| < \delta_k \Longrightarrow |e^w + (4k-2)| < 1/2,$$

and

$$|w - \log(4k-2)| < \delta_k^{'} \Longrightarrow |e^w - (4k-2)| < 1/2.$$

In particular, let us choose sufficiently small $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_1^{'} > 0$ and $\lambda_2^{'} > 0$ such that

$$|w - \log 2| < \delta_0 \Longrightarrow |e^w - 2| < 1/2,$$

$$|w - (\pi i + \log 6)| < \lambda_1 \Longrightarrow |e^w + 6| < 1/2,$$

$$|w - (\pi i + \log 10)| < \lambda_2 \Longrightarrow |e^w + 10| < 1/2,$$

$$|w - \log 6| < \lambda_1^{'} \Longrightarrow |e^w + 6| < 1/2,$$

and

$$|w - \log 10| < \lambda_2^{'} \Longrightarrow |e^w + 10| < 1/2.$$

Next, let us define the following functions:

$$\alpha(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \pi i + \log 6, & \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\ \pi i + \log 10, & \forall z \in B_1; \\ \pi i + \log(4k-2), & \forall z \in B_k,\ k = 2, 3, 4 \ldots; \end{cases}$$

$$\beta(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \pi i + \log 6, & \forall z \in G_1; \\ \pi i + \log 10, & \forall z \in B_1; \\ \pi i + \log(4k-2), & \forall z \in B_k,\ k = 2, 3, 4, \ldots; \\ \log(4k-2), & \forall z \in G_k,\ k = 2, 3, 4, \ldots; \end{cases}$$

$$\gamma(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \log 6, & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \\ \log 10, & \forall z \in G_1; \\ \log(4k-2), & \forall z \in G_k,\ k = 2, 3, 4 \ldots; \end{cases}$$

$$\epsilon_1(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(B_K \cup L_k)\right); \\ \lambda_1, & \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\ \lambda_2, & \forall z \in B_1; \\ \delta_k, & \forall z \in B_k,\ k = 2, 3, 4 \ldots; \end{cases}$$

$$\epsilon_2(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \lambda_1, & \forall z \in G_1; \\ \lambda_2, & \forall z \in B_1; \\ \delta_k, & \forall z \in B_k,\ k = 2, 3, 4, \ldots; \\ \delta_k^{'}, & \forall z \in G_k,\ k = 2, 3, 4, \ldots; \end{cases}$$

and

$$\epsilon_3(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \lambda_1^{'}, & \forall z \in B_k;\ k = 1, 2, 3, \ldots \\ \lambda_2^{'}, & \forall z \in G_1; \\ \delta_k^{'}, & \forall z \in G_k,\ k = 2, 3, 4, \ldots. \end{cases}$$

Clearly, the functions $\alpha(z),\ \beta(z)$ and $\gamma(z)$ are piecewise constant functions, so they are continuous on the set E, and analytic in int.E. Also, since E is a Carleman set, so there exist entire functions $f_1(z),\ g_1(z)$ and $h_1(z)$ such that

$$\forall z \in E,\ \ |f_1(z) - \alpha(z)| \leq \epsilon_1(z),\ |g_1(z) - \beta(z)| \leq \epsilon_2(z) \text{ and } |h_1(z) - \gamma(z)| \leq \epsilon_3(z).$$

Consequently, we get transcendental entire functions $f(z) = e^{f_1(z)},\ g(z) = e^{g_1(z)}$ and $h(z) = e^{h_1(z)}$ which respectively satisfy the following:

$$\begin{aligned} &|f(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ &|f(z) + 6| < 1/2, && \forall z \in G_k,\ k = 1, 2, 3 \ldots; \\ &|f(z) + 10| < 1/2, && \forall z \in B_1 \\ &|f(z) + (4k - 2)| < 1/2, && \forall z \in B_k,\ k = 2, 3, 4 \ldots; \end{aligned} \tag{7.2.16}$$

$$\begin{aligned} &|g(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ &|g(z) + 6| < 1/2, && \forall z \in G_1; \\ &|g(z) + 10| < 1/2, && \forall z \in B_1; \\ &|g(z) + (4k - 2)| < 1/2, && \forall z \in B_k,\ k = 2, 3, 4, \ldots; \\ &|g(z) - (4k - 2)| < 1/2, && \forall z \in G_k,\ k = 2, 3, 4 \ldots; \end{aligned} \tag{7.2.17}$$

and

$$\begin{aligned} &|h(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ &|h(z) - 6| < 1/2, && \forall z \in B_k, k = 1, 2, 3, \ldots; \\ &|h(z) - 10| < 1/2, && \forall z \in G_1; \\ &|h(z) - (4k - 2)| < 1/2, && \forall z \in G_k, k = 2, 3, 4, \ldots; \end{aligned} \tag{7.2.18}$$

As in Lemma 7.2.1, from (7.2.16), (7.2.17), and (7.2.18), we can say that Fatou set of each of the functions f, g and h contains the Carleman set E. Also, from (7.2.16), (7.2.17), and (7.2.18), it is easy to to say that G_k and B_k are contained in the pre-periodic components of $F(f)$, $F(g)$ and $F(h)$. However, from (7.2.16) and (7.2.17), we can say that domains B_1 and B_2 lie in the periodic component of the Fatou sets of both of the functions f and g but from (7.2.18), we can say that domains G_1 and G_2 lie in the periodic component of the Fatou set of function h.

As in Lemma 7.2.1, it is easy to check that domains G_k and B_k for all $k = 1, 2, 3, \ldots$ are contained in the pre-periodic domains of $F(f \circ g), F(g \circ f), F(f \circ h), F(g \circ h), F(h \circ f), F(h \circ g), F(f \circ g \circ h), F(f \circ h \circ g), F(g \circ f \circ h), F(g \circ h \circ f), F(h \circ f \circ g)$ and $F(h \circ g \circ f)$. □

Lemma 7.2.3 ([120, Theorem 1.1]). *There exist three different transcendental entire functions f, g and h, and infinitely many domains which lie in different periodic components of $F(f), F(g), F(h), F(f \circ g), F(g \circ f), F(f \circ h), F(g \circ h), F(h \circ f), F(h \circ g), F(f \circ g \circ h), F(f \circ h \circ g), F(g \circ f \circ h), F(g \circ h \circ f), F(h \circ f \circ g)$ and $F(h \circ g \circ f)$.*

Proof. Let E, ϵ, δ, δ_k , $\delta_k^{'}$, δ_0, λ_1 and λ_2 be as defined as in Lemmas 7.2.1 and 7.2.3. For each $k = 1, 2, \ldots$, choose $\xi_k > 0$, $\zeta_k > 0$ and $\eta_k > 0$ such that

$$|w - \log(4k + 6)| < \xi_k \Longrightarrow |e^w - (4k + 6)| < 1/2,$$

$$|w - \log(4k + 2)| < \zeta_k \Longrightarrow |e^w - (4k + 2)| < 1/2,$$

and

$$|w - (\pi i + \log(4k + 2))| < \eta_k \Longrightarrow |e^w + (4k + 2)| < 1/2.$$

Let us define the following functions:

$$\alpha(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \log(4k+2), & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \\ \pi i + \log(4k+2), & \forall z \in G_k,\ k = 1, 2, 3, \ldots; \end{cases}$$

$$\beta(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \log(4k+6), & \forall z \in B_k,\ k = 1, 3, 5, \ldots; \\ \log(4k-2), & \forall z \in B_k,\ k = 2, 4, 6, \ldots; \\ \pi i + \log(4k+6), & \forall z \in G_k,\ k = 1, 3, 5, \ldots; \\ \pi i + \log(4k-2), & \forall z \in G_k,\ k = 2, 4, 6, \ldots; \end{cases}$$

$$\gamma(z)=\begin{cases}\log 2, & \forall z\in G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right);\\ \log(4k+6), & \forall z\in G_k,\ k=1,2,3,\dots,n-1;\\ \log 6, & \forall z\in G_n;\\ \pi i+\log(4k+6), & \forall z\in B_k,\ k=1,2,3,\dots,n-1;\\ \pi i+\log 6, & \forall z\in B_n.\end{cases}$$

$$\epsilon_1(z)=\begin{cases}\delta_0, & \forall z\in G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right);\\ \zeta_k, & \forall z\in B_k,\ k=1,2,3,\dots;\\ \eta_k, & \forall z\in G_k,\ k=1,2,3,\dots;\end{cases}$$

$$\epsilon_2(z)=\begin{cases}\delta_0, & \forall z\in G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right);\\ \xi_k, & \forall z\in B_k,\ k=1,3,5\dots;\\ \delta_k', & \forall z\in B_k,\ k=2,4,6\dots;\\ \delta_k', & \forall z\in G_k,\ k=1,2,3,\dots;\\ \delta_k, & \forall z\in G_k,\ k=2,4,6\dots;\end{cases}$$

and

$$\epsilon_3(z)=\begin{cases}\delta_0, & \forall z\in G_0\cup\bigcup_{k=1}^{\infty}(B_K\cup L_k\cup M_k);\\ \delta_k, & \forall z\in G_k,\ k=1,2,3,\dots,n-1;\\ \lambda_1, & \forall z\in G_n;\\ \delta_k, & \forall z\in B_k,\ k=1,2,3,\dots,n-1;\\ \lambda_1, & \forall z\in B_n.\end{cases}$$

Clearly, the functions $\alpha(z),\ \beta(z)$ and $\gamma(z)$ are piece wise constant functions, so they are continuous on the set E, and analytic in Int.E. Also, since E is a Carleman set, so there exist an entire functions $f_1(z),\ g_1(z)$ and $h_1(z)$ such that

$$|f_1(z)-\alpha(z)|\le\epsilon_1(z),\ |g_1(z)-\beta(z)|\le\epsilon_2(z) \text{ and } |h_1(z)-\gamma(z)|\le\epsilon_3(z),\forall z\in E.$$

Consequently, we get transcendental entire functions $f(z)=e^{f_1(z)},\ g(z)=e^{g_1(z)}$ and

$h(z) = e^{h_1(z)}$ which respectively satisfy the following:

$$\begin{aligned}
&|f(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
&|f(z) - (4k+2)| < 1/2, && \forall z \in B_k,\ k = 1, 2, 3\ldots;\\
&|f(z) + (4k+2)| < 1/2, && \forall z \in G_k,\ k = 1, 2, 3, \ldots;
\end{aligned} \tag{7.2.19}$$

$$\begin{aligned}
&|g(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
&|g(z) - (4k+6)| < 1/2, && \forall z \in B_k,\ k = 1, 3, 5, \ldots;\\
&|g(z) - (4k-2)| < 1/2, && \forall z \in B_k,\ k = 2, 4, 6, \ldots;\\
&|g(z) + (4k+6)| < 1/2, && \forall z \in G_k,\ k = 1, 3, 5, \ldots;\\
&|g(z) + (4k-2)| < 1/2, && \forall z \in G_k,\ k = 2, 4, 6, \ldots;
\end{aligned} \tag{7.2.20}$$

and

$$\begin{aligned}
&|h(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
&|h(z) - (4k+6)| < 1/2, && \forall z \in G_k, k = 1, 2, 3, \ldots, n-1;\\
&|h(z) - 6| < 1/2, && \forall z \in G_n;\\
&|h(z) + (4k+6)| < 1/2, && \forall z \in B_k, k = 1, 2, 3, \ldots, n-1;\\
&|h(z) + 6| < 1/2, && \forall z \in B_n;\\
&|h(z) - (4k+2)| < 1/2, && \forall z \in B_k,\ k > n;\\
&|h(z) + (4k+2)| < 1/2, && \forall z \in G_k,\ k > n.
\end{aligned} \tag{7.2.21}$$

As in Lemma 7.2.1 and 7.2.3, from (7.2.19), (7.2.20), and (7.2.21), we can see that the Fatou set of each of the functions f, g and h contains the Caleman set E. Also, from (7.2.19), (7.2.20), we can say that G_k and B_k are periodic components of period 2 of the function f, and g. Likewise, as defined in (7.2.21) , domains G_k and $B_k, (k \leq n)$ are periodic components of period n under the function h and G_k and B_k are periodic component of period 2 for $k > n$. So, all these domains lie in periodic component of Fatou sets $F(f)$, $F(g)$ and $F(h)$.

Next, we examine the dynamical behavior of composites of any two and three of the functions f, g and h.

Dynamical behavior of $f \circ g$:

$$
\begin{aligned}
&|(f \circ g)(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
&|(f \circ g)(z) + (4k+6)| < 1/2, && \forall z \in B_k,\ k = 1,3,5,\ldots;\\
&|(f \circ g)(z) + (4k-2)| < 1/2, && \forall z \in B_k,\ k = 2,4,6,\ldots;\\
&|(f \circ g)(z) - (4k+6)| < 1/2, && \forall z \in G_k,\ k = 1,3,5,\ldots;\\
&|(f \circ g)(z) - (4k-2)| < 1/2, && \forall z \in G_k,\ k = 2,4,6,\ldots.
\end{aligned}
\tag{7.2.22}
$$

Dynamical behavior of $g \circ f$:

$$
\begin{aligned}
&|(g \circ f)(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
&|(g \circ f)(z) + (4k+6)| < 1/2, && \forall z \in B_k,\ k = 1,3,5,\ldots;\\
&|(g \circ f)(z) + (4k-2)| < 1/2, && \forall z \in B_k,\ k = 2,4,6,\ldots;\\
&|(g \circ f)(z) - (4k+6)| < 1/2, && \forall z \in G_k,\ k = 1,3,5,\ldots;\\
&|(g \circ f)(z) - (4k-2)| < 1/2, && \forall z \in G_k,\ k = 2,4,6,\ldots.
\end{aligned}
\tag{7.2.23}
$$

The composition rules (7.2.22) and (7.2.23) shows that the domains G_k and $B_k,\ (k = 1,2,3,\ldots)$ for each $k \in \mathbb{N}$ is a periodic domain of period 2 which belongs to the periodic components of $F(f \circ g)$.

Dynamical behavior of $f \circ h$:

$$
\begin{aligned}
&|(f \circ h)(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
&|(f \circ h)(z) - (4k+6)| < 1/2, && \forall z \in B_k,\ k = 1,2,3,\ldots,n-1;\\
&|(f \circ h)(z) - 6| < 1/2, && \forall z \in B_n;\\
&|(f \circ h)(z) + (4k+6)| < 1/2, && \forall z \in G_k,\ k = 1,2,3,\ldots,n-1;\\
&|(f \circ h)(z) + 6| < 1/2, && \forall z \in G_n;\\
&|(f \circ h)(z) + (4k+2)| < 1/2, && \forall z \in B_k,\ k > n;\\
&|(f \circ h)(z) - (4k+2)| < 1/2, && \forall z \in G_k, k > n.
\end{aligned}
\tag{7.2.24}
$$

The composition rule (7.2.24) tells us that each G_k and $B_k,\ (k > n)$ is a periodic domain of period 1, and each G_k and B_k for $k = 1,2,3,\ldots,n$ is a periodic component of period n for even n.

Dynamical behavior of $h \circ f$:

$$
\begin{aligned}
&|(h\circ f)(z)-2|<1/2, && \forall z\in G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right);\\
&|(h\circ f)(z)+(4k+6)|<1/2, && \forall z\in B_k,\ k=1,2,3,\dots,n-1;\\
&|(h\circ f)(z)+6|<1/2, && \forall z\in B_n;\\
&|(h\circ f)(z)-(4k+6)|<1/2, && \forall z\in G_k,\ k=1,2,3\dots,n-1;\\
&|(h\circ f)(z)-6|<1/2, && \forall z\in G_n;\\
&|(h\circ f)(z)+(4k+2)|<1/2, && \forall z\in B_k,\ k>n;\\
&|(h\circ f)(z)-(4k+2)|<1/2, && \forall z\in G_k, k>n.
\end{aligned}
\tag{7.2.25}
$$

From composition rule 7.2.25, we can say that each G_k and B_k, $(k>n)$ is a periodic component of period , and each G_k and B_k for $k=1,2,3,\dots,n$ is a periodic component of period n for odd n.

Dynamical behavior of $g\circ h$:

$$
\begin{aligned}
&|(g\circ h)(z)-2|<1/2, && \forall z\in G_0\cup\left(\bigcup_{k=1}^{\infty}(L_k\cup M_k)\right);\\
&|(g\circ h)(z)-(4k+2)|<1/2, && \forall z\in B_k,\ k=1,3,5\dots,n-1;\\
&|(g\circ h)(z)-(4k+10)|<1/2, && \forall z\in B_k,\ k=2,4,6,\dots,n-1;\\
&|(g\circ h)(z)-10|<1/2, && \forall z\in B_n;\\
&|(g\circ h)(z)+(4k+2)|<1/2, && \forall z\in G_k,\ k=1,3,5\dots,n-1;\\
&|(g\circ h)(z)+(4k+10)|<1/2, && \forall z\in G_k,\ k=2,4,6,\dots,n-1;\\
&|(g\circ h)(z)+10|<1/2, && \forall z\in G_n;\\
&|(g\circ h)(z)+(4k-2)|<1/2, && \forall z\in B_k,\ k>n \text{ and } k \text{ is even};\\
&|(g\circ h)(z)+(4k+6)|<1/2, && \forall z\in B_k,\ k>n \text{ and } k \text{ is odd};\\
&|(g\circ h)(z)-(4k-2)|<1/2, && \forall z\in G_k,\ k>n \text{ and } k \text{ is even};\\
&|(g\circ h)(z)-(4k+6)|<1/2, && \forall z\in G_k,\ k>n \text{ and } k \text{ is odd}.
\end{aligned}
\tag{7.2.26}
$$

From composition rule (7.2.26), it is easy to say that each G_k and B_k, $(k>n)$ is a periodic components of period 1, and each G_k and B_k for odd $k\leq n$ are periodic components of period 2.

Dynamical behavior of $h\circ g$:

$$
\begin{aligned}
&|(h\circ g)(z)-2|<1/2, && \forall z\in G_0\cup\big(\textstyle\bigcup_{k=1}^{\infty}(L_k\cup M_k)\big);\\
&|(h\circ g)(z)-(4k+10)|<1/2, && \forall z\in B_k,\ k=1,3,5\ldots,n-1;\\
&|(h\circ g)(z)-(4k+2)|<1/2, && \forall z\in B_k,\ k=2,4,6,\ldots,n-1;\\
&|(h\circ g)(z)+(4n+6)|<1/2, && \forall z\in B_n,\ \text{for odd n};\\
&|(h\circ g)(z)-(4n+2)|<1/2, && \forall z\in B_n,\ \text{for even n};\\
&|(h\circ g)(z)+(4k+10)|<1/2, && \forall z\in G_k,\ k=1,3,5\ldots,n-1;\\
&|(h\circ g)(z)+(4k+2)|<1/2, && \forall z\in G_k,\ k=2,4,6,\ldots,n-1;\\
&|(h\circ g)(z)-(4n+6)|<1/2, && \forall z\in G_n,\ \text{for odd n};\\
&|(h\circ g)(z)+(4n+2)|<1/2, && \forall z\in G_n,\ \text{for even n};\\
&|(g\circ h)(z)+(4k-2)|<1/2, && \forall z\in B_k,\ k>n \text{ and } k \text{ is even};\\
&|(h\circ g)(z)-6|<1/2, && \forall z\in B_k,\ k>n \text{ and } k-1=n;\\
&|(h\circ g)(z)+(4k+6)|<1/2, && \forall z\in B_k,\ k>n \text{ and } k \text{ is odd};\\
&|(h\circ g)(z)-(4k-2)|<1/2, && \forall z\in G_k,\ k>n \text{ and } k \text{ is even};\\
&|(h\circ g)(z)+6|<1/2, && \forall z\in G_k,\ k>n \text{ and } k-1=n;\\
&|(h\circ g)(z)-(4k+6)|<1/2, && \forall z\in G_k,\ k>n \text{ and } k \text{ is odd}.
\end{aligned}
\tag{7.2.27}
$$

From (7.2.27), we can say that each G_k and $B_k,\ (k>n)$ is a periodic component of period 1, and each G_k and B_k for even $k\le n$ is a periodic component of period 2.

Dynamical behavior of $f\circ g\circ h$:

$$
\begin{aligned}
&|(f\circ g\circ h)(z)-2|<1/2, && \forall z\in G_0\cup\big(\textstyle\bigcup_{k=1}^{\infty}(L_k\cup M_k)\big);\\
&|(f\circ g\circ h)(z)+(4k+2)|<1/2, && \forall z\in B_k,\ k=1,3,5,\ldots,n-1;\\
&|(f\circ g\circ h)(z)+(4k+10)|<1/2, && \forall z\in B_k,\ k=2,4,6,\ldots,n-1;\\
&|(f\circ g\circ h)(z)+10|<1/2, && \forall z\in B_n;\\
&|(f\circ g\circ h)(z)-(4k+2)|<1/2, && \forall z\in G_k,\ k=1,3,5\ldots,n-1;\\
&|(f\circ g\circ g)(z)-(4k+10)|<1/2, && \forall z\in G_k,\ k=2,4,6,\ldots,n-1;\\
&|(f\circ g\circ g)(z)-10|<1/2, && \forall z\in G_n;\\
&|(f\circ g\circ h)(z)-(4k+6)|<1/2, && \forall z\in B_k,\ k>n\ \text{ and k is odd};\\
&|(f\circ g\circ h)(z)-(4k-2)|<1/2, && \forall z\in B_k,\ k>n \text{ and k is even};
\end{aligned}
$$

$$|(f \circ g \circ h)(z) + (4k+6)| < 1/2, \qquad \forall z \in G_k,\ k > n \text{ and k is odd;}$$
$$|(f \circ g \circ h)(z) + (4k-2)| < 1/2, \qquad \forall z \in G_k,\ k > n \text{ and k is even.} \qquad (7.2.28)$$

The composition rule $(7.2.28)$ tells us that each G_k and $B_k,\ (k > n)$ is a periodic components of period 1, and each G_k and B_k for odd $k \leq n$ is a periodic component of period 1.

Dynamical behavior of $f \circ h \circ g$:

$$\begin{aligned}
&|(f \circ h \circ g)(z) - 2| < 1/2, && \forall z \in G_0 \cup \left(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
&|(f \circ h \circ g)(z) + (4k+10)| < 1/2, && \forall z \in B_k,\ k = 1,3,5,\ldots,n-1;\\
&|(f \circ h \circ g)(z) + (4k+2)| < 1/2, && \forall z \in B_k,\ k = 2,4,6,\ldots,n-1;\\
&|(f \circ h \circ g)(z) + (4n+6)| < 1/2, && \forall z \in B_n \text{ and n is odd;}\\
&|(f \circ h \circ g)(z) + (4n+2)| < 1/2, && \forall z \in B_n \text{ and n is even;}\\
&|(f \circ h \circ g)(z) - (4k+10)| < 1/2, && \forall z \in G_k,\ k = 1,3,5\ldots,n-1;\\
&|(f \circ h \circ g)(z) - (4k+2)| < 1/2, && \forall z \in G_k,\ k = 2,4,6,\ldots,n-1;\\
&|(f \circ h \circ g)(z) + (4n+6)| < 1/2, && \forall z \in G_n \text{ and n is odd;} \qquad (7.2.29)\\
&|(f \circ h \circ g)(z) - (4n+2)| < 1/2, && \forall z \in G_n \text{ and n is even;}\\
&|(f \circ h \circ g)(z) - (4k+6)| < 1/2, && \forall z \in B_k,\ k > n \text{ and k is odd;}\\
&|(f \circ h \circ g)(z) - (4k-2)| < 1/2, && \forall z \in B_k,\ k > n \text{ and k is even;}\\
&|(f \circ h \circ g)(z) + 6| < 1/2, && \forall z \in B_k,\ k > n \text{ k is even and k-1=n;}\\
&|(f \circ h \circ g)(z) + (4k+6)| < 1/2, && \forall z \in G_k,\ k > n \text{ and k is odd;}\\
&|(f \circ h \circ g)(z) + (4k-2)| < 1/2, && \forall z \in G_k,\ k > n \text{ and k is even;}\\
&|(f \circ h \circ g)(z) - 6| < 1/2, && \forall z \in G_k,\ k > n \text{ k is even and k-1=n.}
\end{aligned}$$

The composition rule $(7.2.29)$ tells us that each G_k and $B_k,\ (k > n)$ belong to the periodic components of $F(f \circ h \circ g)$ of period 1. Each G_k and B_k for even k are periodic components of period 1.

Dynamical behavior of $g \circ f \circ h$:

$$
\begin{aligned}
&|(g \circ f \circ h)(z) - 2| < 1/2, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big);\\
&|(g \circ f \circ h)(z) + (4k+2)| < 1/2, && \forall z \in B_k,\ k = 1, 3, 5, \ldots, n-1;\\
&|(g \circ f \circ h)(z) + (4k+10)| < 1/2, && \forall z \in B_k,\ k = 2, 4, 6, \ldots, n-1;\\
&|(g \circ f \circ h)(z) + 10| < 1/2, && \forall z \in B_n;\\
&|(g \circ f \circ h)(z) - (4k+2)| < 1/2, && \forall z \in G_k,\ k = 1, 3, 5 \ldots, n-1;\\
&|(g \circ f \circ h)(z) - (4k+10)| < 1/2, && \forall z \in G_k,\ k = 2, 4, 6, \ldots, n-1;\\
&|(g \circ f \circ h)(z) - 10| < 1/2, && \forall z \in G_n;\\
&|(g \circ f \circ h)(z) - (4k+6)| < 1/2, && \forall z \in B_k,\ k > n \text{ and k is odd};\\
&|(g \circ f \circ h)(z) - (4k-2)| < 1/2, && \forall z \in B_k,\ k > n \text{ and k is even};\\
&|(g \circ f \circ h)(z) + (4k+6)| < 1/2, && \forall z \in G_k,\ k > n \text{ and k is odd};\\
&|(g \circ f \circ h)(z) + (4k-2)| < 1/2, && \forall z \in G_k,\ k > n \text{ and k is even}. \qquad (7.2.30)
\end{aligned}
$$

The composition rule (7.2.30) tells us that each G_k and $B_k,\ (k > n)$ is a periodic components of period 1, and each G_k and B_k for odd $k \leq n$ is a periodic component of period 1.

Dynamical behavior of $g \circ h \circ f$:

$$
\begin{aligned}
&|(g \circ h \circ f)(z) - 2| < 1/2, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big);\\
&|(g \circ h \circ f)(z) + (4k+2)| < 1/2, && \forall z \in B_k,\ k = 1, 3, 5, \ldots, n-1;\\
&|(g \circ h \circ f)(z) + (4k+10)| < 1/2, && \forall z \in B_k,\ k = 2, 4, 6, \ldots, n-1;\\
&|(g \circ f \circ h)(z) + 10| < 1/2, && \forall z \in B_n;\\
&|(g \circ h \circ f)(z) - (4k+2)| < 1/2, && \forall z \in G_k,\ k = 1, 3, 5 \ldots, n-1;\\
&|(g \circ g \circ f)(z) - (4k+10)| < 1/2, && \forall z \in G_k,\ k = 2, 4, 6, \ldots, n-1;\\
&|(g \circ f \circ h)(z) - 10| < 1/2, && \forall z \in G_n;\\
&|(g \circ h \circ f)(z) - (4k+6)| < 1/2, && \forall z \in B_k,\ k > n \text{ and k is odd};\\
&|(g \circ h \circ f)(z) - (4k-2)| < 1/2, && \forall z \in B_k,\ k > n \text{ and k is even};\\
&|(g \circ h \circ f)(z) + (4k+6)| < 1/2, && \forall z \in G_k,\ k > n \text{ and k is odd};\\
&|(g \circ h \circ f)(z) + (4k-2)| < 1/2, && \forall z \in G_k,\ k > n \text{ and k is even}. \qquad (7.2.31)
\end{aligned}
$$

The composition rule (7.2.31) tells us that each G_k and $B_k,\ (k > n)$ is a periodic components of period 1, and each G_k and B_k for odd $k \leq n$ is a periodic component of period 1. Note that composition rules (7.2.30) and (7.2.31) show that dynamics of $g \circ f \circ h$ and $g \circ h \circ f$ coincide.

Dynamical behavior of $h \circ f \circ g$:

$$
\begin{array}{ll}
|(h \circ f \circ g)(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
|(h \circ f \circ g)(z) + (4k+10)| < 1/2, & \forall z \in B_k,\ k = 1,3,5,\dots,n-1;\\
|(h \circ f \circ g)(z) + (4k+2)| < 1/2, & \forall z \in B_k,\ k = 2,4,6,\dots,n-1;\\
|(h \circ f \circ g)(z) - (4n+6)| < 1/2, & \forall z \in B_n \text{ and n is odd};\\
|(h \circ f \circ g)(z) + (4n+2)| < 1/2, & \forall z \in B_n \text{ and n is even};\\
|(h \circ f \circ g)(z) - (4k+10)| < 1/2, & \forall z \in G_k,\ k = 1,3,5\dots,n-1;\\
|(h \circ g \circ g)(z) - (4k+2)| < 1/2, & \forall z \in G_k,\ k = 2,4,6,\dots,n-1;\\
|(h \circ f \circ g)(z) + (4n+6)| < 1/2, & \forall z \in G_n \text{ and n is odd};\\
|(h \circ f \circ g)(z) - (4n+2)| < 1/2, & \forall z \in G_n \text{ and n is even};\\
|(h \circ f \circ g)(z) - (4k+6)| < 1/2, & \forall z \in B_k,\ k > n \text{ and k is odd};\\
|(h \circ f \circ g)(z) - (4k-2)| < 1/2, & \forall z \in B_k,\ k > n \text{ and k is even};\\
|(h \circ f \circ g)(z) + 6| < 1/2, & \forall z \in B_k,\ k > n \text{ k is even and k-1=n};\\
|(h \circ f \circ g)(z) + (4k+6)| < 1/2, & \forall z \in G_k,\ k > n \text{ and k is odd};\\
|(h \circ f \circ g)(z) + (4k-2)| < 1/2, & \forall z \in G_k,\ k > n \text{ and k is even};\\
|(h \circ f \circ g)(z) - 6| < 1/2, & \forall z \in G_k,\ k > n \text{ k is even and k-1=n}.
\end{array}
\quad (7.2.32)
$$

The composition rule (7.2.32) assigned above tells us that each G_k and $B_k,\ (k > n)$ is a periodic component of period 1, and each G_k and B_k for even $k \leq n$ is a periodic component of period 1.

Dynamical behavior of $h \circ g \circ f$:

$$
\begin{array}{ll}
|(h \circ g \circ f)(z) - 2| < 1/2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right);\\
|(h \circ g \circ f)(z) + (4k+10)| < 1/2, & \forall z \in B_k,\ k = 1,3,5,\dots,n-1;\\
|(h \circ g \circ f)(z) + (4k+2)| < 1/2, & \forall z \in B_k,\ k = 2,4,6,\dots,n-1;\\
|(h \circ g \circ f)(z) - (4n+6)| < 1/2, & \forall z \in B_n \text{ and n is odd};\\
|(h \circ g \circ f)(z) + (4n+2)| < 1/2, & \forall z \in B_n \text{ and n is even};
\end{array}
$$

$$
\begin{aligned}
&|(h\circ g\circ f)(z)-(4k+10)|<1/2, && \forall z\in G_k,\ k=1,3,5\dots,n-1;\\
&|(h\circ g\circ f)(z)-(4k+2)|<1/2, && \forall z\in G_k,\ k=2,4,6,\dots,n-1;\\
&|(h\circ g\circ f)(z)+(4n+6)|<1/2, && \forall z\in G_n \text{ and n is odd};\\
&|(h\circ g\circ f)(z)-(4n+2)|<1/2, && \forall z\in G_n \text{ and n is even};\\
&|(h\circ g\circ f)(z)-(4k+6)|<1/2, && \forall z\in B_k,\ k>n \text{ and k is odd};\\
&|(h\circ g\circ f)(z)-(4k-2)|<1/2, && \forall z\in B_k,\ k>n \text{ and k is even};\\
&|(h\circ g\circ f)(z)+6|<1/2, && \forall z\in B_k,\ k>n, \text{ k is even and k-1=n};\\
&|(h\circ g\circ f)(z)+(4k+6)|<1/2, && \forall z\in G_k,\ k>n \text{ and k is odd};\\
&|(h\circ g\circ f)(z)+(4k-2)|<1/2, && \forall z\in G_k,\ k>n \text{ and k is even};\\
&|(h\circ g\circ f)(z)-6|<1/2, && \forall z\in G_k,\ k>n, \text{ k is even and k-1=n.}
\end{aligned}
\tag{7.2.33}
$$

The composition rule (7.2.33) tells us that each G_k and $B_k,\ (k > n)$ is a periodic components of period 1, and each G_k and B_k for even $k \leq n$ is a periodic component of period 1.

From all of the above discussion, we found that all domains G_k and $B_k,\ (k > n)$ are periodic components period 2 for the composition of any two functions, and period 1 for the compositions of three functions. Therefore, these domains lie in the Fatou sets of the functions f, g and h, and their compositions. Also, there are other periodic domains G_k and B_k for $k \leq n$ of different periods of the composition that lie in the periodic components of the Fatou sets of the functions f, g and h, and their compositions. □

It is noted that Lemmas 7.2.1, 7.2.2, 7.2.3 were proved by approximation theory, provide neither any information about possible existence of other wandering (or pre-periodic or periodic) domains for f or g or h nor global nature of the maps. That is, these lemmas can not say whether there exist unexpected wandering (or pre-periodic or periodic) domains for f or g or h. However, from these lemmas, we can say that whatever domains that lie in the wandering (pre-periodic or periodic) components of $F(f), F(g), F(h), F(f\circ g), F(g\circ f), F(f\circ h), F(g\circ h), F(h\circ f), F(h\circ g), F(f\circ g\circ h), F(f\circ h\circ g), F(g\circ f\circ h), F(g\circ h\circ f), F(h\circ f\circ g)$ and $F(h\circ g\circ f)$, they also lie respectively in the wandering (pre-periodic or periodic) components of their successive compositions. For example, set B_1 is a periodic component of period two of the functions f and g as defined in (7.2.19), and (7.2.20), then it is also periodic component of period two of their compositions $f\circ g$, $g\circ f$, $f\circ g\circ f$ and $g\circ f\circ g$ etc. In this context, we can also prove the following two results:

Lemma 7.2.4 ([117, Theorem 3]). *If D is a set which lies in the wandering (or pre-periodic or periodic) component of $F(f), F(g), F(f \circ g)$ and $F(g \circ f)$, then it also lies in wandering (or pre-periodic or periodic) component of $F(f^{n_k} \circ g^{n_{k-1}} \circ \ldots \circ g^{n_1})$ and $F(g^{n_k} \circ f^{n_{k-1}} \circ \ldots \circ f^{n_1})$, where $n_k, \ldots n_1 \in \mathbb{N}$.*

Proof. By the Propositions 7.2.4 and 7.2.5, such a set D exists. Since $F(f) = F(f^n)$ and $F(g) = F(g^n)$ for all $n \in \mathbb{N}$. Therefore, D lies in the wandering (or pre-periodic or periodic) component of $F(f^n)$ and $F(g^n)$ for all $n \in \mathbb{N}$. As D lies in the wandering (pre-periodic or periodic) component of $F(f \circ g)$, it also lies in the wandering (or pre-periodic or periodic) component of $F(f^n \circ g^n)$ for all $n \in \mathbb{N}$. By the same argument we are using here, D also lies in the wandering (or pre-periodic or periodic) component of $F(f \circ g)^n$ for all $n \in \mathbb{N}$. Since $F(f \circ g)^n = F(f \circ g \circ \ldots \circ f \circ g)$ (n -times $f \circ g$), so, D lies in the wandering (or pre-periodic or periodic) component of $F(f^n \circ g^n \circ \ldots \circ f^n \circ g^n)$ (n-times $f^n \circ g^n$) for all $n \in \mathbb{N}$. $n \in \mathbb{N}$ is arbitrary, so we conclude that D lies in the wandering (or pre-periodic or periodic) component of $F(f^{n_k} \circ g^{n_{k-1}} \circ \ldots \circ g^{n_1})$ for all $n_k, \ldots n_1 \in \mathbb{N}$. Similarly, we can show that D lies in the wandering (or pre-periodic or periodic) component of $F(g^{n_k} \circ f^{n_{k-1}} \circ \ldots \circ f^{n_1})$ for all $n_k, \ldots n_1 \in \mathbb{N}$. □

Lemma 7.2.5 ([117, Theorem 4]). *If D is a set which lies in the wandering (or pre-periodic or periodic) component of $F(f), F(g), F(h)$, $F(f \circ g)$, $F(g \circ f)$, $F(f \circ h)$, $F(g \circ h)$, $F(h \circ f)$, $F(h \circ g)$, $F(f \circ g \circ h)$, $F(f \circ h \circ g)$, $F(g \circ f \circ h)$, $F(g \circ h \circ f)$, $F(h \circ f \circ g)$ and $F(h \circ g \circ f)$, then it also lies in the wandering (or pre-periodic or periodic) component of $F(f^{n_k} \circ g^{n_{k-1}} \circ h^{n_{k-2}} \ldots \circ f^{n_1})$, $F(g^{n_k} \circ f^{n_{k-1}} \circ h^{n_{k-2}} \circ \ldots \circ g^{n_1})$ and $F(h^{n_k} \circ f^{n_{k-1}} \circ g^{n_{k-2}} \circ \ldots \circ h^{n_1})$ etc.*

Proof. By Lemmas 7.2.1, 7.2.2, 7.2.3, such a set D exists. By the similar argument of above Lemma 7.2.4, the proof of this lemma follows. □

We prove the Theorem 7.2.1 for a semigroup generated by two or three transcendental entire functions as defined as in Propositions 7.2.4 and 7.2.5 or Lemmas 7.2.1, 7.2.2 and 7.2.3.

Proof of Theorem 7.2.1. Let S be a holomorphic semigroup generated by two or three transcendental entire functions. If S is generated by two transcendental entire functions f and g as defined in the Propositions 7.2.4 and 7.2.5, then by Lemma 7.2.4, there is at least a domain which lies in the wandering (or periodic or perioduc) component of the $F(f^{n_k} \circ g^{n_{k-1}} \circ \ldots \circ g^{n_1})$ and $F(g^{n_k} \circ f^{n_{k-1}} \circ \ldots \circ f^{n_1})$ for all $n_k, \ldots n_1 \in \mathbb{N}$. By definition transcendental semigroup, any $h \in S = \langle f, g \rangle$ can be written in either of the form $h = f^{n_k} \circ g^{n_{k-1}} \circ \ldots \circ g^{n_1}$ or $h = g^{n_k} \circ f^{n_{k-1}} \circ \ldots \circ f^{n_1}$ for all $n_k, \ldots n_1 \in \mathbb{N}$. Therefore, there is a domain which lies in the wandering (or pre-periodic or periodic) component of the Fatou set $F(S)$ of transcendental semigroup S. Since for a transcendental entire function, pre-peridic (or periodic) domains are simply connected and so a domain within simply connected domains is also simply connected. The construction of functions in Propositions 7.2.4 and 7.2.5, the domain which lies in the wandering domains is simply connected. If S is generated by three transcendental entire functions f, g and h as defined in Lemmas 7.2.1, 7.2.2 and 7.2.3, then by Lemma 7.2.5 and similar argument as above, Fatou set $F(S)$ contains a simply connected domain. □

We restricted our proof of Theorem 7.2.1 to the holomorphic semigroup generated by two or three transcendental entire functions. Rigorously, it is not known that the essence of this theorem holds if a semigroup is generated by more than three transcendental entire functions. We can only say intuitively that the essence of this theorem may hold if semigroup S is generated by n-transcendental entire functions. There is another strong aspect of this theorem, which is- if a holomorphic semigroup generated by such type of two or three transcendental entire functions, then the Fatou set is non-empty.

It is noted that if f_1, f_2 and f_3 respectively represent the functions f, g and h of Lemmas 7.2.1, 7.2.2 and 7.2.3, then by construction these are functions of bounded type (that is, these are exponential functions). By the result of Huang [43, Theorem 2], for all $z \in F(S)$, every function $l_m \in S = \langle f_1, f_2, f_3 \rangle$ can be written as $l_m(z) = (f_{i_k} \circ f_{i_{k-1}} \circ f_{i_{k-2}} \ldots \circ f_{i_1})(z)$ does not tend to infinity on $F(S)$ as $m \to \infty$, where $1 \leq i_k \leq 3, 1 \leq k \leq m$. Again note that for such functions, every components of Fatou set $F(S)$ is simply connected and $l_m(B(z_0, R_0))$ is contained in the simply connected component of $F(S)$, where $B(z_0, R_0)$ is a disk with center at z_0 and radius R_0. Theorem 7.2.1 is a very concrete supporting example of the result of Huang ([43, Theorem 2]).

It is noted that Haung has proved that every multiply connected component of $F(S)$ must be wandering and bounded ([43, Theorem 1]). There is connection between certain Fatou component and escaping set of a transcendental semigroup. Such type of connection exhibits non-empty escaping set of a transcendental semigroup. In [63, Theorem 2.4], Kumar and Kumar proved this relation as follows.

Theorem 7.2.2. *For a transcendental semigroup S, a multiply connected Fatou component and a Baker domain are in the escaping set. That is, if $U \subset F(S)$ and U is a multiply connected Fatou component or a Baker domain, then $U \subset I(S)$.*

Proof. Let U be a multiply connected component of Fatou set $F(S)$. The rest of the proof follows from Lemma 4.1.2.

Next, suppose that $U \subset F(S)$ is a Bakar domain of a semigroup S. Let $U_f \subset F(f)$ be a Baker domain of $f \in S$. Since $F(S) \subset F(f)$ for all $f \in S$, we have $U \subset U_f \subset F(f)$ and $U_f \subset I(f)$ for some $f \in S$. Therefore by normality, $U_f \subset I(S)$. □

Chapter 8

FAST ESCAPING SETS OF TRANSCENDENTAL SEMIGROUPS

In this chapter, we study fast escaping sets of transcendental semigroups. In particular, we discuss some fundamental structure and properties of the fast escaping sets of transcendental semigroups. We also show how far the classical dynamical theory of fast escaping sets applies to general settings of transcendental semigroups, and what new phenomena can occur.

8.1 Fast escaping set

There is no formulation of the fast escaping set of a transcendental semigroup. In this section, we define fast escaping set, and try to formulate some other related terms and results.

Let S be a transcendental semigroup. Let us define a set

$$A_R(S) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R, f) \text{ for all } f \in S \text{ and } n \in \mathbb{N}\} \tag{8.1.1}$$

where $M(r, f) = \max_{|z|=r} |f(z)|$ and $M^n(r, f)$ denotes the nth iterates of $M(r, f)$. $R > 0$ can be taken any value such that $M(r, f) > r$ for $r \geq R$. If r is sufficiently large, then $M^n(r, f) \to \infty$ as $n \to \infty$. The set $A_R(S)$ is well defined in semigroup S because for any $f \in S$, $f^n \in S$ for all $n \in \mathbb{N}$. From the condition $|f^n(z)| \geq M^n(R, f)$ for all $f \in S$ and $n \in \mathbb{N}$ of the set $A_R(S)$, we can also say that a point

$z \in \mathbb{C}$ is in $A_R(S)$ if every sequence $(g_n)_{n\in\mathbb{N}}$ in S has a subsequence $(g_{n_k})_{n_k\in\mathbb{N}}$ which increases without bound at least as fast as the n-iterates of the maximum modulus of each g_{n_k}.

Definition 8.1.1 (Fast escaping set). *The fast escaping set $A(S)$ of a transcendental semigroup S consists the set $A_R(S)$ and all its pre-images. In other words, fast escaping set is a set of the form*

$$A(S) = \bigcup_{n\geq 0} f^{-n}(A_R(S)) \tag{8.1.2}$$

where $f^{-n}(A_R(S)) = \{z \in \mathbb{C} : f^n(z) \in A_R(S)\}$ for all $f \in S$ and $n \in \mathbb{N}$.

We can do certain stratification of a fast escaping set which helps to make it more visible, and provides a significant new understanding of the structure and properties of this set. We can write fast escaping set as a countable union of all its labels as we define below.

Definition 8.1.2 (Lth label of fast escaping set). *Let $A(S)$ be a fast escaping set of a transcendental semigroup S. For $L \in \mathbb{Z}$, the set of the form*

$$A_R^L(S) = \{z \in \mathbb{C} : |f^n(z)| \geq M^{n+L}(R) \quad \text{for all } \ f \in S,\ n \in \mathbb{N} \text{ and } \ n + L \geq 0\} \tag{8.1.3}$$

is called Lth level of fast escaping set $A(S)$.

It is noted that the set $A_R(S)$ as defined as above in (8.1.1) is the *0th level* of $A(S)$. As $M^{n+1}(R, f) > M^n(R, f)$ for all $n \geq 0$, so from (8.1.3), we get the following chain of relation

$$\ldots \subset A_R^L(S) \subset A_R^{L-1}(S) \subset \ldots \subset A_R^1(S) \subset A_R(S) \subset A_R^{-1}(S) \subset A_R^{-2}(S) \subset \ldots \subset A_R^{-(L-1)}(S) \subset A_R^{-L}(S) \subset \ldots \tag{8.1.4}$$

From (8.1.2) and (8.1.4), the fast escaping set can also be written as an expanding union of its labels.

$$A(S) = \bigcup_{L\in\mathbb{N}} A_R^{-L}(S). \tag{8.1.5}$$

Again, from Definition 8.1.1, that is, from (8.1.2), if any $z_0 \in A(S)$, then $z_0 \in f^{-n}(A_R(S))$ for some $n \geq 0$. It gives $f^n(z_0) \in A_R(S)$ for all $f \in S$. From (8.1.1), there is $L \in \mathbb{N}$

such that $|f^L(f^n(z_0))| = |f^{n+L}(z_0)| \geq M^n(R, f)$. With this clause, the fast escaping set of a transcendental semigroup S can now be written as

$$A(S) = \{z \in \mathbb{C} : \text{there exists } \ L \in \mathbb{N} \ \text{ such that}$$
$$|f^{n+L}(z)| \geq M^n(R) \text{ for all } \ f \in S, \ \text{and} \ \ n \in \mathbb{N}\}. \quad (8.1.6)$$

We can use any one of the form (8.1.2) or (8.1.5) or (8.1.6) as a definition of a fast escaping set. Note that by definition, the fast escaping set $A(S)$ of any transcendental semigroup S is a subset of escaping set $I(S)$. By Theorem 4.1.3(3), we can say that $I(S)$ may be empty. For any transcendental semigroup S, if $I(S) = \emptyset$, then we must have $A(S) = \emptyset$. It is not known whether there is a transcendental semigroup S such that $I(S) \neq \emptyset$ but $A(S) = \emptyset$. It is noted that in classical transcendental dynamics, both of these sets are non-empty.

Example 8.1.1 ([114, Example 3.1]). *Let S be a transcendental semigroup generated by the functions $f(z) = e^z$ and $g(z) = e^{-z}$. The function $h = g \circ f^n \in S$ is iteratively bounded at any $z \in \mathbb{C}$. Therefore, $I(S) = \emptyset$ and $A(S) = \emptyset$.*

Like escaping set $I(S)$, fast escaping set $A(S)$ is also neither an open nor a closed set if it is non-empty. Similar to Theorem 4.1.3(3), the following assertion is also clear from the definition of fast escaping set.

Theorem 8.1.1 ([114, Theorem 3.1]). *Let S be a transcendental semigroup. Then $A(S) \subset A(f)$ for all $f \in S$, and hence $A(S) \subset \bigcap_{f \in S} A(f)$.*

8.2 Elementary properties of fast escaping set

In this section, we check how far basic properties of a fast escaping set of the classical transcendental dynamics can be generalized to fast escaping set of transcendental semigroup dynamics. In Chapter 4, we examined the contrast between classical and semigroup holomorphic dynamics in the invariant features of Fatou, Julia and escaping sets. In this section, we see the same type of contrast in the fast escaping set. It is noted that in classical transcendental dynamics, the fast escaping set is completely invariant.

We prove the following elementary results that are important regarding the structure of fast escaping set $A(S)$. These results may also have more chances of leading further results concerning the properties and structure of $A(S)$. Indeed, it shows certain connection and contrast between classical and semigroup dynamics, and it is also a nice generalization of classical transcendental dynamics to semigroup dynamics.

Theorem 8.2.1 ([114, Theorem 4.1]). *Let S is a transcendental semigroup such that $A(S) \neq \emptyset$. Then the following are hold.*

1. *$A(S)$ is S-forward invariant.*
2. *$A(S)$ is independent of R.*
3. *$J(S) = \partial A(S)$.*
4. *$J(S) \subset \overline{A(S)}$.*
5. *$A(S) \cap J(S) \neq \emptyset$.*

Proof. (1). By Definition 8.1.2, we can write $A_R^L(S) \subset \{z \in \mathbb{C} : |z| \geq M^L(R),\ L \geq 0\}$. Therefore, for any $z_0 \in A_R^L(S)$, $f(z_0) \in \{z \in \mathbb{C} : |f(z)| \geq M^{L+1}(R),\ L \geq 0\} = A_R^{L+1}(S)$ for all $f \in S$, and $n \in \mathbb{N}$. This shows that $f(A_R^L(S)) \subset A_R^{L+1}(S)$ for all $f \in S$. However from relation (8.1.4), $A_R^{L+1}(S) \subset A_R^L(S)$. Hence, we have $f(A_R^L(S)) \subset A_R^L(S)$. This fact together with equation (8.1.5), we can say that $A(S)$ is S-forward invariant.

(2) Choose $R_0 > R$, then from (8.1.4), we have $A_{R_0}^L(S) \subset A_R^L(S)$ for all $L \in \mathbb{Z}$, and so $\bigcup_{L\in\mathbb{N}} A_{R_0}^{-L}(S) \subset \bigcup_{L\in\mathbb{N}} A_R^{-L}(S)$. Since there is $m \in \mathbb{N}$ such that $M^m(R) > R_0$, and so

$$\bigcup_{L\in\mathbb{N}} A_R^{-L}(S) \subset \bigcup_{L\in\mathbb{N}} A_R^{m-L}(S) = \bigcup_{L\in\mathbb{N}} A_{M^m(R)}^{-L}(S) \subset \bigcup_{L\in\mathbb{N}} A_{R_0}^{-L}(S).$$

From above two inequality, we have

$$\bigcup_{L\in\mathbb{N}} A_{R_0}^{-L}(S) = \bigcup_{L\in\mathbb{N}} A_R^{-L}(S) = A(S).$$

This proves $A(S)$ is independent of R.

(3) We prove this statement by showing $Int.(A(S)) \subset F(S)$ and $Ext.(A(S)) \subset F(S)$ where $Int.$ and $Ext.$ are respectively represent interior and exterior of $A(S)$. $A(S)$ is S-forward invariant, so $f^n(A(S)) \subset A(S)$ for all $f \in S$, and $n \in \mathbb{N}$. Suppose $z \in Int.(A(S))$, then there is a neighborhood V of z such that $z \in V \subset A(S)$. Since $A(S)$ contains no periodic points, so $|f^{n+L}(z)| \geq M^n(R)$ for all $f \in S$, and $n \in \mathbb{N}$, and hence $(f^n)_{n\in\mathbb{N}}$ is normal on V by Montel's theorem. Thus $z \in F(S)$. This proves $Int.(A(S)) \subset F(S)$.

By Theorem 3.2.3 of [15], the closure and complement of $A(S)$ are also forward invariant. Therefore, from $f^n(A(S)) \subset A(S)$, we can write $f^n(\mathbb{C} - \overline{A(S)}) \subset \mathbb{C} - \overline{A(S)}$. for all $n \in \mathbb{N}$. $\mathbb{C} - \overline{A(S)} = Ext.(A(S))$. By the assumption of non-empty $A(S)$, $\overline{A(S)}$

is also a non-empty closed set. By definition, $F(S)$ is a largest open set on which S is normal family, so we must $\mathbb{C} - \overline{A(S)} = Ext.(A(S)) \subset F(S)$.

(4) The proof follows from (3).

(5) By Theorem 8.1.1, $A(S) \subset A(f)$ for all $f \in S$. A Fatou component $U \subset F(S)$ is also a component of $F(f)$ for each $f \in S$.

Case (i): If U is a multiply connected component of $F(S)$, then by [85, Theorem 2 (a)], $\overline{U} \subset A(f)$ for all $f \in S$. Again, by Theorem 8.1.1, $\overline{U} \subset A(S)$. This shows that $\partial U \subset A(S)$. $\partial U \subset J(f)$ for all $f \in S$. By [75, Theorem 4.2], we write $\partial U \subset J(S)$. This proves $A(S) \cap J(S) \neq \emptyset$.

case (ii): If U is simply connected component of $F(S)$ that meets $A(S)$, then by [89, Theorem 1.2 (b)], $\overline{U} \subset A(f)$ for all $f \in S$. Therefore, as in case (i), $\overline{U} \subset A(S)$. By [89, Corollary 4.6], if $F(S)$ has only simply connected components, then $\partial A_R^L(S) \subset J(S)$ where $\partial A_R^L(S)$ is L-th label of $F(S)$. From (8.1.5), we conclude that $A(S) \cap J(S) \neq \emptyset$. $\square$

There are many classes of functions from which we get $I(f) \subset J(f)$, and for such functions, we must have $A(f) \subset J(f)$. Kumar and Kumar [61, Theorem 4.5] prove that $I(S) \subset J(S)$ if transcendental semigroup S is of finite or bounded type. We prove the following similar result.

Theorem 8.2.2 ([114, Theorem 4.2])**.** *Let S be a bounded or finite type transcendental semigroup. Then $A(S) \subset J(S)$ and $J(S) = \overline{A(S)}$.*

Proof. For each $f \in S$, Eremenko and Lyubich [34] proved that $I(f) \subset J(f)$. Poon [75, Theorem 4.2] proved that $J(S) = \overline{\bigcup_{f \in S} J(f)}$. Therefore, for any $f \in S$, we have $A(S) \subset A(f) \subset J(f) \subset J(S)$. The second part follows from $A(S) \subset J(S)$ together with Theorem 8.2.1 (4). $\square$

There are many functions in the class $\mathscr{B}$, the escaping set $I(f)$ consists of uncountable family of curves tending to infinity. For example, function $\lambda \sin z + \gamma$ with $\lambda, \gamma \in \mathbb{C}$ belongs to the class $\mathscr{S} \subset \mathscr{B}$, and its escaping set is an uncountable union of curves tending to infinity, the so-called Cantor bouquet. For the function $f(z) = \lambda e^z, 0 < \lambda < 1/e$, the Fatou set is completely invariant attracting basin and Julia set is a Cantor bouquet consisting of uncountably many disjoint simple curves, each of which has finite end point and other endpoint is ∞. The escaping set of such a function consists of open curves (without endpoints) together with some of their end points. It is noted that for such a function, each point in the escaping set can be connected to ∞ by a curve in the escaping set. For such functions, every point in such a curve belongs to fast escaping set except possibly a finite endpoint. More generally, let f be a finite composition of functions of finite order in the class $\mathscr{B}$ and let $z_0 \in I(f)$. Then z_0 can be connected to ∞ by a simple curve $\Gamma \subset I(f)$ such that $\Gamma \setminus \{z_0\} \subset A(f)$ (see, for instance, [83, Theorem 1.2]).

There is a chance of similar result in semigroup dynamics if semigroup S is generated by the transcendental functions of finite order in the class $\mathscr{B}$. If so, then every $f \in S$ is a finite composition of the functions of finite order in the class $\mathscr{B}$, and for each of such function f, $A(f)$ consists of curves $\Gamma \setminus \{z_0\}$ with exception of some of the end points. Since $A(S) \subset A(f)$ for each $f \in S$, then $A(S)$ may consist of curves $\Gamma \setminus \{z_0\}$ with exception of some of the end points.

8.3 On the Lth labels of $A(S)$

In this section, we concentrate on L-th label $A_R^L(S)$ of fast escaping set $A(S)$. Fast escaping set can be written as expanding union of L-th labels, so, we hope that certain structure and properties of each L-th label may determine structure and properties of fast escaping set. Analogous to classical transcendental dynamics [89], unlike the set $A(S)$, each of its label is a closed set.

Theorem 8.3.1 ([114, Theorem 5.1]). *Let $L \in \mathbb{Z}$, and let S be a transcendental semigroup such that $A(S) \neq \emptyset$. Then the set $A_R^L(S)$ is closed and unbounded for each $L \in \mathbb{Z}$.*

Proof. By Definition 8.1.2, we can write $A_R^L(S) \subset A_R^L(f)$ for all $f \in S$. This implies that $A_R^L(S) \subset \bigcap_{f \in S} A_R^L(f)$. For each $L \in \mathbb{Z}$, $A_R^L(f)$ is a closed and unbounded set and also by [89, Theorem 1.1] each component of $A_R^L(f)$ is closed and unbounded for all $f \in S$. Therefore, $\bigcap_{f \in S} A_R^L(f)$ is also a closed and unbounded set, and each of its component is closed and unbounded. $A_R^L(S)$ is a component of $\bigcap_{f \in S} A_R^L(f)$, so, it must be closed and unbounded. □

On the light of Theorem 8.3.1, and equation (8.1.5), we obtained a new structure of the fast escaping set $A(S)$, a countable union of closed and unbounded sets $A_R^L(S)$. This result also provides a solution of Eremenko's Conjecture 2.4.1 in transcendental semigroup dynamics. This generalizes the result of classical transcendental dynamics to transcendental semigroup dynamics.

Labels of fast escaping set $A(S)$ can be used to show if U is a Fatou component in $A(S)$, then boundary of U is also in $A(S)$. There are variety of results on simply connected and multiply connected Fatou components. Each of the Fatou component of transcendental semigroup is either a stable (periodic) or unstable (wandering (non-periodic)) domain. For any Fatou component U, we prove the following result which is analogous to [89, Theorem 1.2] of classical transcendental dynamics.

Theorem 8.3.2 ([114, Theorem 5.2])**.** *Let U be a Fatou component of transcendental semigroup S that meets $A_R^L(S)$, where $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ for all $f \in S$ and $L \in \mathbb{N}$. Then*

1. $\overline{U} \subset A_R^{L-1}(S)$*;*
2. *if U is simply connected, then $\overline{U} \subset A_R^L(S)$*

Proof. Since $U \cap A_R^L(S) \neq \emptyset$. The fact $A_R^L(S) \subset A_R^L(f)$ for all $f \in S$ implies that $U \cap A_R^L(f) \neq \emptyset$ for all $f \in S$. Therefore, from [89, Theorem 1.2 (a)], we always have $\overline{U} \subset A_R^{L-1}(f)$ for all $f \in S$. Hence, $\overline{U} \subset A_R^{L-1}(S)$. The second part also follows similarly by using [89, Theorem 1.2 (b)]. □

By part (2) of Theorem 8.1.1, we can conclude that $\overline{U} \subset A_R^L(S)$ for all simply connected component U of $F(S)$. Therefore, if all components of $F(S)$ are simply connected, then we must $\partial A_R^L(S) \subset J(S)$ and hence interior of $A_R^L(S)$ is contained in $F(S)$. This theorem also generalizes the result of classical transcendental dynamics to transcendental semigroup dynamics. That is, whatever Fatou component (simply or multiply connected) U of $F(S)$ intersecting $A(S)$, there is always $\overline{U} \subset A(S)$. Again, another question may raise. Such a Fatou component U is periodic or wandering? Note that in classical transcendental dynamics, such a Fatou component is always wandering ([89, Corollary 4.2]). For transcendental semigroup dynamics, such a Fatou component is again wandering domain. For, if $U \cap A(S) \neq \emptyset$, then $U \cap A(f) \neq \emptyset$ for all $f \in S$. In this case, U is wandering domain of each $f \in S$, so it is wandering domain of S.

Chapter 9

SUMMARY AND CONCLUSIONS

In this chapter, we give a brief summary, and a conclusion of this research. At the last section, we give an outline on further ideas and open questions related to the investigation of Fatou, Julia, escaping and fast escaping sets, and other related ideas of holomorphic semigroups.

9.1 Summary

This thesis belongs to the area of holomorphic semigroup dynamics, where semigroups are generated by the holomorphic functions of a complex variable. It is a study of the behavior of the compositions of a finite set of holomorphic functions in the complex plane. In the study, we restricted holomorphic functions either to rational functions or to (transcendental) entire functions. By nature, we focus on the structure and properties of Fatou, Julia, escaping and fast escaping sets of holomorphic semigroups generated by transcendental entire functions. As mentioned in the objective section, we examined to what extent, structure and properties of the Fatou, Julia, escaping and fast escaping sets of classical holomorphic dynamics are generalized and preserved, and what new phenomena occurred.

The techniques of getting goals of the objectives that we developed in this thesis are comparisons of the Fatou, Julia and escaping sets of a holomorphic semigroup and its proper subsemigroups. The general comperison is Theorem 4.1.3, and Lemmas 6.1.1 and 6.1.3. We studied certain holomorphic semigroups, whose Fatou, Julia and escaping sets coincide with its every cyclic subsemigroups (Theorems 4.1.6, 4.2.6 and 4.2.7). On the basis of index of the subsemigroup, we also studied certain holomorphic semigroups, whose Fatou, Julia and escaping sets coincide with its every proper subsemigroups (Theorems 6.1.3 and 6.2.1). We generalized the notion of abelian holomorphic semigroups to the notion of nearly abelian holomorphic semigroups, and then we investigated identical dynamical results of abelian semigroups (Theorems 5.1.2). We also studied some dynamical relation between a holomorphic semigroup and its conjugate semigroups (Theorems 5.2.1, 5.2.2 and 5.2.3). By definition, Fatou and escaping sets of a holomorphic semigroup might be empty. In Chapter-7, we developed some examples of holomorphic semigroups, whose Fatou and escaping sets are non-empty (Theorems 7.1.1 and 7.2.1) by using approximation theory. Finally, we defined fast escaping set of a transcendental semigroup, and then we studied some fundamental structure and properties of a fast escaping set (Theorems 8.1.1, 8.2.1, 8.2.2, 8.3.1, and 8.3.2). This is all, mainly, we did in this research.

9.2 Conclusions

We studied holomorphic dynamics by the tools and techniques of semigroup theory. We also observed that much more could be done by using such kind of algebraic approach. There are few studies that establish a relationship between holomorphic dynamics and certain kind of algebraic structures. Nekrashevych [66, 67] studied such type connection between classical holomorphic dynamics and self-similar groups by using tools and techniques from group theory. Chacon et al. [28] studied holomorphic semigroup dynamics via combinatorial method, and they significantly improve the result of Hinkkanen and Martin [46, Theorem 3.1] by giving a tree structure to the backward orbit and show that almost every path of the tree is dense in Julia set of the rational semigroup. We can also rethink that how these tools and techniques naturally generalize further to the study of the dynamics of holomorphic semigroups.

Almost results discussed in this thesis are quite classical in the sense that we have not studied any applications. In fact, it is quite new area of mathematical research, and even yet there is no intensive research in the field of applications of holomorphic semigroup dynamics. For further research, we have also plan to see some applications of this research study to other field of pure and applied mathematics.

9.3 Recommendations for further work

In this section, we briefly consider six areas of further study, which arise from or are closely related to the work in this thesis. Some of the ideas, open questions and conjectures introduced in this chapter are already work in progress, and there are already existing partial results. We present them in the sense that we made unsuccessful attempts in their proofs. It does not mean that they are complete list of open problems in the field of holomorphic semigroup dynamics. We have made a plane to work on the following open problems. It is not necessary that we should find solution of every question, but we will try as far as possible.

9.3.1 For an alternative definition of an escaping set

For a transcendental semigroup S, we defined escaping set (see, for instance, Definition 4.1.1) by

$$I(S) = \{z \in \mathbb{C} : S \text{ is iteratively divergent at } z\}$$

Kumar and Kumar [61, Definition 2.1] defined it by

$$I(S) = \{z \in \mathbb{C} : \text{every sequence in } S \text{ has a subsequence which diverges to infinity at } z\}$$

These two definitions can not be logically equivalent. Example 4.1.1 is a powerful evident. However, if we could prove the following slightly modified assertion, then above two definitions of escaping sets of a transcendental semigroup can be equivalent in a certain sense, and in our case, this can be an alternative definition of an escaping set of a transcendental semigroup.

Open Problem 9.3.1. *Let S be a holomorphic semigroup and $z \in \mathbb{C}$. Then $z \in I(S)$ if and only if every non-convergent sequence in S has a subsequence which diverges to ∞ at z.*

We only need to prove *only if* part, and if part of this assertion is proved in Theorem 4.1.2.

9.3.2 Eremenko's conjecture for a non-cyclic holomorphic semigroup

The following assertion can be the Eremenko's conjecture for a non cyclic holomorphic semigroup.

Conjecture 9.3.1. *Let S be a transcendental semigroup, and let $I(S)$ be its escaping set such that $I(S) \neq \emptyset$. Then every component of $I(S)$ is unbounded.*

Theorem 4.1.6 is a partial solution to Conjecture 9.3.1. In this theorem, escaping set of a semigroup S coincide with escaping set of its every cyclic subsemigroup, that is, escaping set $I(f)$ for every $f \in S$. It is noted that the semigroup in this theorem is generated by such types transcendental entire functions whose escaping sets satisfy the Eremenko's conjecture 2.4.1, and hence for such a transcendental semigroup, Conjecture 9.3.1 holds trivially.

For a general transcendental semigroup S, Conjecture 9.3.1 can be settled down if we could prove the following assertions:

Open Problem 9.3.2. *Let S be a transcendental semigroup, and $A(S)$ be a fast escaping set such that $A(S) \neq \emptyset$. Then every component of $A(S)$ is unbounded.*

Open Problem 9.3.3. *Let S be a transcendental semigroup, and $I(S)$ be an escaping set such that $I(S) \neq \emptyset$. Then $I(S) \subset A(f)$ for all $f \in S$.*

9.3.3 Baker's conjecture for a non-cyclic holomorphic semigroup

The following assertion can be the Baker's conjecture for a non cyclic holomorphic semigroup.

Conjecture 9.3.2. *If a semigroup S generated by transcendental entire functions of order less than 1/2, or has order at most 1/2, minimal type, then all components of the Fatou set $F(S)$ are bounded.*

We have not made any attempt for this Conjecture 9.3.2 in this thesis, but we mentioned Baker Conjecture of classical holomorphic dynamics (Conjecture 4.2.2), and some attempts of its proof at the end of Chapter 2.3. This conjecture in holomorphic semigroup dynamics can be proved if we could prove the following assertions:

Open Problem 9.3.4. *Let S be a semigroup generated by transcendental entire functions of order less than 1/2, or has order at most 1/2, minimal type. Let D be a component of $F(S)$ such that all limit functions of convergent subsequences of composition sequencees of S are finite in D. Then D is bounded.*

Open Problem 9.3.5. *Let S be a semigroup generated by transcendental entire functions of order less than 1/2, or has order at most 1/2, minimal type. Let D be a component of $F(S)$ such that there is a component U of $F(S)$ such that $f(U) \subset U$ and $f^n(D) \subset U$ for all $f \in S$ for some $n \in \mathbb{N}$. Furthermore, $f^m(z) \to \infty$ as $m \to \infty$, locally uniformly for all $z \in U$. Then D is bounded.*

As described in Section 2.5, there is also a recent investigation that shows a kind of connection between Eremenko's Conjecture 2.4.1 and Baker's Conjecture 2.3.1 in classical transcendental dynamics (for example, Proposition 2.5.4 of this thesis). We have also a plan to see how far that connection can be generalized in semigroup dynamics, and what new phenomena can occur.

9.3.4 Generalization of Lemmas 7.2.1, 7.2.2 and 7.2.3

We proved Lemmas 7.2.1, 7.2.2 and 7.2.3) for three transcendental entire functions. The following assertion can be a generalization of these lemmas.

Open Problem 9.3.6. *For any $n \in \mathbb{N}$, there exist n different transcendental entire functions and infinitely many domains which lie in different wandering (or pre-periodic or periodic) components of individual functions and their different compositions.*

9.3.5 Example of non-cyclic transcendental semigroup with multiply connected Fatou components

Bergweiler [20, Theorem 1] proved that there exists an entire function which has both a simply and a multiply connected wandering domains. In [117, Theorem1] (Theorem 7.2.1, in this thesis), we generalized this result of Bergweiler, in a certain sense, to a semigroup generated by three transcendental entire functions. In particular, we proved that there exists a non-cyclic transcendental semigroup which has a simply connected wandering (pre-periodic, priodic) Fatou component. The Bergweiler's result of classical transcendental dynamics should be generalized to semigroup dynamics if we could prove the following assertions:

Open Problem 9.3.7. *There is a non trivial transcendental semigroup S such that the Fatou set $F(S)$ has at least a multiply connected component.*

Open Problem 9.3.8. *There is a non trivial transcendental semigroup S such that the Fatou set $F(S)$ has both a simply and a multiply connected component.*

9.3.6 Connection of (partial) fundamental sets with Fatou components

In Theorem 7.1.1, we proved that partial fundamental set is in the Fatou set, and Fundamental set is in the escaping set of a holomorphic semigroup. There may certain connection of these sets with Fatou components. We purpose the connection in the following assertions:

Open Problem 9.3.9. *Let U be a fundamental set for a transcendental semigroup S. Then it is contained in either a Baker domain or in a multiply connected component of the Fatou set $F(S)$.*

Open Problem 9.3.10. *Let U be a partial fundamental set for a transcendental semigroup S. Then it is contained in either a wandering domain or in a pre-periodic domain of the Fatou set $F(S)$.*

REFERENCES

[1] Anderson, J. M., & Hinkkanen, A. (1998). Unbounded domains of normality. *Proc. Amer. Math. Soc.*, **126** (11): 3243-3252.

[2] Baker, I. N. (1958). Ganzer funktionen. *Math. Z.*, **69**: 121-163.

[3] Baker, I. N. (1963). Multiply connected wandering domain of normality in iteration theory. *Math. Z.*, **81**: 206-214.

[4] Baker, I. N. (1968). Entire functions. *Math. Z.*, **79**: 243-249.

[5] Baker, I. N. (1968). Repulsive fixed points of entire functions. *Math. Z.*, **104**: 252-256.

[6] Baker, I. N. (1969). Limit functions and set of normality in iteration theory. *Ann. Acad. Sc. Fenn. Ser. A. I. Math.*, **467**: 1-11.

[7] Baker, I. N. (1970). *Completely invariant domain of entire functions*, Math. essay, Ohio University Press, Athens, Ohio.

[8] Baker, I. N. (1975). The domain of normality of entire function. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, **1**: 277-283.

[9] Baker, I. N. (1976). An entire function which has wandering domains. *J. Austral. Math. Soc. (Series-A)* **22**: 173-176.

[10] Baker, I. N. (1981). The iteration of polynomials and transcendental entire functions. *J. Austral. Math. Soc. Series- A*, **30**: 483- 495.

[11] Baker, I .N. (1984). Wandering domains in the iteration of entire functions. *J. Proc. London Math. Soc.* **49** (3): 563-576.

[12] Baker, I. N. (1985). Some entire functions with multiply connected wandering domains. *Ergod. Th. and Dynam. Sys.*, **5** (2): 163 -169.

[13] Baker, I. N. (1987). The wandering domains for maps of the punctured plane. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, **12**: 191-198.

[14] Baker, I. N. (1988). Infinite limits in the iteration of entire functions. *Ergod. Theo. Dymn. System* **5**: 503-507.

[15] Beardon, A. F. (1991). *Iteration of rational functions, Complex analytic dynamical systems*, Springer-Verlag, New York, Inc.

[16] Bergelson, V. (2003). Minimal idempotents and ergodic Ramsey theory: Ergodic Theory of $\mathbb{Z}^d$ action. *Topics in dynamics and ergodic theory: London Math. Soc. Lecture Note Ser.*, **310**: 8-39.

[17] Bergweiler, W. (1991). Periodic points of entire functions: Proof of a conjecture of Baker. *Complex Variable*, **17**: 57 - 72.

[18] Bergweiler, W. (1993). Iteration of meromorphic functions. *Bull. (New Series) Amer. Math. Soc.*, **29** (2) : 131-188.

[19] Bergweiler, W. (1995). On the Julia set of analytic self-map of the punctured plane. *Analysis*, **15**: 251-256.

[20] Bergweiler, W. (2011). An entire functions with simply and multiply connected wandering domains. *Pure Appl. Math. Quarterly*, **7**(2):107-120. *arXiv*:0708.0941v1 [math. DS].

[21] Bergweiler, W. (2012). On the set where iterates of entire function are bounded. *Proc. Amer. Math. Soc.*, **140**: 847-853.

[22] Bergweiler, W., & Hinkkanen, A. (1999). On semi-conjugations of entire functions. *Math. Proc. Camb. Phil. Soc.* **126** (3): 565- 574.

[23] Bergweiler, W., & Morosawa, S. (2002). Semi-hyperbolic entire entire functions. *Non-linearity*, **15** (5): 1673-1684.

[24] Bergweiler, W., Fegella, N., & Rempe, L. (2015). Hyperbolic entire functions with bounded Fatou components, *Comment. Math. Helv.*, **90** (4): 799-829.

[25] Bergweiler, W., & Peter, J. (2013). Escape rate and Hausdorff measure for entire functions. *Math. Z.*, **274**: 551-572.

[26] Bergweiler, W., Rippon, P. J., & Stallard, G.M. (2013). Multiply connected wandering domain of entire functions. *Proc. London. Math. Soc*. **107** (3): 1261-1301.

[27] Bergweiler, W., & Wang, Y. (1998). On the dynamics of composite entire functions. *Ark. Mat.*, **36**: 31-39.

[28] Chacon, G. R., Colucci, R., & D' Angeli, D. (2014). Density of backward paths on the Julia set of a semigroup. *Sarajevo Journal of Mathematics*, **10** (22): 77-85. DOI: 10.5644/SJM.10.1.10.

[29] Conway, J. B. (2002). *Functions of one complex variables (second edition)*. Narosa Publishing House, New Delhi, Madras, Bombay, Calcutta.

[30] Devaney, R. L. (1984). Julia sets and bifurcation diagrams for exponential maps. *Bull. Amer. Math. Soc.*, **11**: 167-171.

[31] Eremenko, A. (1989). On the iteration of entire functions. *Dynamical System and Ergodic Theory, Banach Center Publications*, **23**: 339-334.

[32] Eremenko, A., & Lyubich, M.Y. (1987). Examples of entire functions with pathological dynamics. *J. London Math. Soc.*, **36** (2): 458-468.

[33] Eremenko, A., & Lyubich, M.Y. (1990). The dynamics of analytic transforms. *Leningrad Math. J. I.* **3**: 563-634.

[34] Eremenko, A., & Lyubich, M.Y. (1992). Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier, Grenoble*, **42**: 989-1020.

[35] Fatou, P. (1919). Sur les equations fonctionelles. *Bull. Soc. Math. France*, **47**: 161-271.

[36] Fatou, P. (1920). Sur les equations fonctionelles. *Bull. Soc. Math. France*, **48**: 33-94.

[37] Fatou, P. (1920). Sur les equations fonctionelles. *Bull. Soc. Math. France*, **48**: 208-314.

[38] Fatou, P. (1926). Sur l'iteration des fonctions transcendentes entieres. *Acta Math.*, **47** (4): 337- 370.

[39] Gaier, A. (1987). *Lectures on complex approximation*. Birkhauser.

[40] Gamelin, T. W. (2001). *Complex analysis*. Springer-Verlag, New York Inc.

[41] Grigorochuk, R. I. (1988). Semigroups with cancellations of degree growth. *Mat. Zamekti*, **43** (3): 305-319.

[42] Hahn, L. S., & Epstein, B. (2011). *Classical complex analysis*, Jones and Bertlett publishers, New Delhi, India.

[43] Haung, Z. (2004). The dynamics of semigroups of transcendental meromorphic functions. *Tsinghua Science and Technology* , **9** (4): 472-474. (ISSN 1007-0214 18/21).

[44] Higgins, P. (1992). *Techniques of semigroup theory*. Oxford University Press, Oxford.

[45] Hinkkanen, A. (2008). Entire functions with bounded Fatou components. *Transcendental dynamics and complex analysis*, Cambridge University Press: 1871-2016.

[46] Hinkkanen, A., & Martin, G.J. (1996). The dynamics of semigroups of rational functions-I. *Proc. London Math. Soc.*, **73** (3): 358-384.

[47] Hinkkanen, A., & Martin, G.J. (1996). *Some properties of semigroups of rational functions*. XVth Rolf Nevanlinna Colloquium (Joensuu, 1995), 5358, de Gruyter, Berlin.

[48] Hinkkanen, A., & Martin, G.J. (1996). Julia sete of rational semigroups. *Math. Z.*, **222** (2): 161-169.

[49] Holland, A. S. B. (1973). *Introduction to theory of entire functions*. Academia Press.

[50] Howie, J. M. (1995). *Fundamentals of semigroup theory*, Clarendon Press, Oxford.

[51] Hua, X .H., & Yang, C.C. (1998). *Dynamic of transcendental functions*, Gordon and Breach Science Publishers, Australia, Canada, China, India, Japan, Singapore.

[52] Huang, Z. G., & Cheng, T. (2013). Singularities and strictly wandering domains of transcendental semigroups. *Bull. Korean Math. Soc.*, **50** (1): 343-351. DOI: org/104134/BKMS.2013.50.1.343.

[53] Julia, G. (1918). Memoire sur l' iteration des fractions rationelles. *J. Math. Pures Appl.*, **1** (8): 47-245.

[54] Julia, G. (1918). Sur l'iteration des fonctions rationnnelles. *J. Math. Pures Appl.*, **4** (7): 47-245.

[55] Julia, G. (1922). Sur la permutabilite des fonctions rationnnelles. *Ann. Sci. Ecole Norm. Sup*, **39**: 131-215.

[56] Jura, A. (1978). Determining ideals of a given finite index in a finitely presented semigroup. *Demonstratio Math.*, **11**: 813-827.

[57] Kriete, H., & Sumi, H. (2000). Semihyperbolic transcendental semigroups. *J. Math. Kyoto Uni. (JMKYAZ)*, **40** (2): 205-216.

[58] Kumar, D. (2015). On escaping sets of some families of entire functions and dynamics of composite entire functions. *arXiv*:1406.2453v4 [math. DS].

[59] Kumar, D., Datt, G., & Kumar, S. (2015). Dynamics of composite entire functions. *J. Ind. Math. Soc. (N-S)*, **83** (3-4): 107-114. *arXiv*:1207.5930v5 [math. DS].

[60] Kumar, D.,& Kumar, S. (2015). The dynamics of semigroups of transcendental entire functions I. *Indian J. Pure Appl. Math.*, **46** (1): 11-24. DOI: 10.1007/s13226-015-0104-0. *arXiv*:1302.7249 [math. DS].

[61] Kumar, D., & Kumar, S. (2016). The dynamics of semigroups of transcendental entire functions II. *Indian J. Pure Appl. Math.*, **47** (3): 409-423. DOI: 10.1007/s13226-016-0173-8. *arXiv*: 1401.0425v3 [math. DS].

[62] Kumar, D., Kumar, S., & Poon, K. K. (2016). Escaping set and Julia set of transcendental semigroups. *Advances in Theo. and Appl. Math.***11** (4): 423-436. ISSN 0973-4554. *arXiv*:1410.2747 [math. DS].

[63] Kumar, D., & Kumar, S. (2017). Semigroups of transcendental entire functions and their dynamics. *Proc. Indian Acad. Sci. (Math. Sci.)* **127** (2): 349-360. DOI: 10.1007s12044-016-0298-z. *arXiv*: 1405.0224v5 [math. DS].

[64] Maltcev, V., & Ruskuc, M. (2013). On hopfian cofinite subsemigroups. arXiv: 1307. 6929 v1 [math. GR].

[65] Milnor, J. (1999). *Dynamics in one complex variable, Introductory Lectures*. F. Wierweg and Sohn. ISBN: 3-528-03130-1.

[66] Nekrashevych, V. (2005). *Self similar groups*, Mathematical Surveys and Monographs, 117. American Mathematical Society, Providence, RI.

[67] Nekrashevych, V. (2008). *Symbolic dynamics and self similar groups*, Holomorphic dynamics and renormalizations, 2573. Field Inst. Commun. 53, American Mathematical Society, Providence, RI.

[68] Morosawa, S., Nishimura, Y., Taniguchi, M., & Ueda, T. (1998). *Holomorphic Dynamics*, Cambridge University Press, Cambridge, UK .

[69] Osborne, J. (2012). The structure of spider's web fast escaping sets. *Bul. London Math. Soc.*, **44** (3): 503-519.

[70] Osborne, J. (2013). Connectedeness propertties of set where iterates of entire functions are bounded. *Math. Proc. Camb. Philos. Soc.*, **155** (3): 391-410.

[71] Osborne, J. W. (2013). Spider's web and locally connected Julia sets of transcendental entire functions. *Ergod. Theo. and Dynam. Sys.*, **33** (4): 1146-1161.

[72] Osborne, J. W., Rippon, P. J., & Stallard, G. M. (2015). Connectedness properties of the set where iterates of entire function are unbounded. *arXiv*: 1504.05611v1[math. DS].

[73] Osborne, J. W., & Sixsmith, D. J. (2015). On the set where iterates of entire function are neither escaping nor bounded. *arXiv*:1503.08077v1[math. DS].

[74] Osborne, J. W., Rippon, P. J., & Stallard, G. M. (2016). The iterated minimum modulus and conjectures of Baker and Eremenko. *arXiv*: 1609.06644v1 [math. CV].

[75] Poon, K. K. (1998). Fatou-Julia theory on transcendental semigroups. *Bull. Austral. Math. Soc.*, **58**: 403-410.

[76] Poon K. K. (1999). Fatou-Julia theory on transcendental semigroups II. *Bull. Austral. Math. Soc.* **55**: 257-262.

[77] Poon, K. K., & Yang, C.C. On the Fatou set of two permutable entire functions. *Pre-print*.

[78] Rempe, L. (2007). On a question of Eremenko concerning escaping components of entire functions. *Bull. London Math. Soc.*, **39**: 661-666.

[79] Rempe, L. (2009). Rigidity of escaping dynamics for transcendental entire functions. *Acta Math.*, **203**: 235-267.

[80] Rempe, L. (2010). The escaping set of the exponential. *Ergodic Theo. Dyn. Sys.*, **30**: 595-599.

[81] Rempe, L. (2011). Connected escaping sets of exponential maps. *Ann. Acad. Sci. Fenn. Math.*, **36**: 71-80.

[82] Rempe, L. (2016). Arc-like continua, Julia set of entire function and Eremenko conjecture. *arXiv*: 1610. 06278v2[math. DS].

[83] Rempe, L., Rippon, P. J., & Stallard, G.M. (2010). Are Devaney hairs fast escaping? *Diff. eq. and Appl.*, **16**: 739-762.

[84] Rempe, L., Rottenfusser, G., Ruckert, J., & Schleicher, D. (2011). Dynamic rays of bounded type entire functions. *Ann. Math.*, **173** (1): 77-125.

[85] Rippon, P. J., & Stallard, G. M. (2005). On questions of Fatou and Eremenko. *Proc. Amer. Math. Soc.*, **133** (4): 1119-1126. DOI: S 0002-9939(04))07805-0.

[86] Rippon, P. J., & Stallard, G. M. (2008). Functions of small growth with no unbounded Fatou components. *arXiv*: 0801.3626v1 [math. CV].

[87] Rippon, P. J., & Stallard, G. M. (2011). Boundaries of escaping Fatou components. *Proc. Amer. Math. Soc.*, **139** (8): 2807-2821.

[88] Rippon, P. J., & Stallard, G. M. (2011). Slow escaping points of meromorphic functions. *Trans. Amer. Math. Soc.*, **363**: 4171-4201.

[89] Rippon, P. J., & Stallard, G. M. (2012). Fast escaping points of entire functions. *Proc. London Math. Soc.*, **105** (3 & 4): 787-720.

[90] Rippon, P. J., & Stallard, G. M. (2014). Regularity and fast escaping points of entire functions. *International Mathematics Research Notices, IMRN 2014*: 5203-5229.

[91] Rippon, P. J., & Stallard, G. M. (2014). A sharp growth condition for fast escaping spider's web. *arXiv*: 1208. 3371v1 [math. DS].

[92] Ruskuc N. (1998). On large subsemigroups and finiteness conditions of semigroups. *Proc. London Math. Soc.*, **76** (3): 383-405.

[93] Schleicher, D. (1998). *Dynamics of entire functions*. Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 295-339. DOI:$10.1007978 - 3 - 642 - 13171 - 4\_5$.

[94] Schleicher, D., & Zimmer, J. (2003). Escaping points of exponential maps. *J. London Math. Soc.*, **67** (2): 380-400. DOI: 10.1112/S0024610702003897.

[95] Singh, A. (2004). *Properties and structures of Fatou sets and Julia sets*. Unpublished Ph.D. Thesis, University of Jammu, Jammu, India.

[96] Singh, A. P. (2003). On the dynamics of composition of entire functions. *Math. Proc. Camb. Phil. Soc.*, **134** (1): 129-138.

[97] Singh, A. P., & Wang, Y. (2006). Julia sets of permutable holomorphic maps. *Sciences in China, Series A: Mathematics*, **49** (11): 1715-1721. DOI: 10.1007/s11425-006-2063-8.

[98] Sixsmith, D. J. (2011). Entire functions for which the escaping set is a spider's web. *Math. Proc. Camb. Phil. Soc.*, **151** (3): 551-571.

[99] Sixsmith, D. J. (2012). Simply connected fast escaping Fatou components. *Pure Appl. Math. Q.*, **8** (4): 1029-1046.

[100] Sixsmith, D. J. (2013). *Topics in transcendental dynamics*. Unpublished Ph.D. Thesis, The Open University, UK.

[101] Sixsmith, D. J. (2015). Maximally and non-maximally fast escaping points of transcendental entire Functions. *Math. Proc. Camb. Phil. Soc.*, **158** (2): 365-383. DOI: https://doi.org/10.1017/S0305004115000018

[102] Stallard, G. M. (1991). *Some problems in the iteration of meromorphic functions*. Unpublished Ph. D. Thesis, Imperial Collage London, UK.

[103] Stankewitz, R. (1999). Completely invariant Julia sets of polynomial semigroups. *Proc. Amer. Math. Soc.*, **127** (10): 2889-2898. DOI:S0002-9939(99)04857-1.

[104] Stankewitz, R. (2007). Completely invariant sets of normality for rational semigroups. *Complex Variables, Theory and Application: An Iternational Journal*, **40** (3): 199-210. DOI: org/10.1080/17476930008815219.

[105] Stankewitz, R., Sugawa, T., & Sumi, H. (2007). Some counter examples in dynamics of rational semigroups. *arXiv* :0708.3434v1 [math DS].

[106] Stankewitz, R. & Sumi, H. (2011). Dynamical properties and structure of Julia sets of postcritically bounded polynomial semigroups. *arXiv*: 0708.3187v4 [math. DS].

[107] Subedi, B. H., & Singh, A. (2016). A review on the structure and properties of the escaping sets of transcendental entire function. *The Nep. Math. Sci. Report*, **34** (1 & 2): 65-78. DOI: https://doi.org/10.3126/nmsr.v34i1-2.30019.

[108] Subedi, B. H., & Singh, A. (2018). A short comparison of classical complex dynamics and holomorphic semigroup dynamics. *arXiv*: 1803.08368v1 [math. DS].

[109] Subedi, B. H., & Singh, A. (2018). Invariant properties of Fatou set, Julia set and escaping set of holomorphic semigroup. *arXiv*: 1803.09662v1 [math. DS].

[110] Subedi, B. H., & Singh, A. (2018). Escaping sets of hyperbolic semigroups. *The Nep. Math. Sci. Report*, **35** (1 & 2): 45-52. DOI: https://doi.org/10.3126/nmsr.v35i1-2.29979. arXiv:1803.10381 v1 [math. DS].

[111] Subedi, B. H., & Singh, A.(2018). On completely invariant Julia set of transcendental semigroup. *arXiv*:1804.03554v1 [math. DS].

[112] Subedi, B. H., & Singh, A. (2018). Completely invariant escaping set of transcendental semigroup. *arXiv*:1804. 11252 v1 [math. DS].

[113] Subedi, B. H., & Singh, A. (2018). A study of Fatou set, Julia set and escaping sets in nearly abelian transcendental semigroup. *arXiv*:1808. 00875 v1 [math. DS].

[114] Subedi, B. H., & Singh, A. (2018). Fast escaping set of transcendental semigroup. *Universal Journal of Applied Mathematics*, **7** (1):1-7. DOI: 10.13189/ujam.2019.070101. *arXiv*: 1809. 06743 v1 [math. DS].

[115] Subedi, B. H., & Singh, A. (2019). Fatou, Julia, and escaping sets in holomorphic (sub)semigroup dynamics. *Turk. J. Math.*, **43** (2): 930-940. DOI: 10. 3906/mat-1810-133. *arXiv*: 1807.04499v1 [math. DS].

[116] Subedi, B. H. (2019). Wandering domains of three transcendental entire functions and their compositions. *The Indian Journal of Mathematics*, **61** (3): 329-342. *arXiv*: 1803.08368v1 [math. DS].

[117] Subedi, B. H. (2019). Transcendental semigroup that has simply connected Fatou components. *Journal of Institute of Engineering, Tribhuvan University*, **15** (1): 122-127. DOI: https://doi.org/10.3126/jie.v15i1.27721. *arXiv*: 1804. 01213 v1 [math. DS].

[118] Subedi, B. H., & Singh, A. (2019). Fatou, Julia, and escaping sets of conjugate holomorphic semigroups. *The Nep. Math. Sci. Report*, **36** (1-2): 61-66. DOI: https://doi.org/10.3126/nmsr.v36i1-2.29971. *arXiv*:1808.03275v1 [math DS].

[119] Subedi, B. H., & Singh, A. (2020). Dynamics on the pre-periodic components of the Fatou set of three transcendental entire functions and their compositions. *Nepal Journal of Science and Technology* (NJST), **19** (1): 161-166. DOI: https://doi.org/10.3126/njstv19i1.29796.

[120] Subedi, B. H., & Singh, A. (2020). Dynamics on the periodic components of the Fatou set of three transcendental entire functions and their compositions. *The Journal of the Nepal Mathematical Society*, **3** (1):37-46. DOI: https://doi.org/10.3126/jnms.v3i1.33002.

[121] Subedi, B. H., & Singh, A. (2020). A case study of holomorphic semigroups. *The Journal of the Nepal Mathematical Society*, **3** (2):35-40. DOI: https://doi.org/10.3126/jnms.v3i2.33958. *arXiv*: 1901.07364 v1 [math. DS].

[122] Subedi, B. H., & Singh, A. (2020). A comparision of of classical holomorphic dynamics and holomorphic semigroup dynamics -II. *The Nep. Math. Sci. Report*, **37** (1-2): 86-95. DOI: https://doi.org/10.3126/nmsr.v37i1-2.34098.

[123] Sumi, H. (1998). On dynamics of hyperbolic rational semigroup. *J. Math. Kyoto Univ. (JMKYAZ)*, **37** (4): 717-733.

[124] Tomar, G. (2017). On the dynamics of composition of transcendental entire functions in angular region-II. *The Journal of Indian Mathematical Society*, **84** (3-4): 287-296. ISSN (Online): 2455-6475.

[125] Tuen, W. Ng. (2001). Permutable entire functions and their Julia sets. *Math. Proc. Camb. Phil. Soc.*, **131**: 129-138.

[126] Wang, X., Hua, X., Yang, C.C., & Yang, D. (2006). Dynamics of permutable transcendental entire functions. *Rocky Mountain J. Math.*, **36** (6): 2041-2055.

[127] Wang, X., & Haung, Z. (2004). On nearly abelian polynomial semigroups. *Proc. Amer. Math. Soc.*, **133** (1): 83-89. DOI: S 0002-9939(04)07669-5.

APPENDIX-A

List of Scientific Publications

2020

1. B. H. Subedi, & A. Singh. 3n+1 Problem and its Dynamics. *Nepal Journal of Mathematical Sciences*,**1** (2020): 43-50. DOI: https://doi. org/10.3126/njmathsci.v1i0.34159.
2. B. H. Subedi, & A. Singh. A case study of holomorphic semigroups. *The Journal of the Nepal Mathematical Society*, **3** (2) (2020): 35-40. DOI: https://doi. org/10.3126/jnms.v3i2.33958.
3. B. H. Subedi, & A. Singh. A comparison of classical holomorphic dynamics and holomorphic semigroup dynamics -II. *The Nep. Math. Sci. Report*, **37** (1-2) (2020): 86-95. DOI: https://doi.org/10.3126/nmsr.v37i1-2.34098.
4. B. H. Subedi, & A. Singh. Dynamics on the pre-periodic components of the Fatou set of three transcendental entire functions and their compositions. *Journal of Science and Technology* (NJST), **19** (2020): 161-166. DOI: https://doi.org/10.3126/njstv19i1.29796.
5. B. H. Subedi, & A. Singh. Dynamics on the periodic components of the Fatou set of three transcendental entire functions and their compositions. *Journal of the Nepal Mathematical Society*, **3**(1) (2020): 37-46. DOI: https://doi.org/10.3126/jnms.v3i1.33002.

2019

1. B. H. Subedi, & A. Singh. Fatou, Julia, and escaping sets of conjugate holomorphic semigroups. *Nep. Math. Sci. Report*, **36** (1-2) (2019): 61-66. DOI: https://doi.org/10.3126/nmsr.v36i1-2.29971.

2. B. H. Subedi, & A. Singh. Fatou, Julia, and escaping sets in holomorphic (sub)semigroup dynamics. *Turk. J. Math.*, **43** (2) (2019): 930-940. DOI: 3906/mat-1810-133.

3. B. H. Subedi, & A. Singh. Fast escaping set of transcendental semigroup. *Universal Journal of Applied Mathematics*, **7** (1) (2019) : 1-7. DOI: 10.13189/ujam.2019.070101.

4. B. H. Subedi. Transcendental semigroup that has simply connected Fatou components. *Journal of Institute of Engineering, Tribhuvan University*, **15** (1) (2019): 122-127. DOI: https://doi.org/10.3126/jie.v15i1.27721.

5. B. H. Subedi. Wandering domains of three transcendental entire functions and their compositions. *Indian Journal of Mathematics*, **61** (3) (2019): 329-342.

2018

1. B. H. Subedi,& A. Singh. Escaping sets of hyperbolic semigroups. *Nep. Math. Sci. Report*, **35** (1 & 2) (2018): 45-52. DOI: https://doi.org/10.3126/nmsr.v35i1-2.29979. ISSN 2392-411X.

2. B. H. Subedi. Structurally similar sets of transcendental entire functions. *Journal of Advanced College of Engineering and Management*, **4** (2018): 11-16. DOI: https://doi.org/10.3126/jacem.v4i0.23175. ISSN 2392-4853.

3. B. H. Subedi. A survey on the structure and properties of escaping set under entire semigroups. *Journal of Academic Research, NPA, TU*, **1** (2017): 115-120. ISSN 2616-0269.

2017

1. B. H. Subedi. Escaping set of transcendental semigroup. *Proceedings of the National Conference on History and Recend Trends in Mathematics*, (2017): 29-32. ISSN 2594-3375.

2. B. H. Subedi, & A. Singh. The subsets of escaping sets of transcendental entire functions. *Proceedings of the National Conference on Mathematics and Applications*, (2017): 18-23. ISSN 2565-4969.

2016

1. B. H. Subedi, & A. Singh. A riview on the structure and properties of the escaping set of transcendental entire functions. *Nep. Math. Sci. Report*, **34** (1 & 2) (2016): 65-78. DOI: https://doi.org/10.3126/nmsr.v34i1-2.30019. ISSN 2392-411X.

2. B. H. Subedi. Escaping set and Eremenko's conjecture. *EUREKA, An Int. J. Math.*, **3** (2016): 9-14. ISSN 2392-4233.

3. B. H. Subedi. Around the field of complex dynamics. *J. math. Edu.*, **1** (2) (2016): 40-47.

2015

1. B. H. Subedi, & A. Singh. Spider's web structure of escaping set and fast escaping set. *Procceedings of the Seminar on Recent Development of Mathematics and its Applications*, (2015): 49-55.

APPENDIX-B

List of Attended Conferences and Seminars

2020

1. Presentation of a poster on *Relation between Fatou, Julia and escaping sets of a holomorphic semigroup and its proper subsemigroups* at the Recent Trends in Nonlinear Science Conference held at the Centre de Recerca Mathematica (CRM), Barecelona, Spain from February 3-7, 2020.

2. Participation at the *Bifurcation and finiteness problems in ordinary differential equation* Conference held at the Centre de Recerca Mathematica (CRM), Barecelona, Spain from February 17-21, 2020.

3. Seminar presentation on *Fatou, Julia and escaping sets in holomorphic subsemigroup dynamics* at complex dynamics recearch group of the Centre de Recerca Mathematica (CRM), Barecelona, Spain on February 24, 2020.

2019

1. Presentation of an invited talk on *A study of holomorphic semigroups* in Second International conference on Applications of Mathematics to Nonlinear Science (AMNS -2019), June 27-30, 2019, Pokhara, Nepal.

2018

1. Participation at the *CIMPA-Nepal Research School on Dynamical System*, November 24 to December 2, 2018. Organized by CIMPA France and Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal.

2017

1. Participation at the *ICTS Program on Geometry, Groups and Dynamics (GGD-2017)*, November 6-24, 2017. Organized by International Centre for Theoretical Sciences of TATA Institute of Fundamental Research (ICTS-TIFR), Bangalore, India.

2. Participation at the *International Conference in Class Groups of Number Fields and Related Topics*, September 4-7, 2017. Organized by Harish-Chandra Research Institute (HRI), Allahabad, India.

3. Participation at the *CIMPA Research School on Recent Trends in Non-commutative algebras*, June 19-30, 2017. Organized by CIMPA France and IISER, Pune, India.

4. Participation and Delivered a Talk on *Escaping Set of Transcendental Semigroup* in National Conference on History and Recent Trends of Mathematics, June 2-4, 2017. Organized by Department of Mathematics, Balmiki Campus, Kathmandu, Nepal.

5. Participation and deliver a talk on *On subset of escaping set* in National conference in Mathematics and its Applications- 2017 (NCMA -2017), January 11-17, 2017. Organized by Nepal Mathematical Society in Chitawan, Nepal.

2016

1. Participation at the *CIMPA-INDIA Research School in Geometric Flows*, December 01-12, 2016. Organized by CIMPA France and Department of Mathematics, Jadavpur University, Kolkata, India.

2. Participation at the *Second International Workshop and Conference on Commutative Algebra*, October 17 -28, 2016. Organized by Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal.

3. Participation and Delivered a talk on *On the Partition of Fast Escaping Set* in International Conference on Applications Mathematics to Non-Linear Science (AMNS – 2016), May 26 – 29, 2016. Organized by Nepal Mathematical Society and Association of Nepalese Mathematicians in America (ANMA) in Kathmandu, Nepal.

2015

1. Participation and Delivered a talk on *Spider's Web Structure of Escaping Set* in Second Nepal-Japan Joint Workshop on Some Aspects of Fundamental Mathematics and its Applications, November 23-26, 2015. Organizesd by Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal.

2. Participation and delivered a Talk on *A Survey on the Structure and Properties of Escaping Set under Entire Semigroups* in First International Conference and Workshop on Commutative Algebra, April 20- 26, 2015. Organized by Central Department of Mathematics and Department of Mathematics Education, Tribhuvan University, Kirtipur, Kathmandu, Nepal.

APPENDIX-C

Copy of Certificates of Attended Seminars and Conferences

Central Department of Mathematics
and
Department of Mathematics Education
Tribhuvan University

CERTIFICATE

This Certificate is awarded to

Mr/Ms. Bishnu Hari Subedi *for having participated and delivered talk(s) entitled* A survey on the structure of Escaping set under entire semigroup *in the*

First International Workshop and Conference on Commutative Algebra

held in the Central Department of Mathematics, T.U. from 20^{th} *to* 26^{th} *April, 2015.*

Prof. Roger Wiegand University of Nebraska-Lincoln USA	Dr. Ajaya Singh Member Organizing Committee	Prof. Prakash Muni Bajracharya M.Phil.(Math) Co-ordinator Tribhuvan University

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INTERNATIONAL CENTRE *for* THEORETICAL SCIENCES

ICTS

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Geometry, Groups and Dynamics (GGD) - 2017

CERTIFICATE OF PARTICIPATION

This certifies that

Bishnu Hari Subedi

has successfully participated in the ICTS program 'Geometry, Groups and Dynamics (GGD) – 2017' held in ICTS campus, Bangalore, India from 06 November 2017 to 24 November 2017.

Krishnendu Gongopadhyay
On behalf of the Organizers

Central Department of Mathematics, Tribhuvan University, Nepal

in cooperation with

The International Centre of Theoretical Physics (ICTP),
International Mathematical Union (IMU)
and
University Grants Commission (UGC), Nepal

present this

Certificate of Participation

To

Bishnu Hari Subedi

Central Department of Mathematics, Tribhuvan University, Nepal

for actively participating
at the CIMPA Research School on Dynamical Systems, Kathmandu, Nepal
October 25 – November 5, 2018

CIMPA

KATHMANDU NEPAL

Prof. Dr. Sergei B Kuksin
International Coordinator

Dr. Shree Ram Khadka
National Coordinator

Date: November 5, 2018, Kathmandu, Nepal

INTERNATIONAL MATHEMATICAL UNION IMU

The Abdus Salam
ICTP International Centre for Theoretical Physics

Second International Conference on Applications of Mathematics to Nonlinear Sciences (AMNS-2019)

June 27-30, 2019, Pokhara, Nepal

http://anmaweb.org/AMNS-2019

This certificate is awarded to

Bishnu Hari Subedi

for presenting a talk

A Study of Holomorphic Semigroups

in an invited session of

Second International Conference on Applications of Mathematics to Nonlinear Sciences (AMNS-2019)

on June 27-30, 2019, Pokhara, Nepal.

Dr. Naveen Vaidya
AMNS-2019 Conference Convener

Conveners

*Naveen K. Vaidya, President of ANMA
San Diego State University, USA
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CERTIFICATE OF PRESENTATION

I hereby certify that

Bishnu Hari Subedi

has delivered a *Poster Presentation* at the **Recent Trends in Nonlinear Science** conference, held at the Centre de Recerca Matemàtica from February 3rd to 7th, 2020 by presenting the work entitled

Relation between Fatou, Julia and Escaping Sets of a Holomorphic Semigroup and its Proper Subsemigroups

DanceNet
Dinámica, Atractores y Nolinealidad. Caos y Estabilidad

Professor Joan Torregrosa
DANCE coordinator

February 7th, 2020
Centre de Recerca Matemàtica, Bellaterra

APPENDIX-D

Copy of Published Papers

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 930 – 940
© TÜBİTAK
doi:10.3906/mat-1810-133

Fatou, Julia, and escaping sets in holomorphic (sub)semigroup dynamics

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Abstract: We investigate under what conditions the Fatou, Julia, and escaping sets of a transcendental semigroup are respectively equal to the Fatou, Julia, and escaping sets of their subsemigroups. We define the partial fundamental set and fundamental set of a holomorphic semigroup, and on the basis of these sets, we prove that the Fatou and escaping sets of a transcendental semigroup S are nonempty.

Key words: Transcendental semigroup, escaping set, finite index, cofinite index, Rees index, partial fundamental set, fundamental set

1. Introduction

We confine our study to the Fatou, Julia, and escaping sets of a holomorphic semigroup and its subsemigroup. A semigroup S is a very classical algebraic structure with a binary composition that satisfies the associative law. Semigroups arose naturally from the general mappings of a set into itself. Hence, a set of holomorphic functions on complex plane $\mathbb{C}$ or Riemann sphere $\mathbb{C}_\infty$ naturally forms a semigroup. Here, we take a set A of holomorphic functions and construct a semigroup S that consists of all elements that can be expressed as a finite composition of elements in A. We call such a semigroup S the *holomorphic semigroup* generated by the set A. A nonempty subset T of a holomorphic semigroup S is a *subsemigroup* of S if $f \circ g \in T$ for all $f,\ g \in T$.

For simplicity, we denote the class of all rational functions on $\mathbb{C}_\infty$ by $\mathscr{R}$ and the class of all transcendental entire functions on $\mathbb{C}$ by $\mathscr{E}$. Let $\mathscr{F} = \{f_\alpha : \alpha \in \Delta\} \subseteq \mathscr{R}$ or $\mathscr{E}$. The *holomorphic semigroup* generated by $\mathscr{F}$ is denoted by

$$S = \langle f_\alpha \rangle.$$

The index set Δ is allowed to be infinite in general unless otherwise stated. It is easy to see that S is a collection of holomorphic functions, and is closed under functional composition. S is called a *rational semigroup* or a *transcendental semigroup* depending on whether $\mathscr{F} \subseteq \mathscr{R}$ or $\mathscr{F} \subseteq \mathscr{E}$. A holomorphic semigroup S is *abelian* if $f_\alpha \circ f_\beta = f_\beta \circ f_\alpha$ for all generators f_α and f_β of S.

A semigroup generated by finitely many holomorphic functions $f_i, (i = 1, 2, \dots, n)$ is called a *finitely generated holomorphic semigroup*, and we write $S = \langle f_1, f_2, \dots, f_n \rangle$. If S is generated by only one holomorphic function f, then S is called a *cyclic semigroup*, and we write $S = \langle f \rangle$. In this case, each $g \in S$ can be written

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2010 *AMS Mathematics Subject Classification:* 37F10, 30D05

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as $g = f^n$, where f^n is the nth iterate of f with itself. We say that $S = \langle f \rangle$ is a *trivial semigroup*. By the definition of holomorphic semigroup, we at once get the following result.

Proposition 1.1 *Let $S = \langle f_\alpha \rangle$ be a holomorphic semigroup. Then for every $f \in S$, f^m (for all $m \in \mathbb{N}$) can be written as $f^m = f_{\alpha_1} \circ f_{\alpha_2} \circ f_{\alpha_3} \circ \cdots \circ f_{\alpha_p}$, where $\alpha_i \in \{\alpha : \alpha \in \Delta\}$, for some $p \in \mathbb{N}$.*

A family $\mathscr{F}$ of holomorphic functions forms a *normal family* in a domain D if every sequence $(f_\alpha) \subseteq \mathscr{F}$ has a subsequence (f_{α_k}) which is uniformly convergent or divergent on all compact subsets of D. If there is a neighborhood U of a point $z \in \mathbb{C}$ such that $\mathscr{F}$ is a normal family in U, then we say that $\mathscr{F}$ is normal at z. We say that a holomorphic function f is *iteratively divergent* at $z \in \mathbb{C}$ if

$$f^n(z) \to \infty \text{ as } n \to \infty.$$

A semigroup S is *iteratively divergent* at z if every $f \in S$ is iteratively divergent at z. A semigroup S is said to be *iteratively bounded* at z if there is an element $f \in S$ which is not iteratively divergent at z.

Like in classical complex dynamics (that is, based on the Fatou-Julia-Eremenko theory of a holomorphic function), the Fatou, Julia, and escaping sets in the settings of a holomorphic semigroup are defined as follows:

Definition 1.1 (Fatou, Julia, and escaping sets) *The Fatou set of the holomorphic semigroup S is defined by*

$$F(S) = \{z \in \mathbb{C} : S \text{ is normal at } z\},$$

and the Julia set $J(S)$ of S is the complement of $F(S)$. If S is a transcendental semigroup, the escaping set of S is defined by

$$I(S) = \{z \in \mathbb{C} : S \text{ is iteratively divergent at } z\}.$$

We call each point of the set $I(S)$ an escaping point.

It is obvious that $F(S)$ is the largest open subset (of $\mathbb{C}$ or $\mathbb{C}_\infty$) on which the semigroup S is normal. And its complement $J(S)$ is a closed set for any semigroup S. However, the escaping set $I(S)$ is neither an open nor a closed set (if it is nonempty) for any transcendental semigroup S. Any maximally connected subset U of the Fatou set $F(S)$ is called a *Fatou component*. If $S = \langle f \rangle$, then the Fatou, Julia, and escaping sets are respectively denoted by $F(f), J(f)$, and $I(f)$.

It is possible that the Fatou, Julia, or escaping set of a holomorphic semigroup may be equal, respectively, to the Fatou, Julia, or escaping set of a proper subsemigroup.

Definition 1.2 (Finite index and cofinite index) *A subsemigroup T of a holomorphic semigroup S is said to be of finite index if there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1, where $S^1 = S \cup \{Identity\}$, such that*

$$S = (f_1 \circ T) \cup (f_2 \circ T) \cup \ldots \cup (f_n \circ T). \tag{1.1}$$

The smallest n that satisfies 1.1 is called the index of T in S. Similarly, a subsemigroup T of a holomorphic semigroup S is said to be of cofinite index if there exists finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that for any $f \in S$, there is $i \in \{1, 2, \ldots, n\}$ such that

$$f_i \circ f \in T. \tag{1.2}$$

The smallest n that satisfies 1.2 is called the cofinite index of T in S.

Note that the size of a subsemigroup T of a semigroup S is measured in terms of index. If a subsemigroup T has a finite index or cofinite index in the semigroup S, then we say T is a finite indexed subsemigroup or a cofinite indexed subsemigroup, respectively.

For any holomorphic function f,

$$CV(f) = \{w \in \mathbb{C} : w = f(z) \text{ for some } z \text{ such that } f'(z) = 0\}$$

(where $f^{'}$ represents derivative of f with respect to z) is the set of *critical values* of f. The set $AV(f)$ consisting of all $w \in \mathbb{C}$ such that there exists a curve $\Gamma : [0, \infty) \to \mathbb{C}$ so that $\Gamma(t) \to \infty$ and $f(\Gamma(t)) \to w$ as $t \to \infty$ is the set of *asymptotic values* of f and

$$SV(f) = \overline{(CV(f) \cup AV(f))}$$

is the set of *singular values* of f. If $SV(f)$ is finite, then f is said to be of *finite type*. If $SV(f)$ is bounded, then f is said to be of *bounded type*. The sets

$$\mathscr{S} = \{f : f \text{ is of finite type}\}$$

and

$$\mathscr{B} = \{f : f \text{ is of bounded type}\}$$

are respectively known as *Speiser class* and *Eremenko-Lyubich class*.

In [8, Theorem 5.1], Poon proved that the Fatou and Julia sets of a finitely generated abelian transcendental semigroup S is the same as the Fatou and Julia sets of each of its particular functions if the semigroup S is generated by finite type transcendental entire functions. In [13, Theorems 3.3], we proved that the escaping set of a transcendental semigroup S is the same as the escaping set of each of its particular functions if the semigroup S is generated by finite type transcendental entire functions. In this paper, we prove the following assertion:

Theorem 1.1 *If a subsemigroup T has finite index or cofinite index in an abelian transcendental semigroup S, then $I(S) = I(T), J(S) = J(T)$ and $F(S) = F(T)$.*

In Section 2, we define Rees index in semigroups. We then prove Theorem 1.1 for a subsemigroup T having finite Rees index.

From [11, Theorem 3.1 (1) and (3)], we can say that Fatou and escaping sets of holomorphic semigroup may be empty. The result [8, Theorem 5.1] is one of the case of nonempty Fatou set and that of [13, Theorem 3.3] is a case of the nonempty escaping set of transcendental semigroup. We obtain another case of nonempty Fatou and escaping sets on the basis of the following definition.

Definition 1.3 (Partial fundamental set and fundamental set) *A set U is called a partial fundamental set for the semigroup S if*

1. $U \neq \emptyset$,
2. $U \subset R(S)$,
3. $f(U) \cap U = \emptyset$ *for all* $f \in S$.

If in addition to (1)*,* (2)*, and* (3)*, U satisfies the property*

4. $\bigcup_{f\in S} f(U) = R(S)$*,*

then U is called a fundamental set for S.

The set $R(S)$ is defined and discussed in Remark 4.1 of Section 4. On the basis of Definition 1.3, we obtain the following result.

Theorem 1.2 *Let S be a holomorphic semigroup and U a partial fundamental set for S. Then $U \subset F(S)$. If, in addition, S is a transcendental semigroup and U is a fundamental set, then $U \subset I(S)$.*

The organization of this paper is as follows: In Section 2, we briefly review the notion of finite index subsemigroups and cofinite index subsemigroups with suitable examples, we review some results from rational (sub)semigroup dynamics, and we extend the same in transcendental (sub)semigroup dynamics. We introduce the Rees index of a subsemigroup, and we prove the dynamical similarity of a holomorphic semigroup and its subsemigroup. In Section 3, we prove Theorem 1.1, and we also prove it without the abelian condition for the subsemigroup having finite Rees index. In Section 4, we define discontinuous semigroups, and on the basis of this notion, we discuss partial fundamental sets and fundamental sets, and then we prove Theorem 1.2.

2. Results from general holomorphic (sub)semigroup dynamics

There are various notions of how large a substructure is inside of an algebraic object in order that the two structures share certain properties. One such a notion is *index*, and it plays an important role in general group theory and semigroup theory. It is used to measure the difference between a group (semigroup) and a subgroup (subsemigroup). It occurs in many important theorems of the group theory and semigroup theory. The notions of finite index, cofinite index and Rees index of subsemigroup have been used to gauge the size of subsemigroup. If the subsemigroup T is big enough in semigroup S, then S and T share many properties. In this context, Theorem 1.1 states that if T has finite index or cofinte index in S, then both S and T share the same Fatou, Julia, and escaping sets. In the semigroup theory, the cofinite index is also known as *Grigorochuk index*, and this index was introduced by Grigorochuk [3] in 1988. Maltcev and Ruskuc [7, Theorem 3.1] proved that for every element f of a finitely generated semigroup S and every proper cofinite indexed subsemigroup T, one has $f \circ T \neq S$. Note that if the semigroup is a group, the notion of finite index and cofinite index coincide. The subsemigroup T of a finitely generated semigroup S consisting of all words of finite length (compositions of a finite number of holomorphic functions) has a finite index and a cofinite index in S.

From Definition 1.2, the finite index and cofinite index of subsemigroups of the following examples will be clear.

Example 2.1 *A subsemigroup*

$$T = \langle \sin\sin z,\ \cos\cos z,\ \sin\cos z,\ \cos\sin z\rangle$$

of the transcendental semigroup $S = \langle \sin z, \cos z\rangle$ has finite index 3 and cofinite index 2.

Example 2.2 *A subset $T = \{$words (compositions) begining with $f\}$ of a holomorphic semigroup $S = \langle f, g\rangle$ is clearly a subsemigroup of S. Then T has an infinite index but cofinite index 1 in S.*

Note that in Example 2.2, S is finitely generated but T is not. Since any generating set of T must contain $\{f \circ g^n : n \geq 1\}$. The only cofinite subsemigroup of T is T itself. Hence, T has cofinite index 1 in S.

Example 2.3 *Let $S = \langle f \rangle$ where f is a holomorphic function. Then the subsemigroup $T = \langle f^n : n \in \mathbb{N} \rangle$. has finite index n in S and cofinite index 1 in S.*

Note that in Example 2.3, the subsemigroup T has n different translates in S, which are $T, f \circ T, \ldots, f^{n-1} \circ T$. Here, the only cofinite subsemigroup of T is T itself. If we choose the subsemigroup of S to be S itself, then there are infinitely many translates of S, namely, $h \circ S = h \circ \langle f \rangle$ for all $h \in S$. So S has an infinite index in itself. Again, it has cofinite index 1 in itself.

Using Theorem 3.1 of [11], we can prove the following assertion:

Lemma 2.1 *For any subsemigroup T of a holomorphic semigroup S, we have $F(S) \subset F(T), J(S) \supset J(T)$.*

Proof We prove that $F(S) \subset F(T)$. The next inclusion follows taking the complements. By Theorem 3.1 of [11], $F(S) \subset \cap_{f \in S} F(f)$, and $F(T) \subset \cap_{g \in T} F(g)$ for any subsemigroup T of the semigroup S. Since any $g \in T$ is also in S; thus, by the same Theorem 3.1 of [11], we also have $F(S) \subset F(g)$ for all $g \in T$ and hence, $F(S) \subset \cap_{g \in T} F(g)$. Now for any $z \in F(S)$, we have $z \in \cap_{g \in T} F(g)$ for all $g \in T$. This implies that $z \in F(g)$ for all $g \in T$. This proves $z \in F(T)$ and hence, $F(S) \subset F(T)$. □

Hinkannen and Martin [4, Theorem 2.4] proved that if a subsemigroup T has a finite index or a cofinite index in the rational semigroup S, then $F(S) = F(T)$ and $J(S) = J(T)$. In the following theorem, we prove the same result in the case of a general holomorphic semigroup. Note that by a general holomorphic semigroup, we mean either a rational semigroup or a transcendental semigroup.

Theorem 2.1 *If a subsemigroup T has a finite index or a cofinite index in the holomorphic semigroup S, then $F(S) = F(T)$ and $J(S) = J(T)$.*

Proof From Lemma 2.1, $F(S) \subset F(T)$ for any holomorphic semigroup S. If S is a rational semigroup, the result follows from [4, Theorem 2.4]. We prove the reverse inclusion, if S is a transcendental semigroup.

Let the subsemigroup T of a semigroup S has finite index n. Then by Definition 1.2, there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that

$$S = f_1 \circ T \cup f_2 \circ T \cup \ldots \cup f_n \circ T.$$

Then for any $g \in S$, there is an $h \in T$ such that $g = f_i \circ h$. Choose a sequence $(g_j)_{j \in \mathbb{N}}$ in S. Then each g_j is of the form $g_j = f_i \circ h_j$, where $h_j \in T$ and $1 \leq i \leq n$. Here, we may assume the same i for all j. Hence, without loss of generality, we may choose a subsequence (g_{j_k}) of (g_j) such that $g_{j_k} = f_i \circ h_{j_k}$ for particular f_i, where (h_{j_k}) is a subsequence of (h_j) in T. Since on $F(T)$, the sequence (h_{j_k}) has a convergent subsequence so do the sequences (g_{j_k}) and (g_j) in $F(S)$. This proves that $F(T) \subset F(S)$.

Let the subsemigroup T of a semigroup S have cofinite index n. Then by Definition 1.2, there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that for every $f \in S$, there is $i \in \{1, 2, \ldots, n\}$ such that $f_i \circ f \in T$. Let us choose a sequence $(g_j)_{j \in \mathbb{N}}$ in S. Then, for each j, there is an i with $1 \leq i \leq n$ such that $f_i \circ g_j = h_j \in T$. Let $z \in F(T)$. Then the sequence (h_j) has a convergent subsequence in T, and hence so does the sequence (g_j) in $F(S)$. This proves that $F(T) \subset F(S)$. □

Next, we see a special subsemigroup of a holomorphic semigroup that yields a cofinite index.

Definition 2.1 (Stablizer, wandering component and stable domains) *For a holomorphic semigroup S, let U be a component of the Fatou set $F(S)$ and U_f be a component of the Fatou set containing $f(U)$ for some $f \in S$. The set of the form*

$$S_U = \{f \in S : U_f = U\}$$

is called the stabilizer of U on S. If S_U is nonempty, we say that a component U satisfying $U_f = U$ is a stable basin for S. The component U of $F(S)$ is said to be wandering if the set $\{U_f : f \in S\}$ contains infinitely many elements. That is, U is a wandering domain if there is sequence $(f_i)_{i\in\mathbb{N}}$ of elements of S such that $U_{f_i} \neq U_{f_j}$ for $i \neq j$.

Note that for any rational function f, we always have $U_f = U$. So U_S is nonempty for a rational semigroup S. However, if f is transcendental, it is possible that $U_f \neq U$. Hence, S_U may be empty for a transcendental semigroup S. Bergweiler and Rohde [1] proved that $U_f - U$ contains at most one point which is an asymptotic value of f if f is an entire function.

Lemma 2.2 *Let S be a holomorphic semigroup. Then the stabilizer S_U (if it is nonempty) is a subsemigroup of S and $F(S) \subset F(S_U)$, $J(S) \supset J(S_U)$.*

Proof Let $f, g \in S_U$. Then by Definition 2.1, $U_f = U$ and $U_g = U$, where U_f and U_g are components of the Fatou sets containing $f(U)$ and $g(U)$, respectively. Then $f(U) \subseteq U_f = U$ and $g(U) \subseteq U_g = U \implies (f \circ g)(U) = f(g(U)) \subseteq f(U_g) = f(U) \subseteq U_f = U$. Since $(f \circ g)(U) \subseteq U_{f\circ g}$, so either $U_{f\circ g} \subseteq U$ or $U \subseteq U_{f\circ g}$. The only possibility in this case is $U_{f\circ g} = U$. Hence, $f \circ g \in S_U$, which proves that S_U is a subsemigroup of S. The proofs of $F(S) \subset F(S_U)$, $J(S) \supset J(S_U)$ follow from Lemma 2.1. □

There may be a connection between having no wandering domains and the stable basins of cofinite index. We have established the connection in the following theorem for a general holomorphic semigroup S.

Theorem 2.2 *Let S be a holomorphic semigroup with no wandering domains. Let U be any component of the Fatou set. Then the forward orbit $\{U_f : f \in S\}$ of U under S contains a stabilizer of U of cofinite index.*

Proof If S is a rational semigroup, see, for instance, the proof of [4, Theorem 6.1]. If S is a transcendental semigroup, we sketch our proof in the following way.

We are given that U is a nonwandering component of the Fatou set $F(S)$. So U has a finite forward orbit $U_1, U_2, \ldots, U_n$ (say) with $U_1 = U$.
Case (i): If for every $i = 1, 2, \ldots n$, there is $f_i \in S$ such that $f_i(U_i) \subseteq U_1$, then by Lemma 2.2 the stabilizer $S_{U_1} = \{f \in S : U_{1f} = U_1\}$ is a subsemigroup of S. For any $f \in S$, there is f_i for each $i = 1, 2, \ldots, n$ such that $U_{1_{f_i\circ f}} = U_1$. This shows that $f_i \circ f \in S_{U_1}$. Therefore, U_1 is a required stable basin such that the stabilizer S_{U_1} has a cofinite index in S.
Case (ii): If, for every $j = 2, \ldots n$, there is $f_j \in S$ such that $f_j(U_j) \subseteq V$, where $V = U_j$ such that $j \geq 2$, then the number of components of forward orbits of V is strictly less than that of U. In this way, we can find a component $W = U_i$ for some $i \leq n$ whose forward orbit has fewest components. For every component W_g of the forward orbit of W, there is $f \in S$ such that $f(W_g) \subseteq W$. That is, $W_{g\circ f} = W$, and it follows that W is a required stable basin such that the stabilizer S_W has a cofinite index. □

Let S be a holomorphic semigroup and $f \in S$. Then $S \circ f$ and $f \circ S$ are subsemigroups of S. Note that $S \circ f$ and $f \circ S$ may not be finitely generated even if the semigroup S is. If $S \circ f = \langle f_1, f_2, \ldots, f_n \rangle$ where $f_i \in S$ for $i = 1, 2, \ldots n$, then $f_i = g_i \circ f$, where $g_i \in S$. For any $g \in S$, we have $g^n \circ f \in S \circ f$ for all $n \geq 1$ but not every $g^n \circ f \in \langle f_1, f_2, \ldots, f_n \rangle$. From this fact, we came to know that the notion of cofinite index fails to preserve the basic finiteness (finitely generated) condition of a subsemigroup. That is, if T is a subsemigroup of cofinite index in semigroup S, then S being finitely generated may not always imply that T is finitely generated. There is another notion of index which preserves the finiteness condition of a subsemigroup.

Definition 2.2 (Rees index) *Let S be a semigroup and T be a subsemigroup. The Rees index of T in S is defined as $|S-T|+1$, where $|S-T|$ represents the cardinality of $S-T$. In this case, T is a large subsemigroup of S, and S is a small extension of T.*

The Rees index was first introduced by Jura [5] in the case where T is an ideal of the semigroup S. In such a case, the Rees index of T in S is the cardinality of factor semigroup S/T. From Definition 2.2, it is clear that the Rees index of T in S is the size of the complement $S - T$. For a subsemigroup to have finite Rees index in its parent semigroup is a fairly restrictive property, and it occurs naturally in semigroups (for instance, all ideals in the additive semigroup of positive integers are of finite Rees index). Note that Rees index does not generalize group index, and even the notion of finite Rees index does not generalize finite group index. That is, if G is an infinite group and H is a proper subgroup, the group index of H in G may be finite even though the Rees index is infinite. In fact, let G be an infinite group and H is a subgroup of G. Then H has finite Rees index in G if and only if $H = G$.

Next, we investigate how similar a semigroup S and its large subsemigroup T are. One basic similarity (proved first by Jura [5]) is the following result.

Lemma 2.3 *Let T be a large subsemigroup of a semigroup S. Then S is finitely generated if and only if T is finitely generated.*

Proof See for instance [10, Theorem 1.1]. □

On the basis of Lemma 2.3, we obtain the following dynamical similarity of a holomorphic semigroup and its subsemigroup.

Theorem 2.3 *Let T be a large subsemigroup of a finitely generated holomorphic semigroup S. Then $F(S) = F(T)$ and $J(S) = J(T)$.*

Proof We prove that $F(S) = F(T)$. The other equality follows by taking complements. By Lemma 2.1, it is clear that $F(S) \subset F(T)$. Hence, it is sufficient to prove that $F(T) \subset F(S)$. By Lemma 2.3, T is finitely generated. Let $X = \{f_1, f_2, \ldots, f_n\} \subset S$ be a generating set of T. Clearly, S is generated by the set $Y = X \cup (S - T)$. Every sequence (f_i) in $F(T)$ (where $f_i = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}$, and $i_n \in \{1, 2, \ldots, n\}$) has a convergent subsequence. Now each element g_m of a sequence (g_m) in S can be written as $g_m = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n} \circ h_{j_1} \circ h_{j_2} \circ \ldots \circ h_{j_k}$, where $S - T = \{h_1, h_2, \ldots, h_k\} \subset S$ and $j_k \in \{1, 2, \ldots, k\}$. Since $S - T$ is finite, a convergent sequence in $F(T)$ can be extended to a convergent sequence in $F(S)$. Thus, every sequence (g_m) in $F(S)$ has a convergent subsequence. Hence $F(T) \subset F(S)$. □

3. Proof of Theorem 1.1

We now prove a result analogous to Lemma 2.1 in the case of an escaping set of a transcendental semigroup.

Lemma 3.1 *For any subsemigroup T of a transcendental semigroup S, we have $I(S) \subset I(T)$.*

Proof By Theorem 3.1 of [11], $I(S) \subset \cap_{f \in S} I(f)$ and $I(T) \subset \cap_{g \in T} I(g)$ for any subsemigroup T of S. Since $T \subset S$, the same theorem implies that $I(S) \subset I(g)$ for all $g \in T$. Hence, $I(S) \subset \cap_{g \in T} I(g)$. Now for any $z \in I(S)$, we have $z \in \cap_{g \in T} I(g)$ for all $g \in T$. This implies that $z \in I(g)$ for all $g \in T$. By Definition 1.1, we have $g^n(z) \to \infty$ as $n \to \infty$ for all $g \in T$. This proves that $z \in I(T)$ and hence, $I(S) \subset I(T)$. □

Lemma 3.2 *Let S be a transcendental semigroup. Then*

1. *int.$(I(S)) \subset F(S)$ and ext.$(I(S)) \subset F(S)$, where int. and ext. respectively denote the interior and exterior of $I(S)$.*
2. *$\partial I(S) = J(S)$, where $\partial I(S)$ denotes the boundary of $I(S)$.*

Proof We refer to Lemma 4.2 and Theorem 4.3 of [6]. □

Note that Lemma 3.2 is an extension of Eremenko's result [2], $\partial I(f) = J(f)$, of classical transcendental dynamics to more general semigroup dynamics. We prove the following assertion which can be an alternative definition of escaping set.

Lemma 3.3 *If $z \in \mathbb{C}$ is an escaping point of a transcendental semigroup S, then every nonconvergent sequence in S has a divergent subsequence at z.*

Proof Let $z \in \mathbb{C}$ be an escaping point of a transcendental semigroup S. Let $f \in S$. Then by Definition 1.1, there is a sequence $(g_n)_{n \in \mathbb{N}}$ in S representing $g_1 = f$, $g_2 = f^2, \ldots, g_n = f^n, \ldots$ (say) such that $g_n(z) \to \infty$ as $n \to \infty$ or there is a sequence in S which contains $(g_n)_{n \in \mathbb{N}}$ as a subsequence such that $g_n(z) \to \infty$ as $n \to \infty$. More generally, every nonconvergent sequence in S has a subsequence which diverges infinity at z. □

We are now ready to prove Theorem 1.1.

Proof [Proof of Theorem 1.1] We prove $I(S) = I(T)$. The fact that $J(S) = J(T)$ is obvious from Lemma 3.2 (2). That $F(S) = F(T)$ is also obvious. By Lemma 2.1, we always have $I(S) \subset I(T)$ for any subsemigroup T of S. For proving this theorem, it is enough to show the reverse inclusion $I(T) \subset I(S)$.

Let a subsemigroup T of a semigroup S have finite index n. Then, by Definition 1.2, there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that

$$S = f_1 \circ T \cup f_2 \circ T \cup \ldots \cup f_n \circ T.$$

Then, for any $g \in S$, there is $h \in T$ such that $g = f_i \circ h$. Choose a sequence $(g_j)_{j \in \mathbb{N}}$ in S. Then each g_j is of the form $g_j = f_i \circ h_j$, where $h_j \in T$, $1 \leq i \leq n$. Here, we may assume the same i for all j. Let $z \in I(T)$. Then by Lemma 3.3, every nonconvergent sequence $(h_j)_{j \in \mathbb{N}}$ in T has a divergent subsequence $(h_{j_k})_{j_k \in \mathbb{N}}$ at the point z. That is, $h^n_{j_k}(z) \to \infty$ as $n \to \infty$ for all j_k. In this case, every sequence $(g_j)_{j \in \mathbb{N}}$ in S has a subsequence $(g_{j_k})_{k \in \mathbb{N}}$, where $g_{j_k} = f_i \circ h_{j_k}$ with $h^n_{j_k}(z) \to \infty$ as $n \to \infty$. Since S is an abelian transcendental semigroup, $g_{j_k} = f_i \circ h_{j_k} = h_{j_k} \circ f_i$. Thus, we may write $g_{j_k}^n(z) = h_{j_k}^n(f_i(z)) \to \infty$ as $n \to \infty$. This shows that $f_i(z) \in I(S)$. If f_i = identity for a particular i, we are done. If f_i is not identity, then it is an element of an abelian transcendental semigroup S, and in this case $I(S)$ is backward invariant by [12, Theorem 2.6]. Thus, we must have $z \in I(S)$. Therefore, $I(T) \subset I(S)$.

Let a subsemigroup T of a semigroup S have cofinite index n. Then by Definition 1.2, there exists a finite collection $\{f_1, f_2, \dots, f_n\}$ of elements of S^1 such that for every $f \in S$, there is $i \in \{1, 2, \dots, n\}$ such that $f_i \circ f \in T$. Let us choose a sequence $(g_j)_{j\in\mathbb{N}}$ in S. Then for each j, there is a i with $1 \leq i \leq n$ such that $f_i \circ g_j = h_j \in T$. Let $z \in I(T)$. Then by Lemma 3.3, every nonconvergent sequence $(h_j)_{j\in\mathbb{N}}$ in T has a divergent subsequence $(h_{j_k})_{j_k\in\mathbb{N}}$ at the point z. This follows that sequence $(f_i \circ g_j)$ has a divergent subsequence $(f_i \circ g_{j_k})$ (say) at z. Since S is abelian, we can write that $(f_i \circ g_{j_k})(z) = (g_{j_k} \circ f_i)(z) = g_{j_k}(f_i(z)) = h_{j_k}(z)$. Now for any $z \in I(T)$, $h_{j_k} \in T$, we must have $h_{j_k}^n(z) = g_{j_k}^n(f_i(z)) \to \infty$ as $n \to \infty$. This implies that $f_i(z) \in I(S)$. If f_i = identity for a particular i, we are done. If f_i is not the identity, then of it is an element of abelian transcendental semigroup S. Then as in the first part, we write that $I(T) \subset I(S)$. □

The abelian hypothesis can be deleted from Theorem 1.1 if we use the Rees index. Thus, we have the following generalization of Theorem 1.1.

Theorem 3.1 *If a subsemigroup T of a finitely generated transcendental semigroup S has a finite Rees index, then $I(S) = I(T), J(S) = J(T)$ and $F(S) = F(T)$.*

Proof If we prove $I(S) = I(T)$, then the equality $J(S) = J(T)$ will follow from Lemma 3.2 (2). The inclusion $I(S) \subset I(T)$ follows from Lemma 3.1. Thus, we prove $I(T) \subset I(S)$.

By Theorem 2.3, T is finitely generated. Let $X = \{f_1, f_2, \dots, f_n\} \subset S$ be a generating set of T. Clearly, S is generated by the set $Y = X \cup (S - T)$. By Lemma 3.3, every nonconvergent sequence (f_i) in T (where $f_i = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}$, and $i_n \in \{1, 2, \dots, n\}$) has a divergence subsequence (f_{n_k}) at each point of $I(T)$. Now each element g_m of the sequence (g_m) in S can be written as $g_m = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n} \circ h_{j_1} \circ h_{j_2} \circ \cdots \circ h_{j_k}$, where $S - T = \{h_1, h_2, \dots, h_k\} \subset S$ is a finite set and $j_k \in \{1, 2, \dots, k\}$. This shows that a divergent sequence in $I(T)$ can be extended to a divergent sequence in $I(S)$. Thus, every sequence (g_m) in $I(S)$ has a divergent subsequence. Hence $I(T) \subset I(S)$. □

4. Proof of Theorem 1.2

It is known that for certain holomorphic semigroups, the Fatou sets and the escaping sets might be empty. In this section, we discuss the notion of discontinuous semigroup. This notion yields a partial fundamental set and a fundamental set. We prove Theorem 1.2 by showing that a partial fundamental set is in the Fatou set $F(S)$ and that a fundamental set is in the escaping set $I(S)$.

Definition 4.1 (Discontinuous semigroup) *A semigroup S is said to be discontinuous at a point $z \in \mathbb{C}$ if there is a neighborhood U of z such that $f(U) \cap U = \emptyset$ for all $f \in S$ or equivalently, translates of U by distinct elements of S (S-translates) are disjoint. The neighborhood U of z is also called a nice neighborhood of z.*

Remark 4.1 *Given a holomorphic semigroup S, there are two natural subsets associated with S.*

1. *The regular set $R(S)$ that consists of points $z \in \mathbb{C}$ at which S is discontinuous.*

2. *The limit set* $L(S)$ *that consists of points* $z \in \mathbb{C}$ *for which there is a point* z_0, *and a sequence* (f_n) *of distinct elements of* S *such that* $f_n(z_0) \to z$ *as* $n \to \infty$.

A set $X \subset \mathbb{C}$ is S-invariant or invariant under S if $f(X) = X$ for all $f \in S$. It is clear that both of the sets $R(S)$ and $L(S)$ are S-invariant. If U is a nice neighborhood, then $U \subset R(S)$. Thus, $R(S)$ is an open set, whereas the set $L(S)$ a closed set, and $R(S) \cap L(S) = \emptyset$. Recall that a set U is a *partial fundamental set* for the semigroup S if (1) $U \neq \emptyset$, (2) $U \subset R(S)$, (3) $f(U) \cap U = \emptyset$ for all $f \in S$. If in addition to (1), (2), and (3), U satisfies the property (4) $\bigcup_{f \in S} f(U) = R(S)$, then U is called a *fundamental set* for S. We say that $x, y \in \mathbb{C}$ are S- equivalent if there is an $f \in S$ such that $f(x) = y$. Condition (3) asserts that no two points of U are S-equivalent under semigroup S, and condition (4) asserts that every point of $R(S)$ is equivalent to some point of U. Note that if we replace (3) by $f^{-1}(U) \cap U = \emptyset$ for all $f \in S$, we say U is a backward partial fundamental set for S; if, in addition, U satisfies $\bigcup_{f \in S} f^{-1}(U) = R(S)$, then we say U is a backward fundamental set. Note that Theorems 1.2 and 4.1 hold if we have given (partial) backward fundamental set in the statements. Similar to the results of Hinkkanen and Martin [4, Lemma 2.2] in the case of a rational semigroup, we prove the following in the case of transcendental semigroup S.

Proof [Proof of Theorem 1.2] Let S be a holomorphic semigroup. The set U is a nonempty open set, and $f(U) \cap U = \emptyset$ for all $f \in S$ by Definition 4.1. The statement $f(U) \cap U = \emptyset$ for all $f \in S$ implies that S omits U on U. Since U is open, it contains more than two points. Then by Montel's theorem, S is normal on U. Therefore, $U \subset F(S)$.

Let S be a transcendental semigroup. To prove $U \subset I(S)$, we have to show that $f^n(z) \to \infty$ as $n \to \infty$ for all $f \in S$ and for all $z \in U$. The condition $f(U) \cap U = \emptyset$ for all $f \in S$ implies that $f^n(U) \cap U = \emptyset$, since $f \in S$ implies $f^n \in S$. Also, U is a fundamental set, so by Definition 1.3 (4), we have $\bigcup_{f \in S} f(U) = R(S)$. By Remark 4.1(2), there are no points in U which appear as the limit points under distinct $(f_m)_{m \in \mathbb{N}}$ in S. That is, (f_m) has a divergent subsequence (f_{m_k}) at each point of U. Thus, by [11, Theorem 2.2], for any $z \in U$, $f^n(z) \to \infty$ as $n \to \infty$ for any $f \in (f_m)$. This shows that $U \subseteq I(S)$. □

Finally, we generalize Theorem 1.2 in the following form. We give a short sketch of the proof. For a more detailed proof, we refer to [9, Theorem 2.1].

Theorem 4.1 *Let* U_1 *and* U_2 *be two (partial) fundamental sets for transcendental semigroups* S_1 *and* S_2, *respectively. Suppose furthermore that* $\mathbb{C} \setminus U_1 \subset U_2$ *and* $\mathbb{C} \setminus U_2 \subset U_1$. *Then the semigroup* $S = \langle S_1, S_2 \rangle$ *is discontinuous, and* $U = U_1 \cap U_2$ *is a (partial) fundamental set for the semigroup* S.

Proof [Sketch of the proof] Let U_1, U_2 and S_1, S_2 be as given in the theorem. It is clear from Theorem 1.2 that $F(S_1) \neq \emptyset$, $F(S_2) \neq \emptyset$; also $I(S_1) \neq \emptyset$ and $I(S_2) \neq \emptyset$ if U_1 and U_2 are fundamental sets of S_1 and S_2 respectively. Note that $U \neq \emptyset$ by the assumption. Clearly, $f(U) \cap U = \emptyset$ for every $f \in S$. This proves S is discontinuous and that U is a (partial) fundamental set for S. □

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References

[1] Bergweiler W, Rohde S. Omitted value in the domains of normality. Proc Amer Math Soc 1995; 123: 1857-1858.

[2] Eremenko A. On the iterations of entire functions. Dynamical System and Ergodic Theory, Banach Center Publication 1989; Vol. 23: 339-345.

[3] Grigorochuk RI. Semigroups with cancellations of degree growth. Mat Zamekti 1988; 43, No. 3: 305-319.

[4] Hinkkanen A, Martin GJ. The dynamics of semigroups of rational functions- I. Proc London Math Soc 1996; (3) 73: 358-384.

[5] Jura A. Determining ideals of a given finite index in a finitely presented semigroup. Demonstratio Math. 1978; 11: 813-827.

[6] Kumar D, Kumar S. The dynamics of semigroups of transcendental entire functions-II. arXiv: 1401.0425 v3 [math.DS], 22 May 2014.

[7] Maltcev V, Ruskuc, M. On hopfian cofinite subsemigroups. arXiv: 1307. 6929 v1 [math. GR], 26 Jul. 2013.

[8] Poon KK. Fatou-Julia theory on transcendental semigroups. Bull Austral Math Soc 1998; Vol. 58: 403-410.

[9] Poon KK. Fatou-Julia theory on transcendental semigroups II. Bull Austral Math Soc 1999; Vol. 55: 257-262.

[10] Ruskuc N. On large subsemigroups and finiteness conditions of semigroups. Proc London Math Soc 1998; (3) 76 : 383-405.

[11] Subedi BH, Singh A. A short comparison of classical complex dynamics and holomorphic semigroup dynamics. arXiv: 1803.08368 v1 [math. DS], 22 March 2018.

[12] Subedi BH, Singh A. Invariant properties of Fatou set, Julia set and escaping set of holomorphic semigroup. arXiv: 1803.09662 v1 [math. DS], 23 March 2018.

[13] Subedi BH, Singh A. Escaping sets of hyperbolic semigroups. The Nepali Mathematical Sciences Report 2018; 35 (1): 45-52.

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WANDERING DOMAINS OF THREE TRANSCENDENTAL ENTIRE FUNCTIONS AND THEIR COMPOSITIONS

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We prove that there exist three transcendental entire functions that can have infinite number of domains which lie in the wandering component of the Fatou set of each of these functions and their compositions. This result is a generalization of a result of [5].

1. Introduction

We denote the complex plane by $\mathbb{C}$, extended complex plane by $\mathbb{C}_\infty$ and set of integers greater than zero by $\mathbb{N}$. We assume the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is an entire function unless stated otherwise. For any $n \in \mathbb{N}$, f^n always denotes the nth iterate of f. If $f^n(z) = z$ for some smallest $n \in \mathbb{N}$, then we say that z is a *periodic point* of *period* n. In particular, if $f(z) = z$, then z is a *fixed point* of f. If $|(f^n)'(z)| < 1$, where $'$ represents complex differentiation of f^n with respect to z, then z is called an *attracting periodic point*. A family $\mathcal{F} = \{f : f$ is meromorphic on some domain X of $\mathbb{C}_\infty\}$ forms *normal family* if every sequence $(f_i)_{i\in\mathbb{N}}$ of functions contains a subsequence which converges uniformly to a finite limit or converges to ∞ on every compact subset D of X.

The *Fatou set* of f, denoted by $F(f)$, is the set of points $z \in \mathbb{C}$ such that sequence $(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighborhood of z. By definition, Fatou set is open and may or may not

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be empty. Fatou set is non-empty for every entire function with attracting periodic points. A connected component of the Fatou set is called a *Fatou component.* The complement of $F(f)$ denoted by $J(f)$ is called a *Julia set*.

If $U \subset F(f)$ (a Fatou component), then $f(U)$ lies in some component V of $F(f)$ and $V - f(U)$ is a set which contains at most one point (see for instance [4]). Let $U \subset F(f)$ such that $f^n(U)$ for some $n \in \mathbb{N}$, is contained in some component of $F(f)$, which is usually denoted by U_n. A Fatou component U is called a *pre-periodic* if there exist integers n, m with $n > m \geq 0$ such that $U_n = U_m$. In particular, if $U_n = U_0 = U$ (that is, $f^n(U) \subset U$) for some smallest positive integer $n \geq 1$, then U is called a *periodic Fatou component* of period n and $\{U_0, U_1, \dots, U_{n-1}\}$ is called the *periodic cycle* of U. A component of Fatou set $F(f)$ which is not pre-periodic is called an *wandering domain*. That is, a Fatou component U is a wandering domain if $U_m \neq U_n$, for all positive integers $m \neq n$.

Our particular interest of this paper is that whether there are more than two transcendental entire functions that can have similarity between the dynamics of their compositions and dynamics of each of these individual functions. Dynamics of two transcendental entire functions and their compositions were studied by Singh [7]. He constructed some examples of transcendental entire functions, where dynamics of individual functions is similar to the dynamics of their compositions. In particular, Singh proved that there exist two transcendental entire functions and a domain which lies in the wandering component of individual functions and also lies in the wandering component of their compositions (Theorem 2). Later, Dinesh Kumar et al. [5] extended this result to the possibility of having infinitely many domains satisfying the condition in Singh's result. In this paper, we investigate three transcendental entire functions such that each of individual functions as well as their every composition consists of infinite number of domains which lie in the wandering component of each of functions and their compositions. In particular, we prove the following result.

Theorem 1.1. *There exist three different transcendental entire functions f, g and h and infinitely many domains which lie in different wandering components of $F(f), F(g), F(h), F(f \circ g), F(g \circ f), F(f \circ h), F(g \circ h), F(h \circ f), F(h \circ g), F(f \circ g \circ h), F(f \circ h \circ g), F(g \circ f \circ h), F(g \circ h \circ f), F(h \circ f \circ g)$ and $F(h \circ g \circ f)$.*

2. Carleman Set

To work out a proof of Theorem 1.1, we need a notion in approximation theory of entire functions. In our case, we can use the notion of Carleman set from which we obtain approximation of any holomorphic function by entire functions.

Definition 2.1 (Carleman set). Let F be a closed subset of $\mathbb{C}$ and $C(F) = \{f : F \longrightarrow \mathbb{C} : f$ is continuous on F and holomorphic in the interior $F^{\circ} of F\}$. Then F is called a *Carleman set* (for $\mathbb{C}$) if for any $g \in C(F)$ and any positive continuous function ϵ on F, there exists entire function h such that $|g(z) - h(z)| < \epsilon$, for all $z \in F$.

The following important characterization of Carleman set was proved by Nersesjan in 1971, but we cite this from ([3], Theorem 4, p. 157).

Theorem 2.1. *Let F be a closed proper subset of $\mathbb{C}$. Then F is a Carleman set for $\mathbb{C}$ if and only if F satisfies the following conditions*:

(1) $\mathbb{C}_\infty - F$ *is connected*;

(2) $\mathbb{C}_\infty - F$ *is locally connected at* ∞;

(3) *for every compact subset K of $\mathbb{C}$, there is a neighborhood V of ∞ in $\mathbb{C}_\infty$ such that no component of F°, where F° represents interior of F, intersects both K and V.*

Note that the space $\mathbb{C}_\infty - F$ is connected if and only if each component Z of open set $\mathbb{C} - F$ is unbounded. This fact together with Theorem 2.1 can be a nice tool for checking whether a set is a Carleman set for $\mathbb{C}$. The sets given in the following examples are Carlemen sets for $\mathbb{C}$.

Example 2.1 ([3], Example page 133). The set $E = \{z \in \mathbb{C} : |z| = 1, \text{Re}z \geq 0\} \cup \{z = x : x > 1\} \cup \big(\bigcup_{n=3}^{\infty} \{z = re^{i\theta} : r > 1, \theta = \pi/n\}\big)$ is a Carleman set for $\mathbb{C}$ by Theorem 2.1.

Example 2.2 ([7], Set S, page 131). The set $E = G_0 \cup \big(\bigcup_{k=1}^{\infty} (G_k \cup B_K \cup L_k \cup M_k)\big)$, where

$$G_0 = \{z \in \mathbb{C} : |z-2| \leq 1\};$$

$$G_k = \{z \in \mathbb{C} : |z-(4k+2)| \leq 1\} \cup \{z \in \mathbb{C} : \text{Re}z = 4k+2,\ \text{Im}z \geq 1\} \cup \{z \in \mathbb{C} : \text{Re}z = 4k+2,\ \text{Im}z \leq -1\}, \quad (k = 1, 2, 3, \ldots);$$

$$B_k = \{z \in \mathbb{C} : |z+(4k+2)| \leq 1\} \cup \{z \in \mathbb{C} : \text{Re}z = -(4k+2), \text{Im}z \geq 1\} \cup \{z \in \mathbb{C} : \text{Re}z = -(4k+2),\ \text{Im}z \leq -1\}, \quad (k = 1, 2, 3, \ldots);$$

$$L_k = \{z \in \mathbb{C} : \text{Re}z = 4k\}, \quad (k = 1, 2, 3, \ldots);$$

and

$$M_k = \{z \in \mathbb{C} : \text{Re}z = -4k\}, \quad (k = 1, 2, 3, \ldots)$$

is a Carleman set for $\mathbb{C}$ by Theorem 2.1.

3. Proof of the Main Result (Theorem 1.1)

For a given transcendental entire function, we can find an entire function such that their composition can have a wandering domain as shown in the following result by Baker and Singh ([1], Theorem 3).

Proposition 3.1. *Let g be a transcendental entire function having at least one fixed point. Then there exists an entire function f such that $g \circ f$ has a wandering domain.*

The dynamics of one composition may help in the study of the dynamics of the other composition. In particular, if one composition has wandering domain, then so has the other composition. In this regard, Bergweiler and Wang [2] proved the following result which is a kind of generalization of Proposition 3.1.

Proposition 3.2. *Let f and g be entire functions. Then $f \circ g$ has wandering domains if and only if $g \circ f$ has wandering domains.*

It is also to be noted that certain classes of entire functions do not have wandering domains (see for instance [2], Theorem 3). It is known that the dynamics of $f \circ g$ are very similar to the dynamics of $g \circ f$. Singh [7] interested to know whether there is similarity between dynamics of individual entire functions and their compositions. However, in reality, it does not hold in general. With the help of Carleman set of Example 2.2, Singh ([7], Theorem 2) proved the following result which shows that certain amount of similarity might hold.

Proposition 3.3. *There exists two transcendental entire functions f and g and a domain U such that U lies in the wandering component of the $F(f), F(g), F(f \circ g)$ and $F(g \circ f)$.*

In fact, Singh also proved other results regarding the dynamics of two individual functions and their compositions (see for instance [7], Theorems 1, 3 and 4). Tomar [8] extended the result of Singh (Proposition 3.3) to the following results.

Proposition 3.4. *There exist two transcendental entire functions f and g and infinitely many domains in the angular region which lie in the wandering component of the $F(f), F(g), F(f \circ g)$ and $F(g \circ f)$.*

Note that Singh ([6], Theorems 3.2.1-3.2.6) studied different components of the Fatou set of a transcendental entire function in an angular region by using approximation theory of entire functions, in particular, with the help of Carleman set.

Dinesh Kumar et al. ([5], Theorem 2.2) extended these results to the following assertion.

Proposition 3.5. *There exist two different transcendental entire functions f and g and infinitely many domains which lie in different wandering component of the $F(f), F(g)$, $F(f \circ g)$ and $F(g \circ f)$.*

Theorem 1.1 is an extension of Propositions (3.3), (3.4), and (3.5). These propositions can be extended to the existence of more than two different transcendental entire functions such that each individual functions and their compositions may have infinitely many domains which lie in different wandering component of each of the functions and their compositions. We proceed for the following long proof of Theorem 1.1.

Proof of Theorem 1.1. Let

$$E = G_0 \cup \Big(\bigcup_{k=1}^{\infty} (G_k \cup B_K \cup L_k \cup M_k) \Big),$$

where G_0, G_k, B_k, L_k and M_k are sets as defined in Example 2.2. Then E is a Carleman set for $\mathbb{C}$. By the continuity of exponential function, for given $\epsilon > 0$, there exists $\delta > 0$, may depend on a given point w_0, such that

$$|w - w_0| < \delta \Longrightarrow |e^w - e^{w_0}| < \epsilon.$$

Let us choose $\epsilon = \frac{1}{2}$. Then there exist sufficiently small $\delta_k > 0,\ \delta_k^{'} > 0$ and $\delta_k^{''} > 0$ such that

$$|w - (\pi i + \log(4k+6))| < \delta_k \Longrightarrow |e^w + (4k+6)| < \frac{1}{2},\ (k = 1, 2, 3, \ldots);$$

$$|w - \log(4k-2)| < \delta_k^{'} \Longrightarrow |e^w - (4k+6)| < \frac{1}{2},\ (k = 1, 2, 3, \ldots);$$

and

$$|w - \log(4k-6)| < \delta_k^{''} \Longrightarrow |e^w - (4k-6)| < \frac{1}{2},\ (k = 3, 4, 5, \ldots);$$

In particular, let us choose sufficiently small $\delta_0 > 0, \lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$|w - \log 2| < \delta_0 \Longrightarrow |e^w - 2| < \frac{1}{2};$$

$$|w - (\pi i + \log 6)| < \lambda_1 \Longrightarrow |e^w + 6| < \frac{1}{2};$$

and

$$|w - (\pi i + \log 10)| < \lambda_2 \Longrightarrow |e^w + 10| < \frac{1}{2}.$$

Next, let us define the following functions:

$$\alpha(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \pi i + \log 6, & \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\ \pi i + \log(4k+6), & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{cases}$$

$$\beta(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \log(4k-2), & \forall z \in G_k,\ k = 2, 3, 4, \ldots; \\ \pi i + \log 6, & \forall z \in G_1; \\ \pi i + \log(4k+6), & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{cases}$$

$$\gamma(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \log(4k-6), & \forall z \in G_k,\ k = 3, 4, 5, \ldots; \\ \pi i + \log 6, & \forall z \in G_2; \\ \pi i + \log 10, & \forall z \in G_1; \\ \pi i + \log(4k+6), & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{cases}$$

Let us define the following functions:

$$\epsilon_1(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \lambda_1, & \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\ \delta_k, & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{cases}$$

$$\epsilon_2(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \delta_k^{'}, & \forall z \in G_k,\ k = 2, 3, 4, \ldots; \\ \lambda_1, & \forall z \in G_1; \\ \delta_k, & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{cases}$$

and

$$\epsilon_3(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ \delta_k^{''}, & \forall z \in G_k,\ k = 3, 4, 5, \ldots; \\ \lambda_1, & \forall z \in G_2; \\ \lambda_2, & \forall z \in G_1; \\ \delta_k, & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{cases}$$

Clearly, the functions $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ are piecewise constant functions, so they are continuous on the set E and holomorphic in E°. Also, since E is a Carleman set, so there exist entire functions $f_1(z)$, $g_1(z)$ and $h_1(z)$ such that, for all $z \in E$, $|f_1(z) - \alpha(z)| \leq \epsilon_1(z)$, $|g_1(z) - \beta(z)| \leq \epsilon_2(z)$ and $|h_1(z) - \gamma(z)| \leq \epsilon_3(z)$.

Consequently, we get transcendental entire functions $f(z) = e^{f_1(z)}$, $g(z) = e^{g_1(z)}$ and $h(z) = e^{h_1(z)}$ which respectively satisfy the following:

$$\begin{aligned} &|f(z) - 2| < \tfrac{1}{2}, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big); \\ &|f(z) + 6| < \tfrac{1}{2}, && \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\ &|f(z) + (4k+6)| < \tfrac{1}{2}, && \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{aligned} \tag{3.1}$$

$$\begin{aligned} &|g(z) - 2| < \tfrac{1}{2}, && \forall z \in G_0 \cup \big(\textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big); \\ &|g(z) - (4k-2)| < \tfrac{1}{2}, && \forall z \in G_k,\ k = 2, 3, 4, \ldots; \\ &|g(z) + 6| < \tfrac{1}{2}, && \forall z \in G_1; \\ &|g(z) + (4k+6)| < \tfrac{1}{2}, && \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &|h(z) - 2| < \tfrac{1}{2}, && \forall z \in G_0 \cup \textstyle\bigcup_{k=1}^{\infty}(L_k \cup M_k); \\ &|h(z) - (4k-6)| < \tfrac{1}{2}, && \forall z \in G_k, k = 3, 4, 5, \ldots; \\ &|h(z) + 6| < \tfrac{1}{2}, && \forall z \in G_2; \\ &|h(z) + 10| < \tfrac{1}{2}, && \forall z \in G_1; \\ &|h(z) + (4k+6)| < \tfrac{1}{2}, && \forall z \in B_k, k = 1, 2, 3, \ldots; \end{aligned} \tag{3.3}$$

From (3.1), (3.2) and (3.3), we can say that each of the functions f, g and h map $G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)$ into smaller disk $|z - 2| < \frac{1}{2}$ contained in G_0 and each of these function is a contracting mapping. So, $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$ contains fixed points z_1, z_2 and z_3 (say) such that

$$f^n(G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)) \longrightarrow z_1, \text{ as } n \longrightarrow \infty;$$

$$g^n(G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)) \longrightarrow z_2, \text{ as } n \longrightarrow \infty;$$

$$h^n(G_0 \cup \bigcup_{k=1}^{\infty}(L_k \cup M_k)) \longrightarrow z_3, \text{ as } n \longrightarrow \infty.$$

The fixed points z_1, z_2 and z_3 are respectively the attracting fixed points for the functions f, g and h, so $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$ lies in attracting cycle and hence $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$ is a subset of each of the Fatou set $F(f), F(g)$ and $F(h)$. Also, $J(f) \neq \mathbb{C}$, $J(g) \neq \mathbb{C}$ and

$J(h) \neq \mathbb{C}$ and so Julia set of each of the function f, g and h does not contain interior point and hence Fatou set of each of these function contains all interior points. Fatou set of each of the function f, g and h contains Carleman set E.

Also from (3.1), we can say that function f maps each G_k into smaller disk contained in B_1 and each B_k into smaller disk contained in B_{k+1}. In fact, G_k and B_k are contained in the wandering components of Fatou set $F(f)$. Again from (3.2), we can say that function g maps each of the domains G_k into the smaller disk contained in G_{k-1}, $(k = 2, 3, 4, \ldots)$, G_1 into smaller disk contained in B_1 and B_k, $(k = 1, 2, 3, \ldots)$ into the smaller disks contained in B_{k+1}. In fact, G_k and B_k are contained in the wandering components of the Fatou set $F(g)$. Likewise, from (3.3), we can say that domains G_k and $B_k, (k = 1, 2, 3, \ldots)$ are contained in the wandering components of $F(h)$.

Next, we examine the dynamical behavior of compositions of the functions f, g and h. The composite of any two and all of three of these functions satisfy the following.

Dynamical behavior of $f \circ g$:

$$\begin{array}{ll} |(f\circ g)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ |(f\circ g)(z) + 6| < \frac{1}{2}, & \forall z \in G_k,\ k = 2, 3, 4, \ldots; \\ |(f\circ g)(z) + 10| < \frac{1}{2}, & \forall z \in G_1; \\ |(f\circ g)(z) - (4k+10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{array} \tag{3.4}$$

The composition rule (3.4) shows that the domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ belong to $F(f\circ g)$ and in fact, each G_k and B_k is contained in the wandering components of $F(f \circ g)$.

Dynamical behavior of $g \circ f$:

$$\begin{array}{ll} |(g\circ f)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ |(g\circ f)(z) + 10| < \frac{1}{2}, & \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\ |(g\circ f)(z) + (4k+10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{array} \tag{3.5}$$

From the composition rule (3.5), we can say that the domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ belong to $F(g \circ f)$ and in fact, each G_k and B_k belongs to the wandering component of $F(g \circ f)$.

Dynamical behavior of $f \circ h$:

$$
\begin{array}{ll}
|(f \circ h)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(f \circ h)(z) + 14| < \frac{1}{2}, & \forall z \in G_1; \\
|(f \circ h)(z) + 10| < \frac{1}{2}, & \forall z \in G_2; \\
|(f \circ h)(z) + 6| < \frac{1}{2}, & \forall z \in G_k,\ k = 4, 5, 6, \ldots; \\
|(f \circ h)(z) + (4k + 10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots;
\end{array} \tag{3.6}
$$

As defined in above composition rule (3.6), the domains $G_0 \cup \left(\bigcup_{k=1}^{\infty} (L_k \cup M_k)\right)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ belong to $F(f \circ h)$ and in fact, each G_k and B_k, for all $k = 1, 2, 3, \ldots$ belongs to the wandering components of $F(f \circ h)$.

Dynamical behavior of $h \circ f$:

$$
\begin{array}{ll}
|(h \circ f)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(h \circ f)(z) + 10| < \frac{1}{2}, & \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\
|(h \circ f)(z) + (4k + 10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots;
\end{array} \tag{3.7}
$$

From the composition rule (3.7), we can say that the domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and B_k, for all $k = 1, 2, 3, \ldots$ belong to $F(h \circ f)$ and in fact, each G_k and B_k, for all $k = 1, 2, 3, \ldots$ is contained in the wandering components of $F(h \circ f)$.

Dynamical behavior of $g \circ h$:

$$
\begin{array}{ll}
|(g \circ h)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(g \circ h)(z) + 14| < \frac{1}{2}, & \forall z \in G_1; \\
|(g \circ h)(z) + 10| < \frac{1}{2}, & \forall z \in G_2; \\
|(g \circ h)(z) + 6| < \frac{1}{2}, & \forall z \in G_k;\ k = 3, 4, 5, \ldots; \\
|(g \circ h)(z) + (4k + 10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots;
\end{array} \tag{3.8}
$$

As defined in the above composition rule (3.8), the domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ belong to $F(g \circ h)$

and in fact, each G_k and B_k, for all $k = 1, 2, 3, \ldots$ is contained in wandering components of $F(g \circ h)$.

Dynamical behavior of $h \circ g$:

$$\begin{array}{ll}
|(h\circ g)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(h\circ g)(z) + 10| < \frac{1}{2}, & \forall z \in G_k, k = 1, 2; \\
|(h\circ g)(z) + 6| < \frac{1}{2}, & \forall z \in G_3, ; \\
|(h\circ g)(z) - (4k - 10)| < \frac{1}{2}, & \forall z \in G_k,\ k = 4, 5, 6, \ldots; \\
|(h\circ g)(z) + (4k + 10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots;
\end{array} \tag{3.9}$$

As defined in the above composition rule (3.9), the domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ belong to $F(h \circ g)$ and in fact, each G_k and B_k, for all $k = 1, 2, 3, \ldots$ is contained in wandering components of $F(h \circ g)$.

Dynamical behavior of $f \circ g \circ h$:

$$\begin{array}{ll}
|(f\circ g\circ h)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(f\circ g\circ h)(z) + 18| < \frac{1}{2}, & \forall z \in G_1; \\
|(f\circ g\circ h)(z) + 14| < \frac{1}{2}, & \forall z \in G_2; \\
|(f\circ g\circ h)(z) + 10| < \frac{1}{2}, & \forall z \in G_3; \\
|(f\circ g\circ h)(z) + 6| < \frac{1}{2}, & \forall z \in G_k,\ k = 4, 5, 6, \ldots; \\
|(f\circ g\circ h)(z) + (4k + 10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots;
\end{array} \tag{3.10}$$

The composition rule (3.10) assigned above tells us that domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ lie in $F(f \circ g \circ h)$ and in fact, each G_k and B_k, for all $k = 1, 2, 3, \ldots$ is contained in the wandering component of $F(f \circ g \circ h)$.

Dynamical behavior of $f \circ h \circ g$:

$$\begin{array}{ll}
|(f\circ h\circ g)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\
|(f\circ h\circ g)(z) + 14| < \frac{1}{2}, & \forall z \in G_k, \text{ for } k = 1, 2; \\
|(f\circ h\circ g)(z) + 10| < \frac{1}{2}, & \forall z \in G_3, \text{ for } k = 1, 2; \\
|(f\circ h\circ g)(z) + 6| < \frac{1}{2}, & \forall z \in G_k,\ k = 4, 5, 6, \ldots; \\
|(f\circ h\circ g)(z) + (4k + 10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots;
\end{array} \tag{3.11}$$

The composition rule (3.11) assigned above tells us that domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ lie in $F(f \circ g \circ h)$ and in fact, each G_k and B_k, for all $k = 1, 2, 3, \ldots$ is contained in the wandering component of $F(f \circ h \circ g)$.

Dynamical behavior of $g \circ f \circ h$:

$$\begin{array}{ll} |(g\circ f\circ h)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big); \\ |(g\circ f\circ h)(z) + 18| < \frac{1}{2}, & \forall z \in G_1; \\ |(g\circ f\circ h)(z) + 14| < \frac{1}{2}, & \forall z \in G_2; \\ |(g\circ f\circ h)(z) + 10| < \frac{1}{2}, & \forall z \in G_k,\ k = 3, 4, 5, \ldots; \\ |(g\circ f\circ h)(z) + (4k+10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{array} \tag{3.12}$$

The composition rule (3.12) assigned above tells us that domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ lie in $F(g \circ f \circ h)$ and in fact, each G_k and B_k, for all $k = 1, 2, 3, \ldots$ is contained in the wandering component of $F(g \circ f \circ h)$.

Dynamical behavior of $g \circ h \circ f$:

$$\begin{array}{ll} |(g\circ h\circ f)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big); \\ |(g\circ h\circ f)(z) + 14| < \frac{1}{2}, & \forall z \in G_k,\ k = 1, 2, 3, \ldots; \\ |(g\circ h\circ f)(z) + (4k+10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{array} \tag{3.13}$$

The composition rule (3.13) assigned above tells us that domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ lie in $F(g \circ h \circ f)$ and in fact, each G_k and B_k, for all $k = 1, 2, 3, \ldots$ is contained in the wandering component of $F(g \circ h \circ f)$.

Dynamical behavior of $h \circ f \circ g$:

$$\begin{array}{ll} |(h\circ f\circ g)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big); \\ |(h\circ f\circ g)(z) + 14| < \frac{1}{2}, & \forall z \in G_1; \\ |(h\circ f\circ g)(z) + 10| < \frac{1}{2}, & \forall z \in G_k,\ k = 2, 3, 4, \ldots; \\ |(h\circ f\circ g)(z) + (4k+10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \ldots; \end{array} \tag{3.14}$$

The composition rule (3.14) assigned above tells us that domains $G_0 \cup \big(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\big)$, G_k and B_k, $(k = 1, 2, 3, \ldots)$ lie in $F(h \circ f \circ g)$

and in fact, each G_k and B_k, for all $k = 1, 2, 3, \dots$ is contained in the wandering component of $F(h \circ f \circ g)$.

Dynamical behavior of $h \circ g \circ f$:

$$\begin{array}{ll} |(h \circ g \circ f)(z) - 2| < \frac{1}{2}, & \forall z \in G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right); \\ |(h \circ g \circ f)(z) + 14| < \frac{1}{2}, & \forall z \in G_k,\ k = 1, 2, 3, \dots; \\ |(h \circ g \circ f)(z) + (4k + 10)| < \frac{1}{2}, & \forall z \in B_k,\ k = 1, 2, 3, \dots; \end{array} \tag{3.15}$$

The composition rule (3.15) assigned above tells us that domains $G_0 \cup \left(\bigcup_{k=1}^{\infty}(L_k \cup M_k)\right)$, G_k and B_k, $(k = 1, 2, 3, \dots)$ lie in $F(h \circ g \circ f)$ and in fact, each G_k and B_k, for all $k = 1, 2, 3, \dots$ is contained in the wandering component of $F(h \circ g \circ f)$.

From all of the above discussion, we found that the domains G_k and B_k, for all $k = 1, 2, 3, \dots$ are contained in the wandering domains of the functions f, g, h and their compositions. □

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References

[1] I. N. Baker and A. P. Singh, Wandering domains in the iteration of compositions of entire functions, Ann. Acad. Sci. Fenn. Math., Series A. I. Math., **20**(1995), 149-153.

[2] W. Bergweiler and Y. Wang, On the dynamics of composite entire functions, Ark. Mat., **36**(1998), 31-39.

[3] A. Gaier, Lectures on Complex Approximation, Birkhauser, 1987.

[4] M. E. Herring, Mapping properties of Fatou components, Ann. Acad. Sci. Fenn. Math., **23**(1998), 263-274.

[5] D. Kumar, G. Datt and S. Kumar, Dynamics of composite entire functions, J. Indian Math. Soc., **82**(3-4)(2015), 107-114.

[6] A. Singh, Properties and structure of Fatou sets and Julia sets, Ph.D. Thesis, Univ. Jammu, Jammu and Kashmir, India, 2004.

[7] A. P. SINGH, On the dynamics of composition of entire functions, Math. Proc. Camb. Phil. Soc., **134**(2003), 129-138.

[8] G. TOMAR, On the dynamics of composition of transcendental entire functions in angular region-II, J. Indian Math. Soc., **84**(3-4)(2017), 287-296.

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ESCAPING SETS OF HYPERBOLIC SEMIGROUPS

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Abstract: *In this paper, we mainly study hyperbolic semigroups from which we get non-empty escaping sets and Eremenko's conjecture remains valid. We prove that if each generator of bounded type transcendental semigroups is hyperbolic, then the semigroups are themselves hyperbolic and all components of escaping sets are unbounded.*

Key Words: Escaping sets, Eremenko's conjecture, transcendental semigroups, post singularly bounded (finite) semigroups, hyperbolic semigroups.
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1. Introduction

Throughout this paper, we denote the *complex plane* by $\mathbb{C}$ and the set of integers greater than zero by $\mathbb{N}$. We assume the function $f : \mathbb{C} \to \mathbb{C}$ is *transcendental entire function* unless otherwise stated. For any $n \in \mathbb{N}$, f^n always denotes the nth *iterates* of f. The *escaping set* of f is defined by

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$$

and any point $z \in I(S)$ is called an *escaping point*. For a transcendental entire function f, the escaping set $I(f)$ was first studied by A. Eremenko [2]. He showed that

(1) $I(f) \neq \emptyset$,
(2) the boundary of this set is a Julia set $J(f)$ (that is, $J(f) = \partial I(f)$),
(3) $I(f) \cap J(f) \neq \emptyset$, and
(4) $\overline{I(f)}$ has no bounded component.

Furthermore,

5. $I(f^n) = I(f)$ for all $n \in \mathbb{N}$.
6. $I(f)$ is completely invariant under f.

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In view of the statement (4), he posed a question:

Is every component of $I(f)$ unbounded?

This question is considered as an important open problem of transcendental dynamics, and nowadays, it is famous as *Eremenko's conjecture.* Note that the complement of the Julia set $J(f)$ in $\mathbb{C}$ is the *Fatou set* $F(f)$. A connected maximal open subset of $F(f)$ is called *Fatou component.*

For any holomorphic function f, we call

$$C(f) = \{z \in \mathbb{C} : f'(z) = 0\}$$

(where f' represents derivative of f with respect to z) by the set of *critical points* and

$$CV(f) = \{w \in \mathbb{C} : w = f(z) \text{ such that } f'(z) = 0 \text{ for some } z\}$$

by the set of *critical values.* The set $AV(f)$ consisting of all $w \in \mathbb{C}$ such that there exists a curve $\Gamma : [0, \infty) \to \mathbb{C}$ so that $\Gamma(t) \to \infty$ and $f(\Gamma(t)) \to w$ as $t \to \infty$ is the set of *asymptotic values* of f and

$$SV(f) = \overline{(CV(f) \cup AV(f))}$$

is the set of *singular values* of f. If $SV(f)$ is finite, then f is said to be of *finite type.* If $SV(f)$ is bounded, then f is said to be of *bounded type.* The sets

$$\mathscr{S} = \{f : f \text{ is of finite type}\}$$

and

$$\mathscr{B} = \{f : f \text{ is of bounded type}\}$$

are respectively known as *Speiser class* and *Eremenko-Lyubich class.*

The main concern of this paper is to study of escaping sets of transcendental semigroups. So, we start our formal study from the notion of transcendental semigroups. The set $\text{Hol}(\mathbb{C})$ denotes a set of all holomorphic functions of $\mathbb{C}$. If $f \in \text{Hol}(\mathbb{C})$, then f is either a polynomial or a transcendental entire function. The composite of two entire functions is an entire function. So, this fact makes the set $\text{Hol}(\mathbb{C})$ a semigroup with semigroup operation being the functional composition.

Definition 1.1 (Transcendental semigroup). Let $A = \{f_i : i \in \mathbb{N}\} \subset \text{Hol}(\mathbb{C})$ be a set of transcendental entire functions $f_i : \mathbb{C} \to \mathbb{C}$. A *transcendental semigroup S* is a semigroup generated by the set A with semigroup operation being the functional composition. We denote this semigroup by

$$S = \langle f_1, f_2, f_3, \cdots, f_n, \cdots \rangle \quad \text{or simply by} \quad S = \langle f_i \rangle.$$

Here, each $f \in S$ is a transcendental entire function and S is closed under functional composition. Thus $f \in S$ is constructed through a composition of the finite number of functions f_{i_k}, (where $i_k \in \{1, 2, 3, \ldots, m\}$ for some $m \in \mathbb{N}$). That is,

$$f = f_{i_1} \circ f_{i_2} \circ f_{i_3} \circ \cdots \circ f_{i_m}.$$

A semigroup S generated by finitely many functions f_i, $(i = 1, 2, 3, \dots, n)$ is called *finitely generated transcendental semigroup* and write

$$S = \langle f_1, f_2, \dots, f_n \rangle.$$

If S is generated by only one transcendental entire function f, then S is called*cyclic or trivial transcendental semigroup* and write

$$S = \langle f \rangle$$

In this case, each $g \in S$ can be written as $g = f^n$, where f^n is the nth iterates of f with itself. The transcendental semigroup S is *abelian* if

$$f_i \circ f_j = f_j \circ f_i$$

for all generators f_i and f_j of S.

We say that a family $\mathscr{F}$ of holomorphic functions is a *normal family* in $\mathbb{C}$ if every sequence $(f_i) \subseteq \mathscr{F}$ has a subsequence (f_{i_k}) which is uniformly convergent or divergent on all compact subsets of $\mathbb{C}$. If there is a neighborhood U of a point $z \in \mathbb{C}$ such that $\mathscr{F}$ is a normal family in U, then we say that $\mathscr{F}$ is normal at z. If $\mathscr{F}$ is a semigroup S such that it is normal family in a neighborhood U of a point $z \in \mathbb{C}$, we say S is normal at z. We say that a function f is *iteratively divergent* at $z \in \mathbb{C}$ if $f^n(z) \to \infty$ as $n \to \infty$. A semigroup S is *iteratively divergent* at z if

$$f^n(z) \to \infty \text{ as } n \to \infty$$

for all $f \in S$. A semigroup S is said to be *iteratively bounded* at z if there is an element $f \in S$ which is not iteratively divergent at z.

Like in iteration theory of a single transcendental entire function, the Fatou set, Julia set and escaping set in the settings of transcendental semigroups are defined as follows:

Definition 1.2 (**Fatou set, Julia set and escaping set**). Let S be a transcendental semigroup. The *Fatou set* of S is defined by

$$F(S) = \{z \in \mathbb{C} : S \text{ is normal at } z\},$$

and the *Julia set* $J(S)$ of S is the complement of $F(S)$. The *escaping set* of S is defined by

$$I(S) = \{z \in \mathbb{C} : S \text{ is iteratively divergent at } z\}$$

We call each point of the set $I(S)$ by an *escaping point*.

If $S = \langle f \rangle$, then the Fatou set, Julia set and escaping set are respectively denoted by $F(f), J(f)$ and $I(f)$. So, Definition 1.2 generalizes the Fatou set, Julia set and escaping set of a single transcendental entire function. For simplicity, we call the dynamics of $S = \langle f \rangle$ by classical transcendental dynamics and the dynamics of $S = \langle f_i \rangle$ for at least $i = 1, 2$ by transcendental semigroup dynamics.

2. Some Fundamental Features of Escaping Set

The following immediate relation between $I(S)$ and $I(f)$ for any $f \in S$ will be clear from the Definition 1.2 of escaping sets.

Theorem 2.1. *$I(S) \subset I(f)$ for all $f \in S$ and hence $I(S) \subset \bigcap_{f \in S} I(f)$.*

Proof. Let $z \in I(S)$, then $f^n(z) \to \infty$ as $n \to \infty$ for all $f \in S$. By which we mean $z \in I(f)$ for any $f \in S$. This immediately follows the second inclusion. □

We dealt this Theorem 2.1 in the case of a transcendental semigroup S even though it holds for polynomial semigroups. Note that the above same type of relation (Theorem 2.1) holds between $F(S)$ and $F(f)$. However opposite relation holds between the sets $J(S)$ and $J(f)$. Poon [9, Theorem 4.1, Theorem 4.2] proved that the Julia set $J(S)$ is perfect and $J(S) = \overline{\bigcup_{f \in S} J(f)}$ for any transcendental semigroup S. From Theorem 2.1, we can say that the escaping set may be empty. For example, the escaping set of the semigroup $S = \langle f, g \rangle$ generated by functions $f(z) = e^z$ and $g(z) = e^{-z}$ is empty (that is, the particular function $h = g \circ f^k \in S$ (say) is iteratively bounded at any $z \in I(f)$). Note that $I(f) \neq \emptyset$ in classical iteration theory ([2, Theorem 1]). Dinesh Kumar and Sanjay Kumar [5, Theorem 2.5] have also mentioned the following transcendental semigroup S, where $I(S)$ is an empty set.

Proposition 2.1. *The transcendental entire semigroup $S = \langle f_1,\ f_2 \rangle$ generated by two functions f_1 and f_2 from respectively two parameter families $\{e^{-z+\gamma} + c$ where $\gamma, c \in \mathbb{C}$ and $Re(\gamma) < 0,\ Re(c) \geq 1\}$ and $\{e^{z+\mu} + d,$ where $\mu, d \in \mathbb{C}$ and $Re(\mu) < 0,\ Re(d) \leq -1\}$ of functions has empty escaping set $I(S)$.*

There are several classes transcendental semigroups whose escaping sets are non-empty. The following examples [8, Examples 3.2 and 3.3] and [5, Examples 2.6 and 2.7] are evident.

Example 2.1. Let $S = \langle f, g \rangle$, where $f(z) = e^z + \lambda$ and $g(z) = e^z + \lambda + 2\pi i$ for all $\lambda \in \mathbb{C} - \{0\}$. Then $I(S) = I(f) \neq \emptyset$.

Example 2.2. Let $S = \langle f, g \rangle$, where $f(z) = \lambda \sin z$ and $g(z) = \lambda \sin z + 2\pi$ for all $0 < |\lambda| < 1$. Then $I(S) = I(f) \neq \emptyset$.

Example 2.3. Let $S = \langle f, g \rangle$, where $f(z) = e^{\lambda z}$ and $g(z) = e^{s\lambda z} + 2\pi i/\lambda$ for all $\lambda \in \mathbb{C} - \{0\}$ and $s \in \mathbb{N}$. Then $I(S) = I(f) \neq \emptyset$.

Example 2.4. Let $S = \langle f, g \rangle$, where $f(z) = \lambda \sin z$ for all $\lambda \in \mathbb{C} - \{0\}$ and $g(z) = f^n + 2\pi$ for all $n \in \mathbb{N}$. Then $I(S) = I(f) \neq \emptyset$.

From all of these examples, we can get non-empty escaping sets. Dinesh Kumar and Sanjay Kumar [5, Theorem 3.4] generalized these examples to the following result.

Proposition 2.2. *Let $S = \langle f, g \rangle$ be a transcendental semigroup generated by periodic function f with period p and another function g defined by $g = f^n + p,\ n \in \mathbb{N}$. Then $I(S) = I(f)$.*

In the case of non-empty escaping set $I(S)$, Eremenko's result [2], $\partial I(f) = J(f)$ of classical transcendental dynamics can be generalized to semigroup dynamics. The following results is due to Dinesh Kumar and Sanjay Kumar [5, Lemma 4.2 and Theorem 4.3] which gives the generalized answer in semigroup settings.

Proposition 2.3. *Let S be a transcendental entire semigroup such that $I(S) \neq \emptyset$. Then*

(1) *$int(I(S)) \subset F(S)$ and $ext(I(S)) \subset F(S)$, where int and ext respectively denote the interior and exterior of $I(S)$.*
(2) *$\partial I(S) = J(S)$, where $\partial I(S)$ denotes the boundary of $I(S)$.*

This last statement is equivalent to $J(S) \subset \overline{I(S)}$. If $I(S) \neq \emptyset$, then we [11, Theorem 4.6] proved the following result which is a generalization of Eremenko's result $I(f) \cap J(f) \neq \emptyset$ [2, Theorem 2] of classical transcendental dynamics to holomorphic semigroup dynamics.

Theorem 2.2. *Let S be a transcendental semigroup such that $F(S)$ has a multiply connected component. Then $I(S) \cap J(S) \neq \emptyset$*

Eremenko and Lyubich [3] proved that if transcendental function $f \in \mathscr{B}$, then $I(f) \subset J(f)$, and $J(f) = \overline{I(f)}$. Dinesh Kumar and Sanjay Kumar [5, Theorem 4.5] generalized these results to a finitely generated transcendental semigroup of bounded type as shown below.

Proposition 2.4. *For every finitely generated transcendental semigroup $S = \langle f_1, f_2, \ldots, f_n \rangle$ in which each generator f_i is of bounded type, then $I(S) \subset J(S)$ and $J(S) = \overline{I(S)}$.*

Proof. Eremenko and Lyubich's result [3] shows that $I(f) \subset J(f)$ for each $f \in S$ of bounded type. Poon's result shows [9, Theorem 4.2] that $J(S) = \overline{\bigcup_{f \in S} J(f)}$. Therefore, (from Definition 1.2 of escaping set and theorem 2.1) for every $f \in S$, we can write,

$$I(S) \subset I(f) \subset J(f) \subset J(S).$$

The next part follows from the facts $J(S) \subset \overline{I(S)}$ and $I(S) \subset J(S)$. □

3. Escaping sets of Hyperbolic Semigroups

The definitions of critical values, asymptotic values and singular values as well as post singularities of transcendental entire functions can be generalized to arbitrary setting of transcendental semigroups.

Definition 3.1 (**Critical point, critical value, asymptotic value and singular value**). A point $z \in \mathbb{C}$ is called *critical point* of S if it is a critical point of some $g \in S$. A point $w \in \mathbb{C}$ is called a *critical value* of S if it is a critical value of some $g \in S$. A point $w \in \mathbb{C}$ is called an *asymptotic value* of S if it is an asymptotic value of some $g \in S$. A point $w \in \mathbb{C}$ is called a *singular value* of S if it is a singular value of some $g \in S$. For a semigroup S, if all $g \in S$ belongs to $\mathscr{S}$ or $\mathscr{B}$, we call S a semigroup of class $\mathscr{S}$ or $\mathscr{B}$ (finite or bounded type).

Definition 3.2 (**Post singularly bounded (or finite) semigroup**). A transcendental semigroup S is said to be *post-singularly bounded* (*post-singularly finite*) if each $g \in S$ is post-singularly bounded (or post-singularly finite). Post singular set of post singularly bounded semigroup S is the set

$$P(S) = \overline{\bigcup_{f \in S} f^n(SV(f))}$$

Definition 3.3 (**Hyperbolic semigroup**). An transcendental entire function f is said to be *hyperbolic* if the post-singular set $P(f)$ is a compact subset of $F(f)$. A transcendental semigroup S is said to be *hyperbolic* if each $g \in S$ is hyperbolic (that is, if $P(S)$ is a compact subset of $F(S)$).

Note that if transcendental semigroup S is hyperbolic, then each $f \in S$ is hyperbolic. However, the converse may not true. The fact $P(f^k) = P(f)$ for all $k \in \mathbb{N}$ shows that f^k is hyperbolic if f is hyperbolic. The following result has been shown by Dinesh Kumar and Sanjay Kumar [5, Theorem 3.16] where Eremenko's conjecture holds.

Proposition 3.1. *Let $f \in \mathscr{B}$ periodic with period p and hyperbolic. Let $g = f^n + p$, $n \in \mathbb{N}$. Then $S = \langle f, g \rangle$ is hyperbolic and all components of $I(S)$ are unbounded.*

Example 3.1. $f(z) = e^{\lambda z}$ is hyperbolic entire function for each $\lambda \in (0, \frac{1}{e})$. The semigroup $S = \langle f, g \rangle$ where $g = f^m + p$, and $p = \frac{2\pi i}{\lambda}$ is hyperbolic transcendental semigroup.

We generalize Proposition 3.1 to finitely generated hyperbolic semigroups by adding some extra conditions. This result will be the good source of non-empty escaping set transcendental semigroup where, the Eremenko's conjecture holds in semigroup setting.

Theorem 3.1. *Let $S = \langle f_1, f_2, \ldots, f_n \rangle$ is an abelian bounded type transcendental semigroup in which each f_i is hyperbolic for $i = 1, 2, \ldots, n$. Then semigroup S is hyperbolic and all components of $I(S)$ are unbounded.*

Lemma 3.1. *Let f and g be transcendental entire functions. Then $SV(f \circ g) \subset SV(f) \cup f(SV(g))$.*

Proof. See for instance [1, Lemma 2]. □

Lemma 3.2. *Let f and g are permutable transcendental entire functions. Then $f^m(SV(g)) \subset SV(g)$ and $g^m(SV(f)) \subset SV(f)$ for all $m \in \mathbb{N}$.*

Proof. We first prove that $f(SV(g)) \subset SV(g)$. Then we use induction to prove $f^m(SV(g)) \subset SV(g)$.

Let $w \in f(SV(g))$. Then $w = f(z)$ for some $z \in SV(g)$. In this case, z is either a critical value or an asymptotic value of function g.

First let us suppose that z is a critical value of g. Then $z = g(u)$ with $g'(u) = 0$. Since f and g are permutable functions, so

$$w = f(z) = f(g(u)) = (f \circ g)(u) = (g \circ f)(u).$$

Also,

$$(f \circ g)^{'}(u) = f^{'}(g(u))g^{'}(u) = 0.$$

This shows that u is a critical point of $f \circ g = g \circ f$ and w is a critical value of $f \circ g = g \circ f$. By permutability of f and g, we can write

$$f^{'}(g(u))g^{'}(u) = g^{'}(f(u))f^{'}(u) = 0$$

for any critical point u of $f \circ g$. Since $g^{'}(u) = 0$, then either $f^{'}(u) = 0 \Rightarrow u$ is a critical point of f or $g^{'}(f(u)) = 0 \Rightarrow f(u)$ is a critical point of g. This shows that $w = g(f(u))$ is a critical value of g. Therefore, $w \in SV(g)$.

Next, suppose that z is an asymptotic value of function g. We have to prove that $w = f(z)$ is also asymptotic value of g. Then there exists a curve $\gamma : [0, \infty) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $g(\gamma(t)) \to z$. So, $f(g(\gamma(t))) \to f(z) = w$ as $t \to \infty$ along γ. Since $f \circ g = g \circ f$, so

$$f(g(\gamma(t))) \to f(z) = w \Rightarrow g(f(\gamma(t))) \to f(z) = w$$

as $t \to \infty$ along γ. This shows w is an asymptotic value of g. This proves our assertion.

Assume that $f^k(SV(g)) \subset SV(g)$ for some $k \in \mathbb{N}$ with $k \leq m$. Then

$$f^{k+1}(SV(g)) = f(f^k(SV(g))) \subset f(SV(g)) \subset SV(g)$$

Therefore, by induction, for all $m \in \mathbb{N}$, we must have $f^m(SV(g)) \subset SV(g)$. The next part $g^m(SV(f)) \subset SV(f)$ can be proved similarly as above.

□

Lemma 3.3. *Let f and g are two permutable hyperbolic transcendental entire functions. Then $f \circ g$ is also hyperbolic.*

Proof. We have to prove that $P(f \circ g)$ is a compact subset of the Fatou set $F(f \circ g)$. From [7, Lemma 3.2], we can write $F(f \circ g) \subset F(f) \cap F(g)$. This shows that $F(f \circ g)$ is a subset of $F(f)$ and $F(g)$. So this Lemma will be proved if we prove $P(f \circ g)$ is a compact subset of $F(f) \cup F(g)$. By the definition of post singular set of transcendental entire function, we can write

$$\begin{aligned}
P(f \circ g) &= \overline{\bigcup_{m \geq 0} (f \circ g)^m(SV(f \circ g))} \\
&= \overline{\bigcup_{m \geq 0} f^m(g^m(SV(f \circ g)))} \qquad \text{(by using permutabilty of } f \text{ and } g\text{)} \\
&\subset \overline{\bigcup_{m \geq 0} f^m(g^m(SV(f) \cup f(SV(g)))} \qquad \text{(by Lemma 3.1)} \\
&= \overline{\bigcup_{m \geq 0} f^m(g^m(SV(f))) \cup g^m(f^{m+1}(SV(g)))} \\
&\subset \overline{\bigcup_{m \geq 0} f^m(SV(f)))} \cup \overline{\bigcup_{m \geq 0} g^m(SV(g)))} \qquad \text{(by Lemma 3.2)} \\
&= P(f) \cup P(g)
\end{aligned}$$

Since f and g are hyperbolic, so $P(f)$ and $P(g)$ are compact subset of $F(f)$ and $F(g)$. Therefore, the set $P(f) \cup P(g)$ must be compact subset of $F(f) \cup F(g)$. □

Proof of the Theorem 3.1. Any $f \in S$ can be written as

$$f = f_{i_1} \circ f_{i_2} \circ f_{i_3} \circ \cdots \circ f_{i_m}.$$

By permutability of each f_i, we can rearrange f_{i_j} and ultimately represented by

$$f = f_1^{t_1} \circ f_2^{t_2} \circ \ldots \circ f_n^{t_n}$$

where each $t_k \geq 0$ is an integer for $k = 1, 2, \ldots, n$. Lemma 3.3 can be applied repeatably to show each of $f_1^{t_1}, f_2^{t_2}, \ldots, f_n^{t_n}$ is hyperbolic. Again by repeated application of same lemma, we can say that

$$f = f_1^{t_1} \circ f_2^{t_2} \circ \ldots \circ f_n^{t_n}$$

is itself hyperbolic and so the semigroup S is hyperbolic. Next part follows from [12, Theorem 3.3] by the assumption of this theorem. □

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References

[1] W. Bergweiler and Y. Wang, *On the dynamics of composite entire functions*, Ark. Mat. **36** (1998), 31-39.

[2] A. Eremenko, *On the iterations of entire functions*, Dyn. Sys. and Ergod. Theo. , Banach Center Publication, Warsaw, Poland, vol. 23 (1989), 339-344.

[3] A. Eremenko and M.Y. Lyubich, *Dynamical properties of some Classes of entire functions*, Ann. Inst. Fourier, Grenoble, **42** (1992), 989-1020.

[4] H. Kriete and H. Sumi, *Semi-hyperbolic transcendental semi-groups*, Math. Kyoto Uni. (JMKYAZ) **402** (2000), 205-216.

[5] D. Kumar and S. Kumar, *The dynamics of semigroups of transcendental entire functions-II*, arXiv: 1401.0425 v3 [math.DS], 22 May 2014.

[6] D. Kumar and S. Kumar, *Escaping set and Julia set of transcendental semi-groups*, arXiv: 141.2747 v3 [math. DS], 10 October 2014.

[7] D. Kumar, *On escaping set of some families of entire functions and dynamics of composite entire functions*, arXiv: 1406. 2453 v4 [math. DS] 4 Oct. 2015.

[8] D. Kumar and S. Kumar, *Semigroups of transcendental entire functions and their dynamics*, arXiv: 1405.0224 v5 [math.DS], 7 Oct. 2015.

[9] K. K. Poon, *Fatou-Julia theory on transcendental semigroups*, Bull. Austral. Math. Soc. Vol- 58 (1998), 403-410.

[10] L. Rempe, *On a question of Eremenko concerning escaping component of entire function.*, Bull. Lond. Math. Soc. **39** (2007), 661-666.

[11] B. H. Subedi and A. Singh, *A short comparison of classical complex dynamics and holomorphic semi-group dynamics*, arXiv:1803. 08368 v1 [math. DS], 22 March 2018.

[12] B. H. Subedi and A. Singh, *Invariant properties of Fatou set, Julia set and escaping set of holomorphic semigroup*, arXiv:1803. 09662 v1 [math. DS], 23 March 2018.