Tribhuvan University

Institute of Science and Technology

Numerical Solutions of 2D Incompressible Navier-Stokes Equations in Variable Viscosity Case

A Thesis
submitted in partial fulfillment of the requirements
for
the degree of Master of Philosophy (Math)
by

Dhak Bahadur Thapa

to the
Central Department of Mathematics
Kirtipur

April, 2016

Tribhuvan University
Institute of Science and Technology
Central Department of Mathematics

Student's Declaration

I hereby declare that I am the only author of this work and that no sources other than that listed here have been used in this work.

.....
(**Dhak Bahadur Thapa**)
Date:.....

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I hereby recommend that this thesis prepared under my supervision by **Dhak Bahadur Thapa** entitled **Numerical Solutions of 2D Incompressible Navier-Stokes Equations in Variable Viscosity Case** during period prescribed by the rules and regulation of this institute in partial fulfillment of the requirements for the degree of M. Phil in Mathematics be processed for the evaluation.

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We certify that the **Research Committee of the Central Department of Mathematics**, TU, Kirtipur approved this research work entitled **Numerical Solutions of** $2D$ **Incompressible Navier Stokes Equations in Variable Viscosity Case** done by *Mr. Dhak Bahadur Thapa* in the scope and generality as a thesis in the partial fulfillment for the requirement of the degree of M. Phil. in Mathematics.

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Dedication

To my respected father Mr. Dan Bahadur Thapa, mother Mrs. Bhima Thapa, father in law Mr. Bal Kumar Thapa and mother in law Mrs. Durga Thapa.

Abstract

In this work, we consider a mathematical model of slider bearings with parallel and non-parallel plates based on 2D Navier Stokes equations with variable viscosity without stresses and external forces. The model is simplified and reduced to one dimensional viscous incompressible Couette flow and solved numerically employing a second order finite difference scheme (Crank Nicolson). The consistency, stability and convergence of the numerical scheme are also studied. The numerical solution is compared with the exact solution and presented graphically. Finally, numerical procedures for the original realistic 2D model are outlined.

Contents

List of Figures

Chapter 1

Introduction

1.1 Preliminaries

The Navier-Stokes equations are mathematical model aimed at describing the motion of an incompressible viscous fluids like water, glycerine, oil and under certain circumstances, air also. They were introduced in 1822 by the French engineer *Claude Louis Marie Henri Navier* and successively re-obtained with different arguments, by a number of authors including *Augustin Louis Cauchy*in 1823, *Simeon Denis Poisson* in 1829, *Adhemar Jean Claude Barre de Saint-Venant* in 1837, and, finally, *George Gabriel Stokes* in 1845.

The Navier-Stokes equations are now regarded as the universal basis of fluid mechanics, no matter how complex and unpredictable the behavior of its solutions may be. It is also known to be the only hydrodynamic equation that is compatible with the isotropy and linearity of the stress-strain relation. Navier's original derivation was not influential and the equation was re-derived at least four times, by Cauchy, by Poisson, by Saint-Venant, and by Stokes. Each new derivation either ignored or denigrated his predecessor's contribution. Each had his own way to justify the equation. Each judged differently the kind of motion and the nature of the system to which it was applied.

All five authors of the Navier-Stokes equations shared a molecular ontology,

but they differed considerably over the extent to which their derivations involved molecular assumptions. A wide spectrum of methodological attitudes existed at the time. At one extreme was Poisson, who insisted on the necessity of discrete sums over molecules. At the other extreme was Cauchy who combined infinitesimal geometry and spatial symmetry arguments to define strains and stresses and to derive equations of motion without referring to molecules. Yet the opposition was not radical. Poisson used Cauchy's concept and Cauchy eventually provided his own molecular derivations. Others compromised between the molecular and the molar approaches. Navier started with molecular forces but quickly jumped to the macroscopic level by considering virtual works. Saint-Venant insisted that a clear definition of the concept of stress could only be molecular but nevertheless provided a purely macroscopic derivation of the Navier-Stokes equation. Stokes obtained the general form of the stresses in a fluid by a Cauchy type of argument but he justified the linearity of the stresses with respect to deformations by reasoning on molecules *(Theory of Continua)*.

These methodological differences largely explain why Navier's successors ignored or criticized his derivation of the Navier-Stokes equations. His short-cuts from the molecular to the macroscopic levels seemed arbitrary or even contradictory. Cauchy and Poisson simply ignored Navier's contribution to fluid dynamics. Saint-Venant and Stokes both gave credit to Navier for the equation but believed an alternative derivation to be necessary.

The many authors of the Navier-Stokes equations also differed in the types of application they envisioned. Navier and Saint-Venant had pipe and channel flow in mind. Cauchy and Poisson's interests were more philosophical than practical. Cauchy did not even intend the equation to be applied to real fluids. He derived it for a perfectly inelastic solid and noted its identity with Fourier heat equation in the limiting case of slow motion.

1.1.1 Scope

Even though for some time their applications in the real life was not fully recognized, now they became the foundations of many branches of applied sciences including Meteorology, Oceanography, Geology, Oil Industry, Aero-dynamics, Ship and Car Industries, Flim Industries, Biology, Medicine and many more. In each of the above areas, these equations have collected many undisputed successes which definitely place them among the most accurate, simple and beautiful models of mathematical physics.

Fluid mechanics is a very intricate and intriguing discipline of the applied sciences. It is, therefore, not surprising that the mathematics involved in the study of its properties can be extremely complex and difficult. Complexities and difficulties may be more or less challenging depending on the mathematical model chosen to describe the physical situation.

1.1.2 Challenges

Apart from these outstanding successes, up to the present time, a number of unresolved basic mathematical questions remain open mostly for $3D$ flows. The most celebrated is, Whether global solution for $3D$ regular flow for data of arbitrary size exists (*global regularity problem*).

Since the 20^{th} century, this problem has challenged several mathematicians who have not been able to furnish a definite answer. In fact, till today, $3D$ regular flows are known to exist either for all times but for data of small size, or for the data of arbitrary size but for a finite interval of time only. The *global regularity problem* has become so intriguing that in the year 2000, it was decided to put as one of the seven dollar 1 Millennium Prize Problems of the Clay Mathematical Institute.

Moreover, the Navier-Stokes equations present other fundamental open questions. It is not known whether in the $3D$ case, the associated initial-boundary value problem is (in an appropriate function space) well-posed in the sense of Hadamard. In other word, in $3D$ whether the solutions to this problem exist

for all times are unique and depend continuously upon the data without being necessarily regular *(uniqueness)*.

Further, other outstanding question for both $2D$ and $3D$ flow is describing the steady-state flow, which states that for the multiply-connected flow region $\mathbb{R}$, does a solution exist under a given velocity distribution at the boundary of $\mathbb{R}$ that can satisfy the physical requirement of conservation of mass (*boundary value problem*).

Actually in the $2D$ case, the first two problems i.e. global regularity problem and uniqueness have been solved, while the third one remains still open. Nevertheless, there is hope that proving or disproving the first two problems in $3D$ will require completely fresh and profound ideas that will open new avenues to the understanding of turbulence.

1.1.3 Limitation

Navier Stokes Equations are restricted to the case of Newtonian fluids only. Besides, this study is limited to $2D$ incompressible Navier-Stokes equations for the variable viscosity case.

1.1.4 Objectives

This paper aims to study the physicsl properties of the fluids, derivation of the Navier-Stokes equations and exploration of the procedures for the numerical solutions (Finite Difference Method) in variable viscosity case.

1.1.5 Basic Definitions

1. Fluid: The substance which flows with the application of a shear stress no matter, how small it may be. Equivalently, it is a substance in which the inter-molecular force of attraction is weak and extremely small. Liquids and gases are together known as fluid.
2. Viscosity: The property of fluids by virtue of which they oppose the continuous deformation under external shearing force is called viscosity.

3. Steady-State Flow: Flow where flow parameters like density, velocity and pressure fields are time-independent.
4. Unsteady Flow: Flow where one or more fluid parameters changes with time.
5. 3D-flow: Flow where velocity and pressure fields depend on three spatial variables.
6. Planar or 2D flow: Flow where velocity and pressure fields depend only on two spatial variables belonging to a portion of a plane, and the component of the velocity orthogonal to that plane is identically zero.
7. Stream Line Flow: The curve path of the fluid element in steady motion where the tangent at any point of it gives the direction of the resultant velocity.
8. Local Solution: Solution where velocity and pressure fields exist only for a finite interval of time.
9. Global Solution: Solution where velocity and pressure fields exist for all positive times.
10. Regular Solution: Solution where velocity and pressure fields satisfy the Navier-Stokes equations and the corresponding initial and boundary conditions in the ordinary sense of differentiation and continuity.
11. Control Volume: Finite arbitrary volume of the fluid which flows and to which the principles of conservation of mass, momentum and energy holds true. It is denoted by Ω and its boundary surface by $\delta\Omega$.
12. Compressible fluids: The fluid whose volume changes with pressure, eg. Air.
13. Incompressible fluids: Fluids whose volume does not change with the pressure, eg. Water.
14. Newtonian Fluids: The fluid whose shear stress is non zero and is directly proportional to the time rate of shear strain(velocity gradient). The constant of proportionality is the coefficient of viscosity, eg. Mustard oil.

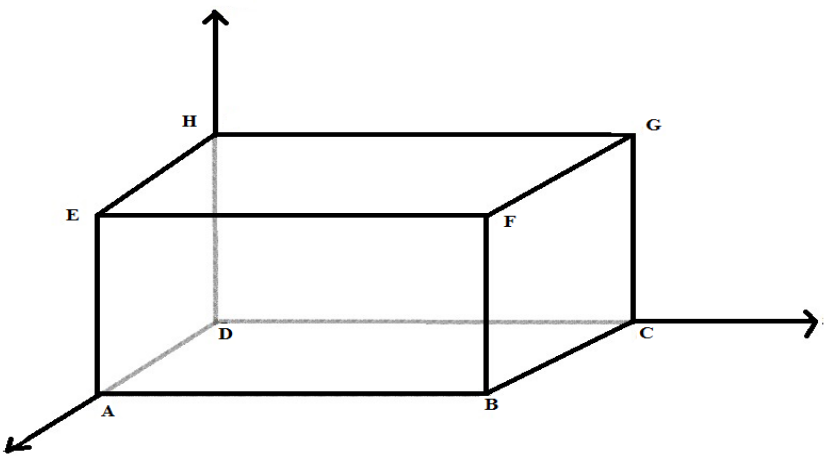


Figure 1.1: Control Volume in rectangular parallelopiped form.

Figure 1.2: Control volume in oval shape

15. Non-Newtonian Fluids: The fluid whose viscosity depends on time rate of shear strain and even on time, eg. Glass.

16. Conservation Form: The fluid flow equation which can be obtained directly by applying the conservation principles to the fluid elements at a particular point in its course of motion. It may also be defined as the divergence of some fluid flow fluxes.

17. Non-Conservation/Convective Form: The fluid flow equation in differential or in integral form obtained from the fluid elements at any time of its motion.

18. Governing Equation: In general, a pair of partial differential equations whose general solutions are not known analtically are known as governing equations.

1.1.6 Continuum Hypothesis

To study the fluid flow behaviour, certain assumptions are made.

1. The volume of the fluid is large enough as compared to that of a fluid element. This enables that the density of fluid particle is independent of volume. In other words, the mass of a fluid particle is a smooth function of the volume.
2. The volume of a fluid particle is indefinitely smaller as compared to the whole volume occupied by the fluid.

These assumptions form a basis of the *continuum hypothesis*. The continuum hypothesis states that

1. The fluid particle is a material point, and
2. The density of fluid is continuous function of space and time.

Under the continuum hypothesis, we assume that the part of the fluid under observation consists of infinitely many material points, and we expect that the motion of this continuum is described by partial differential equations. Fluids are made of molecules. The cohesive forces keep them attracted to each other. However, the molecules are in constant motion. The distance a molecule travel before hitting other is called *Mean Free Path* denoted by λ. Note that

$$\lambda \propto T \text{ and } \lambda \propto \frac{1}{p},$$

where, T is temperature and p is the pressure.

If we look at microscopic length scale, the fluid molecules are found to be moving in the space bouncing off each other and the wall of the container. At this length scale, fluids are discontinuous spatially. At a larger length scale, fluids may appear in continuous phase. The length scale at which the fluid appears

to be continuous is called *characteristic length* denoted by L. For instance, if we observe a tiny liquid droplet of diameter $1mm$, then the collection of such droplets constitute a fluid which is taken as a continuous matter. This is the assumption of continuum hypothesis. The characteristic length in this example is $L = 1mm$. Further, we may think the continuity of fluid as the physical phenomenon where a solute may be dissolved by a solvent. If the solute is dissolved by the solvent without stiring it within a fixed interval of time, then the solvent is considered as a continuous matter.

We note that

$$\lambda \propto L.$$

i.e.

$$\lambda = K_n L,$$

i.e.

$$K_n = \frac{\lambda}{L},$$

where the constant of proportionality K_n is called *Knudsen Number*. The continuity of a fluid can be observed in terms of Knudsen Number as

- If $K_n \leq 0.001$, fluid is considered as continuum.
- If $0.001 < K_n < 0.1$, rarefaction effect starts influencing the fluid properties.
- If $K_n = 0.1$, continuum assumptions starts to breakdown.
- If $K_n \geq 10$, continuum approach breaks completely.

1.1.7 Reynolds number

In the 1880, Osborne Reynolds carried out visualisation studies of flow in a pipe. He noticed out that well ordered laminar flow degenerated into a chaotic motion when the velocity in the pipe reached a certain value. It is due to a certain non-dimensional parameter causing complexity of the flow, called *Reynolds number* denoted by Re. Reynolds number represents the ratio between inertial forces and

the viscous forces. Thus, for a flow in pipe, Reynolds number is given by

$$Re = \frac{\rho \vec{v} D}{\mu},$$

where D is the diameter of the pipe and $\vec{v}$ is average velocity of fluid in the pipe. For small values of Re, the flow is found to be laminar. As the value of Re increases, the flow becomes more complicated and finally turbulent. Thus, a flow is turbulent if the inertial force in it is extremely large than the viscous force. This shows that inertial force is responsible for the dynamic redistribution of the flow structures. When it dominates the viscous force in a flow, i.e. when $Re >> 1$, the flow becomes turbulent.

1.2 Properties of Fluids

In the study of fluid mechanics, certain physical properties of the fluids are involved. Some of them are

- Density: It is defined as the mass of the material contained per unit of its volume. It is represented by ρ and is given by

  $$\rho = \frac{m}{V}.$$

  In the limiting case, if δm and δV denote mass and volume of a fluid element respectively, then

  $$\rho = lim_{\delta V \to 0} \left(\frac{\delta m}{\delta V} \right).$$

  The density of liquids vary with the temperature and pressure. But there are some fluids whose density is not change with the temperature and pressure. Such fluids are termed as *incompressible fluid.* Similarly, the density of some fluids change with the temperature and pressure. Such fluids are termed as *compressible fluids.*

- Specific Gravity: The ratio of the density of a substance to the density of a particular reference substance is termed as specific gravity of that substance. It is a non-dimensional quantity.

- Specific Weight: The weight of a substance per unit or its volume is termed as specific weight. It is denoted by ω^*. Thus,

$$\omega^* = \rho g,$$

where, ρ = density of the substance and g= acceleration due to gravity.

- Viscosity: The property of a fluid by virtue of which it opposes the continuous deformation of the fluid under the action of an external force is termed as viscosity.

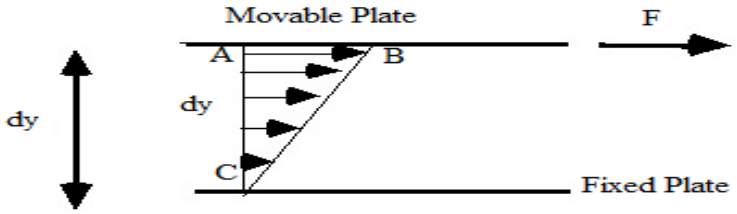


Figure 1.3: Flow in parallel plates

Let us consider a thin layer of fluid of thickness dy kept between two plates as shown in the figure. The lower plate is kept stationary where as the upper plate is moved parallel to the lower by the application of a tangential shearing stress F applied in the direction parallel to the lower plate. If $d\vec{v}$ denotes the velocity of the upper plate relative to the lower in time dt and $d\theta$ be the angular displacement of the upper plate,
Horizontal displacement of upper plate $= d\theta dy = d\vec{v}dt$. This implies

$$\frac{d\theta}{dt} = \frac{d\vec{v}}{dy}. \tag{1.1}$$

If S be the area of the upper plate in contact with the fluid, then the applied shear stress is

$$\tau = \frac{F}{S}.$$

For the Newtonian fluid, the external shear stress is proportional to the time rate of shear strain, we have

$$\tau \propto \frac{d\theta}{dt},$$

i.e.

$$\tau = \mu \frac{d\theta}{dt}, \tag{1.2}$$

where μ is the constant of proportionality called *coefficient of dynamic viscosity or Newtonian viscosity or simply coefficient of viscosity.* From equations (1.1) and (1.2), we get

$$\tau = \mu \frac{d\vec{v}}{dy}.$$

This relation is true for Newtonian fluid s only.

- Kinematic Viscosity: It is the ratio of the dynamic viscosity to the density of the fluid. It is denoted by ν. Thus,

$$\nu = \frac{\mu}{\rho}.$$

- Intensive and Extensive Quantities
  1. Extensive quantity: The physical quantity whose value is proportional to the size of the system. It is additive in nature. For example; mass, momentum, kinetic energy, heat content, electrical charge etc.
  2. Intensive quantity: The physical quantity whose value does not depend on the size of the system. It is not additive in nature. For example; temperature, pressure, electrical potential etc.

     Extensive quantities can be modeled in a very general way by measures. The content of an extensive quantity q in a set $A \subset \Omega \subset \mathbb{R}^d$ is $Q = \mu(A)$, for A measurable with μ as the measure of q. The main property we are interested in here is additivity,

$$\mu\left(\cup A_i\right) = \sum \mu(A_i),$$

for a countable family of disjoint measurable sets $\{Ai\}$. The measure in this respect deals mainly with density, i.e.

$$\mu(A) = \int_A f(x) d\lambda(x),$$

where $f : \Omega \to \mathbb{R}$ is the density function and λ is the Lebesgue measure.

1.2.1 Kinematics of Fluid Flow

Kinematics is the study of the motion of a fluid, without considering the forces which cause this motion, that is without considering the equations of motion. In fluid dynamics, we need to convert the kinematics of a mass-point to the kinematics of a fluid particle. Generally, such problem of fluid motion is dealt with two approaches (a) *Lagrangian Approach* and (b) *Eulerian Approach.*

In Lagrangian approach, any fluid particle is selected in space and its motion is observed throughout its course. If a particle be at $P(x_0, y_0, z_0)$ at time t_0, then its position after time t is given by

$$x = x(x_0, y_0, z_0, t), \quad y = y(x_0, y_0, z_0, t), \quad z = z(x_0, y_0, z_0, t).$$

The velocity components of the particles along co-ordinate axes are given by

$$u = \frac{\partial x}{\partial t}, \quad v = \frac{\partial y}{\partial t}, \quad z = \frac{\partial z}{\partial t}.$$

In Eulerian approach, any point in the space occupied by the fluid is selected and the changes of hydrodynamical parameters occuring at that point during the course of time is observed. Since a point is choosen in this approach, the velocity components of a fluid element changes with time. So fluid motion is characterized by the formation of velocity field. The velocity components along the axes will be the function of spatial co-ordinates and time.

$$u = u(x, y, z, t), v = v(x, y, z, t), w = w(x, y, z, t).$$

Hence the velocity field at any point (x, y, z, t) is given by

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k},$$

where u, v, w are the velocity components along axes.

1.2.2 Reynold's Transport Theorem:

The sum of the changes of some extensive property (say L) defined over a control volume Ω is equal to what is lost (or gained) through the boundaries $\partial\Omega$ of the volume plus what is created/consumed by sources and sinks inside the control volume. Mathematically,

$$\frac{d}{dt}\int_{\Omega} L dV = -\int_{\partial\Omega} L\vec{v}.ndA - \int_{\Omega} Q dV$$

where,

$\vec{v}$ =velocity of the fluid;

Q =amount of fluid in the sources and sinks.

1.2.3 Mass Conservation (Continuity Equation)

It states that

Rate of mass accumulation within Ω = Rate of mass flow into Ω
$-$Rate of mass flow out of Ω.

Let $P(x, y, z)$ be a position of a fluid element. Consider a rectangular control volume in space at this point with edges dx, dy, dz units respectively as shown in the figure. Let u, v, w be the components of velocity and ρ be the fluid density at the point P, then the mass of the fluid entering per unit time through the face $PQRS$ along X direction is

$$(\rho u)\, dy\, dz.$$

The mass of fluid flowing out per unit time of the face $ABCD$ along x direction is

$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right] dy\, dz.$$

Hence, the mass of the fluid left in the control volume per unit time due to the flow in x direction is

$$-\frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz.$$

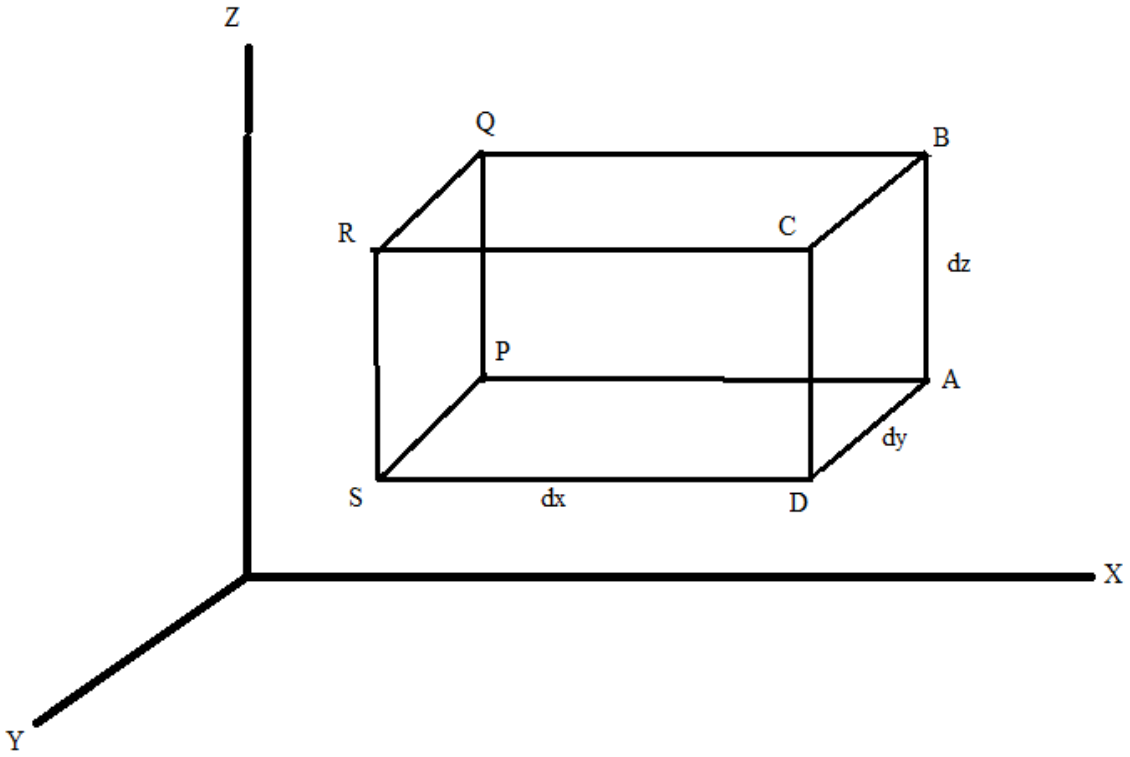


Figure 1.4: Fluid flow in rectangular parallelopiped

Similarly, the mass of the fluid left in the control volume per unit time due to the flow in y and z directions are respectively

$$-\frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dz,$$

and

$$-\frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz.$$

Hence, total mass of the fluid inside the control volume per unit time is

$$-\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]dx\,dy\,dz.$$

Now, the original fluid mass inside the control volume is $\rho\,dx\,dy\,dz$ and its increase per unit time is

$$\frac{\partial \rho}{\partial t}dx\,dy\,dz.$$

By the priciple of mass conservation, we have

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right].$$

i.e.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0. \tag{1.3}$$

This is the *Equation of Continuity* for $3D$ compressible unsteady flow.

For the steady-state motion, $\frac{\partial \rho}{\partial t} = 0$. So, the continuity equation for $3D$ steady flow becomes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0. \tag{1.4}$$

For incompressible fluids, $\rho =$ constant. The equation of continuity (1.4) for $3D$ incompressible steady flow becomes,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.5}$$

The mass conservation equation (equation of continuity) for $2D$ unsteady compressible flow is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0.$$

And, mass conservation equation (equation of continuity) for $2D$ steady incompressible flow is

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0.$$

1.2.4 Conservation of Momentum

It states that:

Accumulation of momentum within Ω =Rate of momentum flow into $\Omega-$
Rate of momentum flow out of $\Omega+$
Forces acting on Ω faces+
Body forces within Ω.

Consider a fluid element as our flow model. Newton second law states that the force vector acting on a control volume is equal to the rate of change of momentum.

Consider a control mass system for which the mass doesn't change with the time. Hence the rate of change of momentum is mass times the acceleration experienced by the fluid in the control volume. Thus, the x-component of the force (F_x) is given by

$$F_x = ma_x,$$

where m stands for mass and a_x stands for x-component of the acceleration.

We note that two types of forces act on the control volume, one is *body forces* that act on the control volume from a distance and don't depend upon the geometry of the body such as gravitational and electromagnetic forces and other, *surface forces* that act directly on the control surfaces of the fluid element. Since the later one acts on the surface elements, the geometry is an important parameter in deciding the forces acting on it. The surface forces further be subdivided into *normal* and *shear* forces. The normal forces are caused by the normals to the body surface whereas the shear forces are caused by the shear stresses.

Let us consider a elementary fluid element $OABCGHE$ having a corner at origin and surfaces parallel to the axes with edges dx, dy, dz units respectively. Let $\vec{v} = (u, v, w)$ be the velocity components of the moving fluid along the axes and ρ be its local density.

Then,

$$\text{Body force acting in x-direction} = \rho f_x \, dx \, dy \, dz,$$

where f_x is the acceleration in the x-direction. For the surface forces, if τ_{ij} denotes the stress tensor acting in j-direction on a plane whose normal vector is in the i-direction,

Shear force acting on the face $OABC$ in the x- increasing direction=τ_{yx}.
Shear force acting on the face $EFGH$ in the x- increasing direction=$\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy$.
So, net force in x-increasing direction due to this pair of shear stresses on the surfaces $OABC$ and $EFGH$ is

$$\tau_{yx} dx \, dz - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dx \, dz,$$

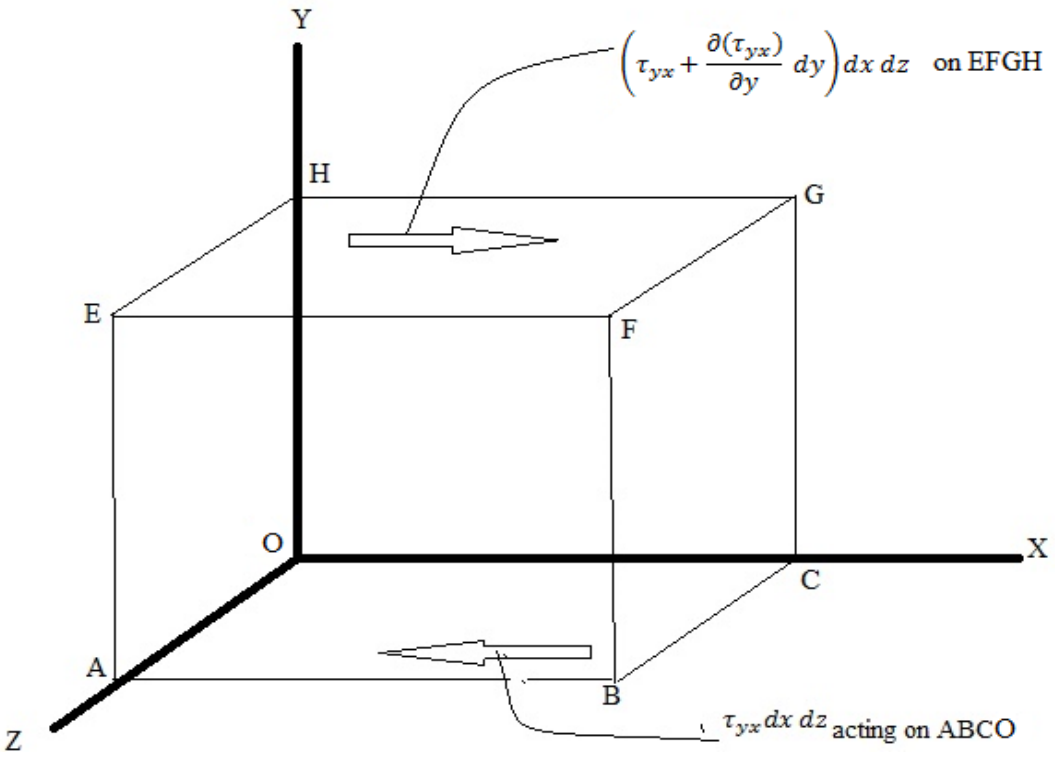


Figure 1.5: Small moving fluid element with forces in x-direction

i.e.

$$-\frac{\partial \tau_{yx}}{\partial y} dx\, dy\, dz.$$

Similarly, net force in x-increasing direction due to the stresses τ_{xx} and τ_{xz} are

$$-\frac{\partial \tau_{xx}}{\partial x} dx\, dy\, dz$$

and

$$-\frac{\partial \tau_{zx}}{\partial z} dx\, dy\, dz$$

respectively. So, total stress due to all three pairs of opposite faces in x-direction is

$$\left(-\frac{\partial \rho}{\partial t} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dx\, dy\, dz.$$

This gives,

$$F_x = \rho f_x dx\, dy\, dz + \left(-\frac{\partial \rho}{\partial t} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) dx\, dy\, dz.$$

But from $F_x = m.a_x$, we have

$$F_x = \rho \frac{Du}{Dt}\, dx\, dy\, dz,$$

where,

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}.$$

Therefore, we have

$$\rho \frac{Du}{Dt} = \rho f_x + \left(-\frac{\partial \rho}{\partial t} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right), \tag{1.6}$$

where $\rho\, dx\, dy\, dz$ is the mass of the fluid element flowing in the control volume. This is the x-component of the momentum equation for a viscous flow. In the same way, we can find the y and z-components of the momentum equations as

$$\rho \frac{Dv}{Dt} = \rho f_y + \left(-\frac{\partial \rho}{\partial t} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right), \tag{1.7}$$

and,

$$\rho \frac{Dw}{Dt} = \rho f_z + \left(-\frac{\partial \rho}{\partial t} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right), \tag{1.8}$$

The equations (1.6) to (1.8) are called momentum equations which can be written in vector form as

$$\rho \frac{D\vec{v}}{Dt} = \rho \vec{F} - \nabla p + \nabla . \partial_j \tau_{ji} + \rho \vec{F} \tag{1.9}$$

The vector form of the continuity equation is

$$\frac{D\rho}{Dt} + \rho \nabla . \vec{v} = \frac{\partial \rho}{\partial t} + \nabla . (\rho \vec{v}) = 0.$$

Using this form, we can write,

$$\rho \frac{Du}{Dt} = \frac{\partial (\rho u)}{\partial t} + \nabla . (\rho u \vec{v}).$$

Then, the equation (1.6) can be expressed in conservative form as

$$\frac{\partial(\rho u)}{\partial t} + \nabla.(\rho u\vec{v}) = -\frac{\partial p}{\partial t} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x.$$

Similarly, we can write the other two components of momentum equations in conservative form. These equations are still not appropriate since the informations about the stress system in the fluid is not known. Newton was the first person who suggested that stress is proportional to the time rate of shear strain. Further, the stress in some fluids is related to the velocity gradient. Such type of fuctional relationship between the flow quantities is called *constitutive relation.* The fluids which follow the above relation between stress and strain rates are called *Newtonian fluids.* For Newtonian fluids, Stokes had proved that

$$\tau_{ij} = \lambda\delta_{ij}\frac{\partial v_k}{\partial x_k} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \rho\delta_{ij},$$

where p is the thermodynamic pressure or hydrostatic stress, μ is the molecular viscosity and λ is the bulk coefficient of viscosity. If p_m is the mechanical pressure, then by definition

$$p_m = \frac{\tau_{ij}}{3}.$$

Stoke was the first person who hypothesized a relationship between λ and μ as

$$3\lambda + 2\mu = 0,$$

which is obtained by equating the thermodynamic pressure with the mechanical pressure. Using the constitutive relation, the final form of momentum equation for compressible flow are

u-momentum equation:

$$\begin{aligned}\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\lambda\nabla.\vec{v} + 2\mu\frac{\partial u}{\partial x}\right) + \\ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \rho f_x.\end{aligned} \tag{1.10}$$

v-momentum equation:

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho vw)}{\partial z}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left(\lambda\nabla.\vec{v}+2\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\rho f_y. \tag{1.11}$$

And, w-momentum equation:

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho uw)}{\partial x}+\frac{\partial(\rho vw)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left(\lambda\nabla.\vec{v}+2\mu\frac{\partial w}{\partial z}\right)+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\rho f_z. \tag{1.12}$$

We can simplify these equations for incompressible flow i.e. $\rho =$ constant. For the incompressible flow, the continuity equation is given by

$$\nabla.(\vec{v})=0.$$

Using this relation, the momentum equations (1.10) to (1.12) in vector form for incompressible flow reduce into,

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}.\nabla)\vec{v}=-\frac{\nabla p}{\rho}+\nu\nabla^2\vec{v}+\vec{F}. \tag{1.13}$$

The momentum equations in $2D$ flow can be obtained from the equations (1.10) and (1.11) in usual form as:

- u-momentum equation in $2D$ flow:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho vu)}{\partial y}=-\frac{\partial P}{\partial x}-\frac{\partial\tau_{xx}}{\partial x}-\frac{\partial\tau_{xy}}{\partial y}+\rho f_x,$$

- v-momentum equation in $2D$ flow:

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}=-\frac{\partial P}{\partial y}-\frac{\partial\tau_{xy}}{\partial x}-\frac{\partial\tau_{yy}}{\partial y}+\rho f_y,$$

where τ_{xx}, τ_{yy} and τ_{xy} are viscous stresses acting across the pairs of opposite faces of the rectangular control volume. The term *viscous stress* is the rate at which a

deforming fluid element is opposed by the fluid viscosity. In a simple shear flow, Stoke's law shows that the viscous shear stress τ_{xy} is

$$\tau_{xy} = -\mu \frac{\partial v}{\partial y},$$

where μ is the coefficient of viscosity, v is the velocity of the fluid in y direction. For many common fluids we have,

$$\tau \propto \frac{\partial \theta_x}{\partial t}.$$

For Newtonian fluids, the general stress-strain relations can be expressed as the viscous stresses being linearly related to the strain rates, with the constant of proportionality being the coefficient of viscosity μ. In $2D$ flow, we obtain the viscous stresses as

$$\tau_{xx} = -2\mu \frac{\partial u}{\partial x}, \tau_{yy} = -2\mu \frac{\partial v}{\partial y}, \tau_{xy} = -\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right).$$

With this expression, the u- momentum equation becomes:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)\right] + \rho f_x.$$

and, the v-momentum equation becomes,

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)\right] + \frac{\partial}{\partial y}\left[2\mu \left(\frac{\partial v}{\partial y} \right)\right] + \rho f_x.$$

1.2.5 The Navier-Stokes Equations

The above set of equations that describe a real fluid motion are collectively known as the Navier-Stokes equations. In $2D$, the compressible unsteady Navier Stokes equations are,

1. Continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0.$$

2. u-momentum equation:
$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}=-\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\rho f_x.$$

3. v-momentum equation:
$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho uv)}{\partial y}+\frac{(\rho v^2)}{\rho y}=-\frac{\partial P}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[2\mu\left(\frac{\partial v}{\partial y}\right)\right]+\rho f_y.$$

In $3D$ flow, the compressible unsteady Navier Stokes equations are as follows.

1. Equation of Continuity :
$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0.$$

2. u-Momentum Equation:
$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho uu)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}=-\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\rho f_x.$$

3. v-Momentum Equation:
$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho uv)}{\partial x}+\frac{(\rho v^2)}{\partial y}+\frac{\partial(\rho vw)}{\partial z}=-\frac{\partial P}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[2\mu\left(\frac{\partial v}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\rho f_y.$$

4. w-Momentum Equation:
$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho uw)}{\partial x}+\frac{(\rho vw)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}=-\frac{\partial P}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\left(\frac{\partial w}{\partial z}\right)\right]+\rho f_z.$$

In vector notation, the Navior-Stokes Equations can be expressed as

- Momentum Equations:
$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}.\nabla\vec{v}\right)=-\nabla P+\mu\Delta\vec{v}+\rho f,$$

- Continuity Equation:
$$\mathrm{div}(\vec{v})=0.$$

1.2.6 The Energy Equation

The energy equation is nothing but the first law of thermodynamics stated for a control volume system. The first law of thermodynamics for a control system states that

The rate of change of energy inside the control volume must be due to the heat interaction associated with the control volume plus the work done reversibly due to the boundary displacement by the body and the surface force.

Work done:

The rate of work done by the body forces for the control volume of mass $\rho(dx\,dy\,dz)$ is given by

$$\rho\vec{F}.\vec{v}(dx\,dy\,dz). \tag{1.14}$$

The contribution of the surface forces acting on the element are due to the normal and shear stresses. The rate of work done in a direction is the product of the forces in the direction and component of the velocity in that direction.

In the figure, work done by the body surface on the face $ABCD$ is given by

$$(\tau_{yx}\,dx\,dz)u.$$

Similarly, work done due to the shear stress acting on the face $EFGH$is given by

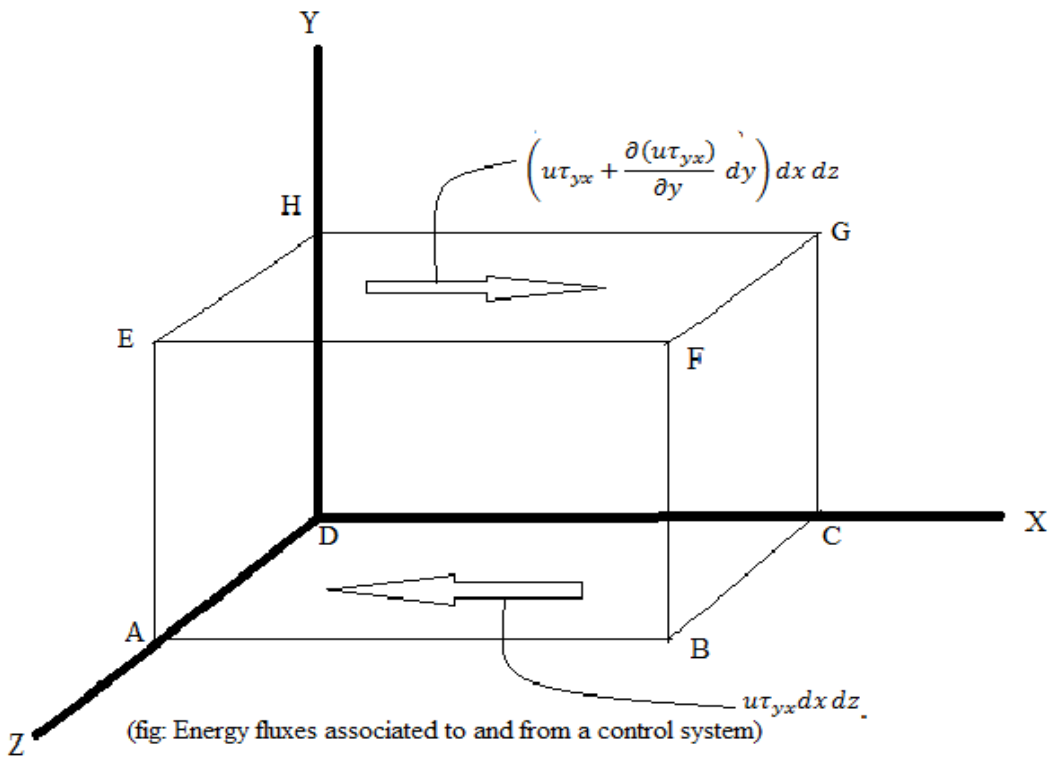
$$\left(u\tau_{yx} + \frac{\partial(u\tau_{yx})}{\partial y}dy\right)dx\,dz.$$

Since the two shear forces act in opposite direction, the net energy flux in the x-direction due to the shear stresses in the x-direction on faces $ABCD$ and $EFGH$ is

$$\frac{\partial(u\tau_{yx})}{\partial y}dx\,dy\,dz.$$

SImilarly, the normal stresses acting on the faces $ADHE$ and $BCGF$ gives rise to the work done term

$$\left[up - \left(up + \frac{\partial(up)}{\partial x}dx\right)\right]dy\,dz.$$



(fig: Energy fluxes associated to and from a control system)

That is,

$$-\frac{\partial(up)}{\partial x}dx\,dy\,dz.$$

Now, considering all the forces as shown in the figure, the net energy flux for the control volume due to these forces is given by

$$\left(-\frac{\partial(up)}{\partial x}+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}\right)dx\,dy\,dz. \tag{1.15}$$

The expressions (1.14) and (1.15) are obtained by considering forces acting in the x-direction only. Considering the surface forces in y and z directions, the total contribution coming from the body and surface forces is given by

$$\left(-\nabla.(p\vec{v})+\frac{\partial}{\partial x_j}(u_i\tau_{ji})\right)dx\,dy\,dz+\rho\vec{F}.\vec{v}\,dx\,dy\,dz. \tag{1.16}$$

Heat Transfer:

The net flux of heat is due to volumetric heating such as absorption or emission of

radiation and heat transfer across the control surface due to thermal conduction. If q denotes the rate of volumetric heat addition per unit mass, then the volumetric heating of the element is given by

$$\rho q \, dx \, dy \, dz. \tag{1.17}$$

In the above figure, heat transferred due to conduction across the face $ADHE$ is given by

$$q_x \, dy \, dz,$$

where q_x is the heat flux in the x-direction per unit time per unit area by the thermal conduction. SImilarly, the heat flux out of element across the face $BCFG$ is given by

$$\left(q_x + \frac{\partial q_x}{\partial x} \, dx \right) dy \, dz.$$

The net heat flux in the x-direction into the fluid element by thermal conduction is

$$q_x \, dy \; dz - \left(q_x + \frac{\partial q_x}{\partial x} dx \right) \; dy \, dz$$

That is ,

$$-\frac{\partial q_x}{\partial x} dx \, dy \, dz. \tag{1.18}$$

Thus, the heating of the fluid element due to thermal conductivity by considering all directional conduction is given by

$$-\frac{\partial (q_j)}{\partial x_j} dv. \tag{1.19}$$

By Newton's law of conductive heat transfer, the directional heat transfer is related with temperature gradient as

$$q_j = -k \frac{\partial T}{\partial x_j},$$

where, k is the thermal conductivity. Hence, the total heat interaction term is obtained by using Newton's law as

$$\left[\rho q + \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{\partial x_j} \right) \right] dx \, dy \, dz. \tag{1.20}$$

Rate of change of energy

The total energy of a moving fluid per unit mass is the sum of its internal energy per unit mass (e) and its kinetic energy per unit mass $\left(\frac{V^2}{2}\right)$, where V denote the velocity of the fluid element. Since we are dealing a moving fluid element, the time rate of change of energy per unit mass is given by the substantial derivative

$$\rho\frac{D}{Dt}\left(e+\frac{V^2}{2}\right)dx\,dy\,dz.$$

Final Form

The final form of the energy equation is obtained by equating the rate of change of energy with the sum of the work done and heat transfer terms which is given by

$$\rho\frac{D}{Dt}\left(e+\frac{V^2}{2}\right)=\left[\rho q+\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right)\right]-\left(\nabla.(p\vec{v})+\frac{\partial}{\partial x_j}(u_i\tau_{ji})\right)+\rho\vec{F}.\vec{v}. \quad (1.21)$$

1.2.7 Energy Equation in Conservation Form

We note that

$$\rho\frac{De}{Dt}=\rho\frac{\partial e}{\partial t}+\rho\vec{v}.\nabla e,$$

and,

$$\nabla.(\rho e\vec{v})=e\nabla.(\rho\vec{v})+\rho\vec{v}.\nabla e.$$

Therefore, we have

$$\rho\frac{De}{Dt}=\frac{\partial(\rho e)}{\partial t}-e\left(\frac{\partial\rho}{\partial t}+\nabla.(\rho\vec{v})\right)+\nabla.(\rho e\vec{v}).$$

Since the quantity in bracket is zero by the continuity equation, we have

$$\rho\frac{De}{Dt}=\frac{\partial(\rho e)}{\partial t}+\nabla.(\rho e\vec{v}).$$

Then the conservation form of the energy equation is

$$\begin{aligned}\frac{\partial(\rho e)}{\partial t} + \nabla.(\rho e\vec{v}) =& \rho q + \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) - p\nabla.\vec{v} + \lambda(\nabla.\vec{v})^2 \\ &+ \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2\right] \\ &+ \mu\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + |\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right].\end{aligned}$$

Chapter 2

Literature Review

Abstract

To reduce tear and wear of machinery lubrication is essential. Lubricants form a layer between two surfaces preventing direct contact and reduce friction between moving parts and hence reduce wear. The choice of lubricant is important for a given application. In this model the lubrication of the Slider Bearing is studied. A simple Slider Bearing has two plates of given profile separated by a gap between the plates is filled with the lubricant. One of the plates is fixed and other is moving horizontally. Due to the viscosity of the lubricant, motion of the plate's results in work done on the lubricant increasing the temperature. This study will be helpful in finding the condition under which the safe operation of the bearing is ensured. That is, in finding the condition under which the temperature of the lubricant is lower than the ignition temperature. When the variable viscosity is considered the case becomes complicated. Further investigations are necessary.

2.0.8 Introduction

The incompressible Navier Stokes equation for two dimensional flow is

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} + \frac{\nabla p}{\rho} = \nu(\Delta \vec{v}) + \vec{f}.$$

together with the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{v}) = 0$$

where the symbol $\vec{v} = (u, w)$ represents the velocity vector, p the pressure and ρ the density, $\vec{f}$ the body force and ν the kinematic viscosity.

Let us consider that the fluid is incompressible i.e.$\rho = \text{const}$ with no external force i.e.$\vec{f} = 0$, so that $\frac{\partial \rho}{\partial t} = 0$. Then these equations reduces into

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} + \frac{\nabla p}{\rho} = \nu(\Delta \vec{v}) \tag{2.1}$$

and

$$\nabla.(\rho \vec{v}) = 0. \tag{2.2}$$

Let $u(x, z, t)$ and $w(x, z, t)$ be the components of velocity of the fluids in horizontal and vertical directions respectively, then the equation (2.1) in expanded form is

$$\frac{\partial \mathbf{u}}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2},\right) \tag{2.3}$$

and

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right). \tag{2.4}$$

And the equation (2.2) is,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{2.5}$$

We impose the following boundary conditions:

$$\begin{aligned} (u, w) &= (0, 0) \text{ at } z = 0; \\ (u, w) &= (u_p, 0) \text{ at } z = l. \end{aligned} \tag{2.6}$$

2.0.9 Slider Bearing with Parallel Plates

Let u_p denote the uniform horizontal velocity of the top plate and l is the separation between the plates along z axis and L be the length of the plates. Suppose,

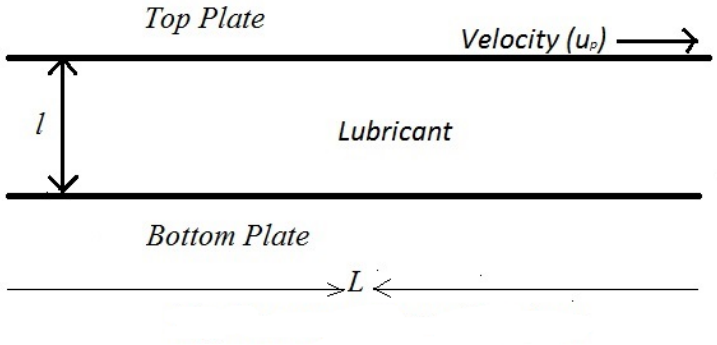


Figure 2.1: Slider Bearing: Parallel Plates

the lower plate is kept constant. Let a typical set of parameters used for the bearing plates be:

$$L = 5\ cm,\ l = 5\mu m, u_p = 1m/sec,\ \rho = 1 \times 10^3\ Kgm^{-3}, \mu = 1 \times 10^{-4} m^2/sec$$

To non dimensionalize the equations (2.3) and (2.4), suppose the parameters are scaled as

$$x = \overline{x}L, z = \overline{z}l, u = \overline{u}u_p, w = \overline{w}\epsilon u_p, t = \frac{\overline{t}L}{u_p}, p = \overline{p}P.$$

From (2.3), we get

$$u_p \frac{\partial \overline{u}}{\partial \overline{t}} \frac{\partial \overline{t}}{\partial t} + \overline{u}u_p \frac{\partial(\overline{u}u_p)}{\partial \overline{x}} \frac{\partial \overline{x}}{\partial x} + \overline{w}\epsilon u_p \frac{\partial(\overline{u}u_p)}{\partial \overline{z}} \frac{\partial \overline{z}}{\partial z} + \frac{1}{\rho} \frac{\partial(\overline{p}P)}{\partial \overline{x}} \frac{\partial \overline{x}}{\partial x} = \frac{1}{\rho} \left\{ \frac{u_p}{L^2} \frac{\partial^2 \overline{u}}{\partial \overline{x}^2} + \frac{u_p}{L^2} \frac{\partial^2(\overline{u})}{\partial \overline{z}^2} \right\}.$$

i.e.

$$\frac{u_p^2}{L} \frac{\partial \overline{u}}{\partial \overline{t}} + \frac{\overline{u}u_p^2}{L} \frac{\overline{u}}{\partial \overline{x}} + \frac{\overline{w}\epsilon u_p^2}{l} \frac{\partial \overline{u}}{\partial \overline{z}} + \frac{1}{\rho} \frac{P}{L} \frac{\partial \overline{p}}{\partial \overline{x}} = \nu \left(\frac{u_p}{L^2} \frac{\partial^2 \overline{u}}{\partial \overline{x}^2} + \frac{u_p}{l^2} \frac{\partial^2 \overline{u}}{\partial \overline{z}^2} \right) \tag{2.7}$$

And from (2.4),

$$\frac{\epsilon u_p^2}{L} . \frac{\partial \overline{w}}{\partial \overline{t}} + \frac{\overline{u}\epsilon u_p^2}{L} . \frac{\partial \overline{w}}{\partial \overline{x}} + \frac{\overline{w}\epsilon^2 u_p^2}{l} \frac{\partial \overline{w}}{\partial \overline{z}} + \frac{1}{\rho} \frac{P}{L} \frac{\partial \overline{p}}{\partial \overline{z}} = \nu \left(\frac{\epsilon u_p}{L^2} \frac{\partial^2 \overline{w}}{\partial \overline{x}^2} + \frac{\epsilon u_p}{l^2} \frac{\partial^2 \overline{w}}{\partial \overline{z}^2} \right). \tag{2.8}$$

Multiplying (2.7) by $\frac{L}{u_p^2}$ and setting $\epsilon = \frac{l}{L}$ is typically 1×10^{-4}, where P is undecided scale factor for the pressure and eliminating the terms which have small coefficients as compared to $\frac{1}{\epsilon}$, we get

$$\frac{P}{\rho u_p^2} \frac{\partial \overline{p}}{\partial \overline{x}} = \frac{\nu L}{l^2 u_p} \frac{\partial^2 \overline{u}}{\partial \overline{z}^2}. \tag{2.9}$$

Let us choose P such that

$$\frac{P}{\rho u_p^2} = \frac{\nu L}{l^2 u_p}$$

i.e.

$$P = \frac{\rho \nu u_p L}{l^2}$$

i.e.

$$P = \frac{\mu u_p L}{l^2} \tag{2.10}$$

where $\mu = \rho\nu$. Then, with this choice of P, equation (2.9) becomes;

$$\frac{\partial \overline{p}}{\partial \overline{x}} = \frac{\partial^2 \overline{u}}{\partial \overline{z}^2} \tag{2.11}$$

And, multiplying equation (2.8) by $\frac{L}{\epsilon u_p^2}$ and setting, $\epsilon = \frac{l}{L}$ and eliminating the terms with coefficients $\leq \frac{1}{\epsilon}$, we get

$$\frac{1}{\rho}\frac{PL}{\epsilon u_p^2}\frac{\partial \overline{p}}{\partial \overline{z}} = 0$$

i.e.

$$\frac{\partial \overline{p}}{\partial \overline{z}} = 0 \tag{2.12}$$

Thus $\overline{p}$ is independent of $\overline{z}$.

Again, from equation (2.5), we obtain

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{w}}{\partial \overline{z}} = 0 \tag{2.13}$$

The boundary condition for $\overline{u}$ and for $\overline{w}$ are:

$$\overline{u} = 0 \text{ at } \overline{z} = 0 \text{ and } \overline{u} = 1 \text{ at } \overline{z} = 1,$$

$$\overline{w} = 0 \text{ at } \overline{z} = 0 \text{ and } \overline{w} = 0 \text{ } at \text{ } \overline{z} = 1.$$

Now, integrating equation (2.12), we have

$$\overline{p} = \phi(\overline{x}),$$

With the boundary conditions ; $\overline{p} = 0$ at $\overline{x} = 0$ and $\overline{p} = 0$ at $\overline{x} = 1$, we get;

$$\phi(0) = \phi(1) = 0. \tag{2.14}$$

Again, Integrating equation (2.11) w.r.t. $\overline{z}$, we obtain,

$$\frac{\partial \overline{u}}{\partial \overline{z}} = \frac{\partial \overline{p}}{\partial \overline{x}}\overline{z} + c_1$$

i.e.

$$\frac{\partial \overline{u}}{\partial \overline{z}} = \frac{\partial \phi(\overline{x})}{\partial \overline{x}}\overline{z} + c_1$$

Again integrating, we get

$$\overline{u} = \frac{\partial \phi(\overline{x})}{\partial \overline{x}}.\frac{z^2}{2} + c_1\overline{z} + c_2 \tag{2.15}$$

where c_1 and c_2 are constants of integration. With the boundary condition $\overline{u} = 0$ at $\overline{z} = 0$ and $\overline{u} = 1$ at $\overline{z} = 1$, we get, $c_2 = 0$ and $c_1 = 1 - \frac{1}{2}\frac{\partial \phi}{\partial \overline{x}}$. Then the expression for $\overline{u}$ from equation (2.14) becomes

$$\overline{u} = \left(\frac{\overline{z}^2 - \overline{z}}{2}\right)\frac{\partial \phi}{\partial \overline{x}} + \overline{z}.$$

Again from equation (2.13),

$$\frac{\partial \overline{w}}{\partial \overline{z}} = -\frac{\partial \overline{u}}{\partial \overline{x}};$$

i.e

$$\frac{\partial \overline{w}}{\partial \overline{z}} = -\frac{\partial}{\partial \overline{x}}\left[\frac{\overline{z}^2 - \overline{z}}{2}\frac{\partial \phi}{\partial \overline{x}} + \overline{z}\right]$$

Integrating w.r.t $\overline{z}$, we get

$$\overline{w} = -\left(\frac{\overline{z}^3}{6} - \frac{\overline{z}^2}{4}\right)\frac{\partial^2 \phi}{\partial \overline{x}^2} + c_3$$

where c_3 is constant of integration. With the boundary conditions; $\overline{w} = 0$ at $\overline{z} = 0$ and $\overline{z} = 1$, we have $c_3 = 0$ and $c_3 = -\frac{1}{12}\frac{\partial^2 \phi}{\partial \overline{x}^2}$. Therefore,

$$\frac{\partial^2 \phi}{\partial \overline{x}^2} = 0.$$

Integrating we get,

$$\frac{\partial \phi}{\partial \overline{x}} = c_4,$$

and so,

$$\phi = c_4\overline{x} + c_5.$$

With the boundary conditions (2.14) we have $\phi = 0$. So, we must have that

$$\overline{p} = 0.$$

Thus, it is shown that the pressure is zero, the bearing with parallel plates can support no load. Therefore, it is not helpful in physical problem. Therefore, we consider the case of non-parallel plates.

2.0.10 Slider Bearing with non-parallel plates.

For the non-parallel plates, we assume that the bottom plate is flat and the top plate is given by the linear function $z = h(x)$ as in figure. Other parameters remain the same as bearing with parallel plates.

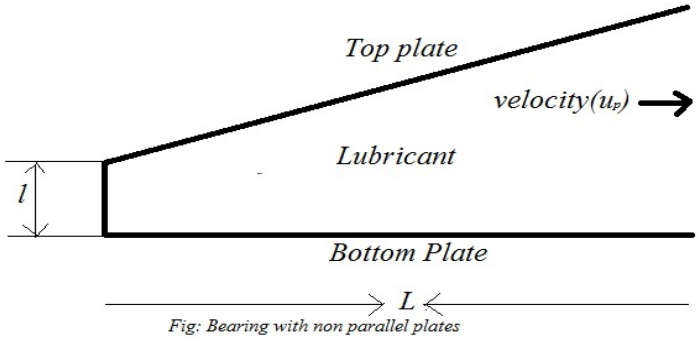


Figure 2.2: Slider Bearing: Non Parallel Plates

We scale the above equations (2.7), (2.8) and (2.9) with $x = \overline{x}L, z = \overline{z}l, u = \overline{u}u_p, w = \overline{w}\epsilon u_p, h(x) = l\overline{h}(\overline{x}), t = \frac{\overline{t}L}{u_p}, p = \overline{p}P$, we obtain the non dimensional equations (2.11), (2.12) and (2.13). Imposing the boundary conditions $\overline{u} = 0$ at $\overline{z} = 0$ and $\overline{u} = 1$ at $\overline{z} = \overline{h}(\overline{x})$, in equation (2.15) we get,

$$\frac{\partial \phi}{\partial \overline{x}}.0 + c_1.0 + c_2 = 0$$
$$\therefore c_2 = 0.$$

and

$$\frac{\partial \phi}{\partial \overline{x}}\frac{\overline{h}^2(\overline{x})}{2} + c_1\overline{h}(\overline{x}) = 1.$$

$$\therefore c_1 = \frac{1}{\overline{h}(\overline{x})} - \frac{1}{2}\frac{\partial\phi}{\partial\overline{x}}\overline{h}(\overline{x}).$$

Thus, the equation (2.15) becomes

$$\overline{u} = \frac{\partial\phi}{\partial\overline{x}}\left(\frac{\overline{z}^2}{2} - \frac{\overline{z}\overline{h}}{2}\right) + \frac{\overline{z}}{\overline{h}}.$$

Denoting $\frac{\partial\phi}{\partial\overline{x}}$ by $\phi'(\overline{x})$, we get

$$\overline{u} = \phi'(\overline{x})\left(\frac{\overline{z}^2}{2} - \frac{\overline{z}\overline{h}}{2}\right) + \frac{\overline{z}}{\overline{h}} \tag{2.16}$$

Now from equation (2.13), we have

$$\begin{aligned}
\frac{\partial\overline{w}}{\partial\overline{z}} &= -\frac{\partial\overline{u}}{\partial\overline{x}} \\
&= -\frac{\partial}{\partial\overline{x}}\left\{\left(\frac{\overline{z}^2 - \overline{z}\overline{h}}{2}\right)\phi'(\overline{x}) + \frac{\overline{z}}{\overline{h}}\right\} \\
&= -\left(\frac{\overline{z}^2 - \overline{z}\overline{h}}{2}\phi''(\overline{x}) - \frac{\overline{z}}{2}\frac{d\overline{h}}{d\overline{x}}\phi'(\overline{x}) + \frac{\overline{h}.0 - \overline{z}\frac{d\overline{h}}{d\overline{x}}}{\overline{h}^2}\right) \\
\therefore \frac{\partial\overline{w}}{\partial\overline{z}} &= -\frac{\overline{z}^2 - \overline{z}\overline{h}}{2}\phi''(\overline{x}) + \frac{\overline{z}}{2}\frac{d\overline{h}}{d\overline{x}}\phi'(\overline{x}) + \frac{\overline{z}\left(\frac{d\overline{h}}{d\overline{x}}\right)}{\overline{h}^2}.
\end{aligned}$$

Integrating w.r.t. $\overline{z}$ we get,

$$\overline{w} = -\left(\frac{\overline{z}^3}{6} - \frac{\overline{z}^2\overline{h}}{4}\right)\phi''(\overline{x}) + \frac{\overline{z}^2}{4}\frac{d\overline{h}}{d\overline{x}}\phi'(\overline{x}) + \frac{\overline{z}^2}{2\overline{h}^2}\frac{d\overline{h}}{d\overline{x}} + c_3. \tag{2.17}$$

By imposing the boundary conditions, $\overline{w} = 0$ at $\overline{z} = 0$, we get

$$c_3 = 0,$$

and using the boundary condition $\overline{w} = 0$ at $\overline{z} = \overline{h}(\overline{x})$, we get

$$\begin{aligned}
&-\left(\frac{\overline{h}^3(\overline{x})}{6} - \frac{\overline{h}^2(\overline{x})}{4}\overline{h}\right)\phi''(\overline{x}) + \frac{\overline{h}^2(\overline{x})}{4}\frac{d\overline{h}}{d\overline{x}}\phi'(\overline{x}) + \frac{\overline{h}^2(\overline{x})}{2\overline{h}^2}\frac{d\overline{h}}{d\overline{x}} = 0. \\
i.e.\ &-\frac{\overline{h}^3(\overline{x})}{12}\phi''(\overline{x}) + \frac{\overline{h}^2(\overline{x})}{4}\frac{d\overline{h}}{d\overline{x}}\phi'(\overline{x}) - \frac{1}{2}\frac{d\overline{h}}{d\overline{x}} = 0. \\
\therefore\ &\frac{\overline{h}^3(\overline{x})}{12}\phi''(\overline{x}) + \frac{\overline{h}^2(\overline{x})}{4}\frac{d\overline{h}}{d\overline{x}}\phi'(\overline{x}) + \frac{1}{2}\frac{d\overline{h}}{d\overline{x}} = 0. \\
\Rightarrow\ &\frac{d}{d\overline{h}}\left(\frac{\overline{z}^3(\overline{x})}{12}\phi'(\overline{x})\right) + \frac{1}{2}\frac{d\overline{h}}{d\overline{x}} = 0.
\end{aligned}$$

Assume a linear profile $\overline{h}(\overline{x}) = \overline{x} + 1$. With this assumption, we get

$$\frac{d}{d\overline{x}}\left(\frac{(\overline{x}+1)^3}{12}\phi'(\overline{x})\right) + \frac{1}{2}\frac{d}{d\overline{x}}(\overline{x}+1) = 0$$

Integrating, we get

$$\begin{aligned}
&\frac{(\overline{x}+1)^3}{12}\phi'(\overline{x}) + \frac{1}{2}(\overline{x}+1) = r_1 \\
&\phi'(\overline{x}) = \frac{(-\frac{1}{2}(\overline{x}+1) + r_1)}{\frac{(\overline{x}+1)^3}{12}} \\
&\therefore \phi'(\overline{x}) = \frac{-6}{(\overline{x}+1)^2} + \frac{12r_1}{(\overline{x}+1)^3} \qquad (2.18)
\end{aligned}$$

Integrating we get,

$$\phi(\overline{x}) = \frac{6}{\overline{x}+1} - \frac{6r_1}{(\overline{x}+1)^2} + r_2.$$

The constants of integration r_1 and r_2 are determined by boundary conditions on ϕ i.e. $\phi = 0$ at $\overline{x} = 0$ and $\overline{x} = 1$. Imposing the boundary conditions, we get

$$\begin{aligned}
6r_1 - r_2 =& 6 \\
\frac{3}{2}r_1 - r_2 =& 3.
\end{aligned} \qquad (2.19)$$

Soliving them, we get

$$r_1 = \frac{2}{3},\ r_2 = -2.$$

Hence we have

$$\begin{aligned}
\phi(\overline{x}) =& \frac{6}{1+\overline{x}} - \frac{4}{(1+\overline{x})^2} - 2 \\
=& \frac{6}{1+\overline{x}} - \frac{4}{(1+\overline{x})^2} - 2 \\
=& \frac{2\overline{x} - 2\overline{x}^2}{(1+\overline{x})^2} \\
=& \frac{2\overline{x}(1-\overline{x})}{(1+\overline{x})^2} \\
\therefore \overline{p} =& \frac{2\overline{x}(1-\overline{x})}{(1+\overline{x})^2}
\end{aligned}$$

which is positive for $\overline{x} \in (0,1)$. Hence, pressure is developed inside the fluid and the bearing supports a load given by the integral of pressure between the limits $\overline{x} = 0$ and $\overline{x} = 1$. Hence,

$$\text{load } (\phi) = \int_0^1 \frac{2\overline{x}(1-\overline{x})}{(1+\overline{x})^2} d\overline{x}.$$

Thus,

$$\text{load} = 6 \ln (2) - 4. \tag{2.20}$$

Substituting the pressure ϕ in equation (2.16) and (2.17), we get the expression of u and w respectively.

2.1 Viscuss Dissipation Caused by Motion

In this section we want to find a critical temperature T_c before the lubricant catches the fire. Suppose that the plates are kept at a constant temperature T_1 and the lubricant catches fire at critical temperature T_c. We rescale the temperature according as

$$T = T_1 + \overline{\theta}(T_c - T_1), \tag{2.21}$$

where $\overline{\theta} \in (0,1)$ is a non-dimensional variable.

2.1.1 Constant Viscosity

Assume that the viscosity of the lubricant is constant with respect to temperature. We have energy balance equation

$$\rho C_p \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) + \phi$$

where,

$$\phi = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \frac{\partial v_i}{\partial x_j}$$

Where ρ and k are density and thermal conductivity. The external energy q_i is neglected because it is not present.

Now, from energy Balance equation , we have

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z}\right)=2\mu\left[\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right),$$

where c_p is the specific heat of the fluid and k is its thermal conductivity. Suppose that the plates are kept at a constant temperature T_1 and lubricant catches fire at critical temperature T_c. We proceed to non-dimensionalize this equation by using the scaling (2.21) and neglecting the terms that are small as compared to $\frac{1}{\epsilon}$, we get the simplified energy equation given by

$$\begin{aligned}\mu\frac{u_p^2}{l^2}\left(\frac{\partial \overline{u}}{\partial \overline{z}}\right)^2 &= -\frac{k}{l^2}\frac{\partial^2\overline{\theta}}{\partial\overline{z}^2}(T_c-T_1).\\ \left(\frac{\partial \overline{u}}{\partial \overline{z}}\right)^2 &= -\frac{k}{\mu u_p^2}\frac{\partial^2\overline{\theta}}{\partial\overline{z}^2}(T_c-T_1).\\ \therefore\left(\frac{\partial \overline{u}}{\partial \overline{z}}\right)^2 &= -B\frac{\partial^2\overline{\theta}}{\partial\overline{z}^2}.\end{aligned}$$

Where,

$$B=\frac{k}{\mu u_p^2}(T_c-T_1).$$

Substituting the value of $\overline{u}$ from the equation (2.15), we get

$$\frac{\partial}{\partial\overline{z}}\left[\frac{(\overline{z}^2-\overline{z})}{2}\frac{d\phi}{d\overline{x}}+\frac{\overline{z}}{\overline{h}}\right]^2=-B\frac{\partial^2\overline{\theta}}{\partial\overline{z}^2}$$

Integrating twice the equation, we get,

$$\frac{z^4}{12}\phi'(\overline{x})^2+\left[\frac{1}{\overline{h}}-\frac{\overline{h}}{2}\phi'(\overline{x})\right]^2\frac{z^2}{2}+\frac{z^3}{3}\phi'(\overline{x})\left[\frac{1}{\overline{h}}-\frac{\overline{h}}{2}\phi'(\overline{x})\right]+c_1\overline{z}+c_2=-B\overline{\theta}. \quad (2.22)$$

The boundary conditions on $\overline{\theta}$ are $\overline{\theta}=0$ at $\overline{z}=0$ and $\overline{z}=\overline{h}(\overline{x})$. Using the boundary condition for $\overline{\theta}$ in the equation(2.22), the expression for $\overline{\theta}$ appears as

$$\begin{aligned}-B\overline{\theta} =&\phi'(\overline{x})^2\frac{(\overline{z}^4-\overline{h}^3\overline{z})}{12}+\left[\frac{1}{\overline{h}}-\frac{\overline{h}}{2}\phi'(\overline{x})\right]^2\left(\frac{\overline{z}^2-\overline{z}\overline{h}}{2}\right)\\ &+\phi'(\overline{x})\left[\frac{1}{\overline{h}}-\frac{\overline{h}}{2}\phi'(\overline{x})\right]\left(\frac{\overline{z}^3-\overline{h}^2\overline{z}}{3}\right).\end{aligned} \quad (2.23)$$

We note that $\overline{\theta}$ satisfies the boundary conditions at $\overline{z} = 0$ and $\overline{z} = \overline{h}(\overline{x})$. So, for the case of linear profile $\overline{h}(\overline{x}) = \overline{x} + 1$ and using (2.18), the critical parameter $\phi'(\overline{x})$ becomes

$$\phi'(\overline{x}) = -\frac{6}{(\overline{x}+1)^2} + \frac{8}{(\overline{x}+1)^3}$$

So, from the equation (2.23), the non-dimensional parameter of the temperature $\overline{\theta}$ can be determined.

Here pressure is not zero and the bearing with non-parallel plates can support load easily. Thus, the expression for temperature of a lubricant in slider bearing is derived. For the case, when viscosity is constant, the expression is

$$B = \frac{k}{\mu u_p^2}(T_c - T_1).$$

This gives conditions on possible values of various parameters of the bearing. However, when viscosity of the lubricant changes with temperature, due to nature of PDE involved, an explicit solution could not be derived.

2.1.2 Variable Viscosity

In the previous works, viscosity was considered as constant. Here, variable viscosity is considered. For the liquid, the viscosity decreases with the temperature. From (3) and (4) we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}(v\frac{\partial u}{\partial x}) + \frac{\partial}{\partial z}(v\frac{\partial u}{\partial x}) \tag{2.24}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial}{\partial x}(\frac{\partial w}{\partial x}) + \frac{\partial}{\partial z}(v\frac{\partial w}{\partial x}) \tag{2.25}$$

Using dimensionless viscosity $\nu = \nu_0\overline{\nu}$ in

$$\frac{\partial \overline{p}}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}\left(\nu\frac{\partial \overline{u}}{\partial \overline{z}}\right), \tag{2.26}$$

we obtain,

$$\frac{\partial \overline{p}}{\partial \overline{z}} = \nu_0\frac{\partial}{\partial \overline{z}}\left(\overline{\nu}\frac{\partial \overline{u}}{\partial \overline{z}}\right), \tag{2.27}$$

Similarly, the energy equation reduces to

$$\overline{\nu}\left(\frac{\partial\overline{u}}{\partial\overline{z}}\right)^2 = -B\frac{\partial^2\overline{\theta}}{\partial\overline{z}^2}, \tag{2.28}$$

where, $B = \frac{k}{\mu u_p^2}(T_c - T_1)$. Now, we want to study the equations (2.24), (2.25) and (2.28) seeking a solution profile with a convergent numerical scheme which is done in chapter 6.

Chapter 3

Introduction to Finite Difference Scheme

We begin our discussion of finite difference schemes by defining a grid of points in the (t, x) plane. Let h and k be positive numbers; then the grid will be the points $(t_n, x_m) = (nk, mh)$ for arbitrary integers n and m. For a function v defined on the grid point, we write v_m^n for the value of v at the grid point (t_n, x_m) where $n = 0, 1, 2, ..., N$ and $m = 0, 1, 2, ..., M$. We also use the notation u_m^n for $u(t_n, x_m)$ when u is defined for continuously varying (t, x). The set of points (t_n, x_m) for a fixed value of n is called **grid level** n. We are interested in grids with small values of h and k. The basic idea of finite difference schemes is to replace derivatives by finite differences. This can be done in many ways; for this let $u(t, x)$ be a solution to the differential equation $u_t + au_x = 0$, then we want to find v_m^n that approximates $u(t_n, x_m)$ i.e. $u(nk, mh)$. That means we want to find v_m^n such that $v_m^n \equiv u(nk, mh)$. Now we replace the derivative of $u(t_n, x_m)$ by finite difference as,

$$u_t(t_n, x_m) = \frac{u(t_n + k, x_m) - u(t_n, x_m)}{k} = \frac{v_m^{n+1} - v_n^n}{k}.$$

It is defined on the basis of forward time. similarly on the basis of forward space, we can write

$$u_x = \frac{u(t_n, x_m + h) - u(t_n, x_m)}{h} = \frac{v_{m+1}^n - v_m^n}{h}.$$

These are valid approximations that can be seen from the formulas

$$u_t(t_n, x_m) = lim_{k\to 0}\frac{u(t_n + k, x_m) - u(t_n, x_m)}{k}$$

and

$$u_t(t_n, x_m) = lim_{k\to 0}\frac{u(t_n + k, x_m) - u(t_n - k, x_m)}{2k}.$$

3.1 Finite Difference Schemes

Consider a differential equation,

$$u_t + au_x = 0 \tag{3.1}$$

We obtain the following five finite difference schemes for equation (3.1).

1. Forward time forward space scheme:

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_m^n}{h} = 0 \tag{3.2}$$

2. forward time backward space scheme:

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = 0 \tag{3.3}$$

3. Forward time central space scheme:

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \tag{3.4}$$

4. Leapfrog or central time central space scheme:

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \tag{3.5}$$

5. Lax Friedrich's scheme:

$$\frac{v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0 \tag{3.6}$$

We refer to scheme (3.2) as the forward-time forward-space scheme because forward difference approximations are used for both the time and space derivatives. Similarly, (3.3) and (3.4) are referred to as the forward-time backward-space scheme and forward- time central-space scheme, respectively. The scheme (3.5) is called the leapfrog scheme and (3.6) is called the Lax-Friedrichs scheme. It is very easy to derive finite difference schemes for a partial differential equations. However, the analysis of finite difference schemes to determine if they are useful approximations to the differential equation requires some powerful mathematical tools. Moreover, to develop very efficient and accurate schemes requires more work than went into obtaining the schemes $(3.2) - (3.6)$. Nonetheless, the finite difference method is notable for the great variety of schemes that can be used to approximate a given partial differential equation. Given this short list of schemes, we are naturally led to the question of which of them are useful and which are not, as indeed some are not. In fact, we first determine which schemes have solutions that approximate solutions of the differential equation at all. Later, we determine which schemes are more accurate than others and also investigate the efficiency of the various schemes. Each of the schemes $(3.2) - (3.6)$ can be written expressing v_m^{n+1} as a linear combination of values of v at levels n and $n-1$. For example, scheme (3.4) can be written as

$$v_m^{n+1} = v_m^n - \frac{a\lambda}{2}(v_{m+1}^n - v_{m-1}^n)$$

where $\lambda = \frac{k}{h}$.

Those schemes that involve at only two levels, e.g.,$n+1$ and n, are called *one-step schemes*. Of the schemes $(3.2) - (3.6)$, all except the leapfrog scheme (3.6) are one-step schemes. Given the initial data v_m^0, a one-step scheme can be used to evaluate v_m^n for all positive values of n. The leapfrog scheme (3.5) is an example of a *multistep scheme*. For a multistep scheme it is not sufficient to specify the values of v_m^0 in order to determine v_m^n for all positive values of n. To specify completely the way of computing a solution to a multistep scheme, either we must specify v on enough time levels so that the scheme can be employed or a

procedure for computing the values of v on these initial time levels should given.

3.1.1 Consistency:

A scheme is said to be useful in numerical analysis, if its solutions approximate the solution of the corresponding partial differential equation and that the approximation improves as the grid spacings h and k tend to zero. One of such characteristics is the consistency of the scheme.

Definition 1. *A one-step finite difference scheme approximating a partial differential equation is a convergent scheme if for any solution to the partial differential equation $u(t,x)$, and solutions to the finite difference scheme v_m^n, such that if $v_0^n \longrightarrow u_0(x)$ as $mh \to x$ then $v_m^n \longrightarrow u(t,x)$ as $(nk, mh) \to (t,x)$ as $(k,h) \to 0$.*

Definition 2. *Given a partial differential equation, $Pu = f$, and a finite difference scheme, $P_{k,h}v_m^n$, we say that the finite difference scheme is consistent with the partial differential equation if for any smooth function $\phi(t,x)$, we have,*

$$P_{k,h}\phi(t,x) \longrightarrow P\phi$$

as $h, k \longrightarrow 0$, the convergence being pointwise convergence at each point (t,x).

A *smooth function* is a function which is sufficiently differentiable in the domain of the context. A difference operator $P_{k,h}$ applied to a function of (t,x) in not necessarily restricted to grid the points. In this resoect, a forward difference in x applied at a point (t,x) is

$$\frac{\phi(t,x+h) - \phi(t,x)}{h}.$$

Example 1. *The Forward-time Forward-space Scheme is consistent with the given PDE* (3.1).

For the one-way wave equation (3.1), the operator P is

$$P = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x}$$

so that

$$P\phi = \phi_t + a\phi_x.$$

For the forward-time forward-space scheme (i), the difference operator $P_{k,h}$ is given by

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + a\frac{\phi_{m+1}^n - \phi_m^n}{h}$$

where

$$\phi_m^n = \phi(nk, mh).$$

We begin with the Taylor series of the function ϕ in t and x about (t_n, x_m). We have the Taylor series expansion,

$$\phi_m^{n+1} = \phi_m^n + k\phi_t + \frac{k^2}{2}\phi_{tt} + O(k^3).$$

$$\phi_{m+1}^n = \phi_m^n + h\phi_x + \frac{h^2}{2}\phi_{xx} + O(h^).$$

where the derivatives on the right-hand side are all evaluated at (t_n, x_m), and so

$$P_{h,k}\phi = \phi_t + a\phi_x + \frac{k}{2}\phi_{tt} + \frac{ah}{2}\phi_{xx} + O(k^2) + O(h^2)$$

Thus,

$$P_{h,k}\phi = P\phi + \frac{k}{2}\phi_{tt} + \frac{ah}{2}\phi_{xx} + O(k^2) + O(h^2) \longrightarrow o$$

as $(k, h) \longrightarrow 0$. Therefore, this scheme is consistent.

When analyzing consistency, we need to use the "big oh" and "little oh" notations . In general, if F and G are functions of some parameter a, we write $F = O(G)$ as $a \longrightarrow 0$ if

$$\left|\frac{F}{K}\right| \leq K$$

for some constant K and all sufficiently small a. Further, we write $F = O(G)$ as $a \longrightarrow 0$ if $\frac{F}{G} \longrightarrow 0$ as $a \to 0$. In particular, a quantity is $O(h^r)$ if it is bounded by a constant multiple of h^r for small h. A quantity is $o(L)$ if it converges to zero at an unspecified rate.

Example 2. *The Lax-Friedrichs Scheme is consistent with the PDE* (3.1).

For the PDE, (3.1), Lax-Friedrichs scheme is,

$$\frac{v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0.$$

The difference operator is given by,

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \frac{1}{2}(\phi_{m+1}^n + \phi_{m-1}^n)}{k} + a\frac{\phi_{m+1}^n - \phi_{m-1}^n}{2h}$$

We use the Taylor series

$$\phi_{m\pm1}^n = \phi_m^n \pm h\phi_x + \frac{h^2}{2}\phi_{xx} \pm \frac{h^3}{6}\phi_{xxx} + O(h^4)$$

where the derivatives are evaluated at (t_n, x_m) and we have

$$\frac{1}{2}(\phi_{m+1}^n + \phi_{m-1}^n) = \phi_m^n + \frac{1}{2}h^2\phi_{xx} + O(h^4)$$

and

$$\phi_{m+1}^n - \phi_{m-1}^n = \phi_x + \frac{1}{6}h^2\phi_{xxx} + O(h^4).$$

Substituting these expressions in the Lax Friedrichs scheme, we obtain,

$$P_{k,h}\phi = \phi_t + a\phi_x + \frac{k}{2}\phi_{tt} - \frac{1}{2}\frac{h^2}{k}\phi_{xx} + \frac{ah^2}{6}\phi_{XXX} + O(h^4 + \frac{h^4}{k} + k^2).$$

So, $P_{k,h}\phi - P\phi \longrightarrow 0$ as $h, k \longrightarrow 0$; as long as $\frac{h^2}{t} \longrightarrow 0$. Therefore, the scheme is consistent.

Example 3. *The forward-time central-space scheme is consistent with the PDE* (3.1).

The forward-time central-space scheme for the given PDE is

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

The difference operator is given by

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + a\frac{\phi_{m+1}^n - \phi_{m-1}^n}{2h}.$$

Now, using Taylors series expansion, we get;

$$\phi_{m+1}^n - \phi_{m-1}^n = \phi_x + \frac{1}{6}h^2\phi_{xxx} + O(h^4).$$

and;

$$\phi_{m\pm1}^n = \phi_m^n \pm h\phi_x + \frac{h^2}{2}\phi_{xx} \pm \frac{h^3}{6}\phi_{xxx} + O(h^4)$$

Using these results in difference operator, we get;

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + a\phi_{m+1}^n - \phi_{m-1}^n 2h$$

3.1.2 Stability

If a scheme is convergent as v_m^n converges to $u(t,x)$ then certainly v_m^n is bounded in some sense. This is the essence of *stability.*

Definition 3. *A stability region is any bounded non-empty region of the first quadrant of* $\mathbb{R}^2$ *that has the origin as an accumulation point. That is, a stability region must contain a sequence* (k_v, h_v) *that converges to the origin as* u *tends to infinity. A common example is a region of the form* $\{(k,h) : 0 < k < ch < C\}$ *for some positive constants* c *and* C*.*

Definition 4. *A finite difference scheme* $P_{k,h}v_m^n = 0$ *for a first-order partial differential equation is stable in a stability region* A *if there is an integer* J *such that for any positive time* T*, there is a constant* C_T *such that*

$$h\sum_{m=-\infty}^{\infty}|v_m^n|^2 \le C_T h\sum_{j=0}^{J}\sum_{m=-\infty}^{\infty}|v_m^j|^2 \text{ for } 0 < n_k < T \text{ with } (k,h) \in A.$$

Definition 5. *The* L^2 *norm of the grid function* ω *over the step size* h *is defined as*

$$\|\omega\|_h = \left(h\sum_{-\infty}^{\infty}|\omega_m|^2\right)^{\frac{1}{2}}.$$

Definition 6. *The* L^2 *norm of the grid function* ω *i.e.*$\|\omega\|_2$ *is a measure of the size of the solution. In terms of* L^2 *norm, the stability condition can be written as*

$$\|v^n\|_h \le \left(C_T\sum_{j=0}^{J}\|v^j\|_h^2\right)^{\frac{1}{2}},$$

which is equivalent to

$$\|v^n\|_h \le C_T^*\sum_{j=0}^{J}\|v^j\|_h \text{ for some constant } C_T *. \tag{3.7}$$

For one step scheme we may take $J = 0$ and $J \ge 1$ for multi-step scheme.

Example 4. *The sufficient condition for stability for the forward-time forward-space scheme for the PDE* $u_t + au_x = 0$ *of the form* $v_m^{n+1} = \alpha v_m^n + \beta v_{m+1}^n$ *is*

$$|a| + |\beta| \le 0.$$

Indeed,

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 &= \sum_{m=-\infty}^{\infty} |\alpha v_m^n + \beta v_{m+1}^n|^2. \\
&\leq \sum_{m=-\infty}^{\infty} |\alpha|^2 |v_m^n|^2 + 2|\alpha||v_m^n||\beta||v_{m+1}^n| + |\beta|^2 |v_{m+1}^n|^2. \\
&\leq \sum_{m=-\infty}^{\infty} |\alpha|^2 |v_m^n|^2 + |\alpha||\beta| \left(|v_m^n|^2 + |v_{m+1}^n|^2\right) + |\beta|^2 |v_{m+1}^n|^2. \\
&= \sum_{m=-\infty}^{\infty} \left(|\alpha|^2 |v_m^n|^2 + |\alpha||\beta||v_m^n|^2\right) + \sum_{m=-\infty}^{\infty} \left(|\alpha||\beta||v_{m+1}^n|^2 + |\beta|^2 |v_{m+1}^n|^2\right). \\
&= \sum_{m=-\infty}^{\infty} \left(|\alpha|^2 + 2|\alpha||\beta| + |\beta|^2\right) |v_m^n|^2. \\
&= (|\alpha| + |\beta|)^2 \sum_{m=-\infty}^{\infty} |v_m^n|^2.
\end{aligned}$$

This shows that we have the relation

$$\sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 \leq (|\alpha| + |\beta|)^2 \sum_{m=-\infty}^{\infty} |v_m^n|^2,$$

and since this applies for all n, we have that

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 &\leq (|\alpha| + |\beta|)^2 \sum_{m=-\infty}^{\infty} |v_m^n|^2 \\
&\leq (|\alpha| + |\beta|)^{2.2} \sum_{m=-\infty}^{\infty} |v_m^{n-1}|^2 \\
&\leq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\leq (|\alpha| + |\beta|)^{2n} \sum_{m=-\infty}^{\infty} |v_m^0|^2.
\end{aligned}$$

If $|\alpha| + |\beta| \leq 1$ in magnitude, then the scheme will be stable.

Definition 7. *The initial value problem for the first-order partial differential equation $Pu = 0$ is well-posed if for any time $T > 0$, there is a constant C_T such that*

any solution $u(t,x)$ satisfies

$$\int_{-\infty}^{\infty} |u(t,x)|^2 dx \leq C_T \int_{\infty}^{\infty} |u(0,x)|^2 dx,$$

for $0 \leq t \leq T$.

3.1.3 The Courant-Friedrichs-Lewy (CFL) Condition

Theorem 3.1.1. *For an explicit scheme for the hyperbolic equation $u_t + au_x = 0$ of the form*

$$v_m^{n+1} = \alpha v_{m-1}^n + \beta v_m^n + \gamma v_{m+1}^n,$$

with $\frac{k}{h} = \lambda$ (a constant), a necessary condition for stability is the Courant-Friedrichs-Lewy (CFL) condition, $|a\lambda| \leq 1$.

Theorem 3.1.2. *There are no explicit, unconditionally stable, consistent finite difference schemes for hyperbolic systems of partial differential equations.*

Example 5. *The backward-time backward-space scheme for the PDE $u_t + au_x = 0$ of the form $\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = 0$ is stable when $a > 0$ and λ is any positive number.*

Indeed, we first write the given scheme as

$$(1 + a\lambda)v_m^{n+1} = v_m^n + a\lambda v_{m-1}^{n+1}.$$

Squaring on both sides, we obtain

$$\begin{aligned}(1 + a\lambda)^2 |v_m^{n+1}|^2 \leq & |v_m^n|^2 + 2a\lambda|v_{m-1}^{n+1}| + a^2\lambda^2|v_{m-1}^{n+1}|^2 \\ \leq & (1 + a\lambda)|v_m^n|^2 + a\lambda(1 + a\lambda)|v_{m-1}^{n+1}|^2\end{aligned}$$

Taking the sum over all values of m, we obtain

$$(1 + a\lambda)^2 \sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 \leq (1 + a\lambda) \sum_{m=-\infty}^{\infty} |v_m^n|^2 + a\lambda(1 + a\lambda) \sum_{m=-\infty}^{\infty} |v_{m-1}^{n+1}|^2$$

$$\text{i.e. } \sum_{m=-\infty}^{\infty} |v_m^{n+1}|^2 \leq \sum_{m=-\infty}^{\infty} |v_m^n|^2$$

This inequality shows that the scheme is stable for every value of λ when a is positive.

3.1.4 Fourier Analysis and Stability

Foutier analysis is the tool used most extensively in the study of stability and well-posedness of a scheme. Fourier analysis can be used on both the real line $\mathbb{R}$ and on the grid of integers Z or hZ, defined by $hZ = \{hm : m \in Z\}$. For a function $u(x)$ defined on the real line $\mathbb{R}$, its Fourier transform $\hat{u}(\omega)$ is defined by

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} u(x) dx.$$

The Fourier transform of u is a function of the real variable ω and is uniquely defined by u. The function $\hat{u}$ is an alternative representation of the function u. Information about certain properties of u can be inferred from the properties of $\hat{u}$. For example, the rate at which u decays for large values of ω is related to the number of derivatives that u has. The Fourier inversion formula is given by

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}(\omega) d\omega.$$

In a similar fashion, if v is a grid function defined for all integers m, its Fourier transform is given by

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} v_{\xi},$$

for $\xi \in [-\pi, \pi]$ and $\hat{v}(-\pi) = -\hat{v}(\pi)$.
The Fourier inversion formula is given by

$$v(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{im\xi} \hat{v}(\xi) d\xi.$$

If the spacing between the grid points is h, we can change variables and define the transform by

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{-imh\xi} v_m h,$$

for $\xi \in \left[\frac{-\pi}{h}, \frac{\pi}{h}\right]$, and then the inversion formula is

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} e^{imhx} \hat{v}(\xi)) d\xi,$$

We also note that L^2 norm of u is

$$\|u\|_2 = \left(\int_{-\infty}^{\infty} |u(x)|^2 dx\right)^{\frac{1}{2}},$$

which is the same as the L^2 norm of $\hat{u}$ i.e., $\|\hat{u}\|_2 = \left(\int_{-\infty}^{\infty} |\hat{u}(\omega)|^2 d\omega\right)^{\frac{1}{2}}$. Thus,

$$\left(\int_{-\infty}^{\infty} |u(x)|^2 dx\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} |\hat{u}(\omega)|^2 d\omega\right)^{\frac{1}{2}} \tag{3.8}$$

Also, for the discrete transform we have equality for the L^2 norm of v and the L^2 norm of $\hat{v}$ which are given by,

$$\|\hat{v}\|_h^2 = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\hat{v}(\xi)|^2 d\xi = h \sum_{-\infty}^{\infty} |v_m|^2 = \|v_m\|_h^2 \tag{3.9}$$

Indeed,

$$\begin{aligned}
\|\hat{v}\|_h^2 &= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} |\hat{v}(\xi)|^2 d\xi. \\
&= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \hat{v}(\xi)\overline{\hat{v}(\xi)} d\xi. \\
&= \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \frac{1}{\sqrt{2\pi}} \overline{\hat{v}(\xi)} \left(\frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{-imh\xi} v_m h\right) d\xi. \\
&= \sum_{m=-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-imh\xi} \overline{\hat{v}(\xi)} d\xi\right) v_m h. \\
&= \sum_{m=-\infty}^{\infty} \overline{v_m} v_m h \\
&= h \sum_{m=-\infty}^{\infty} |v_m|^2 \\
&= \|v_m\|_m^2.
\end{aligned}$$

The relations (3.8) and (3.9) are called *Parseval's relations.* It allows us to replace the stability estimates (3.8)and (3.9) by the equivalent inequality

$$\|\hat{v}^n\|_h \leq C_T^* \sum_{j=0}^{J} \|\hat{v}^j\|_h,$$

for the transform of the grid function.

Using Parseval's relations, we can show that the Fourier transform is defined for all functions in $L^2(\mathbb{R})$ and $L^2(hZ)$. Moreover, we can find the derivatives of the Fourier transform of $u(x)$ and that of its inversion. In fact, the Fourier transform of $u(x)$ is given by

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} u(x) dx.$$

And its inversion transform is

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}(\omega) d\omega.$$

Then the derivative of the inverse transform $u(x)$ is

$$\frac{du(x)}{dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega e^{i\omega x} \hat{u}(\omega) d\omega.$$

The Fourier transform of $\frac{du(x)}{dx}$ is

$$\begin{aligned}
\widehat{\frac{du}{dx}}(\xi) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \frac{du(x)}{dx} dx. \\
=& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega e^{i\omega x} \hat{u}(\omega) d\omega \right) dx.
\end{aligned}$$

Therefore, $\left(\widehat{\frac{du}{dx}}\right)(\omega) = i\omega \hat{u}(\omega)$.

3.1.5 Von Neumann Analysis for Stability

An important application of Fourier analysis is the von Neumann analysis of stability of finite difference schemes. With the use of Fourier analysis, the necessary and sufficient conditions for the stability of finite difference schemes can be stated.

Example 6. *Von Neumann Analysis for Forward time backward space scheme*

The scheme for the PDE $u_t + au_x = 0$ is

$$\frac{v_m^{n+1} - v_m^n}{k} + a \frac{v_m^n - v_{m-1}^n}{h} = 0$$

which can be rewritten as

$$v_m^{n+1} = (1 - a\lambda)v_m^n + a\lambda v_{m-1}^n,$$

where $\lambda = \frac{k}{h}$. The Fourier inversion formula for v^n is

$$v_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i\xi x} \hat{v}^n d\xi.$$

Substituting this in the scheme for v^n and v_{m-1}^n we obtain

$$v_m^{n+1} = (1 - a\lambda) \left(\frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \hat{v}^n d\xi \right) + a\lambda \left(\frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i(m-1)h\xi} \hat{v}^n d\xi \right),$$

$$v_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \left[(1 - a\lambda) + a\lambda e^{-ih\xi} \right] \hat{v}^n(\xi) d\xi.$$

But the inversion formula for v_m^{n+1} is given by

$$v_m^{n+1} = \frac{1}{sqrt2\pi} \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \hat{v}^{n+1}(\xi) d(\xi)$$

Comparing these two formulas for v^{n+1}, and using the fact that the Fourier transform is unique, we obtain that

$$\begin{aligned} \hat{v}^{n+1}(\xi) &= \left[(1 - a\lambda) + a\lambda e^{-ih\xi} \right] \hat{v}^n(\xi) \\ &= g(h\xi)\hat{v}^n(\xi). \end{aligned}$$

where $g(h\xi) = \left[(1 - a\lambda) + a\lambda e^{-ih\xi} \right]$. The factor $g(h\xi)$ is called an *amplification factor*. Now we can generalize this formula as

$$\hat{v}^{n+1}(\xi) = g(h\xi)\hat{v}^n(\xi) = [g(h\xi)]^2 \hat{v}^{n-1}(\xi) = ... = [g(h\xi)]^n \hat{v}^0(\xi).$$

By means of the Fourier transform every one-step scheme can be put in this form and this provides a standard method for studying the wide variety of schemes. All the information about a scheme is contained in its amplification factor. In particular, the stability and accuracy of schemes is easy to determine from the amplification factor.

Example 7. *The stability condition for forward time central space scheme by using Von-Neumann analysis*

The scheme for the equation $u_t = au_{xx}$ is

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = 0.$$

By Parseval's relation,

$$\begin{aligned} h\sum_{-\infty}^{\infty} |v_m^n|^2 &= \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} |\hat{v}^n(\xi)|^2 d\xi. \\ &= \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} |\left[g(h\xi)\right]^n \hat{v}^0(\xi)|^2 d\xi. \\ &= \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} |g(h\xi)|^{2n} |\hat{v}^0(\xi)|^2 d\xi. \end{aligned}$$

Thus we see that the stability inequality

$$h\sum_{m=-\infty}^{\infty} |v_m^n|^2 \leq C_T h \sum_{j=0}^{J} \sum_{m=-\infty}^{\infty} |v_m^j|^2,$$

will hold, with $J = 0$, if $|g(h\xi)|^{2n}$ is suitably bounded. We now evaluate $|g(h\xi)|$.

Setting $\theta = h\xi$;, we have

$$\begin{aligned}
g(\theta) =& (1 - a\lambda) + a\lambda e^{-i\theta} \\
=& 1 - a\lambda + a\lambda cos\theta - ia\lambda sin\theta \\
=& (1 - a\lambda + a\lambda cos\theta) - ia\lambda sin\theta \\
\therefore |g(\theta)|^2 =& (1 - a\lambda + a\lambda cos\theta)^2 + (a\lambda sin\theta)^2 \\
=& \left[1 - a\lambda + a\lambda\left(1 - 2sin^2\left(\frac{\theta}{2}\right)\right)\right]^2 + 4a^2\lambda^2 sin^2\left(\frac{\theta}{2}\right) cos^2\left(\frac{\theta}{2}\right) \\
=& \left[1 - 2a\lambda sin^2\left(\frac{\theta}{2}\right)\right]^2 + 4a^2\lambda^2 sin^2\left(\frac{\theta}{2}\right) cos^2\left(\frac{\theta}{2}\right) \\
=& 1 - 4a\lambda sin^2\left(\frac{\theta}{2}\right) + 4a^2\lambda^2 sin^4\left(\frac{\theta}{2}\right) + 4a^2\lambda^2 sin^2\left(\frac{\theta}{2}\right) cos^2\left(\frac{\theta}{2}\right) \\
=& 1 - 4a\lambda sin^2\left(\frac{\theta}{2}\right) + 4a^2\lambda^2 sin^2\left(\frac{\theta}{2}\right) \\
=& 1 - 4a\lambda(1 - a\lambda) sin^2\left(\frac{\theta}{2}\right)
\end{aligned}$$

We see from this last expression that $|g(\theta)|$ is bounded by 1 if $0 \leq a\lambda \leq 1$; thus by

$$\hat{v}^{n+1}(\xi) = [g(h\xi)]^n \, \hat{v}^0(\xi),$$

we obtain

$$h \sum_{-\infty}^{\infty} |v_m^n|^2 \leq \int_{\frac{-\pi}{h}}^{\frac{\pi}{h}} |\hat{v}^0(\xi)|^2 d\xi$$

and so, the scheme is stable by definition.

Theorem 3.1.3. *A one-step finite difference scheme with constant coefficients is stable in a stability region A if and only if there is a constant K (independent of θ, k, and h) such that*

$$g(\theta, k, h) \leq 1 + Kk \tag{3.10}$$

with $(k, h) \in A$. If $g(\theta, k, h)$ is independent of h and k, the stability condition can be replaced with the restricted stability condition $|g(\theta)| \leq 1$.

This theorem shows that to determine the stability of a finite difference scheme, we need to consider only the amplification factor $g(\theta)$. This observation is due to Von Neumann because of which this analysis is called *Von Neumann Analysis*.

Example 8. *Von-Neumann Analysis for forward-time forward-space scheme*

The forward time and forward space scheme for the PDE $u_t + au_x = 0$ is

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_m^n}{h} = 0.$$

This equation in general form becomes

$$v_m^{n+1} = (1 + a\lambda)v_m^n - a\lambda v_{m+1}^n.$$

where a is positive and $\lambda = \frac{k}{h}$ is constant. The Fourier inverse transform for v_m^n is given by

$$v_m^n = \frac{1}{\sqrt{2\pi}} \int_{\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \hat{v}^n(\xi) d\xi.$$

Using the inversion formula for v_m^n and v_{m+1}^n, the scheme becomes

$$\begin{aligned} v_m^{n+1} =& (1 + a\lambda) \left(\frac{1}{\sqrt{2\pi}} \int_{\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \hat{v}^n(\xi) d\xi \right) - a\lambda \left(\frac{1}{\sqrt{2\pi}} \int_{\frac{\pi}{h}}^{\frac{\pi}{h}} e^{i(m+1)h\xi} \hat{v}^n(\xi) d\xi \right) \\ =& \frac{1}{\sqrt{2\pi}} \int_{\frac{\pi}{h}}^{\frac{\pi}{h}} \left[1 + a\lambda - a\lambda e^{ih\xi}\right] e^{imh\xi} \hat{v}^n(\xi) d\xi. \end{aligned}$$

But the Fourier inversion formula for v_m^{n+1} is

$$v_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{\frac{\pi}{h}}^{\frac{\pi}{h}} e^{imh\xi} \hat{v}^{n+1}(\xi) d\xi.$$

Since the Fourier transform is unique, equating the corresponding integrands, we get

$$\hat{v}^{n+1}(\xi) = \left[1 + a\lambda - a\lambda e^{ih\xi}\right] \hat{v}^n(\xi)$$

i.e.

$$\hat{v}^{n+1}(\xi) = g(h\xi)\hat{v}^n(\xi),$$

where $g(h\xi) = 1 + a\lambda - a\lambda e^{ih\xi}$. If we replace $\theta = h\xi$, then we have

$$g(\theta) = 1 + a\lambda - a\lambda(cos\theta + isin\theta).$$

i.e.

$$g(\theta) = (1 + a\lambda - a\lambda cos\theta) - ia\lambda sin\theta.$$

Then, we have

$$\begin{aligned}
|g(\theta)|^2 =& (1 + a\lambda - a\lambda cos\theta)^2 + (a\lambda sin\theta)^2. \\
=& \left(1 + 2a\lambda sin^2\left(\frac{\theta}{2}\right)\right)^2 + 4a^2\lambda^2 sin^2\left(\frac{\theta}{2}\right) cos^2\left(\frac{\theta}{2}\right). \\
=& 1 + 4a\lambda sin^2\left(\frac{\theta}{2}\right) + 4a^2\lambda^2 sin^4\left(\frac{\theta}{2}\right) + 4a^2\lambda^2 sin^2\left(\frac{\theta}{2}\right) cos^2\left(\frac{\theta}{2}\right) \\
=& 1 + 4a\lambda sin^2\left(\frac{\theta}{2}\right) + 4a^2\lambda^2 sin^2\left(\frac{\theta}{2}\right) \\
=& 1 + 4a\lambda(1 + a\lambda) sin^2\left(\frac{\theta}{2}\right).
\end{aligned}$$

If λ is constant, then we see that $|g(\theta)|$ is greater than 1 for θ not equal to zero. Therefore this scheme is unstable. Hence the scheme is not convergent.Further, if a is negative, then the forward-time forward-space scheme is stable for $-1 \leq a\lambda \leq 0$.

Example 9. *Von Neumann Analysis for forward-time central-space scheme*

The forward time and central space scheme for the PDE $u_t + au_x = 0$ is

$$\frac{v_m^{n+1} - v_m^n}{k} - a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0.$$

Replacing v_m^n by $g^n e^{im\theta}$, the scheme reduces to

$$\begin{aligned}
\frac{g^n e^{im\theta}(g-1)}{k} + a\frac{g^n e^{im\theta}(e^{i\theta} - e^{-i\theta})}{2h} &= 0 \\
\frac{g-1}{k} + \frac{a\left(e^{i\theta} - e^{-i\theta}\right)}{2h} &= 0 \\
(g-1) + i2a\lambda sin\theta &= 0 \\
g = 1 - i2a\lambda sin\theta & \\
\therefore |g|^2 = 1 + 4a^2\lambda^2 sin^2\theta \leq 1 + 4a^2\lambda^2 &
\end{aligned}$$

Since λ is constant, g is independent of h and k and Since $|g(\theta)| > 1$ for $\theta \neq 0$ nor 180 deg, then this scheme is unstable.

3.1.6 Convergence

The most basic property that a scheme must have in order to be useful is that its solutions approximate the solution of the corresponding partial differential equation and that the approximation improves as the grid spacings, h and k tend to zero. We call such a scheme a *convergent scheme.*

Definition 8. *A one-step finite difference scheme approximating a partial differential equation is a convergent scheme if for any solution to the partial differential equation, $u(t,x)$, and solutions to the finite difference scheme, v_m^n, such that v_m^0 converges to $u_0(x)$ as mh converges to x, then v_m^n converges to $u(t,x)$ as (nk, mh) converges to (t,x) as $h,k \to 0$.*

Proving that a given scheme is convergent is not easy in general. However, we say a scheme is convergent, if it is a consistent and stable scheme. Thus, consistency and stability of a scheme determines its convergence.

3.1.7 Order of Accuracy

A scheme for the partial differential equation $Pu = f$ can be written in general as $P_{k,h}v = R_{k,h}f$ in a natural way, where each expression $P_{k,h}v$ and $R_{k,h}f$ evaluated at a grid point (t_n, x_m) involves only a finite sum of terms involving $(v_{m'}^{n'})$, or $(f_{m'}^{n'})$ respectively.

Definition 9. *A scheme $P_{k,h}v = R_{k,h}f$ that is consistent with the differential equation $Pu = f$ is accurate of order p in time and order q in space if for any smooth function $\phi(t,x)$, we have,*

$$P_{k,h}\phi - R_{k,h}P\phi = O(k^p) + O(h^q).$$

We say that such a scheme is accurate of order (p,q).

Example 10. *Order of accuracy of Lax-Wendroff scheme*

The Lax-Wendroff scheme for the PDE $u_t + au_x = f$ is

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} - \frac{a^2k}{2}\left(\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}\right) = \frac{1}{2}\left(f_m^{n+1} + f_m^n\right) - \frac{ak}{4h}\left(f_{m+1}^n - f_{m-1}^n\right).$$

Taking

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + a\frac{\phi_{m+1}^n - \phi_{m-1}^n}{2h} - \frac{a^2k}{2}\left(\frac{\phi_{m+1}^n - 2\phi_m^n + \phi_{m-1}^n}{h^2}\right)$$

and

$$R_{k,h}f = \frac{1}{2}\left(f_m^{n+1} + f_m^n\right) - \frac{ak}{4h}\left(f_{m+1}^n - f_{m-1}^n\right).$$

By Taylor's expansion, we get

$$P_{k,h}\phi = \phi_t + \frac{k}{2}\phi_{tt} + a\phi_x - \frac{a^2k}{2}\phi_{xx} + O(h^2) + O(k^2)$$

and, for $P\phi = \phi_t + a\phi_x = f$ we have,

$$R_{k,h}f = f + \frac{k}{2}f_t - \frac{ak}{2}f_x + O(h^2) + O(k^2).$$

$$R_{k,h}P\phi = \phi_t + a\phi_x + \frac{k}{2}\left(\phi_{tt} + a\phi_{tx}\right) - \frac{ak}{2}\left(\phi_{tx} + a\phi_{xx}\right) + O(h^2) + O(k^2).$$

$$R_{k,h}P\phi = \phi_t + a\phi_x + \frac{k}{2}\phi_{tt} + \frac{a^2k}{2}\phi_{xx} + O(h^2) + O(k^2).$$

Then we have,

$$P_{k,h}\phi - R_{k,h}P\phi = O(h^2) + O(k^2)$$

Hence the Lax-Wendroff scheme is accurate of order $(2, 2)$.

Example 11. *Order of accuracy of Crank Nicolson Scheme*

The Crank Nicolson scheme for the PDE $u_t + au_x = f$ is

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} + v_{m+1}^n - v_{m-1}^n}{4h} = \frac{f_m^{n+1} + f_m^n}{2}.$$

Taking

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + a\frac{\phi_{m+1}^{n+1} - \phi_{m-1}^{n+1} + \phi_{m+1}^n - \phi_{m-1}^n}{4h},$$

and

$$R_{k,h}f = \frac{f_m^{n+1} + f_m^n}{2}.$$

By Taylor's expansion, we have

$$\begin{aligned} P_{k,h}\phi =& \frac{\phi_m^{n+1} - \phi_m^n}{k} + a\frac{\phi_{m+1}^{n+1} - \phi_{m-1}^{n+1} + \phi_{m+1}^n - \phi_{m-1}^n}{4h} \\ =& \phi_t + \frac{k}{2}\phi_{tt} + \frac{k^2}{6}\phi_{ttt} + a\phi_x + \frac{ak}{2}\phi_{tx} + \frac{ak^2}{4}\phi_{ttx} + O(h^2) + O(k^3) \end{aligned}$$

And

$$R_{k,h}f = f_m^n + \frac{k}{2}f_t + \frac{k^2}{4}f_{tt} + O(k^3).$$

If $P\phi = \phi_t + a\phi_x = f$, then

$$R_{k,h}P\phi = \phi_t + a\phi_x + \frac{k}{2}\left(\phi_{tt} + a\phi_{tx}\right) + \frac{k^2}{4}\left(\phi_{ttt} + a\phi_{ttx}\right) + O(k^3).$$

Then, $P_{k,h}\phi$ and $R_{k,h}P\phi$ differ only in the terms containing k^2 and h^2. So,

$$P_{k,h}\phi - R_{k,h}P\phi = O(h^2) + O(k^2).$$

Hence Crank Nikolson scheme is also accurate of order $(2, 2)$.

3.2 Finite Difference Schemes for Second-Order Equations

The definitions of convergence, consistency, and order of accuracy for finite difference schemes hold without modification for second-order equations. The stability definition, however, must be altered slightly.

Definition 10. *A finite difference scheme $P_{k,h}v_m^n = 0$ for a second order in t is stable in a stability region A if there is an integer J and for any positive time T there is a constant C_T such that*

$$h\sum_{m=-\infty}^{\infty} |v_m^n|^2 \leq (1 + n^2)C_T h \sum_{j=0}^{J}\sum_{m=-\infty}^{\infty} |v_m^j|^2$$

for all solutions v_m^n and for $0 < nk < T$ with $(k, h) \in A$.

The extra factor of $(l + n^2)$ is the only change required by the second order equation and reflects the linear growth in t allowed by these equations. In the Von-Neumann analysis of schemes for second-order equations, this definition requires that the amplification factors g_v satisfy

$$|g_v| \leq 1 + Kk$$

and permits two such amplification factors to coalesce near the unit circle. If there are no lower order terms, then the stability condition is $|g_v| \leq 1$ with double roots on the unit circle permitted. The integer J in the definition must always be at most 1, since data must always be given at two time levels for second-order equations.

Theorem 3.2.1. *If the amplification polynomial $\phi(g, \theta)$ for a second-order time dependent equation is explicitly independent of h and k, then the necessary and sufficient condition for the finite difference scheme to be stable is that all roots, $g_v(\theta)$, satisfy the conditions:*

1. $|g_v(\theta)| \leq 1$, *and*
2. *If $|g_v(\theta)| = 1$, then $g_v(\theta)$ must be at most a double root.*

Chapter 4

Analytical and Numerical Solutions: A Comparative Study

In this section, we construct a roadway to our destination of solving two dimensional Navier Stokes equation for variable viscosity case. For this, we first analyse the numerical solutions of the one dimensional unsteady, incompressible Couette flow problem, compare them with the exact solutions using Matlab implimentation and then we march forward to our destination.

4.1 Analytical Study of Couette Flow

4.1.1 Introduction

Couette flow is a viscous flow between two parallel plates seperated by some vertical distance. The upper plate is moved with velocity u_e and the lower plate is kept stationary i.e. its velocity $u = 0$. The flow is two dimensional in xy palne. The flow field between the two plates is to be driven exclusively by the shear stress exerted on the fluid by the moving upper plate so that the velocity profile is formed across the flow.

4.1.2 Derivation

The governing equation for this flow is the x-momentum equation in $2D$ given by

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho v u)}{\partial y} = -\frac{\partial p}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} + \rho f_x.$$

This equation in variable viscosity case is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}(v\frac{\partial u}{\partial x}) + \frac{\partial}{\partial z}(v\frac{\partial u}{\partial x}). \tag{4.1}$$

The model for Couette flow extends from $-\infty$ to $+\infty$ in x-direction. Since there is no begining or end of this flow, the flow field variables must be independent of x; that is $\frac{\partial}{\partial x} = 0$ for all quantities. Moreover, the equation of continuity for steady state flow is

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial \rho v}{\partial y} = 0. \tag{4.2}$$

Since $\frac{\partial(\rho u)}{\partial x} = 0$ for Couette flow, equation (4.2) reduces to

$$\frac{\partial(\rho v)}{\partial y} = 0.$$

i.e.

$$\rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y} = 0. \tag{4.3}$$

At the lower wall, $v = 0$ at $y = 0$. So, the equation (4.3) reduces to

$$\left(\frac{\partial v}{\partial y}\right)_{y=0} = 0 \tag{4.4}$$

Expanding v in Taylor's series about the point $y = 0$, we get

$$v(y) = v(0) + \left(\frac{\partial v}{\partial y}\right)_{y=0} y + \left(\frac{\partial^2 v}{\partial x^2}\right)_{y=0} \frac{y^2}{2} + ... \tag{4.5}$$

At the upper wall, we have

$$v(D) = v(0) + \left(\frac{\partial v}{\partial y}\right)_{y=0} D + \left(\frac{\partial^2 v}{\partial x^2}\right)_{y=0} \frac{D^2}{2} + ... \tag{4.6}$$

Since both $v(D) = 0$ and $v(0) = 0$ as well as $\left(\frac{\partial v}{\partial y}\right)_{y=0} = 0$, the only result from equation (4.6) that makes sense is

$$\left(\frac{\partial^n v}{\partial x^n}\right)_{y=0} = 0, \quad \forall n.$$

This implies that

$$v = 0 \tag{4.7}$$

everywhere. Thus, there is no vertical velocity in Couette flow and it is a major physical characteristics of the flow. Therefore, the streamlines for the Couette flow are straight lines which runs parallel to each other. Also the y-momentum equation is

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \rho f_x.$$

For the Couette flow with no body forces,

$$-\frac{\partial p}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} = 0. \tag{4.8}$$

But, we have

$$\tau_{yy} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y} = 0.$$

With this equation in hand, eqution (4.8) becomes,

$$\frac{\partial p}{\partial y} = 0. \tag{4.9}$$

Thus, there are no pressure gradient terms in both x and y directions. Therefore, for steady two dimensional flow with no body force, we have the x-momentum equation

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \tag{4.10}$$

But for the Couette flow, we have

$$\tau_{xx} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x} = 0,$$

and

$$\tau_{yx} = \lambda\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \mu\frac{\partial u}{\partial y}.$$

Substituting these values in (4.10), we get

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)=0.$$

If the flow is incompressible with constant temperature, this equation reduces to

$$\frac{\partial^2 u}{\partial y^2}=0. \tag{4.11}$$

Similarly, the x-momentum equation for unsteady, incompressible Couette flow is

$$\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 U}{\partial y^2}.$$

It is s parabolic PDE.

4.1.3 Analytic Solution

The governing equation for unsteady incompressible viscous Couette flow is

$$\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 U}{\partial y^2}. \tag{4.12}$$

The equation in non-dimensional form is

$$\rho\frac{\partial\left(\frac{u}{u_e}\right)}{\partial\left(\frac{t}{(D/u_e)}\right)}\left(\frac{u_e^2}{D}\right)=\mu\frac{\partial^2\left(\frac{u}{u_e}\right)}{\partial\left(\frac{y}{D}\right)^2}\left(\frac{u_e}{D^2}\right),$$

i.e.

$$\frac{\partial u'}{\partial t'}=\frac{\mu}{\rho u_e D}\frac{\partial^2 u'}{\partial y'^2} \tag{4.13}$$

Since

$$\frac{\mu}{\rho u_e D}\equiv\frac{1}{Re_D},$$

where Re_D is the Reynolds number based on the height D between the two plates. Thus, equation (4.13) becomes

$$\frac{\partial u'}{\partial t'}=\frac{1}{Re_D}\frac{\partial^2 u'}{\partial y'^2}. \tag{4.14}$$

Equation (4.14) is the PDE for which we require exact and numerical solutions. First, we write the equation (4.14) in the form

$$\frac{\partial u}{\partial t} = \frac{1}{Re_D}\frac{\partial^2 u}{\partial y^2}, \tag{4.15}$$

where u, t and y are identical with that of dashes in equation (4.14). To solve the equation (4.15) analytically, let us consider it in the form,

$$u_t = \kappa u_{xx},\ 0 < x < l,\ t > 0, \tag{4.16}$$

where $\kappa = \frac{1}{Re_D}$. It is one dimensional heat equation. Suppose the solution $u(x,t)$ representing the temperature distribution in a homogeneous rod of length l, satisfies the boundary and initial conditions

$$\begin{aligned} u(0,t) =& 0 = u(l,t),\ t \geq 0 \\ u(x,0) =& f(x),\ (0 \leq x \leq l). \end{aligned} \tag{4.17}$$

We assume a separable solution of (4.16) in the form

$$u(x,t) = X(x)T(t) \neq 0. \tag{4.18}$$

Substituting (4.18) in (4.16) gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\kappa T}\frac{dT}{dt}. \tag{4.19}$$

Since the left-hand side depends only on x and the right-hand side is a function of time t only, result (4.19) can be true only if both sides are equal to the same constant λ. Thus, we obtain two ordinary differential equations

$$\begin{aligned} \frac{d^2X}{dx^2} &= \lambda X, \\ \frac{dT}{dt} &= \lambda \kappa T. \end{aligned} \tag{4.20}$$

For $\lambda > 0$, the only solution of the equations (4.20) consistent with the given boundary conditions is $u(x,t) = 0$. Hence for negative $\lambda = -\alpha^2$,

$$\begin{aligned} \frac{d^2X}{dx^2} + \alpha^2 X &= 0, \\ \frac{dT}{dt} + \kappa \alpha^2 k T &= 0, \end{aligned} \tag{4.21}$$

which admit solutions as

$$X(x) = A\ \mathrm{Cos}(\alpha x) + B\ \mathrm{Sin}(\alpha x) \tag{4.22}$$

and

$$T(t) = C\mathrm{exp}(\kappa\alpha^2 t), \tag{4.23}$$

where $A, B,$ and C are constants of integration. The boundary conditions for $X(x)$ are

$$X(0) = 0 = X(l). \tag{4.24}$$

Using them in (4.22), it turns out that $A = 0$ and $B \neq 0$. Hence,

$$\mathrm{Sin}(\alpha l) = 0, \tag{4.25}$$

which gives the eigenvalues

$$\alpha = \alpha_n = \frac{n\pi}{l}, n = 1, 2, 3, ... \tag{4.26}$$

The value $n = 0$ is excluded because it leads to a trivial solution. Thus, the eigenfunctions are given by

$$X_n(x) = B_n\mathrm{Sin}\left(\frac{n\pi x}{l}\right), \tag{4.27}$$

where B_n are non-zero constants. With $\alpha = \alpha_n = \frac{n\pi}{l}$, we combine (4.23) with (4.27) to obtain the solution for $u_n(x,t)$ as

$$u_n(x,t) = a_n\mathrm{exp}\left[\left(\frac{n\pi}{l}\right)^2 \kappa t\right] sin\frac{n\pi x}{l}, \tag{4.28}$$

where $a_n = B_n C_n$ is a new constant. Thus, (4.23) and (4.27) constitute an infinite set of eigenvalues and eigenfunctions, and the most general solution is obtained by the principle of superposition in the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n\ \mathrm{exp}\left[\left(\frac{n\pi}{l}\right)^2 \kappa t\right] sin\left(\frac{n\pi x}{l}\right). \tag{4.29}$$

Now, the initial condition implies that

$$f(x) = \sum_{n=1}^{\infty} a_n sin\left(\frac{n\pi x}{l}\right) \tag{4.30}$$

which determines a_n as

$$a_n = \frac{2}{l}\int_0^l f(x) sin\left(\frac{n\pi x}{l}\right) dx. \tag{4.31}$$

Thus, the final form of the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l f(x')\sin\left(\frac{n\pi x'}{l}\right)dx'\right]\exp\left[\left(\frac{n\pi}{l}\right)^2\kappa t\right]\sin\left(\frac{n\pi x}{l}\right). \tag{4.32}$$

It follows that the series solution (4.29) satisfies the given boundary and initial conditions. It also satisfies the equation (4.16) because the series is convergent for all x $(0 \leq x \leq l)$ and $t \geq 0$ and can be differentiated term by term. Physically, *the temperature distribution decays exponentially with time t*. This shows a striking contrast to the wave equation, whose solution oscillates in time t. The time scale of decay for the n^{th} mode is

$$T_d \equiv \frac{1}{\kappa}\left(\frac{l}{n\pi}\right)^2,$$

which is directly proportional to l^2 and inversely proportional to the thermal diffusivity κ.

4.1.4 Numerical Analysis

The x-momentum equation for unsteady, incompressible Couette flow is

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} \tag{4.33}$$

The finite difference method for the Crank Nicolson scheme for the equation (4.16) is

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{1}{Re_D}\frac{\frac{1}{2}\left(\left(v_{j+1}^{n+1} + v_{j+1}^n\right) + \frac{1}{2}\left(-2v_j^{n+1} - 2v_j^n\right) + \frac{1}{2}\left(v_{j-1}^{n+1} + u_{j-1}^n\right)\right)}{(\Delta y)^2}$$

i.e.

$$v_j^{n+1} = v_j^n + \frac{\Delta t}{2(\Delta y)^2 Re_D}\left(v_{j+1}^{n+1} + v_{j+1}^n - 2v_j^{n+1} - 2v_j^n + v_{j-1}^{n+1} + v_{j-1}^n\right).$$

i.e.

$$\left[-\frac{\Delta t}{2(\Delta y)^2 Re_D}\right] v_{j-1}^{n+1} + \left[1 + \frac{\Delta t}{(\Delta y)^2 Re_D}\right] v_j^{n+1} + \left[-\frac{\Delta t}{2(\Delta y)^2 Re_D}\right] v_{j+1}^{n+1}$$

$$= \left[1 - \frac{\Delta t}{(\Delta y)^2 Re_D}\right] v_j^n + \frac{\Delta t}{2(\Delta y)^2 Re_D}\left(u_{j+1}^n + v_{j-1}^n\right).$$

This is in the form

$$Av_{j-1}^{n+1} + Bv_j^{n+1} + Av_{j+1}^{n+1} = K_j, \tag{4.34}$$

where,

$$A = -\frac{\Delta t}{2(\Delta y)^2 Re_D} \tag{4.35}$$

$$B = 1 + \frac{\Delta t}{(\Delta y)^2 Re_D} \tag{4.36}$$

and,

$$K_j = \left[1 - \frac{\Delta t}{(\Delta y)^2 Re_D}\right] v_j^n + \frac{\Delta t}{2(\Delta y)^2 Re_D}\left(u_{j+1}^n + v_{j-1}^n\right). \tag{4.37}$$

Now, we solve equation (4.34) on a grid such that the vertical distance D between the plates is divided into N equal parts by administrating $(N+1)$ grid points over the vertical height D, i.e.

$$\Delta y = \frac{D}{N} \tag{4.38}$$

We impose the boundary conditions non-dimensional velocity u as

$$u_1 = 0 \tag{4.39}$$

and

$$u_{N+1} = 1 \tag{4.40}$$

As equation (4.34) represents a system of $(N-1)$ equations in $(N-1)$ unknowns namely, $v_2, v_3, ..., v_{N-1}$, the first equation of the system is

$$Av_1^{n+1} + Bv_2^{n+1} + Av_3^{n=1} = K_2 \tag{4.41}$$

But $u_1 = 0$, we have

$$Bv_2^{n+1} + Av_3^{n=1} = K_2 \tag{4.42}$$

The last equation of the system (4.34) is

$$Av_{N-1}^{n+1} + Bv_N^{n+1} + Av_{N+1}^{n=1} = K_N \tag{4.43}$$

But, $v_{N+1} = 1$, we have

$$Av_{N-1}^{n+1} + Bv_N^{n+1} = K_N - Av_e \tag{4.44}$$

With the equation (4.42) and (4.44), the system of equations (4.34) can be expressed as

$$\begin{pmatrix} B & A & 0 & 0 & .... & 0 & 0 & 0 \\ A & B & A & 0 & .... & 0 & 0 & 0 \\ 0 & A & B & A & .... & 0 & 0 & 0 \\ ..... & ... & .... & .... & ... & ... & & \\ ..... & ... & .... & .... & ... & ... & & \\ ..... & ... & .... & .... & ... & ... & & \\ 0 & 0 & 0 & 0 & .... & A & B & A \\ 0 & 0 & 0 & 0 & .... & 0 & A & B \end{pmatrix} \begin{pmatrix} v_2^{n+1} \\ v_3^{n+1} \\ v_4^{n+1} \\ . \\ . \\ . \\ v_{N-1}^{n+1} \\ v_N^{n+1} \end{pmatrix} = \begin{pmatrix} K_2 \\ K_3 \\ K_4 \\ . \\ . \\ . \\ K_{N-1} \\ K_N - Av_e \end{pmatrix}. \tag{4.45}$$

It is a tridiagonal matrix and can be solved by using Thomas algorithm. By the algorithm, we find the solution for $v_2^{n+1}, v_3^{n+1}, ..., v_{N-1}^{n+1}$, the velocities at the $(n+1)^{th}$ time level. Then, the whole process is repeated for a number of time step until the velocity profile converges to a steady state.

The Setup

For the specific solution, We choose $N + 1$ grid points along vertical, so that the space size is

$$\Delta y = \frac{1}{N}.$$

We choose the initial conditions

$$v_i = 0 \text{ for } i = 1, 2, ..., N$$

at $t = 0$. Next, we choose the time step size (Δt) so as to satisfy the CFL condition,

$$\Delta t \leq \frac{1}{2} Re_D (\Delta y)^2.$$

Since the Crank Nicolson scheme is unconditionally stable, we are free to choose time step size Δt.

Example 12. *Consider a PDE*

$$u_t = a u_{xx}$$

with boundary and initial conditions

$$u(0, t = u(1, t) = 0$$

and

$$u(x, 0) = f(x).$$

The exact solution of the PDE is

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-an^2\pi^2 t} sin(n\pi x).$$

Taking $f(x) = 2sin(2\pi x)$. By the Matlab implimentation, for $Re_D = 10$ *i.e.* $a = \frac{1}{10}$, we obtain the following graphical solutions.

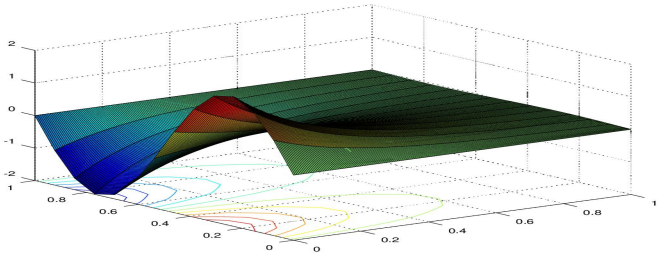


Figure 4.1: Initial conditions

We can observe the various solutions by changing time step size (t), space step size (x) and the Reynolds numbers Re_D. But, we have a restriction in schemes that in every change of the step sizes, the CFL condition should be satisfied. Otherwise, the scheme will be unstable and so, matlab output will not be appropriate.

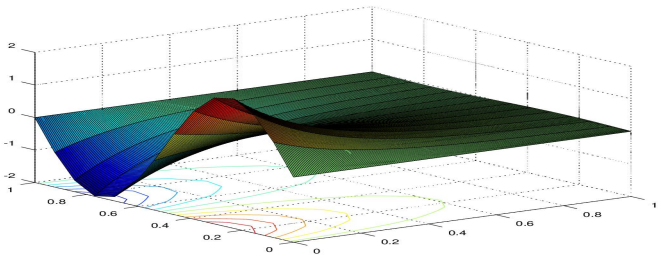


Figure 4.2: velocity profile (Numerical Solution)

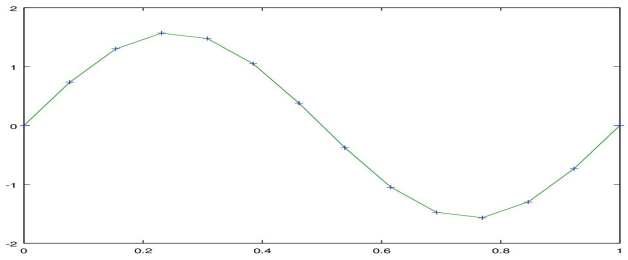


Figure 4.3: Exact solution

Chapter 5

Numerical Solutions of $2D$ Incompressible Navier Stokes Equations in Variable Viscosity Case

In this section, the procedures of solving the two dimensional incompressible Navier Stokes Equations in variable viscosity case by a finite difference scheme is explained.

5.1 The Governing Flow Equations

The Navier Stokes equations neglecting body forces with variable viscosity are
Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{5.1}$$

x-Momentum:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}(\nu\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\nu\frac{\partial u}{\partial x}).$$

i.e.

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho uv) = 0. \tag{5.2}$$

y-Momentum

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial}{\partial x}(\nu\frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(\nu\frac{\partial v}{\partial x}).$$

i.e.

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + p) = 0. \tag{5.3}$$

The system consist of three equations with four four unknowns $\rho, u, v,$ and p. To close the system, one equation is needed. Let us assume a perfect gas. Then, the equation of state is

$$p = \rho RT, \tag{5.4}$$

where R is the specific gas constant. This provides a forth equation, but it also introduces a fifth variable namely, temperature T. So, a fifth equation is required to close the entire system. It must be a thermodynamic relation between the state variables for calorically perfect gas, called *Caloric Equation of State* which would be

$$e = c_v T, \tag{5.5}$$

where e is internal energy of air and c_v is the specific heat at constant volume. The entire system of governing equations can be expressed in conservation form (i.e. divergence of fluxes of some physical quantity) as

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0, \tag{5.6}$$

where U, E and F, the flux terms are column vectors given by

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix} \tag{5.7}$$

$$E = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \end{pmatrix} \tag{5.8}$$

$$F = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \end{pmatrix} \tag{5.9}$$

let us consider a computational domain, a rectangular structured grid with two spatial variables and a time variable. Let $v^n_{i,j}$ denotes the value of the discrete variable v at i^{th} node along x-direction, j^{th} node along y-directin and n^{th} node along t-direction. Further, let us take a plate having length large enough to the mean free path of the oncoming air molecules and sufficient to capture the desired physics. We want low Reynolds number in order to keep computational running times considerably short for the grid we proposed.

5.1.1 The Finite Difference Equation

Here, we need to march in time to a steady state solution together with the flow properties at every (i, j) spatial location. So we need a three dimensional grid. We apply *MacCormack's time marching technique* in two spatial directions. The key steps of the technique are described below.

Let $U^t_{i,j}$ denote a discrete flow field variable corresponding to the variable U at the node (i, j) and time level t of a grid.

- The governing equation (5.6) is re-written as

$$\frac{\partial U}{\partial t} = -\frac{\partial E}{\partial x} - \frac{\partial F}{\partial y}. \tag{5.10}$$

- The flow field variables are advanced at each grid point (i, j) in steps of time by the help of Taylor's series expansion as

$$U^{y+\Delta t}_{i,j} = U^t_{i,j} + \left(\frac{\partial U}{\partial t}\right)_{av} \Delta t, \tag{5.11}$$

  where Δt is the time step size. We note that $U^t_{i,j}$ is the discrete flow field variable which is known at time level t. Indeed, it is known at time level

t either from initial conditions or from the iteration in time t, and the derivative $\left(\frac{\partial U}{\partial t}\right)_{av}$ is obtained as

$$\left(\frac{\partial U}{\partial t}\right)_{av} = \frac{1}{2}\left[\left(\frac{\partial U}{\partial t}\right)_{i,j}^{t} + \left(\frac{\partial \overline{U}}{\partial t}\right)_{i,j}^{t+\Delta t}\right]. \tag{5.12}$$

To obtain a value of $\frac{\partial \overline{U}}{\partial t}_{i,j}^{t+\Delta t}$ so that the solution can be advanced, the following steps are taken.

1. The value of $\left(\frac{\partial U}{\partial t}\right)_{i,j}^{t}$ is calculated using forward spatial differences on the right side of the governing equation (5.10) from the known flow field at time level t.

2. The predicted values of the flow-field variables $\overline{U}_{i,j}^{t}$ can be obtained at time level $(t + \Delta t)$ as

$$\overline{U}_{i,j}^{t+\Delta t} = U_{i,j}^{t} + \left(\frac{\partial U}{\partial t}\right)_{i,j}^{t} \Delta t. \tag{5.13}$$

   Then, the predicted values are obtained by combining steps 1 and 2 as

$$\overline{U}_{i,j}^{t+\Delta t} = U_{i,j}^{t} - \frac{\Delta t}{\Delta x}\left(E_{i+1,j}^{t} - E_{i,j}^{t}\right) - \frac{\Delta t}{\Delta y}\left(F_{i+1,j}^{t} - F_{i,j}^{t}\right) \tag{5.14}$$

3. Using backward spatial differences, the predicted values obtained from step 2 are inserted into the governing equations such that a predicted time derivative $\left(\frac{\partial \overline{U}}{\partial t}\right)_{i,j}^{t+\Delta t}$ can be obtained.

4. Finally, substituting $\left(\frac{\partial \overline{U}}{\partial t}\right)_{i,j}^{t+\Delta t}$ from step 3 into the equation (5.10) to obtain corrected second order accurate values of U at time level $(t + \Delta t)$. As in equation (5.14), steps 3 and 4 are combined as follows.

$$U_{i,j}^{t+\Delta t} = \frac{1}{2}\left[U_{i,j}^{t} + \overline{U}_{i,j}^{t+\Delta t} - \frac{\Delta t}{\Delta x}\left(\overline{E}_{i+1,j}^{t} - \overline{E}_{i,j}^{t}\right) - \frac{\Delta t}{\Delta x}\left(\overline{F}_{i+1,j}^{t} - \overline{F}_{i,j}^{t}\right)\right] \tag{5.15}$$

Steps 1 to 4 are repeated until the flow field variables approach a steady state value; this is the desired steady state solution.

5.1.2 Calculation of Step size in Time and Space

Taking a computational domain of sizes,

$$IMIN = \text{ inflow in } x - \text{direction.}$$

$$IMAX = \text{ outflow in } x - \text{direction.}$$

The step size in x-direction is

$$\Delta x = \frac{L_x}{IMAX - 1}, \tag{5.16}$$

Likewise, the grid normal to the plate's surface are

$$JMIN = \text{inflow in } y - \text{direction.}$$

$$JMAX = \text{outflow in } y - \text{direction.}$$

The step size in the y-direction is

$$\Delta y = \frac{L_y}{JMAX - 1}, \tag{5.17}$$

where L_x and L_y denote the horizontal and vertical heights of the computational domain. The correctness of the grid size is ensured by the *Cell Reynolds Numbers* in x and y-directions, which are defined as

$$Re_x = \frac{\rho_{i,j} u_{i,j} \Delta x}{\mu_{i,j}}, \tag{5.18}$$

and,

$$Re_y = \frac{\rho_{i,j} v_{i,j} \Delta y}{\mu_{i,j}}. \tag{5.19}$$

Since we need stronger velocity gradient to capture the flow field, the step size in y-direction should be less than that in x-direction. For the time step size, the time step is subject to a stability criterion. To determine the size of the time step, the following form of the Courant-Friedrichs-Lewy (CFL) criterion is used.

$$(\Delta t_{CFL})_{i,j} = \left[\frac{|u_{i,j}|}{\Delta x} + \frac{|v_{i,j}|}{\Delta y} + a_{i,j} \sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}} + 2 v'_{i,j} \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \right]^{-1}$$

where,

$$v'_{i,j} = \max\left[\frac{\frac{4}{3}\mu_{i,j}\left(\frac{\gamma\mu_{i,j}}{P_r}\right)}{\rho_{i,j}}\right]$$

$$\Delta t = \min\left[K\left(\Delta t_{CFL}\right)_{i,j}\right],$$

K =Courant number that lies between 0.5 to 0.8.

$a_{i,j}$ =Local speed of sound in m/sec.

$P_r = \frac{\mu c_p}{k} = 0.71$ for calorically perfect air (Called Prandtl number)

5.1.3 Initial and Boundary Conditions

The governing equation is first order in time and second order in space. Therefore, initial and boundary conditions are necessary. For it, we specify the flow properties at each (i, j) location at time $t = 0.0$, at the surface (JMIN=1) and the temperature. Further, the boundary condition profile i.e. U, T and P should be provided at the following positions of the computational domain.

Case 1. At the leading edge $(IMIN, JMIN)$,

Case 2. At the left hand side (except the leading edge) and upper boundaries of the domain,

Case 3. At the surface of the plate,

Case 4. Finally, all properties at the right hand side domain (not including JMIN=1 and JMAX=70) are calculated based on the extrapolation from the two interior points at the same j location. For example, u is determined as

$$u_{(IMAX,j)} = 2u_{(IMAX-1,j)} - u_{(IMAX-2,j)} \tag{5.20}$$

From these known values, the balance of the flow properties at the boundaries can be calculated.

Chapter 6

Conclusion and Discussion

6.1 Conclusion

From this study, we oberved that inspite of complex nature of the Navier-Stokes equations, finite difference scheme is found to be a better tool to study the behaviour of the solutions (velocity profile) of the two dimensional Navier-Stokes equations in variable viscosity case. Though it gives only the numerical solutions, it can approximate the exact solutions with desire error. Only the case is that, we need to choose an appropriate (convergent) scheme for the best approximations, otherwise the solutions diverge. Because of such successes of the method, it is proved to be a good and applicable mathematical tool to deal this problem.

6.1.1 Discussion

Mathematics is an elegant and precise subject. However, when numerical answers are required we sometimes need to rely on approximate methods to predict answers. There are many problems which simply do not have analytical solutions, or may be beyond our current state of knowledge. There are also many problems which are too long (or tedious) to solve by hand. When such problems arise, we can use numerical analysis to reduce the problem to one involving a finite number of unknowns and use a computer to solve the resulting equations.

6.1.2 Further Studies

1. Matlab implimentation of this scheme.
2. Numerical solutions witn matlab implementation of 3D Navier Stokes equations.

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