ORTHOGONALITY IN NORMED LINEAR SPACES

A THESIS SUBMITTED TO THE

CENTRAL DEPARTMENT OF MATHEMATICS
INSTITUTE OF SCIENCE AND TECHNOLOGY
TRIBHUVAN UNIVERSITY, NEPAL

FOR THE AWARD OF
DOCTOR OF PHILOSOPHY
IN MATHEMATICS

BY

BHUWAN PRASAD OJHA

APRIL 2022

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DECLARATION

Thesis entitled **"Orthogonality in Normed Linear Spaces",** submitted to the Central Department of Mathematics, Institute of Science and Technology(IOST), Tribhuvan University, Nepal for the award of the degree of Doctor of Philosophy (Ph.D.), is a research work carried out by me under the supervision of Prof. Dr. Prakash Muni Bajracharya, Central Department of Mathematics, Tribhuvan University. This research is original and has not been submitted earlier in part or full in this or any other form to any university or institute, here or elsewhere, for the award of any degree.

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Bhuwan Prasad Ojha

RECOMMENDATION

This is to recommend that **Mr. Bhuwan Prasad Ojha** has carried out research entitled **"Orthogonality in Normed Linear Spaces"** for the award of Doctor of Philosophy (Ph.D.) in **Mathematics** under my supervision. To my knowledge, this work has not been submitted for any other degree. He has fulfilled all the requirements laid down by the Institute of Science and Technology (IOST), Tribhuvan University, Kirtipur for the submission of the thesis for the award of Ph.D. degree.

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Prof. Dr. Prakash Muni Bajracharya
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Date: 18 June , 2021

TRIBHUVAN UNIVERSITY

INSTITUTE OF SCIENCE AND TECHNOLOGY

CENTRAL DEPARTMENT OF MATHEMATICS

LETTER OF APPROVAL

Date: 18 June , 2021

On the recommendation of **Prof. Dr. Prakash Muni Bajracharya**, this Ph. D. thesis submitted by **Mr. Bhuwan Prasad Ojha**, entitled **Orthogonality in Normed Linear Spaces** is forwarded by Central Department Research Committee (CDRC) to the Dean, IOST, T.U.

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Prof. Dr. Tanka Nath Dhamala
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ABSTRACT

This thesis deals with the orthogonality in normed linear spaces. The goal is to investigate and study different notions of orthogonality in normed spaces. By utilizing the 2-HH norm and bounded linear operators, some notions of orthogonality are introduced and then, different properties of orthogonality in relation to these orthogonalities are studied. We generalize the Robert, Birkhoff-James, and a new orthogonality in terms of the 2-HH norm, and study the main properties of orthogonality. We prove that the Birkoff and Robert orthogonality in terms of the 2-HH norm are equivalent if the underlying space is real inner product space. Further, we prove that the isosceles orthogonality is homogeneous if and only if it is additive. Additionally, we prove that the orthogonality relation of type (I) in terms of 2-HH norm satisfies non-degeneracy, simplification, continuity, and uniqueness properties. Moreover, we prove that the Carlsson orthogonality in terms of bounded linear operators also satisfies non-degeneracy, simplification, and continuity properties. In the case of norm attaining bounded linear operator with disjoint support in a Hilbert space H, we prove that two operators are orthogonal in the sense of Pythagoras if and only if they are orthogonal in the sense of isosceles. In terms of buonded linear operators, we prove that the Pythagorean orthogonality and orthogonality relation of type (I), imply the Birkhoff-James orthogonality, but the converse may not be true. Under the restriction of an element belonging to the norm attainment set, we prove that the orthogonality of images also implies the orthogonality of operators in the Carlsson as well as Robert's sense. Finally, as applications, we prove that the Pythagorean orthogonality implies the best approximation, and the best approximation (resp. $\epsilon-$ best approximation) and Birkhoff orthogonality (resp. $\epsilon-$ Birkhoff orthogonality) are equivalents.

Keywords: Normed linear spaces, Inner-product space, Birkhoff-James orthogonality, Pythagorean orthogonality, p-HH norm, Best approximation

LIST OF SYMBOLS

$\mathbb{N}$	Set of positive integers
$\mathbb{R}$	Set of real numbers or real line
$\mathbb{C}$	Set of complex numbers
$L(X,Y)$	Set of linear operators form X to Y
$B(X,Y)$	Set of bounded linear operators form X to Y
$x \perp_{2-HH-R} y$	x is Robert orthogonal to y via 2-HH norm
$x \perp_{2-HH-B} y$	x is Birkhoff-James orthogonal to y via 2-HH norm
$x \perp_{2-HH-P} y$	x is Pythagorean orthogonal to y via 2-HH norm
$x \perp_{2-HH-i} y$	x is isosceles orthogonal to y via 2-HH norm
$x \perp_{2-HH-N} y$	x is orthogonal to y via 2-HH norm in the sense of new orthogonality
S_X	Unit sphere of X
M_T	Norm attainment set of T
$\mathscr{A}$	Banach algebra
$K(X,Y)$	The set of compact linear operators from X to Y
$< x, y >$	A line segment passes through (x, y)
$[x,y]$	Segment between x and y
M_x	The maximum of the length of non-trival segment contained in the unit sphere
$B(H)$	Set of bounded linear operators in a Hilbert space H
$\|A\|_p$	Schatten p-norm of A
$g(x,y)$	g-angle between two vectors x and y
$B(V,W)$	The space of all adjointable mapping form V to W
$K(B,V)$	Closed linear subspace of $B(V,W)$
$Span(X)$	The set of all finite linear combinations of elements of X

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Chapter 1

INTRODUCTION

The natural definition of orthogonality of elements of an inner-product space is that, $x \perp y$ if and only if the inner-product $(x, y) = 0$. However, in context of a normed linear space, this definition does not work due to the unavailability of the notion of inner-product. As a generalization of orthogonality from inner-product space to normed linear space, the first credit goes to R. C. James in the paper [1] even-though D. B. Robert introduced the first orthogonality in normed linear space in 1934. Robert's definition has the weakness that for some normed linear space at least one of every pair of orthogonal elements would have to be zero [2]. To avoid this difficulty, G. Birkhoff introduced a new concept of orthogonality in normed linear space by stating that " if x is a unit vector of a Banach Space X and $y \in X$, then x is orthogonal to y if and only if the straight line $\{x+\lambda y : \lambda \in \mathbb{K}\}$ is tangent to the unit ball of X at x"[3]. In 1945, James came up with the notion of the Pythagorean and isosceles orthogonalities which characterize inner-product space via their homogeneity and additivity [1].

S. O. Carlsson in 1962, introduced a generalized concepts of orthogonality in normed linear space by indicating that " the isosceles and Pythagorean orthogonality are special cases"[4]. The open problem related to relationship between the Birkhoff-James orthogonality and the Pythagorean orthogonality is discussed in the paper [5]. Motivated by Carlsson's results, two new special case of the Carlsson's orthogonality have been introduced and an attempt has been made to verify some properties of inner-product space in relation to these orthogonalities[6]. In 2010, E. Kikianty and S. S. Dragomir introduced the $p - HH$ norm on the cartesian square of normed linear space by generalizing the previous definition of the Carlsson orthogonality in terms of $2 - HH$ norm, which also generalize the Pythagorean and isosceles orthogonality in terms of 2-HH norm[7]. Motivated by the results of Kikianty and Dragomir, the Birkhoff and new special cases of the Carlsson orthogonality in terms of 2-HH norm have been introduced [8]. Besides, the Robert orthogonality in terms of 2-HH norm has also been introduced in the paper [9]. Bhatia and Semrl came up with new concept of orthogonality in terms of matrices, specially in the case of Birkhoff orthogonality [10]. Sain and Paul linked the Bhatia and Semrl property with norm attaining operators in a finite dimensional normed space which attains its norm on connected closed subset of the unit sphere of X and proved that "if two linear operators are orthogonal in the sense of Birkhoff-James, then there exists an element in the closed connected subset of the unit sphere at which the images of operators are also orthogonal in the sense of Birkhoff-James [11].

In 2013, Paul et al. introduced a concept of strong Birkhoff-James orthogonality by stating that x is said to be strongly orthogonal to y in the sense of Birkhff-James if and only if $\|x\| < \|x + \lambda y\|$ for all $\lambda \neq 0$. The notation $x \perp_{SB} y$ was used to indicate the orthogonality and proved that the strong Birkhoff-James orthogonality implies Birkhoff-James orthogonality, but the converse may not be true. To illustrate this concept, two elements $(1, 0)$ and $(0, 1)$ are taken in $l_\infty(\mathbb{R}^2)$, showing that $(1, 0)$ and $(0, 1)$ are orthogonal in the sense of Birkhoff-James but not strongly orthogonal to each other [12]. To study the difference of orthogonality in the complex case in comparison of real case Paul et al. in 2018 came of with new concept of Birkhoff-James orthogonality introducing new definitions on complex reflexive Banach space [13]. Recently, Bottazzi et al. have introduced a new generalization of earlier results on orthogonality of bounded linear operators. They discussed about Birkhoff-James, isosceles and Robert orthogonality in Banach spaces in terms of bounded linear operators [14].

Motivated by the results of Bottazzi et al., this research is focused on generalizing the earlier definition of the Carlsson orthogonality in terms of bounded linear operators with verifying some properties, like non-degeneracy, continuity and homogeneity property of an inner product space in relation to the Carlsson orthogonality for bounded linear operators. Kikianty and Dragomir mentioned without proof in their paper that the Pythagorean and isosceles orthogonality via 2-HH norm satisfies non-degeneracy, symmetry and continuity property [7]. In this thesis, all these properties have been proved when the norm on X is induced by an inner product. The equivalence of homogeneity and additivity of isosceles orthogonality via 2-HH norm has been proved , the proof of which has been omitted in the paper by Kikianty and Dragomir. This thesis also generalizes the special case of the Carlsson orthogonality in terms of 2-HH norm in real normed linear space and it has been proved that the orthogonality is unique in any normed space X . The Birkhoff and Robert orthogonality via 2-HH norm are found equivalent, whenever the underlying space is real inner product space [15].

1.1 Statement of the problem

The statements of problem are as follows:

- To study orthogonality in normed linear space by generalizing the notion of orthogonality;
- To study orthogonality in terms of p-HH norm;
- To study orthogonality in terms of bounded linear operators in Banach spaces;
- Use of orthogonality in the theory of best approximation.

1.2 Objective of the study

The objective of this study is

1. To study different types of orthogonality like: Birkhoff-James, Carlsson, Pythagorean, Robert, and isosceles by using functional and operators in Banach space and Hilbert spaces,
2. To establish correlation between various orthogonalities,
3. To generalize different orthogonalities via p-HH norm,
4. To generalize the Carlsson orthogonality in terms of operators and study some famous properties of inner-product space in relation to this orthogonality,
5. To link new two spacial cases of the Carlsson orthogonalityi in terms of p-HH norm and bounded linear operators,
6. To introduce some different relations between orthogonality and best approximation in normed linear spaces.

1.3 Outline of the thesis

This thesis is devoted to the study of orthogonality in normed linear space with generalization of different orthogonalities in terms of 2-HH norm in a real normed space and bounded linear space operators in Banach space and Hilbert space. The work begins with chapter 1 providing introduction of orthogonality as well as its development by different mathematicians at different span of time, statement of the problem, objectives, and outlines of the thesis.

Chapter 2 is written with necessary literature of orthogonality needed to introduce new result of the thesis. Some generalizations regarding to the Robert, isosceles, Birkhoff-Jame and new orthogonality in terms of 2-HH norms are introduced and studied. Depending on the generalized definition of the Carlsson orthogonality, a new generalization in terms of bounded linear operators is also introduced with verification of some properties of inner product space. The most important concept introduced in this chapter is the equivalency of Birkhoff-James orthogonality with isosceles orthogonality, however the Birkhoff- James orthogonality is not equivalent with the Pythagorean orthogonality in the case of norm attaining bounded linear operators with disjoint support in Hilbert space.

In chapter 3, the new results of research have been included involving two new particular cases of the Carlsson orthogonality with verifying some important properties. It has also been proved that the new particular case of the Carlsson orthogonality implies the Birkhoff-James orthogonality by taking $y = \frac{x}{1-\alpha}$, but the converse may not be true. The most important concept proved in that section is the new orthogonality through the medium of 2-HH norm that is unique in any normed space with the help of proving

$$h(\mu) := \int_0^1 \|(1-t)y + \frac{\mu}{2}(tx)\|^2 dt + \int_0^1 \|(1-t)y - \frac{\mu}{2}(tx)\|^2 dt$$

is a convex function in $\mathbb{R}$. In the case of new orthogonality in terms of bounded linear operators in Banach space, it implies the Birkhoff-James orthogonality; however, the converse may not be true. For any element belonging to the norm attainment set of T, if the images are orthogonal, then the operators are also orthogonal in the sense of isosceles as well as Birkhoff-James orthogonality.

In chapter 4, as an application of orthogonality in terms of best approximation are involved due to the fact that the problem of best approximation amounts to the problem of minimizing a distance. An attempt has been made to prove some concept related to orthogonality (resp. $\epsilon-$orthogonality) and the best approximation (resp. co-approximation).

The chapter 5 includes the summary, conclusion and recommendation for further work. Finally, in appendix, the publications and certificates of conferences, seminar and workshops have been included.

Chapter 2

ORTHOGONALITY IN NORMED LINEAR SPACES

2.1 Introduction

It is well known fact that a norm in a normed linear space does not necessarily arise from an inner product. Therefore the definition of orthogonality in an inner-product space may not work for any normed space. Due to this, different orthogonality in normed spaces have been introduced and studied through the medium of equivalent propositions to the usual orthogonality, since from 1934. This chapter includes different classical notions of orthogonality which were introduced by different mathematicians at different time period. In order to study orthogonality in normed linear spaces, the main properties of orthogonality in an inner-product space are needed. Due to this reason, some important properties of orthogonality in an inner-product spaces are mentioned as follows:

2.1.1 Main properties of orthogonality in an inner product space:

Let $x, y, z \in X$, where X is an inner-product space. Then the following properties holds [[1], [16]]:

(i) Non-degeneracy: If $x \perp x$, then $x = 0$;

(ii) Simplification: If $x \perp y$, then for any $\lambda \in \mathbb{R}$, $\lambda x \perp \lambda y$;

(iii) Continuity: If $\{x_n\}, \{y_n\} \subset X$ such that $x_n \perp y_n$ for all $n \in \mathbb{N}$, $x_n \to x$ and

$y_n \to y$, then $x \perp y$;

(iv) Homogeneity: If $x \perp y$, then $\lambda x \perp \mu y$ for all $\lambda, \mu \in \mathbb{R}$;

(v) Symmetry: If $x \perp y$, then $y \perp x$;

(vi) Additivity: If $x \perp y$ and $x \perp z$, then $x \perp y + z$;

(vii) If $x \neq 0$, then there exist $\alpha \in \mathbb{R}$ such that $x \perp \alpha x + y$;

(viii) Uniqueness: The above α is unique.

2.2 Robert Orthogonality

The generalization of orthogonality from inner-product space to normed linear space was given by D. B. Robert in 1934 with some limitations. The main weakness of this orthogonality was that, for some normed linear spaces, at least one of orthogonal elements would have to be zero [2]. The definition of orthogonality is as follows:

Definition 1 ([2], [1]). *Let X be a normed linear space. A vector x is orthogonal to y in the sense of Robert if and only if* $\|x + \beta y\| = \|x - \beta y\|$ *identically in* β.

Properties of Robert orthogonality

For any normed linear space X, the Robert orthogonality satisfies the following properties [16]:

(i) Robert orthogonality satisfies non-degeneracy, simplification, and continuity;

(ii) Robert orthogonality is homogeneous and symmetric;

(iii) In any normed linear space, the Robert orthogonality is not additive and does not satisfy the existence property;

(iv) If the norm on X is induced by an inner-product, Robert orthogonality satisfy the existence property.

R. C. James noted that it is interesting to investigate the properties of orthogonality in an inner-product space as applied to normed space and he focused mainly on the existence property in the paper [1] and uniqueness in the paper [17].

Example 1. *[1] Let X be a normed linear space consisting all continuous functions of the form $f = ax + bx^2$, where $\|ax + bx^2\| = \max_{x\in(0,1)} \mid ax + bx^2 \mid$. Then Two elements of X are orthogonal in the sense of Roberts if and only if one is zero; i.e; $\|f + \beta g\| = \|f - \beta g\|$ for all β only if $f = 0$ or $g = 0$.*

Theorem 2.2.1. *[16] Let X be a normed space. Then, the Robert orthogonality is equivalent to any other orthogonality if and only if X is an inner product space.*

For any $(x, y) \in X^2$, Kikianty and Dragomir defined the p- norm on X^2 as follows [7]:

$$\|(x, y)\|_p = \begin{cases} [\|x\|^p + \|y\|^p]^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max\{\|x\|, \|y\|\}, & p = \infty \end{cases} \tag{2.1}$$

From (2.1), it is obvious that $\|(x, y)\|_p = \|(y, x)\|_p$, and therefore p-norm is symmetric. Using the concepts of Hermite-Hadamard's inequality, we have

$$\int_0^1 \|(1 - t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2} = \frac{\|(x, y)\|_p^p}{2} < \infty \tag{2.2}$$

With the help of (2.2), they defined the p-HH norm on X^2 in the following ways [26]:

$$\|(x, y)\|_{p-HH} = \begin{cases} (\int_0^1 \|(1 - t)x + ty\|^p)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \sup_{t\in[0,1]} \|(1 - t)x + ty\|, & \text{if } p = \infty \end{cases} \tag{2.3}$$

For all $x, y \in X$, it is obvious that $\|(x, y)\|_{p-HH} = \|(y, x)\|_{p-HH}$. Therefore the p-HH norm is symmetric. They proved that $(X^2, \|(.,.)\|)$ is a normed linear space because the non-degeneracy and homogeneity of the norm can be derived from (2.3) and the triangle inequality follows from the Minkowski's inequality. If the norm on X is induced by an inner product $(.,.)$, then as a special case of the p-HH norm, it is denoted by 2-HH norm. It is defined in the paper [7] as follows:

$$\|(x, y)\|_{2-HH}^2 = \int_0^1 \|(1 - t)x + ty\|^2 dt. \tag{2.4}$$

For any $p \geq 1$, The p-norm and p-HH norm are equivalent in X^2.

Definition 2. *A real valued function f defined on a non-empty subset $X \subset \mathbb{R}^n$ is called convex if*

(i) the domain X of the function is convex.

(ii) for any $x, y \in X$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{2.5}$$

If the inequality (2.5) is strict whenever $x \neq y$ and $0 < \lambda < 1$, then function f is called strictly convex function.

Let X be normed space in which the norm is induced by an inner-product. We make an attempt to generalize the Robert orthogonality via 2-HH norm, which satisfies non-degeneracy, simplification, continuity, symmetry, and homogeneity property [9].

Definition 3. *[9] Let X be a real normed linear space in which the norm is induced by an inner-product and $\|(1-t)x + \lambda ty\|^2 = \|(1-t)x - \lambda ty\|^2$ a. e. on $[0,1]$. Then x is said to be Robert orthogonal to y through the medium of 2-HH norm, denoted by $x \perp_{2-HH-R} y$ if and only if*

$$\int_0^1 \|(1-t)x + \lambda ty\|^2 dt = \int_0^1 \|(1-t)x - \lambda ty\|^2 dt \tag{2.6}$$

Theorem 2.2.2. *[9] Let X be a real normed linear space in which the norm is induced by an inner-product. Then the Robert orthogonality via 2-HH norm satisfies the non-degeneracy, simplification and continuity.*

Proof. **Nondegeneracy:** If $x \perp_{HH-R} x$. Then

$$\begin{aligned}
\|(x,x)\|^2 &= \int_0^1 \|(1-t)x + t\lambda x\|^2 \, dt \\
&= \int_0^1 \|(1-t)x - t\lambda x\|^2 \, dt \\
&= \int_0^1 \langle (1-t)x - t\lambda x, (1-t)x - t\lambda x \rangle dt \\
&= \|(x,x)\|^2 \int_0^1 (1-t)^2 dt + \|\lambda(x,x)\|^2 \int_0^1 t^2 dt \\
&= \frac{1}{3} \|(x,x)\|^2 (1+\lambda^2)
\end{aligned}$$

It is clear that $\|(x,x)\| = 0 \Rightarrow x = 0$, which gives the non-degeneracy property.

Simplification: If $x\perp_{HH-R}y$ for any $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned}\int_0^1 \|(1-t)\mu x + t\lambda\mu y\|^2\,dt &= |\mu|^2 \int_0^1 \|(1-t)x + t\lambda y\|^2\,dt \\ &= |\mu|^2 \int_0^1 \|(1-t)x - t\lambda y\|^2\,dt \\ &= \int_0^1 \|(1-t)\mu x - t\lambda\mu y\|^2\,dt.\end{aligned}$$

Therefore $\mu x\perp_{HH-R}\mu y$ for any $\mu \in \mathbb{R}$.

Continuity: If $x_n \perp_{2-HH-R} y_n$ with $x_n \to x$ and $y_n \to y$, then

$$\begin{aligned}\int_0^1 \|(1-t)x_n + \lambda t y_n\|^2\,dt &= \int_0^1 \|(1-t)x_n + \lambda t y_n\|^2\,dt \\ \Rightarrow \lim_{n\to\infty}\int_0^1 \|(1-t)x_n + \lambda t y_n\|^2\,dt &= \lim_{n\to\infty}\int_0^1 \|(1-t)x_n + \lambda t y_n\|^2\,dt \\ \Rightarrow \int_0^1 \|(1-t)x + \lambda t y\|^2\,dt &= \int_0^1 \|(1-t)x + \lambda t y\|^2\,dt\end{aligned}$$

This shows that $x \perp_{2-HH-R} y$.

□

Theorem 2.2.3. *[9] Let X be a real normed linear space in which the norm is induced by an inner-product. Then the Robert orthogonality via 2-HH norm is homogeneous, but not symmetric.*

Proof. **Homogeneity:** Let x, y be elements of normed space X, and $\lambda, \mu \in \mathbb{R}$

$$\begin{aligned}\int_0^1 \|(1-t)\lambda x + t\mu y\|^2\,dt &= \int_0^1 \langle (1-t)\lambda x + t\mu y, (1-t)\lambda x + t\mu y\rangle dt \\ &= \|\lambda x\|^2 \int_0^1 (1-t)^2 dt + \|\mu y\|^2 \int_0^1 t^2 dt \qquad (\because x\perp y) \\ &= \frac{1}{3}(\|\lambda x\|^2 + \|\mu y\|^2)\end{aligned}$$

Again,

$$\begin{aligned}\int_0^1 \|(1-t)\lambda x - t\mu y\|^2\, dt &= \int_0^1 \langle (1-t)\lambda x - t\mu y, (1-t)\lambda x - t\mu y\rangle dt \\ &= \|\lambda x\|^2 \int_0^1 (1-t)^2 dt + \|\mu y\|^2 \int_0^1 t^2 dt \qquad (\because x \perp y) \\ &= \frac{1}{3}(\|\lambda x\|^2 + \|\mu y\|^2).\end{aligned}$$

Therefore the Robert Orthogonality via 2-HH norm is homogeneous if the normed space is equipped with an inner-product.

Symmetry: To check the symmetry of $2-HH-R$ orthogonality,

$$\int_0^1 \|(1-t)y + \lambda t x\|^2 = \frac{1}{3}(\|y\|^2 + \lambda^2\|x\|^2), \quad \text{but}$$
$$\int_0^1 \|(1-t)x + \lambda t y\|^2 = \frac{1}{3}(\|x\|^2 + \lambda^2\|y\|^2)$$

$\therefore \quad \int_0^1 \|(1-t)y + \lambda t x\|^2 \neq \int_0^1 \|(1-t)x + \lambda t y\|^2$, showing that $2-HH-R$ orthogonality is not symmetric via $2-HH$ norm.

$\square$

Lemma 2.2.1. *[1] Let X be a normed linear space and $x, y \in X$. Then*

$$\lim_{\mu\to\infty}[\|(\mu+k)x + y\| - \|\mu x + y\|] = k\|x\|$$

Theorem 2.2.4. *[9] Let X be a normed linear space. Then $\forall x \in X, \quad \exists \mu \in \mathbb{R} : \mu x + y \perp_{HH-R} x,$*

Proof. Let $x, y \in X$ such that $x \neq 0$(for the case of x=0, the proof is trivial). Let us define a function $g : \mathbb{R}\times(0,1) \to \mathbb{R}$ by

$$\begin{aligned}g(\mu,t) &= \|(1-t)(\mu x+y) + \lambda t x\| - \|(1-t)(\mu x + y) - \lambda t x\|, \text{ where } \lambda \in \mathbb{R}^+, \mu \in \mathbb{R} \\ &= \|[(1-t)\mu + \lambda t]x + (1-t)y\| - \|[(1-t)\mu - \lambda t]x + (1-t)y\|\end{aligned}$$

and a function $G : \mathbb{R} \to \mathbb{R}$ by

$$G(x) = \int_0^1 g(\mu,t)dt.$$

Now

$$\begin{aligned}\lim_{\mu\to\infty} g(\mu,t) &= \lim_{\mu\to\infty}[\|[(1-t)\mu+\lambda t]x+(1-t)y\| - \|[(1-t)\mu-\lambda t]x+(1-t)y\|] \\ &= (1-t)\lim_{\mu\to\infty}\left[\left\|(\mu+\frac{\lambda t}{1-t})x+y\right\| - \left\|(\mu-\frac{\lambda t}{1-t})x+y\right\|\right]\end{aligned}$$

Let $\mu-\frac{\lambda t}{1-t}=\xi$ so that as $\mu\to\infty$, $\xi\to\infty$. Then $\mu+\frac{\lambda t}{1-t}=\xi+\frac{2\lambda t}{1-t}$

$$\begin{aligned}\therefore\quad \lim_{\mu\to\infty} g(\mu,t) &= \lim_{\xi\to\infty}[\left\|(\xi+\frac{2\lambda t)}{1-t})x+y\right\| - \|\xi x+y\|] \\ &= (1-t)\frac{2\lambda t}{1-t} \qquad \text{(by using Lemma 1.1)} \\ &= 2\lambda t\,\|x\|\end{aligned}$$

Hence $\lim_{\mu\to\infty} G(\mu) = \lim_{\mu\to\infty}\int_0^1 g(\mu,t)dt = \int_0^1 \lim_{\mu\to\infty} g(\mu,t)dt$ and by continuity of g,

$$\begin{aligned}\lim_{\mu\to\infty} G(\mu) &= \int_0^1 2\lambda t\,\|x\|\,dt \\ &= \|\lambda x\| > 0\end{aligned}$$

Also for any $t\in(0,1)$,

$$\begin{aligned}\lim_{\mu\to\infty} g(-\mu,t) &= \lim_{\mu\to\infty}[\|[(1-t)(-\mu)+\lambda t]x+(1-t)y\| - \|[(1-t)(-\mu)-\lambda t]x+(1-t)y\|] \\ &= \lim_{\mu\to\infty}[\|[(1-t)\mu-\lambda t]x-(1-t)y\| - \|[(1-t)\mu+\lambda t]x-(1-t)y\|] \\ &= (1-t)\lim_{\mu\to\infty}\left[\left\|(\mu-\frac{\lambda t}{1-t})x-y\right\| - \left\|(\mu+\frac{\lambda t}{1-t})x-y\right\|\right]\end{aligned}$$

Suppose $\mu+\frac{\lambda t}{1-t}=\xi$ so that as $\mu\to\infty$. $\xi\to\infty$ and $\mu-\frac{2\lambda t}{1-t}=\xi-\frac{2\lambda t}{1-t}$

$$\begin{aligned}\therefore\quad \lim_{\mu\to\infty} g(-\mu,t) &= (1-t)\lim_{\xi\to\infty}\left[\left\|(\xi-\frac{2\lambda t}{1-t})x-y\right\| - \|\xi x-y\|\right] \\ &= (1-t)\frac{(-2\lambda t)}{1-t}\,\|x\| \qquad \text{(by using Lemma 2.2.1)} \\ &= -2\lambda t\,\|x\|\end{aligned}$$

By the continuity of g, we have

$$\lim_{\mu\to\infty} G(-\mu) = \lim_{\mu\to\infty}\int_0^1 g(-\mu,t)dt = \int_0^1 \lim_{\mu\to\infty} g(-\mu,t)dt = \int_0^1 -2\lambda\,\|x\|\,dt = -\lambda\,\|x\| < 0.$$

Since G is continuous, so $\exists \mu_0 \in \mathbb{R} : G(\mu_0) = 0$.

$$\text{Hence} \int_0^1 \|(1-t)(\mu_0 x + y) + \lambda t x\|^2 \, dt = \int_0^1 \|(1-t)(\mu_0 x + y) - \lambda t x\|^2 \, dt.$$

□

2.3 Birkhoff Orthogonality

The concept of Birkhoff orthogonality began in 1935 [3]. In the literature of orthogonality this is known with some other names such as; Birkhoff- James orthogonality and Blaschke Birkhoff-James orthogonality (see [18]). In this paper [3, 17], an orthogonality which satisfies homogeneity but neither symmetric nor additive is defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all λ, is known as Birkhoff orthogonality or Birkhoff-James orthogonality. The geometrical meaning of Birkhoff orthogonality is that if x is an unit vector of a Banach space X and $y \in X$, then x is Birkhoff orthogonal to y means that the straight line $\{x + \lambda y : \lambda \in K\}$ is tangent to the unit ball of X at x. This concept is similar to the statement: suppose two lines l_1 and l_2 intersect at the point m, then $l_1 \perp l_2$ if and only if the distance from a point m of l_2 to a given point n of l_1 is never less than the distance from m and n. [17] For any hyper-plane $H \subset X$, x is said to be orthogonal to H if $\forall x \in H, x \perp h$.

Bhatia and Semrl in [10] generalize the definition of Birkhoff orthogonality in terms of matrix operators. For any matrices A and B they denote the symbol $\|A\|$ for operator norm of A and A is orthogonal to B in the sense of Birkhoff-James iff for any complex number z, $\|A + zB\| \geq \|A\|$. A matrix A is orthogonal to B iff there exist a unit vector $x \in H$ such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx \rangle = 0$ [10]. They also introduced Birkhoff-James orthogonality in [10] as $A \perp B$ if and only if $\|A + zB\|_p \geq \|A\|_p$, where $\|A\|_p$ denotes Schatten p-norm of A defined by $\|A\|_p = [\sum_{j=1}^{n} S_j(A)^p]^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $S_1(A) \geq S_1(A) \geq ...... \geq S_n(A)$ are singular values of A. Taking the special case for $p = 2$, Bhatia and Semrl in [10] also proved that the given orthogonality is equivalent to usual Hilbert space condition $\langle A, B \rangle = 0$, which defines an inner-product on the space of matrices as $\langle A, B \rangle = trace(A^* B)$. The norm associated to this inner product is $\|.\|_2$. In an infinite dimensional case [10], for any bounded operators in a Hilbert space H, $A \perp B$ if and only if there exist a sequence $\{x_n\}$ of unit vectors such in H that $\|Ax\| \to \|A\|$, and $\langle Ax_n, Bx_n \rangle \to 0$. Benitz et al. [19] proved that X is an inner-product space if and only if for any linear operators A and C in a finite dimensional normed space X,

$A \perp C \Leftrightarrow \exists u \in S_X : \|Au\| = \|u\|, Au \perp Cu$, where $S_X = \{x \in X : \|x\| = 1\}$ and "$\perp$" denotes the Birkhoff-James orthogonality.

Theorem 2.3.1. *[19] If S_X is not an ellipse(X is not an inner-product space), then there exists linear operators A and C in X such that $A \perp C$, but there does not exists $u \in S_X$ such that $\|A\| = \|Au\|$ and $Au \perp Cu$.*

Theorem 2.3.2 ([19]). *A real finite dimensional normed space X is an inner-product space if and only if , for $A, C \in L(X)$, $A \perp C \Leftrightarrow \exists x \in S_X : \|A\| = \|Ax\|$, $Ax \perp Cx$.*

Theorem 2.3.3. *[20] The q-angle has the following properties:*

(i) Part of parallelism property: $A_q(x, y) = 0$ iff x and y are linearly dependent.

(ii) Part of homogeneity property: $A_q(Ax, By) = A_q(x, y)$ for every $x, y \in X$ and $A, B \in \mathbb{R} - \{0\}$.

In [20] Chen Zhi-Zhi et al. have given slightly different definition of Birkhoff orthogonality in such a way that; x is Birkhoff orthogonal to y iff $A_q(x, y) = \frac{\pi}{2}$ by using projections of the angles between two vectors x and y in a real two dimensional normed space X.

Definition 4. *[20] The g-angle between two vectors x and y is given by $g(x, y) = \cos^{-1} \frac{g(x,y)}{\|x\| \, \|y\|}$, where $g(x, y) = \frac{1}{2}\|x\|[\tau_+(x, y) + \tau_-(x, y)]$ and $\tau_\pm(x, y) = \lim_{t \to \pm 0} \frac{\|x+ty\| - \|x\|}{t}$. In that case $x \perp_g y$ if $g(x, y) = 0$ or $A_g(x, y) = \frac{\pi}{2}$.*

For any $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ in a two dimensional real normed space X,

$$q(x, y) = \begin{cases} 0 & \text{if x and y are linearly dependent} \\ \|P_{xy}\|^{-1}, & \text{if x and y are linearly independent.} \end{cases}$$

Continuity property: If $x_n \to x$ and $y_n \to y$, then $A_q(x_n, y_n) \to A_q(x, y)$, where $A_q(x, y)$ is q-angle between x and y defined by $A_q(x, y) = \sin^{-1}[q(x, y)]$.

Lemma 2.3.1. *[20] If x is Birkhoff orthogonal to y. Then for any $m, n \in \mathbb{R}$, $\|mx + ny\| \geq \|mx\|$.*

Theorem 2.3.4. *[21] Let $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ be two vectors in a two dimensional real normed space X with basis $\{e_1, e_2\}$. Then x is Birkhoff-orthogonal to y iff $A_q(x, y) = \frac{\pi}{2}$ i.e. $\|P_{xy}\| = 1$.*

2.3.1 Orthogonality on C^*-module

Let A be a C^*-algebra and H be a (left) $\mathscr{A}$-module. Suppose that the linear structure given on $\mathscr{A}$ and H are compatible, that is, $\lambda(ax) = a(\lambda x)$ for every $\lambda \in \mathbb{C}$ and $a \in H$. Then there exists a mapping $\langle ., . \rangle : H \times H \to \mathscr{A}$ with the following properties [22]:

(i) $\langle x, x \rangle \geq 0$ for every $x \in H$,

(ii) $\langle x, x \rangle = 0$ iff $x = 0$,

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in H$,

(iv) $\langle ax, y \rangle = a \langle x, y \rangle$ of every $a \in \mathscr{A}$ and $x, y \in H$,

(v) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for every $x, y, z \in H$

The pair $(H, \langle ., . \rangle)$ is called a (left) pre-Hilbert $\mathscr{A}$-module. The map $\langle ., \rangle$ is called an $\mathscr{A}$-valued inner-product. If the pre-Hilbert $\mathscr{A}$-module $(H, \langle ., . \rangle)$ is complete with respect to the norm $\|x = \| = \|\langle x, x \rangle\|^{\frac{1}{2}}$, then it is called $\mathscr{A}$-Hilbert $\mathscr{A}$-module over $\mathscr{A}$. Rajic et al. in [[22], [21]] introduced a new concept of Birkhoff-James orthogonality in a Hilbert $C*$- modules over $C*$-algebra $\mathscr{A}$ and proved that such orthogonality with respect to $\mathscr{A}$-valued inner product coincide if and only if $\mathscr{A}$ is isomorphic to $\mathbb{C}$. [22] A mapping $T : V \to W$ between $\mathscr{A}$-modules V and W is called adjointable if there exists mapping $T^* : W \to V$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $v \in V, y \in W$. Such a mapping T is bounded, linear and satisfies $T(xa) = T(x)a$ for all $x \in V$ and $a \in \mathscr{A}$. The space of all adjointable mapping from V into W is denoted by $B(V, W)$. Let $\theta_{x,y}(z) = x(y, z)$, where $\theta_{x,y} \in B(V, W)$ and $K(B, V)$ denotes the closed linear subspace of $B(V, W)$ spanned by $\{Q_{xy} : x \in W, y \in V\}$ is called space of compact operators.

Proposition 1. *[22] Let $A, B \in B(H)$.Then $min_{\lambda \in \mathbb{C}} \|A + \lambda B\|^2 = sup_{\|xi\|=1} M_{A,B}(\xi)$, where*

$$M_{A,B}(\xi) = \begin{cases} \|A\xi\|^2 - \frac{|\langle A\xi, B\xi \rangle|^2}{\|B\xi\|^2} & \textit{if}, \quad B\xi \neq 0 \\ \|A\xi\|^2 & \textit{if}, \quad B\xi = 0 \end{cases}$$

Proposition 2. *[22] let $\mathscr{A}$ be a C^*-algebra, and $a, b \in \mathscr{A}$. Then $min_{\lambda \in \mathbb{C}} \|a + \lambda b\|^2 = max_{\varphi \in S(A)} M_{A,B}(\varphi)$, where*

$$M_{a,b}(\varphi) = \begin{cases} \varphi(a^*a) - \frac{|\varphi(a^*b)^2}{\varphi(b^*b)} & \textit{if}, \quad \varphi(b^*b) \neq 0 \\ \varphi(a^*a) & \textit{if}, \quad \varphi(b^*b) = 0 \end{cases}$$

Theorem 2.3.5. *[22] Le V be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ and $x, y \in V$. Then $min_{\varphi\in\mathbb{C}}\|x + \varphi y\|^2 = max_{\varphi\in S(A)}M_{x,y}(\varphi)$, where $M_{x,y}(\varphi) \in \mathscr{A}$ is defined by*

$$M_{x,y}(\varphi) = \begin{cases} \varphi(\langle x, x\rangle) - \frac{|\varphi(\langle x,y\rangle)|^2}{\varphi(\langle y,y\rangle)} & \textit{if}, \quad \varphi(\langle y, y\rangle) \neq 0 \\ \varphi(\langle x, x\rangle) & \textit{if}, \quad \varphi(\langle y, y\rangle) = 0 \end{cases}$$

Theorem 2.3.6. *[22] Let V be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$. Let $x, y \in V$. Then $x\perp_B y \Leftrightarrow \exists\varphi \in S(\mathscr{A}) : \varphi(\langle x, x\rangle) = \|x\|^2$ and $\varphi(\langle x, y\rangle) = 0$.*

Theorem 2.3.7. *[22] Let V be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ and $x, y \in V$. Then*

(i) $x\perp_B y \Leftrightarrow \langle x, x\rangle \perp \langle x, y\rangle \Leftrightarrow \langle x, x\rangle \perp_B \langle y, x\rangle$.

(ii) $x\perp_B y \Rightarrow x\perp_B x \langle x, y\rangle$ *and* $x\perp_B x \langle y, x\rangle$.

Arambasic and Rajic (see in[22]) characterized Hilbert C^*-modules where the Birkhoff orthogonality coincides with the usual orthogonality with respect to inner-product space. By using the Gelfand-Mazur theorem, it can be proved that $\mathscr{A}$ is isomorphic to $\mathbb{C}$ and using this concept, $\mathbb{C}$ is only the unital C^*-algebra in which Birkhoff orthogonality $x\perp_B y$ coincides with $x^*y = 0$ for all elements $x, y \in \mathscr{A}$.

Theorem 2.3.8. *[22] Let $V \neq \{0\}$ be a full Hilbert $\mathscr{A}$-module. then the following statements are equivalent:*

(i) For all $x, y \in V$ the condition ($x\perp_B y \Leftrightarrow \langle x, y\rangle = 0$) is always true.

(ii) $\mathscr{A}$ is isomorphic to $\mathbb{C}$.

Theorem 2.3.9. *[23] Let T be a linear operator on a finite dimensional real normed space X and $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$. If M_T can be partitioned into tow non-empty sets which are contained in complementary subset of X, then there is a linear operator A on X such that $T\perp_B A$ but $Tx \not\perp_B Ax$.*

Theorem 2.3.10. *[23] Let T be a linear operator on a finite dimensional real smooth normed space X. If $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$ is a countable set with more than 2 points. Then for any $x \in M_T$ there is a linear operator A on X such that $T\perp_B A$ but $Tx \not\perp_B Ax$*

Theorem 2.3.11. *[23] Let T be a linear operator on a two dimensional real normed space X, and let $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$. If M_T has more than two components, then for any $x \in M_T$ there is a linear operator A on X such that $T\perp_B A$ but $Tx \not\perp_B Ax$.*

2.3.2 Strong Birkhoff-James Orthogonality

Definition 5. *(Strongly orthogonal set)[12]: A finite set of elements $\{x_1, ..., x_k\}$ is said to be strongly orthogonal set in the sense of Birkhoff-James iff for each $m \in \{1, 2, ..., k\}$*

$$\|x_m\| < \|x_m + \sum_{m=1, m\neq n}^{k} \lambda_n x_n\|,$$

whenever $\lambda_n \neq 0$.

Theorem 2.3.12. *[12] Let X be a normed linear space and $x_0 \in S_x$. If there exists a Hamel basis of X containing x_0 which is strongly orthogonal relative to x_0 in the sense of Birkhoff-James, then x_0 is an extreme point of B_X.*

Theorem 2.3.13. *[12] Let X be a normed linear space and $x_0 \in S_X$ be an exposed point of B_X. Then there exists a Hamel basis of X containing x_0 which is strongly orthogonal relative to x_0 in the sense of Birkhoff-James.*

Theorem 2.3.14. *[12] Let X be a normed linear space and $x_0 \in S_X$. If there exist a Hamel basis of X containing x_0 which is strongly orthogonal relative to x_0 in the sense of Birkhoff-James, then there exists a bounded invertible linear operator A on X such that $\|A\| = \|A_0\| > \|Ay\|$ for all $y \in S_X$ with $y \neq \lambda x_0, \lambda \in S_k$.*

Theorem 2.3.15. *[12] For a normed space X and a point $x \in span(X)$, the following are equivalent:*

(i) *x is an exposed point of B_X.*

(ii) *There is a Hamel basis of X containing x which is strongly orthonormal relative to x in the sense of Birkhoff-James.*

(iii) *There exists a bounded linear operator A on X which attains only at the points of the form λx with $\lambda \in S_k$.*

Theorem 2.3.16. *[12] For a normed linear space X, the following are equivalent:*

(i) *X is strictly convex, and*

(ii) *For each $x \in S_X$, there exists a Hamel basis of X containing x which is strongly orthonormal relative to x in the sense of Birkhoff-James.*

For any bounded linear operator $T, A \in L(X)$, T is said to be Birkhoff-James orthogonal to A if $\|T + \lambda A\| \geq \|T\| \quad$ for all $\lambda \in \mathbb{C}$ and $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$. In the real Banach space X, Sain introduced two sets x^+ and x^- in his paper [24] by

(i) $x^+ = \{y \in X : \|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda \geq 0\}$ and

(ii) $x^- = \{y \in X : \|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda \leq 0\}$

For the complex Banach space, Paul et al. in 2018 introduced the following notations [13] depending on Sain's concept : For any $\gamma \in V$,

(i) $x^+_\gamma = \{y \in X : \|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda = tr, t \geq 0\}$

(ii) $x^-_\gamma = \{y \in X : \|x + \lambda y\| \geq \|x| \quad \text{for all} \quad \lambda = tr, t \leq 0\}$

(iii) $x^{\frac{1}{\gamma}} = \{y \in X : \|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda = tr, t \in \mathbb{R}\}$
where $V = \{\gamma \in \mathbb{C} : |\gamma| = 1, arg(\gamma) \in [0, 2\pi]\}$. Also

(iv) If $\mu = e^{i\pi}\gamma, \quad$ then $\quad x^+_\mu = x^-_\gamma, x^-_\mu = x^+_\gamma \quad$ and $\quad x^{\frac{1}{\mu}} = x^{\frac{1}{\gamma}}$. In the complex Banach space,

(v) $x^+ = \cap \left\{x^+_\gamma : \gamma \in V\right\}, x^- = \cap \left\{x^-_\gamma : \gamma \in V\right\}$ and $x^\perp = \cap \left\{x^{\frac{1}{\gamma}} : \gamma \in V\right\}$

Proposition 3. *[24] Let $x, y \in X$, where X is an complex Banach space and $\gamma \in V$. Then following statements are true*

(i) Either $y \in x^+_\gamma$ or $y \in x^-_\gamma$.

(ii) $x \perp \gamma y \Leftrightarrow y \in x^+_\gamma$ or $y \in x^-_\gamma$.

(iii) $y \in x^+_\gamma \Rightarrow \eta y \in (\xi x)^+_\gamma$ for all $\eta, \xi > 0$.

(iv) $y \in x^+_\gamma \Rightarrow -y \in x^-_\gamma$ and $y \in (-x)^-_\gamma$.

(v) $y \in x^-_\gamma \Rightarrow \eta y \in (\xi x)^-_\gamma$ for all $\eta, \xi > 0$.

(vi) $y \in x^-_\gamma \Rightarrow -y \in x^+_\gamma$ and $y \in (-x)^+_\gamma$.

(vii) $y \in x^+_\gamma \Rightarrow \mu y \in (\mu x)^+_\gamma$ for all $\mu \in \mathbb{C}$.

(viii) $y \in x^-_\gamma \Rightarrow \mu y \in (\mu x)^-_\gamma$ for all $\mu \in \mathbb{C}$.

Proposition 4. *[24] Let $x, y \in X$, where X is a complex Banach space. Then the following are true* *(i)* $x \perp_B y \Leftrightarrow y \in x^+$ *and* $y \in x^-$.

(ii) $y \in x^+ \Rightarrow \eta y(\xi x)^+$ *for all* $\eta, \xi > 0$.

(iii) $y \in x^+ \Rightarrow -y \in x^-$ *and* $y \in (-x)^-$.

(iv) $y \in x^- \Rightarrow -y \in x^+$ *and* $y \in (-x)^+$.

(v) $y \in x^- \Rightarrow \eta y \in (\xi x)^-$ *for all* $\eta, \xi > 0$.

Theorem 2.3.17. *[24] Let X be a reflexive complex Banach space, and Y be any complex Banach space. Let* $T, A \in K(x, y)$. *Then* $T \perp_B A \Leftrightarrow \forall \gamma \in V, \quad \exists \quad x = x(\gamma), y = y(\gamma) \in M_T : Ax \in (Tx)^+_\gamma$ *and* $Ty \in (Ty)^-_\gamma$.

Theorem 2.3.18. *[24] Let X be a complex Banach Space. Let* $x, y \in X$ *and* $r = e^{i\theta}$, *where* $\theta \in [0, 2\pi]$. *If* $y \in x^+_\gamma$, *then either* $y \in x^+_\mu$ *for all* μ *with* $\arg \mu \in [0, \theta]$ *or* $y \in x^+_\mu$ *for all* μ *with* $\arg \mu \in [0, \pi]$.

Theorem 2.3.19. *[24] Let be a linear operator on a finite dimensional complex Banach space X, such that* M_T *is a closed connected subset of* S_X. *Then for* $A \in L(X)$, $T \perp_B A \Leftrightarrow \forall \gamma \in V \quad \exists \quad x = x(\gamma) \in M_T : Tx \perp_\aleph Ax$.

Theorem 2.3.20. *[24] Let T be a linear operator in a finite dimensional complex Banach space X such that* M_T *is a closed connected subset of the unit sphere of X. Then for* $A \in L(X)$, $T \perp_B A \Leftrightarrow \exists \quad \theta \in [0, \pi]$ *and* $x, y \in M_T : Ax \in (Tx)^+_\gamma$ *for all* γ *with* $\arg \gamma \in [\theta - \pi, \theta]$ *and* $Ay \in (Ty)^+_\gamma$ *for all* γ *with* $\arg \gamma \in [\theta, \theta + \pi]$.

2.3.3 Norm parallelism and Birkhoff-James orthogonality

Definition 6. *[25] Let* $x, y \in X$, $K \in \{\mathbb{C}, \mathbb{R}\}$ *and* $T = \{\mu \in K : |\mu| = 1\}$. *Then x is said to be norm parallel to y if* $\|x + \mu y\| = \|x\| + \|y\|$ *for all* $\mu \in T$.

Norm parallelism is symmetric as well as homogeneous; whereas, Birkhoff-James orthogonality is homogeneous but not symmetric in a Banach space. [25] In the case of Hilbert space, two elements are linearly dependent iff they are norm- parallel; however, in normed spaces two linearly dependent vectors are norm-parallel, but the converse may not be true. For instance, $(1, 1)$ and $(1, 0)$ are norm parallel but not linearly dependent. Depending on the concept of Birkhoff-James orthogonality and strong Birkhoff-James orthogonality Paul et al.[25] introduce a new geometric notion of semi-rotund point. For any normed linear space X, $\beta \neq x \in X$ is said the semi-rotund point of X if $\exists y \in X$:

$x \perp_{SB} y$. If for every $x \neq 0 \in X$, x is a semi-rotund point, the normed space X is said to be semi-rotund space. Dragomir introduced the concept of approximate Birkhoff-James orthogonality [**?**] as follows: x is said to be approximate Birkhoff-James orthogonal to y if $\|x+\mu y\| \geq (1-\epsilon)\|x\|$ for all $\mu \in K$ and $\epsilon \in [0,1]$; however, Chmielinski [25, 27] defined approximate Birkhoff-James orthogonality as ; $x \perp_D^{\epsilon} \Leftrightarrow \|x+\mu y\| \geq \sqrt{1-\epsilon^2}\|x\|$ for all $\mu \in K$. The concept of approximate parallelism was developed by Zamani and Moslehian [28] by stating that x is approximately parallel to y if $\inf\{\|x+\lambda y\| : \lambda \in K\} \leq \epsilon\|x\|$ for all $\epsilon \in [0,1]$.

Proposition 5. *[25] let X be a bounded linear operator form a normed space X to normed space Y and $x \in M_T$. Then for any $\epsilon \in [0,1]$ and $y \in X$, we have $x||_y^{\epsilon} \Rightarrow Tx||^{\epsilon}Ty$.*

Theorem 2.3.21. *[25] Let T and A are compact linear operators form a reflexive Banach space X to any normed space Y. Then $T||A \Leftrightarrow \exists \quad x \in M_T \cap M_A : Tx||Ax$.*

Theorem 2.3.22. *[25] If T and A are bounded linear operators form a normed space X to Y. Then $T||A \Leftrightarrow \quad \exists\{x_n\} \in S_X : \lim_{n\to\infty}\|Tx_n\| = \|T\|, \lim_{n\to\infty}\|Ax_n\| = \|A\|$ and $\lim_{n\to\infty}\|Tx_n + \mu Ax_n\| = \|T\| + \|A\|$, for some $\mu \in K$.*

Proposition 6. *[25] Let T be a bounded linear operator form a normed space X into normed space Y and let $x \in M_T$. Then $Tx \perp_D^{\epsilon} Ty \Rightarrow x \perp_D^{\epsilon} y$ for any $\epsilon \in [0,1]$ and $y \in X$.*

Theorem 2.3.23. *[25] let T and A are bounded linear operators from finite dimensional Banach spaces X to Y. Then $T \perp_{SB} A \Leftrightarrow \forall\epsilon > 0, \exists\mu_\epsilon > 0 : \forall|\mu| < \mu_\epsilon, \exists y_\mu \in (\cup_{x\in M_T} B(x,\epsilon)) \cap S_x : \|Ty_\mu + \mu Ay_\mu\| > \|T\|$.*

Theorem 2.3.24. *[25] Let T and A are compact linear operators fron a reflexive Banach space X to any normed space Y be such that $T \perp_B A$ but $T \not\perp_{SB} A$. Then there exists $x \in M_T$ such that $Tx \perp_B Ax$.*

Theorem 2.3.25. *[25] Let T and A are bounded linear operators from a normed space X to Y. If $T \perp_B A$ but $T \not\perp_{SB} A$, then there exists a sequence $\{x_n\}$ in S_X such that $\|Tx_n\| \to \|T\|, Ax_n \to 0$ or there exist a sequence $\{x_n\}$ in S_X and sequence $\{\epsilon_n\}$ in $\mathbb{R}^+$ such that $\|Tx_n\| \to \|T\|, \epsilon_n \to 0$, and $Tx_n \perp_D^{\epsilon_n} Ax_n$.*

2.3.4 Birkhoff-James orthogonality by applying semi-inner product

The concepts of Birkhoff-James orthogonality has been widely used by various researchers since 1935. The latest research on this topic by Sain, Mal, and Paul [29]have studied

Birkhoff-James orthogonality of compact linear operators between Hilbert space and Banach spaces by applying the notion of semi-inner product in normed linear spaces.

Definition 7. *[29] For any normed linear space x, A scalar valued function $(.,.) : X \times X \to K$, where $K \in \{\mathbb{C}, \mathbb{R}\}$ is a semi-inner product if for any $\xi, \eta \in K$ and for any $x, y, z \in X$, it satisfies the following conditions:*

(i) $(\xi x + \eta y, z) = \xi(x, z) + \eta(x, z)$,

(ii) $(x, x) > 0$, *whenever* $x \neq 0$.

(iii) $\mid (x, y) \mid^2 \leq (x, x)(y, y)$,

(iv) $(x, \xi y) = \bar{\xi}(x, y)$.

Every semi-inner product space is a normed space with the norm $\|x\|^2 = (x, x)$ and the norm of any normed space can be generated through a semi-inner product in infinitely many ways. Sain et al. in [29] characterized the Birkhoff-James orthogonality set of any compact linear operators between a reflexive Banach space any Banach spaces. They also proved that there is an relationship between the concept of semi-inner product spaces and the sets $x^+ = \{y \in X : \|x + \gamma y\| \geq \|x\| \text{for} \gamma \geq 0\}$ and $x^+ = \{y \in X : \|x + \gamma y\| \geq \|x\| \text{for} \gamma \leq 0\}$

Theorem 2.3.26. *[29] Let T and A be compact linear operators from a reflexive Banach space X to any Banach space Y. If any one of the following conditions holds;*

(i) M_T *is a connected subset of* S_X.

(ii) M_T *is not connected but* $M_T = D \cup (-D)$, *where D is a non-empty subset of* S_X.

Then $T \perp_B A \Leftrightarrow \exists x \in M_T : Tx \perp_B Ax$.

Theorem 2.3.27. *[29] For a finite-dimensional Banach space X, the following statements are are equivalent.*

(i) For any linear operator T on X, M_T *is the unit sphere of some subspace of X.*

(ii) For any linear operator T on X, $M_T = D_T \cup (-D_T)$, *where* D_T *is connected subset of X.*

(iii) X is an Euclidean space.

Theorem 2.3.28. *[29] Let $x, y \in X$, where X is a normed linear space. Then the following are true.*

(i) $y \in x^+$ *iff there exists a semi-inner product* $(.,.)$ *on X with* $(y, x) \geq 0$.

(ii) $y \in x^-$ *iff there exists a semi-inner product* $(.,.)$ *on X with* $(y, x) \leq 0$.

Theorem 2.3.29. *[29] Let T and A be compact linear operators from a reflexive Banach space X to any Banach space Y be such that $T \perp_B A$. let $\mathscr{O}_Y$ denotes the collection of all semi-inner product on Y. Then*

$$\|T\| = \begin{cases} \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, (Ax, y) \geq 0\} \\ \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, (Ax, y) \leq 0\} \end{cases}$$

Theorem 2.3.30. *[29] Let T and A be bounded linear operators form a normed space X to Y be such that $T \perp_B A$. Ley $\mathscr{O}_Y$ denotes the collection of semi-inner product space on Y. Let $\epsilon > 0$ be arbitrary but fixed after A choice. Then*

(i) $\|T\| = \max\{l_1(\epsilon), l_2(\epsilon)\} = \max\{l_1(\epsilon), l_3(\epsilon)\}$, *where*

(ii) $l_1(\epsilon) = \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, \mid (Ax, y) \mid < \epsilon\}$

(iii) $l_2(\epsilon) = \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, Ax \in (y)^{+\epsilon}\}$

(iv) $l_3(\epsilon) = \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, Ax \in (y)^{-\epsilon}\}$

Theorem 2.3.31. *[29] Let X be normed linear space such that X^* is strictly convex. Let $f, g \in X^*$ be such that $f \perp_B g$. then*

$$\|f\| = \begin{cases} \sup\{f(x) : x \in S_x, g(x) \geq 0\} \\ \sup\{f(x) : x \in S_x, g(x) \leq 0\}. \end{cases}$$

Theorem 2.3.32. *[29] Let T and A are compact linear operators from a reflexive Banach space X to any Banach space Y be such that for each $\lambda \in \mathbb{R}, M_{T+\lambda A} = D_\lambda \cup (-D_\lambda)$, where D_λ is a non-empty connected subset of S_X. Let $\mathscr{O}_Y$ denotes the collection of all semi-inner product space on Y. Then*

$$dist(T, span\{A\}) = \sup\{(Tx, y) : x \in S_X, y \in S_Y, (.,.) \in \mathscr{O}_Y, (Ax, y) = 0\}.$$

Theorem 2.3.33. *[29] Let X be a reflexive Banach space and Y be any Banach space. Let $\mathcal{Z}$ be a finite dimensional subspace of $K(X,Y)$. Let $T \in K(X,Y) \setminus \mathscr{Z}$. Let us further assume that for any $\lambda \in \mathbb{R}$ and for any $A \in \mathscr{Z}$, $M_{T+\lambda A} = D_{\lambda,A} \cup (-D_{\lambda,A})$, where $D_{\lambda,A}$ is non-empty connected subset of S_X. Then there exist $A_0 \in \mathscr{Z}$ such that*

$$dis(T, \mathscr{Z}) = \sup\{(Tx, y) : x \in S_X, y \in S_Y, (A_0x, y) = 0\}.$$

Moreover, A_0 is the best approximation of T in $\mathscr{Z}$.

2.3.5 Modular Birkhoff orthgonality in Banach modules

We have already mentioned that Rajic et al. in[22] studied Birkhoff-James orthogonality in a Hilbert C^*-modules over a C^*-algebra. The most current research as generalization of Birkhoff-James orthogonality from Hilbert space to Banach spaces in [30], Sain and Tanaka studied the stronger version of modular Birkhoff-James orthogonality in the set of bounded and compact linear operators. In order to prove their study they introduced the following notions: $X^{\perp} = \{y \in X : x \perp_B y\}$ and $M_A = \{x \in S_X : \|Ax\| = \|A\|\}$. An element $x \neq 0 \in X$ is said to be smooth point in X if $\mathscr{T}(x) = \{f \in S_X^* : f(x) = \|x\|\}$ is a singleton set. For any Banach space X, an element $x \in X$ is said to be left symmetric in X if for any $y \in X$, $x \perp_B y \Rightarrow y \perp_B x$. Similarly x is said to be right symmetric in X if for any $y \in X$, $y \perp_B x \Rightarrow x \perp_B y$. If x is both left as well as right symmetric, then x is said to be a symmetric point.

Definition 8. *[30] A Banach space X is called a right $\mathscr{A}$-module (where $\mathscr{A}$ is a Banach algebra) if there exists a mapping of $X \times \mathscr{A}$ into X such that for each $a, b \in \mathscr{A}$ and $x \in X$, $x(ab) = (xa)b$ and $\|ax\| \leq \|x\|\|a\|$.*

An element $x \in X$ is said to be right-modular Birkhoff-James orthogonal to $y \in X$, if $x \perp_B ya$ for all $a \in \mathscr{A}$ and left-modular Birkhoff-James orthogonal to y if $x \perp_B ay$ for all $a \in \mathscr{A}$.

Theorem 2.3.34. *[30] Let T and A be compact linear operators form a reflexive real Banach space X to any real Banach space Y such that $M_A = \{\pm x_0\}$ for some $x_0 \in S_X$. Then $A^{\perp}_{B(X)} \Leftrightarrow T(X) \subset (Ax_0)^{\perp}$.*

Definition 9. *[30] A Banach space X is said to be Kadets-Klee if whenever $\{x_n\}$ is a sequence in X and $x \in X$ is such that $\{x_n\}$ converges weekly to x and $\lim\limits_{n\to\infty} \|x_n\| = \|x\|$, then $\lim_{n\to\infty} \|x_n - x\| = 0$.*

Theorem 2.3.35. *[30] Let A be a compact linear operator from a reflexive Kadets-Klee real Banach space to any real Banach space be such that $M_T = \{\pm x_0\}$ for some $x_0 \in S_X$. Then given any bounded linear operator $T \in B(X,Y), A \perp_{B(X)} T \Leftrightarrow T(X) \subset (Ax_0)^{\perp}$.*

Theorem 2.3.36. *[30] Let X, Y be real Banach spaces. Let $A \in B(X,Y)$ be a smooth point in $B(X,Y)$ such that $M_A \neq 0$. Then given any $T \in B(X,Y)$, $A \perp_{B(X)} T \Leftrightarrow T(X) \subset (Ax_0)^{\perp}$, where $M_A = \{\pm x_0\}$.*

Theorem 2.3.37. *[30] Let T and A are compact linear operators from a reflexive complex Banach space X to any complex Banach space Y be such that $M_A = \left\{e^{i\theta}x_0 : \theta \in [0, 2\pi]\right\}$ for some $x_0 \in S_X$. Then given any compact linear operator T, $A \perp_{B(X)} T \Leftrightarrow T(X) \subset (Ax_0)^{\perp}$.*

Theorem 2.3.38. *[30] Let T and A are compact linear operators from a reflexive real Banach Space X to any real Banach space Y be such that $M_A = \{\pm x_0\}$ for some $x_0 \in S_X$. Then given any compact linear operator T, $A \perp^*_{B(Y)} T = \Leftrightarrow Tx_0 = 0$. Moreover,if X is Kadets-Klee, then same is true for any $T \in B(X,Y)$.*

Theorem 2.3.39. *[30] Let T and A are compact linear operators from a reflexive complex Banach space to any complex Banach space Y be such that $M_A = \left\{e^{i\theta}x_0 : \theta \in [0, 2\pi]\right\}$ for some $x_0 \in S_X$. Then given any $T \in K(X,Y), A \perp^*_{B(Y)} T \Leftrightarrow Tx_0 = 0$.*

If A is a bounded linear operator from a normed spaces X to Y, then its adjoint $A^* \in B(Y^*, X^*)$ is defined by $(A^*y^*) = y^*Ax$ for each $x \in X$, $y^* \in Y^*$ and $\|A^*\| = \|x\|$. For any subsets R and S of a Banach space X, $R \perp_B S$ if $x \perp_B y$ for all $x \in R$ and $y \in S$.

Proposition 7. *[30] Let T and A are bounded linear operators from a Banach space X to Y. If $A(x) \perp_B T(X)$, then $A \perp_B T$.*

2.3.6 Birkhoff orthogonality via 2-HH norm

Motivated by various generalizations of Birkhoff orthogonality, we make an attempt to introduce this most popular orthogonality in terms of 2-HH norm which we denote by $x \perp_{2-HH-B} y$ [31].

In the case of 2-HH norm,

$$\begin{aligned}\int_0^1 \|(1-t)x + \lambda ty\|^2 &= \int_0^1 \langle (1-t)x + \lambda ty, (1-t)x + \lambda ty \rangle dt \\ &= \|x\|^2 \int_0^1 (1-t)^2 dt + 2\lambda \langle x, y \rangle \int_0^t t(1-t)dt + \lambda^2 \|y\|^2 \int_0^1 t^2 dt.\end{aligned}$$

$x \perp y$, then

$$\int_0^1 \|(1-t)x + \lambda t y\|^2 = \|x\|^2 \int_0^1 (1-t)^2 dt + \lambda^2 \|y\|^2 \int_0^1 t^2 dt$$
$$= \frac{1}{3}(\|x\|^2 + \|\lambda y\|^2) \tag{2.7}$$

But,

$$\int_0^1 \|(1-t)x\|^2 dt = \|x\|^2 \int_0^1 (1-t)^2 dt = \frac{1}{3}\|x\|^2. \tag{2.8}$$

Since $\|\lambda y\|^2$ is a non-negative quantity, so from relation (2.7) and (2.8), we conclude that

$$\int_0^1 \|(1-t)x + \lambda t y\|^2 \geq \int_0^1 \|(1-t)x\|^2 dt \tag{2.9}$$

Keeping above result in our mind, we can conclude that $x \perp_{2-HH}(B) y$ if the relation (2.9) is satisfied.

In the following theorem, 2-HH-R and 2-HH-B denotes the Robert orthogonality and Birkhoff-James orthogonality via 2-HH norm respectively.

Theorem 2.3.40. *[9] Let $x, y \in X$, where X is a real normed linear space equipped with an inner-product space over the field $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$ and $\mu = \lambda t$. Then 2-HH-R orthogonality implies 2-HH-B orthogonality and conversely.*

Proof. Assume $x \perp_{2-HH-R} y$. Then for any $\mu \in \mathbb{R}$,

$$\begin{aligned}
\int_0^1 \|(1-t)x + \mu y\|^2 dt &= \int_0^1 \|(1-t)x - \mu y\|^2 dt \\
&= \int_0^1 \langle (1-t)x - \mu y, (1-t)x - \mu y \rangle dt \\
&= \int_0^1 [\|(1-t)x\|^2 - (1-t)\langle x, y\rangle - \mu(1-t)\langle y, x\rangle + \|\mu y\|^2] dt \\
&= \int_0^1 \|(1-t)x\|^2 dt + \int_0^1 \|\mu y\|^2 dt \\
&\geq \int_0^1 \|(1-t)x\|^2 dt. \qquad (2.10)
\end{aligned}$$

Therefore x is 2-HH-B orthogonal to y. To prove the converse part, It is enough to show that $x \perp_{2-HH-B} y \Rightarrow \langle x, y \rangle = 0 \Rightarrow x \perp_{2-HH-R} y$.

Let $x \perp_{2-HH-B} y$. Then for any $\lambda \in \mathbb{K}$,

$$\begin{aligned}
&\int_0^1 \|(1-t)x + \lambda t y\|^2 dt \geq \int_0^1 \|(1-t)x\|^2 dt \\
&\Rightarrow \int_0^1 \|(1-t)x\|^2 dt + [\lambda\langle y, x\rangle + \overline{\lambda\langle y, x\rangle}] \int_0^1 t(1-t)dt \\
&+ |\lambda|^2\|y\|^2 \int_0^1 t^2 dt \geq \int_0^1 \|(1-t)x\|^2 dt \\
&\Rightarrow Re[\lambda\langle y, x\rangle] + |\lambda|^2\|y\|^2 \geq 0 \qquad (2.11)
\end{aligned}$$

Now, for $\lambda = \frac{-\langle x,y\rangle}{2\|y\|^2}$, inequality (2.11) becomes $\frac{-|\langle y,x\rangle|^2}{4\|y\|^2} \geq 0$. Therefore, we have $\langle x, y\rangle = 0$.

On the other hand it is easy to show that $\langle x, y\rangle = 0 \Rightarrow x \perp_{2-HH-R} y$.

$\square$

2.4 Isosceles orthogonality

Definition 10. *[1] An element x of a normed linear space X is orthogonal to $y \in X$ in the sense of isosceles if and only if $\|x - y\| = \|x + y\|$.*

For ordinary Euclidean space, the analogy of this definition is two vectors are perpendicular if and only if their sum and difference can be sides of an isosceles triangle.

Properties of isosceles orthogonality:

(i) Isosceles orthogonality satisfy non-degeneracy, simplification and continuity;

(ii) Isosceles orthogonality is symmetric;

(iii) Isosceles orthogonality is unique if and only if the space is strictly convex;

(iv) If the isosceles orthogonality is homogeneous (additive)in X, then X is an inner-product space.

Theorem 2.4.1. *[1] Let $(X, \|.\|)$ be a normed space and $x, y \in X$. Then there exists a number α such that*

$$\|x + (\alpha x + y)\| = \|x - (\alpha x + y)\|$$

Theorem 2.4.2. *[1] If isosceles orthogonality is homogeneous in a normed linear space X, then X is an abstract Euclidean space.*

Theorem 2.4.3. *[1] If isosceles orthogonality is additive in a normed linear space X, then X is an abstract Euclidean space.*

Corollary 2.4.1. *[1] The properties of homogeneity and additivity of isosceles orthogonality are equivalent for a normed linear space X. If the isosceles orthogonality is homogeneous or additive, then for any element x and y there is a unique number α for which $x \perp \alpha x + y$.*

In the paper [32], Kapoor and Prasad discussed about the unique property of isosceles orthogonality and proved that the isosceles orthogonality is unique if and only if the underlying space is strictly convex. The details of uniqueness property was discussed on Alonso's paper [16]. In 1994, Alonso mentioned in the paper [33] that the isosceles orthogonality is said to be unique in a normed linear space whose dimension is at least two if for each two dimensional subspace X_1, and $x \in X_1 \setminus \{0\}$, and each number $\xi > 0$, there exist a unique point (except for the sign) $y \in \xi S_{x_1}$ such that $x \perp_I y$ and it is said to be $\lambda-$unique if for each point $x \neq 0$ and each point y, there exist a unique number λ such that $x \perp_I \lambda x + y$. Ji et al. elaborate the Alonso's results on uniqueness of isosceles orthogonality and proved some important relations related to isosceles orthogonality. The most important concept they proved that, if the isosceles orthogonality is not $\lambda-$unique, then it is not unique and if isosceles orthogonality is not unique, then it is not $\lambda-$unique. Therefore, the uniqueness and $\lambda-$uniqueness if isosceles orthogonality is equivalent [34]

Definition 11. *[34] A real finite dimensional normed linear space is called Minkowski space.If the dimension of Minkowski space is two, then it is called a Minkowski plane. plane.*

Theorem 2.4.4. *[34] Let X be Minkowski or normed plane. If there exists x_1, y_1, y_2 such that $y_1 \neq y_2$, $\|y_1\| = \|y_2\|$, $x \perp_I y_1$ and $x \perp_I y_2$, then the following relations hold:*

$$\|x + y_1\| = \|x - y_1\| = \|x + y_2\| = \|x - y_2\| = \|y_1\| = \|y_2\| \tag{2.12}$$

[34] For any two distinct points $x, y \in X$, the line segment passes through x and y is denoted by (x, y) and the segment between x and y by $[x, y]$. For any point $x \in X \setminus \{0\}$, the maximum of the length of non-trivial segment contained in the unit sphere S_X and

parallel to the line $(-x, x)$ is denoted by M_x. i.e

$$M_x = \sup\left\{\|a-b\| : [a,b] \subseteq S_x, a \neq b, \text{and } \frac{a-b}{\|a-b\|} = \frac{x}{\|x\|}\right\}$$

If there is no non-trivial segment contained in S_X and parallel to the line $(-x, x)$, then $M_x = 0$ [34].

Theorem 2.4.5. *[34] Let X be a Minkowski (or normed plane) and $x \in X$ satisfying $\|x\| > 0$. Then for each number $s \in [0, \frac{2\|x\|}{M_x}]$ $\quad (s \in [0, +\infty$ when $M_x = 0)$, then there exist a unique point $y \imath s S_x$ (except for the sign) such that $x \perp_i y$.*

Corollary 2.4.2. *[34] Let X be a Minkowski (or normed) plane and $x \in X$ satisfying $\|x\| > 0$. Then for each number $0 \leq r \leq \|x\|$, there exists a unique point $y \in rS_X$ (except for the sign) such that $x \perp_I y$.*

Corollary 2.4.3. *[34] Let X be a strictly convex Minkowski(or normed) plane and $x \in X$ satisfying $\|x\| > 0$. Then for any $r \in [0, +\infty)$, there exists a unique point $y \in rS_X$ (except for the sign) such that $x \perp_I y$.*

Theorem 2.4.6. *[34] Let $(X, \|.\|)$ be a normed linear space with $\dim X \geq 2$ and $x, y \in X$ satisfying $\|x\| > 0$ and $0 \leq \|y\| \leq \frac{2\|x\|}{M_x(y)}$ ($0 \leq \|y\| < +\infty$ when $M_x(y) = 0$). Then there exists a unique real number α such that $x \perp_\alpha x + y$.*

Ji and Wu introduced a new geometry constant $D(X)$ to give a quantitative characterization of the difference between Birkhoff-orthogonality and isosceles orthogonality [35]. They showed that 1 and $2(\sqrt{2}-1)$ are the upper and lower bound for D(X), respectively, and characterize the space of which D(X) attains the lower and upper bounds.

Theorem 2.4.7. *[35] For any real normed linear space X with $\dim(X) \geq 2$, $2(\sqrt{2}-1) \leq D(X) \leq 1$ and $D(X) = 1$ if and only if X is Euclidean.*

Theorem 2.4.8. *[35] For any real Banach Space X with $\dim(X) \geq 2$, there exists $e_1, e_2 \in S(X)$ such that $e_1 \perp_I e_2$ and $\inf_{\lambda \in \mathbb{R}}\{\|e_1 + e_2\|\} = 2(\sqrt{2}-1)$ if and only if there exists a two dimensional subspace X_0 of X and $x_0 \in S(X_0)$ such that x_0 is the common end point of two segments of which the length are not less than $\sqrt{2}$.*

Theorem 2.4.9. *[35] Let X be a symmetric Minkowski plane and $\{e_1, e_2\}$ be a pair of axis of X. Then*

(i) $e_1 \perp_R e_2$,

(ii) $e_1 \perp_I e_2$,

(iii) e_1 *and* e_2 *is a pair of conjugate diameters of X,*

(iv) X^* *is also symmetric Minkowski plane and* $\{e_1^*, e_2^*\}$ *is a pair of axes of* X^**, where* e_1^*, e_2^* *is the supporting functional of* e_1, e_2 *respectively.*

Theorem 2.4.10. *[35] Let X be a symmetric Minkowski plane,* e_1, e_2 *be a pair of axes of X, then for all* $x, y \in S(X)$*,* $x = \alpha e_1 + \beta e_2$*,* $x \perp_I y$ *if and only if* $y = \pm(-\beta e_1 + \alpha e_2)$*.*

In 2010, Dragomir and Kikianty came up with new generalization of isosceles orthogonality in terms of 2-HH norm. They gave the definition of orthogonality as follows:

Definition 12. *[7] Let* $(X, \|.\|)$ *be normed space, and* $x, y \in X$*. Then x is said to be isosceles orthogonal to y in terms of 2-HH norm if and only if*

$$\int_0^1 \|(1-t)x + ty\|^2 dt = \int_0^1 \|(1-t)x - ty\|^2 dt.$$

This orthogonality is equivalent to the usual orthogonality, if the space is equipped with an inner-product.

Theorem 2.4.11. *[7] Let X be a normed space. Then the isosceles orthogonality via 2-HH norm is existent.*

Theorem 2.4.12. *[7] Let X be a normed space. Then isosceles orthogonality is unique if and only if X is strictly convex*

Theorem 2.4.13. *[7] Isosceles orthogonality via 2-HH norm is homogeneous in a normed space X if and only if X is an inner product space.*

Theorem 2.4.14. *[36] Let* X *be a normed linear space with the norm induced by an inner-product. Then, the isosceles orthogonality via* $2 - HH$ *norm satisfies the non-degeneracy, continuity and symmetry property.*

Proof. **Non-degeneracy:** If $x \perp_{2-HH-I} y$, then

$$\begin{aligned}
& \int_0^1 \|(1-t)x + tx\|^2 dt = \int_0^1 \|(1-t)x - tx\|^2 dt \\
\Rightarrow \quad & \int_0^1 \|x\|^2 dt = \int_0^1 \|x\|^2 (1-2t)^2 dt \\
\Rightarrow \quad & \|x\|^2 = \frac{2}{3}\|x\|^2 \\
\Rightarrow \quad & \frac{1}{3}\|x\|^2 = 0 \\
\Rightarrow \quad & x = 0. \qquad (2.13)
\end{aligned}$$

Continuity: Let $x_n \to x, y_n \to y$ for all n, and $x_n \perp_{2-HH-I} y_n$. Then,

$$\begin{aligned}
& \int_0^1 \|(1-t)x_n + ty_n\|^2 dt = \int_0^1 \|(1-t)x_n - ty_n\|^2 dt \\
\Rightarrow \quad & \lim_{n\to\infty} \int_0^1 \|(1-t)x_n + ty_n\|^2 dt = \lim_{n\to\infty} \int_0^1 \|(1-t)x_n - ty_n\|^2 dt \\
\Rightarrow \quad & \int_0^1 \|(1-t)x + ty\|^2 dt = \int_0^1 \|(1-t)x - ty\|^2 dt \\
\Rightarrow \quad & x \perp_{2-HH-I} y. \qquad (2.14)
\end{aligned}$$

Symmetry: If $x \perp_{2-HH-I} y$, then

$$\begin{aligned}
& \int_0^1 \|(1-t)x + ty\|^2 dt = \int_0^1 \|(1-t)x - ty\|^2 dt \\
\Rightarrow \quad & \frac{1}{3}(\|x\|^2 + \|y\|^2) = \frac{1}{3}(\|x\|^2 + \|y\|^2) \\
\Rightarrow \quad & \frac{1}{3}(\|y\|^2 + \|x\|^2) = \frac{1}{3}(\|y\|^2 + \|x\|^2) \\
\Rightarrow \quad & \int_0^1 \|(1-t)y + tx\|^2 dt = \int_0^1 \|(1-t)y - tx\|^2 dt \\
\Rightarrow \quad & y \perp_{2-HH-I} x. \qquad (2.15)
\end{aligned}$$

□

Kikianty and Dragomir (2010), proved that the homogeneity and additivity of the Pythagorean orthogonality via $2-HH$ norm is equivalent; however, they also stated the similar result about the isosceles orthogonality with respect to $2-HH$ norm, by omitting the proof. In the following theorem, we give complete proof regarding the equivalency of homogeneity and additivity of the $HH - I$ orthogonality.

Theorem 2.4.15. *[36] Let $x \perp_{2-HH-I} y$. Then, the following are equivalent:*

1. *Isosceles orthogonality via $2 - HH$ norm is homogeneous.*
2. *Isosceles orthogonality via $2 - HH$ norm is additive.*

Proof. $(1) \Rightarrow (2)$. Assume that the isosceles orthogonality via $2 - HH$ norm is homogeneous. We shall show that it is additive. As $2 - HH - I$ orthogonality is homegeneous in a normed sapce X if and only if X is an inner product space, and therefore it is additive. $(2) \Rightarrow (1)$. Conversely assume that additive property holds and $x \perp_{2-HH-I} y$. Since, $2 - HH - I$ orthogonality exists, for any $x, -y$ there exists a $\beta \in \mathbb{R} : x \perp_{2-HH-I} \beta x - y$, and by additivity property, we conclude that $x \perp_{2-HH-I} \beta x$. Hence, $\beta = 0$ whenever $x \neq 0$ and therefore $x \perp_{2-HH-I} -y$. Again by the symmetry and additivity property of isosceles orthogonality via $2 - HH$ norm, we may conclude that $px \perp_{2-HH-I} qy$ for all integers p and q. When $p \neq 0$,

$$\begin{aligned} \int_0^1 \|(1-t)x + t(\frac{q}{p})y\|^2 dt &= \frac{1}{3}(\|x\|^2 + \frac{q^2}{p^2}\|y\|^2) \\ &= \int_0^1 \|(1-t)x - t\frac{q}{p}y\|^2 dt. \end{aligned} \tag{2.16}$$

This shows that $x \perp_{2-HH-I} ky$ for some $k \in \mathbb{Q}$, and by using the continuity of norm, $x \perp_{2-HH-I} ky$ for any real k. Again $2 - HH - I$ orthogonality is symmetric, and therefore we may conclude that it is homogeneous. $\square$

Definition 13. *[14] Let H be a real or complex Hilbert space and $T_1, T_2 \in B(H)$. Then two operators T_1 and T_2 have disjoint support if and only if $T_1 T_2^* = T_2^* T_1 = 0$.*

Bottazi et al. studied the isosceles orthogonality of bounded (positive) linear operators on Hilbert space with some of the related properties, including operators having disjoint support. [14] Let T_1 and T_2 are norm attaining bounded linear operators in a Banach space X. Then, T_1 is said to be isosceles orthogonal to T_2 if for every $h \in M_T$,

$$\|(T_1 - T_2)(h)\| = \|(T_1 + T_2)(h)\|. \tag{2.17}$$

Also, in the same paper, the Pythagorean orthogonal for operators was defined as follows: T_1 is said to be Pythagorean orthogonal to T_2 if for every $h \in M_T$,

$$\|(T_1 - T_2)(h)\|^2 = \|T_1\|^2 + \|T_2\|^2 \text{or} \|(T_1 + T_2)(h)\|^2 = \|T_1\|^2 + \|T_2\|^2. \tag{2.18}$$

Theorem 2.4.16. *[36] Let T_1 and T_2 be norm attaining bounded linear operators with the disjoint support in a Hilbert space H. Then, T_1 is isosceles orthogonal to T_2 if and only if T_1 is Pythagorean orthogonal to T_2.*

Proof. Let $T_1, T_2 \in B(H)$. Assume T_1 is isosceles orthogonal to T_2 and $h \in M_T$. Using the relation (2.21), we have

$$\begin{aligned} \|(T_1 - T_2)(h)\|^2 &= \|(T_1 + T_2)(h)\|^2 \\ &= \|T_1(h)\|^2 + \|T_2(h)\|^2 + 2Re\langle T_1T_2^*h, h\rangle \\ &= \|T_1(h)\|^2 + \|T_2(h)\|^2 \\ &= \|T_1\|^2 + \|T_2\|^2. \end{aligned} \tag{2.19}$$

This shows that T_1 is Pythagorean orthogonal to T_2. Conversely assume that, for any $h \in M_T$,

$$\begin{aligned} \|(T_1 - T_2)(h)\|^2 &= \|T_1(h)\|^2 + \|T_2(h)|^2 \\ &= \|(T_1 + T_2)(h)\|^2. \end{aligned}$$

Therefore,

$$\|(T_1 - T_2)(h)\| = \|(T_1 + T_2)(h)\|.$$

That is,

$$\|T_1 - T_2\| = \|T_1 + T_2\|. \tag{2.20}$$

□

Proposition 8. *[36] Let T_1 and T_2 be bounded linear operators form a Banach space X to Y. Then for any $x \in M_{T_1+T_2} \cap M_{T_1-T_2}$, $T_1(x) \perp_I T_2(x) \Rightarrow T_1 \perp_I T_2$.*

Proof. Let $x \in M_{T_1+T_2} \cap M_{T_1-T_2}$. Suppose $T_1(x) \perp_I T_2(x)$. Then,

$$\begin{aligned} \|T_1(x) - T_2(x)\| &= \|T_1(x) + T_2(x)\| \\ \Rightarrow \quad \|(T_1 - T_2)x\| &= \|(T_1 + T_2)x\| \\ \Rightarrow \quad \|T_1 - T_2\| &= \|T_1 + T_2\| \end{aligned}$$

□

Proposition 9. *[36] Let T_1 and T_2 be bounded linear operators form a Banach space X to*

Y and $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then for any $x \in M_{T_1+T_2} \cap M_{T_1-T_2}$

$$T_1(x_n) \perp_I T_2(x_n) \Rightarrow T_1 \perp_I T_2$$

Proof. Let $x \in M_{T_1+T_2} \cap M_{T_1-T_2}$. Suppose $T_1(x_n) \perp_I T_2(x_n)$. Then,

$$\begin{aligned} \|T_1(x_n) - T_2(x_n)\| &= \|T_1(x_n) + T_2(x_n)\| \\ \Rightarrow \quad \lim_{n\to\infty} \|T_1(x) - T_2(x_n)\| &= \lim_{n\to\infty} \|T_1(x_n) + T_2(x_n)\| \\ \Rightarrow \quad \|T_1(x_n) - T_2(x)\| &= \|T_1(x) + T_2(x)\| \\ \Rightarrow \quad \|T_1 - T_2\| &= \|T_1 + T_2\| \end{aligned}$$

□

2.5 Pythagorean Orthogonality

Definition 14. *[1] Let $(X, \|.\|)$ be normed linear space and $x, y \in X$. An element x is said to be orthogonal to y if and only if*

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

In ordinary Euclidean space, this is analogous to the fact that two vectors are perpendicular if and only if there is right triangle having two vectors as legs. James mentioned in the paper [1] that, in case of normed linear space, Pythagorean and isosceles orthogonality are not equivalent.

Properties of Pythagorean orthogonality [16]

(i) Pythagorean orthogonality is symmetric;

(ii) Pythagorean orthogonality is either homogeneous or additive if and only if the norm is induced by an inner-product;

(iii) Pythagorean orthogonality is unique;

(iv) Pythagorean orthogonality is $\alpha-$unique;

(v) If $X = \mathbb{R}^2$ with a regular octagon as unit sphere then the Pythagorean orthogonality is $S-$unique, however it is not $S-$ unique when S is an square.

Definition 15. *[16] Let $(X, \|.\|)$ be a normed space and $0 \neq x, y \in X$. Then the orthogonality is said to be left (right) unique if there exists only one α such that $x \perp \alpha x + y (\alpha x + y \perp x)$.*

In the case of Pythagorean orthogonality, James proved that the left and right uniqueness are equivalent.

Theorem 2.5.1. *[1] Let $(X, \|.\|)$ be a normed space and $x, y \in X$. Then there exists a number α such that*

$$\|x - (\alpha x + y)\|^2 = \|x\|^2 + \|\alpha x + y\|^2.$$

Corollary 2.5.1. *[1] Let $(X, \|.\|)$ be a normed space and $x, y \in X$. Then there exists a number α such that*

$$\|x + (\alpha x + y)\|^2 = \|x\|^2 + \|\alpha x + y\|^2.$$

Theorem 2.5.2. *[1] If Pythagorean orthogonality is homogeneous in a normed linear space X, then X is an abstract Euclidean space.*

Theorem 2.5.3. *[1] The property of homogeneity and additivity of Pythagorean orthogonality are equivalent for normed linear space.*

Corollary 2.5.2. *[1] If Pythagorean orthogonality is additive in a normed linear space X, then X is an abstract Euclidean space.*

Theorem 2.5.4. *[1] Let $(X, \|.\|)$ be a normed linear space and $x, y \in X$. Then the following are equivalent:*

(i) X is an inner product space;

(ii) x is Pythagorean orthogonal to y implies x is isosceles orthogonal to y;

(iii) x is isosceles orthogonal to y implies x is Pythagorean orthogonal to y.

Theorem 2.5.5. *[1] Let $(X, \|.\|)$ be a normed linear space and $x, y \in X$. Then the following are equivalent:*

(i) X is an inner product space;

(ii) x is Pythagorean orthogonal to y implies x is Birkhoff-James orthogonal to y;

(iii) x is Birkhoff-james orthogonal to y implies x is Pythagorean orthogonal to y.

Dragomir and Kikianty introduced Pythagorean orthogonality in terms of $2-HH$ norm.

Definition 16. *[7] Let $(X, \|.\|)$ be normed linear space. An element $x \in X$ is said to be Pythagorean orthogonal to y in terms of $2-HH$ norm if and only if*

$$\int_0^1 \|(1-t)x+ty\|^2 dt = \frac{1}{3}(\|x\|^2+\|y\|^2).$$

This orthogonality is equivalent to usual orthogonality, when the space is equipped with an inner-product.

Proposition 10. *[7] Let $(X, \|.\|)$ be normed linear space and $x, y \in X$ such that $(1-t)x$ is Pythagorean orthogonal to ty for almost every $t \in [0,1]$, then $x \perp_{HH-P} y$.*

Proposition 11. *[7] Let $(X, \|.\|)$ be a normed space. Then the Pythagorean orthogonality via 2-HH norm is existent.*

Theorem 2.5.6. *[7] Pythagorean orthogonality via 2-HH norm is unique in any normed space X.*

Dragomir and Kikianty mentioned the following theorem in the paper [7] without giving detailed proof. In this thesis we make an attempt to give the detailed proof.

Theorem 2.5.7. *[36] Let X be a normed linear space with the norm induced by an inner-product. Then, the Pythagorean orthogonality via $2-HH$ norm satisfies the non-degeneracy, continuity and symmetry property.*

Proof. **Non-degeneracy:** If $x \perp_{2-HH-p} x$, then

$$\begin{aligned}
& \int_0^1 \|(1-t)x+tx\|^2 dt = \frac{1}{3}(\|x\|^2+\|x\|^2) \\
\Rightarrow \quad & \|x\|^2 = \frac{2}{3}\|x\|^2 \\
\Rightarrow \quad & \frac{1}{3}\|x\|^2 = 0 \\
\Rightarrow \quad & x = 0.
\end{aligned}$$

Continuity: Let $x_n \to x, y_n \to y$ and $x_n \perp_{2-HH-p} y_n$. Then,

$$
\begin{aligned}
& \int_0^1 \|(1-t)x_n + ty_n\|^2 dt = \frac{1}{3}(\|x\|_n^2 + \|y_n\|^2) \\
\Rightarrow \quad & \lim_{n\to\infty} \int_0^1 \|(1-t)x_n + ty_n\|^2 dt = \lim_{n\to\infty} \frac{1}{3}(\|x_n\|^2 + \|y_n\|^2) \\
\Rightarrow \quad & \int_0^1 \|(1-t)x + ty\|^2 dt = \frac{1}{3}(\|x\|^2 + \|y\|^2) \\
\Rightarrow \quad & x \perp_{2-HH-P} y.
\end{aligned}
$$

Symmetry: If $x \perp_{2-HH-P} y$, then

$$
\begin{aligned}
\|(x,y)\|_{2-HH} =& \frac{1}{3}(\|x\|^2 + \|y\|^2) \\
=& \frac{1}{3}(\|y\|^2 + \|x\|^2) \\
=& \int_0^1 \|(1-t)y + tx\|^2 dt \\
=& \|(y,x)\|_{2-HH}.
\end{aligned}
$$

□

In the case of norm attaining bounded linear operators, when the norm is induced by an inner-product, we introduce a relation between the isosceles and Pythagorean orthogonalities by stating that the operator T_1 is isosceles orthogonal to T_2 if and only if T_1 is Pythagorean orthogonal to T_2. Furthermore, we prove an interesting relation that the Pythagorean orthogonality implies the Birkhoff-James orthogonality, but the converse may not be true. To disprove this statement, we take two dimensional matrix operators on a Hilbert space whose norm is obtained by taking trace of the product of the adjoint of an operator with the operator itself.

Bottazi et al. (2020), studied the isosceles orthogonality of bounded (positive) linear operators on Hilbert space with some of the related properties, including operators having disjoint support. Let T_1 and T_2 are norm attaining bounded linear operators in a Banach space X. Then, T_1 is said to be isosceles orthogonal to T_2 if for every $h \in M_T$,

$$\|(T_1 - T_2)(h)\| = \|(T_1 + T_2)(h)\|. \tag{2.21}$$

Also, in the same paper, the Pythagorean orthogonal for operators was defined as follows:

T_1 is said to be Pythagorean orthogonal to T_2 if for every $h \in M_T$,

$$\|(T_1 - T_2)(h)\|^2 = \|T_1\|^2 + \|T_2\|^2 \text{or} \|(T_1 + T_2)(h)\|^2 = \|T_1\|^2 + \|T_2\|^2. \tag{2.22}$$

The following theorem gives the characterization of Birkhoff-James orthogonality for operators.

Theorem 2.5.8. *[36] Let T_1 and T_2 be norm attaining bounded linear operators in a Hilbert space H. If T_1 is Pythagorean orthogonal to T_2, then T_1 is Birkhoff-James orthogonal to T_2, but the converse may not be true*

Proof. Let $T_1, T_2 \in B(H)$ such that T_1 is Pythagorean orthogonal to T_2. Then, by using (2.22) with $h \in M_T$,

$$\begin{aligned} & \|(T_1 + \lambda T_2)(h)\|^2 = \|T_1(h)\|^2 + \|\lambda T_2(h)\|^2 \\ \Rightarrow \quad & \|(T_1 + \lambda T_2)(h)\|^2 \geq \|T_1(h)\|^2 \\ \Rightarrow \quad & \|(T_1 + \lambda T_2)(h)\| \geq \|T_1(h)\|^2 \\ \Rightarrow \quad & \|T_1 + \lambda T_2\| \geq \|T_1\|. \end{aligned} \tag{2.23}$$

This shows that T_1 is Birkhoff-James orthogonal to T_2. $\square$

The following example shows that the converse of above theorem may not be true.

Example 2. *Suppose that H is the two dimensional Hilbert space. Consider the Banach space $B(H)$. Let $T_1 = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$, $T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in B(H)$ and $\lambda \in [0, 1]$. We know that $\|T\|^2 = trace(T^*T)$. Then,*

$$\begin{aligned} T_1 + \lambda T_2 &= \begin{pmatrix} 4+\lambda & 0 \\ 0 & 3+\lambda \end{pmatrix}, \\ (T_1 + \lambda T_2)^* &= \begin{pmatrix} 4+\lambda & 0 \\ 0 & 3+\lambda \end{pmatrix}, \\ (T_1 + \lambda T_2)^*(T_1 + \lambda T_2) &= \begin{pmatrix} (4+\lambda)^2 & 0 \\ 0 & (3+\lambda)^2 \end{pmatrix}. \end{aligned}$$

Thus,

$$\textit{trace}[(T_1 + \lambda T_2)^*(T_1 + \lambda T_2)] = (4+\lambda)^2 + (3+\lambda)^2. \tag{2.24}$$

Since, $\lambda \in [0,1]$,

$$\begin{aligned} & \min[\textit{trace}((T_1+\lambda T_2)^*(T_1+\lambda T_2))] \geq 25 \\ \Rightarrow \quad & \|T_1+\lambda T_2\|^2 \geq 25 \\ \Rightarrow \quad & \|T_1+\lambda T_2\| \geq 5. \end{aligned} \tag{2.25}$$

Similarly we can find $\|T_1\| = 5$. *Therefore, we may conclude that* T_1 *is Birkhoff-James orthogonal to* T_2. *On the other hand,*

$$T_1+T_2 = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix} \textit{and}[(T_1+T_2)^*(T_1+T_2)] = \begin{pmatrix} 25 & 0 \\ 0 & 16 \end{pmatrix}.$$

It follows that $\|T_1+T_2\|^2 = 41$. *However,* $\|T_1|^2 + \|T_2\|^2 = 26$. *Therefore,* T_1 *is not Pythagorean orthogonal to* T_2.

2.6 Carlsson Orthogonality

S. O. Carlsson in 1962 introduced a generalized concept of orthogonality in normed space with indicating that isosceles and Pythagorean orthogonality are special cases [4].

Definition 17. *Let* $(X, \|.\|)$ *be a normed linear space and* $a_k, b_k, c_k, k = 1, ...m$, *a fixed collection of real numbers satisfying*

$$\sum_{k=1}^{m} a_k b_k c_k = 1, \sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0.$$

An element $x \in X$ *is said to be orthogonal to* $y \in X$ *if*

$$\sum_{k-1}^{m} a_k \|b_k x + c_k y\|^2 = 0$$

Theorem 2.6.1. *[4] Let* $(X, \|.\|)$ *be a normed space and* $x, y \in X$. *Then there is a number* α *such that x is Carlsson orthogonal to* $\alpha x + y$.

Definition 18. *[4] Orthogonality is said to have property (H) in normed linear space X if* $x \perp_C y$ *implies that*

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{m} a_k \|n b_k x + c_k y\|^2 = 0 \quad \textit{, where n is a positive integer}$$

Definition 19. *[4] Let $(X, \|.\|)$ be a normed space. The norm on X is said to be Gateaux differentiable if, for every pair of elements $0 \neq x, y \in X$, the limit*

$$N(x, y) = \lim_{h \to 0} \frac{\|x + hy\| - \|x\|}{h} \quad \textit{exists}.$$

Theorem 2.6.2. *[4] If the Carlsson orthogonality has property (H) in X, then the norm of X is Gateaux differentiable and $x \perp_C y$ holds if and only if $N(x, y) = 0$.*

Definition 20. *[4] Let $(X, \|.\|)$ be a normed space and $x, y \in X$. Then x is said to be anti-orthogonal to y if y is orthogonal to x or if and only if*

$$\sum_{K=1}^{m} a_k \|c_k x + b_k y\|^2 = 0.$$

Carlsson proved that if orthogonality has property (H), then it is equivalent to normality and therefore homogeneous. In this regards, it is better to say that anti-orthogonality has property (H). If the orthogonality is replaced by anti-orthogonality, then there is a number α such that $x \perp C\alpha x + y$ and the norm of X is Gateaux differentiable and $x \perp_C y$ fi and only if $N(x, y) = 0$.

Theorem 2.6.3. *[4] If the Carlsson orthogonality has property (H), then it is symmetric and equivalent to normality in X.*

Corollary 2.6.1. *[4] If $\dim X \geq 2$ and the Carlsson orthogonality is homogeneous or additive in X, then X is Euclidean space.*

Theorem 2.6.4. *[4] If X is a normed linear space in which the Carlsson orthogonality has property has property (H), then X is an Euclidean space.*

Corollary 2.6.2. *[4] If X is normed linear space in which the Carlsson orthogonality is homogeneous or additive, then X is Euclidean space.*

Theorem 2.6.5. *Let $a_k \neq 0, b_k, c_k, k = 1, ..., m$, be real numbers such that (b_k, c_k) and (b_r, c_r) are linearly independent for $k \neq r$. If $(X, \|.\|)$ be a normed linear space satisfying the condition*

$$\sum_{k=1}^{m} a_k \|b_k x + c_k y\|^2 = 0 \quad \textit{for all } x, y \in X.$$

Then X is an Euclidean space.

2.6.1 Carlsson Orthogonality in terms of p-HH norm

In 2010, E. Kikianty and S.S. Dragomir played a vital role to introduce $p - HH$ norm on the Cartesian square of normed spaces by generalizing the previous definition of Carlsson orthogonality through the medium of $2 - HH$ norm, which also generalizes the Pythagorean and isosceles orthogonality through te medium of $2 - HH$ [37].

Definition 21. *[37] Let $(X, \|.\|)$ be a normed linear space and a_k, b_k, c_k, $k = 1, 2, 3, ..., m$, $m \in \mathbb{N}$ be real numbers. An element $x \in X$ is said to be Carlsson orthogonal to y in terms of 2-HH norm if and only if*

$$\sum_{k=1}^{m} a_k \int_0^1 \|(1-t)b_k x + tyc_k\|^2 dt = 0$$

with conditions

$$\sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0 \text{ and } \sum_{k=1}^{m} a_k b_k c_k = 1$$

In any inner product space, the Carlsson orthogonality via 2-HH norm is equivalent to the usual orthogonality.

Theorem 2.6.6. *[37] Carlsson orthogonality in terms of $2 - HH$ norm satisfies the non-degeneracy, simplification and continuity.*

Theorem 2.6.7. *[37] Let $(X, \|.\|)$ be a normed space. Then the Carlsson orthogonality via 2-HH norm is existent.*

Theorem 2.6.8. *[37] Let $(X, \|.\|)$ be a normed space in which the Carlsson orthogonality in terms of 2-HH norm is homogeneous (or additive) to the left. Then X is an inner product space.*

Definition 22. *[37] The Carlsson orthogonality in terms of 2-HH norm is said to have property (H) in a normed space X, if x is Carlsson orthogonal to y implies that*

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \sum_{k=1}^{m} a_k \|nb_k(1-t)x + c_k ty\|^2 dt = 0$$

If the Carlsson orthogonality via 2-HH norm is homogeneous (or additive to the left) in X, then it has property (H) and if X is an inner product space, then the Carlsson orthogonality in terms of 2-HH norm is homogeneous (or additive) and therefore it has (H).

Lemma 2.6.1. *[37] Let $(X, \|.\|)$ be a normed space where the Carlsson orthogonality in terms of 2-HH norm has property (H). Suppose that for any $x, y \in X$, there exist $\alpha \in \mathbb{R}$ such the $x \perp_{HH-C} \alpha x + y$. Then*

$$\alpha = \frac{-1}{\|x\|}[\sum_{b_k c_k > 0} a_k b_k c_k N_+(x, y) + \sum_{b_k c_k < 0} a_k b_k c_k N_-(x, y)]$$

Corollary 2.6.3. *[37] If the Carlsson orthogonality via 2-HH norm has property (H), then the norm of X is Gateaux differentiable at x and $x \perp_{HH-C} y$ holds if and only if $N(x, y) = 0$.*

Corollary 2.6.4. *[37] If the Carlsson orthogonality via 2-HH norm has property (H), then it is symmetric and equivalent to B-orthogonality.*

Bottazzi et al. in the paper [14] discussed about Birkhoff-James, isosceles, and Robert orthogonality in Banach spaces in terms of bounded linear operators. Motivated by the results of Bottazzi et al. (2020), we make an attempt to introduce the Carlsson orthogonality for bounded linear operators in Banach Spaces. Furthermore, we verify some properties, like non-degeneracy, continuity, and homogeneity property of an inner-product space in the context of the Carlsson orthogonality in terms of bounded linear operators.

Definition 23. *[36] Let T_1 and T_2 are bounded linear operators on X. Then, the operator T_1 is orthogonal to T_2 in the sense of Carlsson (denoted by $T_1 \perp_C^O T_2$) if for any $h \in M_T$,*

$$\sum_{k=1}^{n} p_k \|(q_k T_1 + r_k T_2)(h)\|^2 = 0, \tag{2.26}$$

satisfying the conditions

$$\sum_{k=1}^{n} p_k q_k r_k = 1, \sum_{k=1}^{n} p_k q_k^2 = \sum_{k=1}^{n} p_k r_k^2 = 0. \tag{2.27}$$

Theorem 2.6.9. *[36] Let T_1 and T_2 be norm attaining bounded linear operators on a Banach space X. If $T_1 = T_2 = T$, then*

$$\sum_{k=1}^{n} p_k \|(q_k T_1 + r_k T_2)(h)\|^2 = 0 \Leftrightarrow T = 0.$$

Proof. Let $T_1, T_2 \in B(X)$ and $h \in M_T$. Assume(2.26) under the condition (2.27). Since,

$T_1 = T_2 = T$, we have

$$
\begin{aligned}
& \sum_{k=1}^{n} p_k \|(q_k T + r_k T)(h)\|^2 = 0 \\
\Rightarrow \quad & \sum_{k=1}^{n} p_k |q_k + r_k|^2 \|Th\|^2 = 0 \\
\Rightarrow \quad & \|Th\|^2 = 0 \\
\Rightarrow \quad & \|Th\| = 0.
\end{aligned}
$$

As $h \in M_T$, we can write $\|Th\| = \|T\|$. Therefore, we may conclude that

$$\|T\| = 0 \Rightarrow T = 0. \tag{2.28}$$

The converse part is obvious. □

Theorem 2.6.10. *[36] Let $\{U_n\}$ and $\{V_n\}$ be sequences of norm attaining bounded linear operators on a Banach space X. Then,*

$$U_n \perp_C^O V_n \Rightarrow U \perp_C^O V.$$

Proof. In the case of linear operators, boundedness and continuity are equivalent. By the continuity of the U_n's and V_n's, we can write

$$\lim_{n\to\infty} U_n(h) = U(h) \quad \text{and} \quad \lim_{n\to\infty} V_n(h) = V(h).$$

Since, U_n is Carlsson orthogonal to V_n and $h \in M_T$, we have

$$\sum_{k=1}^{n} p_k \|(q_k U_n + r_k V_n)(h)\|^2 = 0 \tag{2.29}$$

under the condition (2.27). It follows that

$$
\begin{aligned}
& \lim_{n\to\infty} \sum_{k=1}^{n} p_k \|(q_k U_n + r_k V_n)(h\|^2 = 0 \\
\Rightarrow \quad & \sum_{k=1}^{n} p_k \|(q_k U + r_k V)(h)\|^2 = 0. \\
\Rightarrow \quad & U \perp_C^O V.
\end{aligned}
$$

Therefore, the Carlsson orthogonality satisfies continuity property of an inner-product space. $\square$

Bottazzi et al, (2020) defined the disjoint support as follows: Let H be a real or complex Hilbert space and $T_1, T_2 \in B(H)$. Then, two operators T_1 and T_2 have disjoint support if and only if

$$T_1 T_2^* = T_2^* T_1 = 0. \tag{2.30}$$

Theorem 2.6.11. *[36] Let T_1 and T_2 be norm attaining bounded linear operators on a Hilbert space H with disjoint support. Then, $T_1 \perp_C^O T_2$ implies that $T_2 \perp_C^O T_1$.*

Proof. Let $T_1, T_2 \in B(H)$ and $h \in M_T$. Then, $\|(T_1 + T_2)(h)\| = \|T_1 + T_2\|$. Since, T_1 and T_2 have disjoint support then, equation (2.30) holds. Suppose $T_1 \perp_C^O T_2$. Then, for $h \in M_T$, Equation (2.26) under Condition (2.27) can be written as

$$\sum_{k=1}^{n} p_k[\langle q_k T_1 h + r_k T_2 h, q_k T_1 h + r_k T_2 h\rangle] = 0.$$

It follows that

$$\begin{aligned}
& \sum_{k=1}^{n} a_k[\|q_k T_1\|^2 + \|r_k T_2 h\|^2 + q_k r_k \langle T_1 h, T_2 h\rangle + q_k r_k \langle T_2 h, T_1 h\rangle] = 0 \\
\Rightarrow \quad & \sum_{k=1}^{n} p_k[\|q_k T_1 h\|^2 + \|r_k T_2 h\|^2 + 2 q_k r_k Re\, \langle T_2^* T_1 h, h\rangle] = 0 \\
\Rightarrow \quad & \sum_{k=1}^{n} p_k[\|q_k T_1 h\|^2 + \|r_k T_2 h\|^2 = 0 \\
\Rightarrow \quad & \sum_{k=1}^{n} p_k q_k^2 \|T_1\|^2 + a_k r_k^2 \|T_2\|^2 = 0.
\end{aligned} \tag{2.31}$$

Similarly, if $T_2 \perp_C^O T_1$, then we have

$$\sum_{k=1}^{n} p_k q_k^2 \|T_2\|^2 + p_k r_k^2 \|T_1\|^2 = 0. \tag{2.32}$$

Replacing the role of constants in equations (2.31) and (2.32), we can conclude that $T_2 \perp_C^O T_1$. $\square$

Theorem 2.6.12. *[36] Let T_1 and T_2 be bounded linear operators from a Banach space X to Y. Then for any $x \in M_{q_k T_1 + r_k T_2}$, if the images of T_1 and T_2 are orthogonal in the* *sense of Carlsson orthogonality in terms of bounded linear operator, then the operators are also orthogonal.*

Proof. Let $x \in M_{q_kT_1+r_kT_2}$ and $T_1(x) \perp_C^O T_2(x)$. Then

$$\sum_{k=1}^{n} p_k \|q_kT_1(x) + r_kT_2(x)\|^2 = 0 \tag{2.33}$$

satisfying the conditions

$$\sum_{k=1}^{n} p_kq_kr_k = 1, \quad \sum_{k=1}^{n} p_kq_k^2 = \sum_{k=1}^{n} p_kr_k^2 = 0 \tag{2.34}$$

Since $x \in M_{q_kT_1+r_kT_2}$, we must have $\|q_kT_1(x) + r_kT_2(x)\| = \|(q_kT_1 + r_kT_2)x\| = \|q_kT_1 + r_kT_2\|$. Therefore from relation (2.33) and (2.34), we may conclude that

$$\sum_{k=1}^{n} p_k \|q_kT_1 + r_kT_2\|^2 = 0$$

Which shows that T_1 is Carlsson orthogonal to T_2 in terms of operators. □

Chapter 3

ORTHOGONALITY IN TERMS OF p-HH NORM

3.1 Some special cases of Carlsson's orthogonality

Pythagorean and isosceles orthogonalities have been generalized by S.O. Carlsson in 1962. These orthogonalities are obtained by assigning particular values of constants in a generalized Carlssons orthogonality. In this section we will show how isosceles and Pyhagorean orthogonalities can be obtained from Carlsson orthogonality and introduce two new orthogonality relations in relation to Carlsson's orthogonality.

Theorem 3.1.1. *[4] Pythagorean and isosceles orthogonalities are special cases of the Carlsson orthogonality in normed linear space.*

Proof. For Pythagorean orthogonality, We have

$$\sum_{k=1}^{3} a_k ||b_k x + c_k y||^2 = 0$$
$$\Rightarrow \quad a_1 \|b_1 x + c_1 y\|^2 + a_2 \|b_2 x + c_2 y\|^2 + a_3 \|b_3 x + c_3 y\|^2 = 0$$

Put $a_1 = -1$, $a_2 = a_3 = 1$, $b_1 = b_2 = 1$, $b_3 = 0$ and $c_1 = -1$, $c_2 = 0$, $c_3 = 1$. Then

$$\begin{aligned} 0 &= a_1 \|b_1 x + c_1 y\|^2 + a_2 \|b_2 x + c_2 y\|^2 + a_3 \|b_3 x + c_3 y\|^2 \\ &= (-1) \|x - y\|^2 + \|x\|^2 + \|y\|^2 \\ \Rightarrow &\quad \|x - y\|^2 = \|x\|^2 + \|y\|^2 \end{aligned} \tag{3.1}$$

Now,

$$\sum_{k=1}^{3} a_k b_k c_k = 1, \sum_{k=1}^{3} a_k b_k^2 = 0, \sum_{k=1}^{3} a_k c_k^2 = 0.$$

For isosceles orthogonality,

$$\begin{aligned}
&\sum_{k=1}^{2} a_k ||b_k x + c_k y||^2 = 0 \\
&\Rightarrow a_1 \|b_1 x + c_1 y\|^2 + a_2 \|b_2 x + c_2 y\|^2 = 0
\end{aligned}$$

Put $a_1 = \dfrac{1}{2}$, $a_2 = \dfrac{-1}{2}$, $b_1 = b_2 = 1$, $c_1 = 1$ and $c_2 = -1$. Then

$$\begin{aligned}
&0 = a_1 \|b_1 x + c_1 y\|^2 + a_2 \|b_2 x + c_2 y\|^2 \\
&= \frac{1}{2} \|x + y\|^2 + \frac{-1}{2} \|x - y\|^2 \\
&\Rightarrow \|x + y\|^2 = \|x - y\|^2 \\
&\therefore \|x + y\| = \|x - y\| \qquad (3.2)
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{k=1}^{2} a_k b_k c_k &= a_1 b_1 c_1 + a_2 b_2 c_2 \\
&= \frac{1}{2} \times 1 \times 1 + \frac{-1}{2} \times 1 \times -1 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{2} a_k b_k^2 &= a_1 b_1^2 + a_2 b_2^2 \\
&= \frac{1}{2} \times 1 + \frac{-1}{2} \times 1 \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\sum_{k=1}^{2} a_k C_k^2 &= a_1 C_1^2 + a_2 C_2^2 \\
&= \frac{1}{2} \times 1 + \frac{-1}{2} \times 1 \\
&= 0
\end{aligned}
$$

□

Motivated by this theorem, we make an attempt define two new special cases of the Carlsson orthogonality, which we have mentioned as follows:

Proposition 12. *[6] let $(X, \|.\|)$ be a real normed space in which the norm is induced by an inner product. Then $x \in X$ is orthogonal to $y \in X$ if and only if*

$$2\|x+y\|^2 + i\|x+iy\|^2 = 2\|x-y\|^2 + i\|x-iy\|^2 \tag{3.3}$$

Proof. Let $x, y \in X$. Assume $x \perp y$. Then

$$\sum_{k=1}^{m} a_k \|b_k x + c_k x\|^2 = 0$$

satisfying

$$\sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0 \quad \text{and} \sum_{k=1}^{m} a_k b_k c_k = 1 \tag{3.4}$$

In particular if n=4, we have

$$
\begin{aligned}
&\sum_{k=1}^{4} a_k \|b_k x + c_k x\|^2 = 0 \\
\Rightarrow & a_1\|b_1 x + c_1 y\|^2 + a_2\|b_2 x + c_2 y\|^2 + a_2\|b_3 x + c_3 y\|^2 = 0
\end{aligned} \tag{3.5}
$$

Taking the values of constants in equation (3.5) as: $a_1 = b_1 = c_1 = b_2 = b_3 = 1, a_2 = \frac{i}{2}, c_3 = i, a_3 = c_3 = -1$ satisfying condition (3.4) with $n = 4$ leads to the desired result. The converse part is obvious. □

Corollary 3.1.1. *[6] The orthogonality relation (3.3) satisfies the non-degeneracy, simplification and continuity.*

Proof. Non-degeneracy: If $x \perp y$, then

$$\begin{aligned} 2\|x+x\|^2 + i\|x+ix\|^2 &= 2\|x-x\|^2 + i\|x-ix\| \\ \Rightarrow 8\|x\|^2 &= 0 \\ \Rightarrow x &= 0 \end{aligned}$$

Simplification: If $x \perp y$, then

$$\begin{aligned} 2\|x+y\|^2 + i\|x+iy\|^2 &= 2\|x-y\|^2 + i\|x-iy\|^2 \\ \Rightarrow 2\|\lambda x+\lambda y\|^2 + i\|\lambda x+i\lambda y\|^2 &= 2\|\lambda x-\lambda y\|^2 + i\|\lambda x-i\lambda y\|^2 \\ \Rightarrow &\lambda x \perp \lambda y. \end{aligned}$$

Let $\{x_n\}$ and $\{y_n\}$ be sequence in X such that $x_n \to x$ and $y_n \to y$ and $x_n \perp y_n$. Then

$$\begin{aligned} 2\|x_n+y_n\|^2 + i\|x_n+iy_n\|^2 &= 2\|x_n-y_n\|^2 + i\|x_n-iy_n\|^2 \\ \Rightarrow \lim_{n\to\infty} 2\|x_n+y_n\|^2 + i\|x_n+iy_n\|^2 &= \lim_{n\to\infty} 2\|x_n-y_n\|^2 + i\|x_n-iy_n\|^2 \\ \Rightarrow 2\|x+y\|^2 + i\|x+iy\|^2 &= 2\|x-y\|^2 + i\|x-iy\|^2 \\ \Rightarrow x &\perp y. \end{aligned}$$

□

Proposition 13. *[6] let $(X, \|.\|)$ be a real normed space in which the norm is induced by an inner product. Then $x \in X$ is orthogonal to $y \in X$ if and only if*

$$\|x+\frac{y}{2}\|^2 + \|x-\frac{y}{2}\|^2 = \frac{1}{2}\|\sqrt{2}x+y\|^2 + \|x\|^2 \tag{3.6}$$

Proof. Let $x, y \in X$. Assume $x \perp y$. Then

$$\sum_{k=1}^{m} a_k\|b_k x + c_k x\|^2 = 0$$

satisfying

$$\sum_{k=1}^{m} a_k b_k^2 = \sum_{k=1}^{m} a_k c_k^2 = 0 \quad \text{and} \quad \sum_{k=1}^{m} a_k b_k c_k = 1 \tag{3.7}$$

In particular if n=4, we have

$$\begin{aligned}
&\sum_{k=1}^{4} a_k\|b_kx + c_kx\|^2 = 0\\
\Rightarrow & a_1\|b_1x + c_1y\|^2 + a_2\|b_2x + c_2y\|^2 + a_2\|b_3x + c_3y\|^2 = 0 \qquad (3.8)
\end{aligned}$$

Taking the values of constants in equation (3.8) as: $a_1 = a_2 = b_1 = b_2 = c_3 = 1, a_3 = a_4 = c_2 = \frac{-1}{2}, c_1 = \frac{1}{2}, b_3 = b_4 = \sqrt{2}, c_4 = 0$ satisfying condition (3.7) with $n = 4$ leads to the desired result. Conversely assume (3.6) holds and it can be easily shown that x is orthogonal to y. □

Corollary 3.1.2. *[6] Orthogonality relation (3.6) satisfy the non-degeneracy, simplification and continuity.*

Proof. Non-degeneracy: If $x \perp x$, then

$$\begin{aligned}
&\|x + \frac{x}{2}\|^2 + \|x - \frac{x}{2}\|^2 = \frac{1}{2}\|\sqrt{2}x + x\|^2 + \|x\|^2\\
\Rightarrow & \|x\| = 0\\
\Rightarrow & x = 0
\end{aligned}$$

Simplication: If $x \perp y$, then

$$\begin{aligned}
&\|x + \frac{y}{2}\|^2 + \|x - \frac{y}{2}\|^2 = \frac{1}{2}\|\sqrt{2}x + y\|^2 + \|x\|^2\\
\Rightarrow & \|\lambda x + \frac{\lambda y}{2}\|^2 + \|\lambda x - \frac{\lambda y}{2}\|^2 = \frac{1}{2}\|\sqrt{2}\lambda x + \lambda y\|^2 + \|\lambda x\|^2\\
\Rightarrow & \lambda x \perp \lambda y
\end{aligned}$$

Continuity: Let $\{x_n\}$ and $\{y_n\}$ be sequence in X such that $x_n \to x$ and $y_n \to y$ for all $n \in \mathbb{N}$ and $x_n \perp y_n$, then

$$\begin{aligned}
&\|x_n + \frac{y_n}{2}\|^2 + \|x_n - \frac{y_n}{2}\|^2 = \frac{1}{2}\|\sqrt{2}x_n + y_n\|^2 + \|x_n\|^2\\
\Rightarrow & \lim_{n\to\infty} \|x_n + \frac{y_n}{2}\|^2 + \|x_n - \frac{y_n}{2}\|^2 = \lim_{n\to\infty} \frac{1}{2}\|\sqrt{2}x_n + y_n\|^2 + \|x_n\|^2\\
\Rightarrow & \|x + \frac{y}{2}\|^2 + \|x - \frac{y}{2}\|^2 = \frac{1}{2}\|\sqrt{2}x + y\|^2 + \|x\|^2\\
\Rightarrow & x \perp y.
\end{aligned}$$

□

Lemma 3.1.1. *[6] Let X be a normed linear space, in which the norm on X is induced by an inner product. Then the orthogonality relation (3.6) is homogeneous.*

Proof. If $x \perp y$, then we have,

$$\left\|x+\frac{1}{2}y\right\|^2+\left\|x-\frac{1}{2}y\right\|^2=\frac{1}{2}\left\|\sqrt{2}x+y\right\|^2+\|x\|^2$$

Now,

$$\begin{aligned}
&\left\|\lambda x+\frac{1}{2}\mu y\right\|^2+\left\|\lambda x-\frac{1}{2}\mu y\right\|^2-\frac{1}{2}\left\|\sqrt{2}\lambda x+\mu y\right\|^2-\|\lambda x\|^2=0\\
&\Rightarrow \quad \langle\lambda x+\frac{1}{2}\mu y,\lambda x+\frac{1}{2}\mu y\rangle+\langle\lambda x-\frac{1}{2}\mu y,\lambda x-\frac{1}{2}\mu y\rangle\\
&-\frac{1}{2}\langle\sqrt{2}\lambda x+\mu y,\sqrt{2}\lambda x+\mu y\rangle-\langle\lambda x,\lambda x\rangle=0\\
&\Rightarrow \quad -2\sqrt{2}\lambda\mu\langle x,y\rangle=0
\end{aligned}$$

Therefore the given orthogonality is homogeneous if and only if the space is inner product space.

□

Lemma 3.1.2. *[8] Let $(X,\|.\|)$ be a real normed space in which the norm on X is induced by an inner- product. Then orthogonality relation (3.6) implies Birkhoff orthogonality if $y=\frac{x}{1-\alpha}$, but the converse may not be true.*

Proof. Suppose $x \perp y$. Then by definition,

$$\begin{aligned}
&\left\|x+\frac{1}{2}y\right\|^2+\left\|x-\frac{1}{2}y\right\|^2=\frac{1}{2}\left\|\sqrt{2}x+y\right\|^2+\|x\|^2\\
&\Rightarrow \left\|x+\frac{1}{2}y\right\|^2+\left\|x-\frac{1}{2}y\right\|^2\geq\|x\|^2\\
&\Rightarrow \left\|x+\frac{1}{2}y-x+\frac{1}{2}y\right\|^2\geq\|x\|^2\\
&\Rightarrow \|y\|^2\geq\|x\|^2 \ldots\ldots. \qquad (1)
\end{aligned}$$

Since $y = \frac{x}{1-\alpha}$ so that $y = x + \alpha y$. Therefore form the relation (1)

$$\begin{aligned}
&\|x + \alpha y\|^2 \geq \|x\|^2 \\
&\Rightarrow \|x + \alpha y\| \geq \|x\| \\
&\Rightarrow x \perp_B y.
\end{aligned}$$

□

To disprove the converse part, consider $X = (\mathbb{R}^2, \|.\|_1)$, where $\|.\|_1 = \sum_{k=1}^{2} |x_k|$ for some $x = (x_1, x_2) \in X$. Let $x = (-2, 1), y = (2, 2)$. and $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}
\|x + \alpha y\|_1 &= \|(2, 1) + \alpha(2, 2)\|_1 \\
&= \|-2 + 2\alpha, 1 + 2\alpha\|_1 \\
&= |-2 + 2\alpha| + |1 + 2\alpha| \\
&\geq 3 \\
&= \|x\|_1
\end{aligned}$$

But

$$\begin{aligned}
\left\|x + \frac{1}{2}y\right\|^2 + \left\|x - \frac{1}{2}y\right\|^2 &= \left\|(-2, 1) + \frac{1}{2}(2, 2)\right\|^2 + \left\|(-2, 1) - \frac{1}{2}(2, 2)\right\|^2 \\
&= \|(-2, 1) + (1, 1)\|^2 + \|(-2, 1) - (1, 1)\|^2 \\
&= 18
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\left\|\sqrt{2}x + y\right\|^2 + \|x\|^2 &= \frac{1}{2}\left\|\sqrt{2}(-2, 1) + (2, 2)\right\|^2 + \|(-2, 1)\|^2 \\
&= \frac{1}{2}\left\|(-2\sqrt{2} + 2, \sqrt{2} + 2)\right\|^2 + 9 \\
&= \frac{1}{2}(0.828 + 3.4142)^2 + 9 \\
&= 17.99
\end{aligned}$$

which shows that x is not orthogonal to y in the sense of orthogonality relation (3.6).

If the underlying space X is a real inner product space and the relation (3.6) holds a. e on $[0, 1]$. Then using the concept of 2-HH norm, can define orthogonality relation (3.6) in terms of 2-HH norm as follows:

Definition 24. *Let $(X.\|.\|)$ be a real normed space and $x, y \in X$. Then x is said to be orthogonal to y in the sense of $2-HH$ norm if and only if*

$$\int_0^1 \|(1-t)x+\frac{1}{2}ty\|^2dt+\int_0^1 \|(1-t)x-\frac{1}{2}ty\|^2dt = \frac{1}{2}\int_0^1 \|\sqrt{2}(1-t)x+ty\|^2dt+\int_0^1 \|(1-t)x\|^2dt \tag{3.9}$$

To verify the this, we have the following relations:

$$\begin{aligned}\int_0^1 \|(1-t)x+\frac{1}{2}ty\|^2dt+\int_0^1 \|(1-t)x-\frac{1}{2}ty\|^2dt &= \frac{1}{3}\|x\|^2+\frac{1}{12}\|y\|^2+\frac{1}{3}\|x\|^2+\frac{1}{12}\|y\|^2\\ &= \frac{2}{3}\|x\|^2+\frac{1}{6}\|y\|^2.\end{aligned}$$

And

$$\begin{aligned}\frac{1}{2}\int_0^1 \|\sqrt{2}(1-t)x+ty\|^2dt+\int_0^1 \|(1-t)x\|^2dt &= \frac{1}{2}(\frac{2}{3}\|x\|^2+\frac{1}{3}\|y\|^2)+\frac{1}{3}\|x\|^2\\ &= \frac{2}{3}\|x\|^2+\frac{1}{6}\|y\|^2.\end{aligned}$$

3.2 Results on new orthogonality

Definition 25. *[15] Let $(X.\|.\|)$ be a normed linear space. A vector $x \in X$ is said to be 2-HH-N orthogonal to $y \in X$ (denoted by $x \perp_{2-HH-N} y$) if and only if*

$$\int_0^1 \|(1-t)x+\frac{1}{2}ty\|^2dt+\int_0^1 \|(1-t)x-\frac{1}{2}ty\|^2dt = \frac{2}{3}\|x\|^2+\frac{1}{6}\|y\|^2 \tag{3.10}$$

E. Kikianty and S.S. Dragomir in [7] proved that " the Pythagorean orthogonality via 2-HH norm is unique in any normed space X". To prove this, they use the following Lemma by omitting the proof. We give a detailed proof of Lemma as they stated the paper [7].

Lemma 3.2.1. *[15] Let $x, y \in X$, where X is normed liner space. Let h be a function on $\mathbb{R}$ defined by*

$$h(\mu) := \int_0^1 \|(1-t)y+\mu(tx)\|^2dt.$$

Then h is a convex function on $\mathbb{R}$*, and for any* $r \in (0,1)$ *and* $\mu_1, \mu_2 \in \mathbb{R}$ *where* $h(\mu_1) \neq h(\mu_2)$*, we have*

$$h[r\mu_1 + (1-r)\mu_2] < rh(\mu_1) + (1-r)h(\mu_2).$$

Proof. Let $r \in (0,1)$ and $\mu_1, \mu_2 \in \mathbb{R}$ such that $h(\mu_1) \neq h(\mu_2)$. Then

$$\begin{aligned}
& h[r\mu_1 + (1-r)\mu_2] \\
&= \int_0^1 \|(1-t)y + [r\mu_1 + (1-r)\mu_2](tx)\|^2 dt \\
&= \int_0^1 \|(1-t)y + r\mu_1(tx) + \mu_2(tx) - r\mu_2(tx)\|^2 dt \\
&= \int_0^1 \|(1-t)y + r\mu_1(tx) + \mu_2(tx) - r\mu_2(tx) - r(1-t)y + r(1-t)y\|^2 dt \\
&= \int_0^1 \|r[(1-t)y + \mu_1(tx)] + (1-r)[(1-t)y + \mu_2(tx)]\|^2 dt \\
&\leq \int_0^1 r^2 \|(1-t)y + \mu_1(tx)\|^2 dt + (1-r)^2 \int_0^1 \|(1-t)y + \mu_2(tx)\|^2 dt \\
&\quad + 2r(1-r) \int_0^1 \|(1-t)y + \mu_1(tx)\| \, \|(1-t)y + \mu_2(tx)\| dt \\
&= \int_0^1 r\|(1-t)y + \mu_1(tx)\|^2 dt + (1-r) \int_0^1 \|(1-t)y + \mu_2(tx)\|^2 dt \\
&\quad + (r^2 - r) \int_0^1 [\|(1-t)y + \mu_1(tx)\|^2 + \|(1-t)y + \mu_2(tx)\|^2 \\
&\quad - 2\|(1-t)y + \mu_1(tx)\| \, \|(1-t)y + \mu_2(tx)\|] dt \\
&= rh(\mu_1) + (1-r)h(\mu_2) - r(1-r) \int_0^1 [\|(1-t)y + \mu_1(tx)\| - \|(1-t)y + \mu_2(tx)\|]^2 dt \\
&\leq rh(\mu_1) + (1-r)h(\mu_2)
\end{aligned}$$

Which shows that h is a convex function. Since $h(\mu_1) \neq h(\mu_2)$, then the inequality will be strict and therefore

$$h[r\mu_1 + (1-r)\mu_2] < rh(\mu_1) + (1-r)h(\mu_2).$$

□

As a similar concept of Lemma 3.2.1, we also prove the following Lemma which is useful to prove the uniqueness property of new orthogonality via 2-HH norm.

Lemma 3.2.2. *[15] Let $(X.\|.\|)$ be a normed space and $x, y \in X$. Let h be a function defined on $\mathbb{R}$ by*

$$h(\mu) := \int_0^1 \|(1-t)y + \frac{\mu}{2}(tx)\|^2 dt + \int_0^1 \|(1-t)y - \frac{\mu}{2}(tx)\|^2 dt$$

Then h is a convex function on $\mathbb{R}$ and for any $r \in (0,1)$, and $\mu_1, \mu_2 \in \mathbb{R}$ where $h(\mu_1) \neq h(\mu_2)$, we have

$$h[r\mu_1 + (1-r)\mu_2] < rh(\mu_1) + (1-r)h(\mu_2).$$

Proof. Suppose h(μ)=f(μ)+g(μ), where

$$\begin{aligned} f(\mu) &= \int_0^1 \|(1-t)y + \frac{\mu}{2}(tx)\|^2 dt \quad \text{and} \\ g(\mu) &= \int_0^1 \|(1-t)y - \frac{\mu}{2}(tx)\|^2 dt \end{aligned}$$

First we show that f(k) is a convex function. Let $r \in (0,1)$ and $\mu_1, \mu_2 \in \mathbb{R}$ such that

$h(\mu_1) \neq h(\mu_2)$.

$$
\begin{aligned}
& f[r\mu_1 + (1-r)\mu_2] \\
&= \int_0^1 \|(1-t)y + \frac{1}{2}[[r\mu_1 + (1-r)\mu_2](tx)\|^2 dt \\
&= \int_0^1 \|(1-t)y + \frac{1}{2}r\mu_1(tx) + \frac{1}{2}\mu_2(tx) - \frac{1}{2}r\mu_2(tx)\|^2 dt \\
&= \int_0^1 \|(1-t)y + \frac{r\mu_1}{2}(tx) + \frac{\mu_2}{2}(tx) - r\frac{\mu_2}{2}(tx) + r(1-t)y - r(1-t)y\|^2 dt \\
&= \int_0^1 \|r[(1-t)y + \frac{\mu_1}{2}(tx)] + (1-r)[(1-t)y + \frac{\mu_2}{2}(tx)]\|^2 dt \\
&\leq r^2 \int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\|^2 dt + (1-r)^2 \int_0^1 \|(1-t)y + \frac{\mu_2}{2}(tx)\|^2 dt \\
&\quad + 2r(1-r) \int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\| \, \|(1-t)y + \frac{\mu_2}{2}(tx)\| dt \\
&= r \int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\|^2 dt + (1-r) \int_0^1 \|(1-t)y + \frac{\mu_2}{2}(tx)\|^2 dt \\
&\quad + r^2 \int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\|^2 dt + (1-r)^2 \int_0^1 \|(1-t)y + \frac{\mu_2}{2}(tx)\|^2 dt \\
&\quad - r \int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\|^2 dt - (1-r) \int_0^1 \|(1-t)y + \frac{\mu_2}{2}(tx)\|^2 dt \\
&\quad + 2r(1-r) \int_0^1 \|(1-t)y + \frac{\mu_1}{2}(tx)\| \, \|(1-t)y + \frac{\mu_2}{2}(tx)\| dt \\
&= rf(\mu_1) + (1-r)f(\mu_2) + (r^2 - r) \int_0^1 [\|(1-t)y + \frac{\mu_1}{2}(tx)\| - \|(1-t)y + \frac{\mu_1}{2}(tx)\|]^2 dt \\
&= rf(\mu_1) + (1-r)f(\mu_2) - (r - r^2) \int_0^1 [\|(1-t)y + \frac{\mu_1}{2}(tx)\| - \|(1-t)y + \frac{\mu_1}{2}(tx)\|]^2 dt \\
&\leq rf(\mu_1) + (1-r)f(\mu_2)
\end{aligned}
$$

Which shows that f is a convex function. Similarly, for the function

$$
g(\mu) = \int_0^1 \|(1-t)t - \frac{\mu}{2}(tx)\|^2 dt,
$$

we can show that

$$
g[r\mu_1 + (1-r)\mu_2] \leq rg(\mu_1) + (1-r)g(\mu_2)
$$

and we conclude that g is also a convex function. Also, we know that the sum of two convex functions is also convex. Then $h(\mu) = f(\mu) + g(\mu)$ is convex. Since $h(\mu_1) \neq$

$h(\mu_2)$, then the inequality will be strict and therefore

$$h[r\mu_1 + (1-r)\mu_2] < rh(\mu_1) + (1-r)h(\mu_2)$$

$\square$

Theorem 3.2.1. *[15] 2-HH-N orthogonality is unique in any normed space X.*

Proof. The proof has a similar idea to that of Kapoor and Prasad [pp. 406] and Kikianty and Dragomir [pp. 41]. Suppose 2-HH-N orthogonality is not unique. Then we must have elements $x \neq 0$ and $y \in X$, and a $\lambda > 0$ such that $x \perp_{2-HH-N} y$ and $x \perp_{2-HH-N} \lambda x + y$. Define a convex function

$$h(\mu) = \int_0^1 \|(1-t)y + \frac{\mu}{2}(tx)\|^2 dt + \int_0^1 \|(1-t)y - \frac{\mu}{2}(tx)\|^2 dt.$$

Now,

$$\begin{aligned} h(1) &= \int_0^1 \|(1-t)y + \frac{1}{2}(tx)\|^2 dt + \int_0^1 \|(1-t)y - \frac{1}{2}(tx)\|^2 dt \\ &= \frac{2}{3}\|y\|^2 + \frac{1}{6}\|x\|^2 \\ &= h(0) + \frac{1}{6}\|x\|^2 \end{aligned} \tag{3.11}$$

Setting $\beta = \frac{2(1-t)\lambda}{t}$ and note that

$$\begin{aligned} h(\beta) &= \int_0^1 \|(1-t)y + \frac{1}{2}.\frac{2(1-t)\lambda}{t}(tx)\|^2 dt + \int_0^1 \|(1-t)y - \frac{1}{2}.\frac{2(1-t)\lambda}{t}(tx)\|^2 d \\ &= \int_0^1 \|(1-t)y + (1-t)\lambda x\|^2 dt + \int_0^1 \|(1-t)y - (1-t)\lambda x\|^2 dt \\ &= \int_0^1 \|(1-t)(y+\lambda x)\|^2 dt + \int_0^1 \|(1-t)(y-\lambda x)\|^2 dt \\ &= \frac{\|y+\lambda x\|^2}{3} + \frac{\|y-\lambda x\|^2}{3} \end{aligned} \tag{3.12}$$

and,

$$\begin{aligned} h(\beta+1) &= \int_0^1 \|(1-t)(y+\lambda x) + \frac{tx}{2}\|^2 dt + \int_0^1 \|(1-t)(y-\lambda x) - \frac{tx}{2}\|^2 dt \\ &= \frac{\|y+\lambda x\|^2}{3} + \frac{\|y-\lambda x\|^2}{3} + \frac{\|x\|^2}{6} \\ &= h(\beta) + \frac{\|x\|^2}{6} \end{aligned} \tag{3.13}$$

Now, suppose that $0 < \beta < 1$ and note that $h(1) \neq h(0)$ (since $x \neq 0$), Lemma 3.2.2 gives

$$h(\beta) < \beta h(1) + (1-\beta)h(0) \tag{3.14}$$

Also $h(\beta+1) \neq h(\beta)$ (since $x \neq 0$) and with the help of Lemma 3.2.2

$$\begin{aligned} h(1) &< \beta h(\beta) + (1-\beta)h(\beta+1) \\ &= \beta h(\beta) + (1-\beta)[h(\beta) + \frac{\|x\|^2}{6}] \\ &= \beta h(\beta) + (1-\beta)[h(\beta) + h(1) - h(0)] \\ \Rightarrow h(\beta) &> \beta h(1) + (1-\beta)h(0), \end{aligned}$$

which contradicts (3.14). Now consider the case $\beta > 1$, we have

$$\begin{aligned} h(1) &\leq \frac{\beta-1}{\beta}h(0) + \frac{1}{\beta}h(\beta) \\ &= h(0) + \frac{1}{\beta}[h(\beta) - h(0)] \\ \Rightarrow h(1) - h(0) &\leq \frac{1}{\beta}[h(\beta) - h(0)] \\ \Rightarrow \frac{\|x\|^2}{6} &\leq \frac{1}{\beta}[h(\beta) - h(0)] \end{aligned}$$

Since $x \neq 0$, then, $h(\beta) \neq h(0)$, and using the Lemma 3.2.2, we have

$$h(1) < \frac{\beta-1}{\beta}h(0) + \frac{1}{\beta}h(\beta) \tag{3.15}$$

Also $h(1) \neq h(\beta+1)$ and Lemma 3.2.2 gives us

$$\begin{aligned} h(\beta) &< \frac{1}{\beta}h(1) + \frac{\beta-1}{\beta}h(\beta+1) \\ &= \frac{1}{\beta}h(1) + \frac{\beta-1}{\beta}[h(\beta) + h(1) - h(0)] \\ \Rightarrow h(1) &> \frac{1}{\beta}h(\beta) + \frac{\beta-1}{\beta}h(0), \end{aligned}$$

which contradicts the relation (3.15). For the case $\beta = 1$, we have

$$\begin{aligned} h(2) &= h(1) + \frac{\|x\|^2}{6} \\ &= h(0) + \frac{\|x\|^2}{3} \end{aligned}$$

This shows that $h(2) \neq h(0)$ (since $x \neq 0$). Then we have

$$\begin{aligned} h(1) &< \frac{1}{2}h(0) + \frac{1}{2}h(2) \\ &= \frac{1}{2}[h(0) + h(0) + \frac{\|x\|^2}{3}] \\ \Rightarrow h(1) &< h(0) + \frac{\|x\|^2}{6}, \end{aligned}$$

which contradicts (3.11). Thus in all cases we get a contradiction. Hence 2-HH-N orthogonality is unique in any normed space. $\square$

Definition 26. *Let T_1 and T_2 be norm attaining bounded linear operators in a Banach space X. An operator $T_1 \in B(X)$ is said to be orthogonal to $T_2 \in B(X)$ if and only if for any $h \in M_T$,*

$$\|(T_1 + \frac{1}{2}T_2)(h)\|^2 + \|(T_1 - \frac{1}{2}T_2)(h)\|^2 = \frac{1}{2}\|(\sqrt{2}T_1 + T_2)(h)\|^2 + \|T_1(h)\|^2. \quad (3.16)$$

Theorem 3.2.2. *Let T_1 and T_2 be bounded linear operators in a real Hilbert space H. Then, orthogonality relation (3.16) implies the Birkhoff-James orthogonality, but the converse may not be true.*

Proof. Let $T_1, T_2 \in B(H)$ and $h \in M_T$. Assume that T_1 is orthogonal to T_2. Then, we have

$$\begin{aligned} & \|(T_1 + \frac{1}{2}T_2)(h)\|^2 + \|(T_1 - \frac{1}{2}T_2)(h)\|^2 = \frac{1}{2}\|(\sqrt{2}T_1 + T_2)(h)\|^2 + \|T_1(h)\|^2 \\ \Rightarrow \quad & \|T_2(h)\|^2 \geq \|T_1(h)\|^2. \end{aligned}$$

Setting $T_2 = \frac{T_1}{1-\alpha}$, so that $T_2 = T_1 + \alpha T_2$ and we get

$$\|(T_1 + \alpha T_2)(h)\|^2 \geq \|T_1(h)\|^2.$$

This implies that

$$\|(T_1 + \alpha T_2)(h)\| \geq \|T_1(h). \quad (3.17)$$

Hence, T_1 is Birkhoff-James orthogonal to T_2. To disprove the above statement, we can take operators $T_1, T_2 \in B(H)$ as described in the example of (Theorem. 2.5.8), showing that T_1 is Birkhoff-James orthogonal to T_2. On the other hand,

$$\|(T_1 + \frac{1}{2}T_2)\|^2 + \|(T_1 - \frac{1}{2}T_2)\|^2 = 51. \tag{3.18}$$

However,

$$\frac{1}{2}\|(\sqrt{2}T_1 + T_2)\|^2 + \|T_1^2\| = 27 + 7\sqrt{2}, \tag{3.19}$$

showing that T_1 is not orthogonal to T_2. □

Theorem 3.2.3. *Let $T_1, T_2 \in B(X)$. Then for any $x \in M_{T_1+T_2} \cap M_{T_1-T_2}$ and $T_1(x) + T_2(x) \perp_B T_2(x)$ and $T_1(x) - T_2(x) \perp_B T_2(x)$ implies $T_1 \perp_I T_2$.*

Proof. Since $T_1(x)+T_2(x) \perp_B T_2(x)$, then we have $\|T_1(x)+T_2(x)\| \leq \|T_1(x)+T_2(x)+\lambda T_2(x)\|$ for all $\lambda \in \mathbb{K}$. Taking $\beta = 1 + \lambda$, we have

$$\begin{aligned}
\|T_1(x) + T_2(x)\| &\leq \|T_1(x) + T_2(x) + (\beta - 1)T_2(x)\| \\
&= \|T_1(x) + T_2(x) + \beta T_2(x) - T_2(x)\| \\
&= \|T_1(x) + \beta T_2(x)\| \\
\Rightarrow \|(T_1 + T_2)x\| &\leq \|(T_1 + \beta T_2)x\|
\end{aligned}$$

In particular for $\beta = -1$, we have $\|(T_1 + T_2)x\| \leq \|(T_1 - T_2)x\|$. Since $x \in M_{T_1+T_2} \cap M_{T_1(x)+T_2(x)}$, then we must have $\|(T_1+T_2)\| \leq \|(T_1-T_2)\|$. Similarly if $T_1(x)-T_2(x) \perp_B T_2(x)$, we obtain $\|T_1 - T_2\| \leq \|T_1 + T_2\|$ and therefore by combining these inequalities we get the desired result. □

Corollary 3.2.1. *Let $T_1, T_2 \in B(X)$ and assume that $T_1 + T_2 \perp_B T_2$ and $T_1 - T_2 \perp_B T_2$. Then $T_1 \perp_I T_2$.*

Proof. Since $T_1 + T_2 \perp_B T_2$, then we have $\|T_1 + T_2\| \leq \|T_1 + T_2 + \lambda T_2\|$ for all $\lambda \in \mathbb{K}$. Taking $\beta = 1 + \lambda$, we have

$$\begin{aligned}
\|T_1 + T_2\| &\leq \|T_1 + T_2 + (\beta - 1)T_2\| \\
&= \|T_1 + T_2 + \beta T_2 - T_2\| \\
&= \|T_1 + \beta T_2\|
\end{aligned}$$

In particular for $\beta = -1$, we have $\|T_1 + T_2\| \leq \|T_1 - T_2\|$. Similarly if $T_1 - T_2 \perp_B T_2$, we obtain $\|T_1 - T_2\| \leq \|T_1 + T_2\|$ and therefore by combining these inequalities we get the desired result. □

Proposition 14. *Let T be a bounded linear operator from a normed linear space X to Y and* $\{x_n\}$ *,* $\{y_n\}$ *be sequence in X and* $x \in M_T$ *such that* $x_n \to x, y_n \to y, Tx_n \to Tx,$*. Then for any* $y \in X$ *and* $\epsilon \in [0,1)$*,* $Tx_n \perp_B^\epsilon Ty_n \Rightarrow x \perp_B^\epsilon y$*.*

Proof. Let $x \in M_T$ and $\epsilon \in [0,1)$. Since $Tx_n \perp_B^\epsilon Ty_n$, then we have for any $\lambda \in \mathbb{K}$,

$$\begin{aligned}
(1-\epsilon)\|Tx_n\| &\leq \|Tx_n + \lambda Ty_n\| \\
\Rightarrow (1-\epsilon)\|Tx_n\| &\leq \|T\|\|x_n + \lambda y_n\| \\
\Rightarrow (1-\epsilon)\lim_{n\to\infty}\|Tx_n\| &\leq \|T\|\lim_{n\to\infty}\|x_n + \lambda y_n\| \\
\Rightarrow (1-\epsilon)\|Tx\| &\leq \|T\|\|x + \lambda y\| \qquad (3.20)
\end{aligned}$$

Since $x \in M_T$, then we must have $\|Tx\| = \|T\|\|x\|$, and therefore the inequality (3.20) can be written as $(1-\epsilon)\|T\|\|x\| \leq \|T\|\|x+\lambda y\|$. Hence $\|(1-\epsilon)\|x\| \leq \|x + \lambda y\|$ is the desired result. □

Chapter 4

BEST APPROXIMATION IN NORMED LINEAR SPACES

4.1 Introduction

In the case of normed linear space, the concept of best approximation was developed by M. Nicolescu in 1938 [38] and the more detail were discussed in the paper by M. K. Krein [39]. Within the framework of normed linear spaces the problem of best approximation amounts to the problem of minimizing a distance, hence it is geometrized, and thus in its study one can use arguments based on geometric intuition [38].

Definition 27 ([40], [41]). *Let G be a subset of a normed linear space X. A point $g_0 \in G$ is said to be best approximation for $f \in X$, if and only if for all $g \in G$,*

$$\|f - g_0\| \leq \|f - g\|$$

and is said to be best co-approximation

$$\|g_0 - g\| \leq \|f - g\|$$

The set of all best approximation of $f \in X$ in G is denoted by $P_G(f)$ and best co-approximation by $R_G(f)$. For a subspace G of X, Mazaheri and Modarres in the paper

[42] define $\widehat{G}$ and $\widetilde{G}$ as follows:

$$\widehat{G} = P_G^{-1}(0) = \{f \in X : \|f\| = d(f, G)\} = \{f \in X : x \perp G\} \quad \text{and}$$
$$\widetilde{G} = R_G^{-1}(0) = \{f \in X : \|g\| \leq \|f - g\| \quad \forall g \in G\} = \{f \in X : G \perp f\}$$

Lemma 4.1.1 ([42],[43]). *Let G be a linear subspace of normed space X. Then*

(i) G is Chebychev subspace if and only if $X = G \oplus \widehat{G}$,

(ii) G is proximal subspace if and only if $X = G + \widehat{G}$.

Lemma 4.1.2 ([42],[43]). *Let G be a linear subspace of a normed space X and $x \in X \backslash G$. Then for any element $g_0 \in G$, the following statements are equivalent:*

(i) $g_0 \in R_G(x)$,

(ii) For each $g \in G$ there exist a functional $f^g \in X^*$ such that $\|f^g\| = 1$, $f^g(x) = f^g(g_0)$ and $f^g(g) = \|g\|$.

Lemma 4.1.3 ([42], [44]). *Let X be a real Banach space and G be proximinal subset of X. Suppose $P_G : X \to G$ is the best approximation operator on G. Then for all $x \in X$ and $g \in G$, the following inequality holds:*

$$\|z - g\| \leq \|x - g\| z \in P_G(x) \tag{4.1}$$

If the inequality (4.1) is true, then $P_G(x) \subseteq R_G(x)$ for every $x \in X$.

Proposition 15. *[42] Let X be a Banach space and G be linear subspace of X. Then $\widetilde{G}$ is a linear closed subspace of X.*

Theorem 4.1.1. *[42] Let X be a smooth Banach space and G be a co-proximinal subspace of X, then $\widetilde{G}$ is a proximinal subspace of X.*

Theorem 4.1.2. *[42] Let X be a smooth Banach space and G be a co-Chebychev subspace of X, then $\widetilde{G}$ is a Chebychev subspace of X.*

Definition 28. *[40] Let G be a subset of a normed linear space X and $\epsilon > 0$, then a point $g_0 \in G$ is said to be $\epsilon-$best approximation for $f \in X$ if*

$$\|f - g_0\| \leq \|f - g\| + \epsilon \quad (\textit{for all } g \in G)$$

and $g_0 \in G$ is said to be $\epsilon-$co-approximation for $f \in X$ if

$$\|g_0 - g\| \leq \|f - g\| + \epsilon \quad (\textit{for all } g \in G) \tag{4.2}$$

The set of all $\epsilon-$approximation of f in G is denoted by $P_G(f, \epsilon)$ and the set of all $\epsilon-$co-approximation of f in G is denoted by $R_G(f, \epsilon)$.

Definition 29. *[40] Let $(X, \|.\|)$ be a normed space, $f, g \in X$ and $\epsilon > 0$. Then f is said to be $\epsilon-$orthogonal to g if and only if*

$$\|f\| \leq \|f + \alpha g\| + \epsilon \quad \textit{for all scalar } \alpha \ with \ \mid \alpha \mid \leq 1.$$

If G_1 and G_2 are subsets of X, then $G_1 \perp_\epsilon G_2$ if and only if $g_1 \perp_\epsilon g_2$ for all $g_1 \in G$ and $g_2 \in G$.

Theorem 4.1.3. *[40] Let G be a subspace of a normed space X and $\epsilon > 0$. Then for all $f \in X$,*

$$g_0 \in P_G(f, \epsilon) \Leftrightarrow f - g_0 \perp_\epsilon G$$
$$g_0 \in R_G(f, \epsilon) \Leftrightarrow G \perp_\epsilon f - g_0$$

Lemma 4.1.4. *[40] Let G be a subspace of a normed space X, then*

(i) If $\epsilon > 0, f, g \in X$ and $f \perp_\epsilon g$, then $f \perp_\delta g$ for all $\delta \geq \epsilon$.

(ii) If $f, g \in X$ and $f \perp g$, then $f \perp_\epsilon g$ for all $\epsilon > 0$.

(iii) If $f \in X$ and $\epsilon > 0$, then $0 \perp_\epsilon f$ and $f \perp_\epsilon 0$.

(iv) If $f \perp_\epsilon g$ and $\mid \beta \mid < 1$, then $\beta f \perp_\epsilon \beta g$.

(v) If $f \in X, \epsilon > 0$ and $\delta \geq \epsilon$, then

$$g_0 \in P_G(f, \epsilon) \Rightarrow g_0 \in P_G(f, \delta)$$
$$g_0 \in R_G(f, \epsilon) \Rightarrow g_0 \in R_G(f, \delta).$$

Mazaheri and Zadeh proved in the paper [45] that the isometric operators on normed linear space preserves all approximation property. They proved that every linear operator preserving approximation is an isometry multiplied by a constant.

Definition 30. *[45] Let$(X, \|.\|)$ and $(Y, \|.\|)$ be two normed linear spaces. A mapping $T : X \to Y$ is called preserving approximation if and only if for all subspace G of X and all $x \in X$,*

$$T(P_G(x)) = P_{T(G)}(T(x)) \tag{4.3}$$

Similarly T is called preserving co-approximation if and only if for app subspace G of X and all $x \in X$,

$$T(R_G(x)) = R_{T(G)}(T(x)) \tag{4.4}$$

Lemma 4.1.5. *[45] Let $T : X \to X$ is an isometry operator, where X be a normed linear space, then for all subspace G of X and $x \in X$,*

$$\begin{aligned} T(P_G(x)) &= P_{T(G)}(T(x)) \quad \textit{and} \\ T(R_G(x)) &= R_{T(G)}(T(x)) \end{aligned}$$

Corollary 4.1.1. *[45] Let $(X, \|.\|)$ be a normed space. Every isometry operator $T : X \to X$ is preserving approximation (resp. Co-approximation).*

Theorem 4.1.4. *[45] Suppose $T : X \to Y$ be a linear map from a normed space X into Y preserving approximation (resp. Co-approximation).*

(i) Suppose G is a subspace of X, then G is proximinal (resp. co-proximinal)of X if and only if $T(G)$ is proximinal (resp. co-proximinal).

(ii) Suppose G is a subspace of X, then G is Chebyshev (resp. co-Chevyshev)of X if and only if $T(G)$ is Chebyshev (resp. co-Chevyshev).

(iii) If T is linear, then for all $x, y \in X$, $x \perp y \Rightarrow T(x) \perp T(y)$.

(iv) For all subset G of X, $T(\widehat{G}) = \widehat{T(G)}$, (res. $T(\widetilde{G}) = \widetilde{T(G)}$).

(v) Suppose G is a subspace of X, then G is orthogonality complement in X if and only if $T(G)$ is orthogonality complement in Y.

(vi) Suppose G is a subspace of X, if T is a continuous and onto preserve approximation map, then G is quasi Chebyshev if and only if $T(G)$ is quasi Chebyshev.

Definition 31. *[45] Let $(X, \|.\|)$ and $(Y, \|.\|)$ be a normed linear spaces and $\epsilon > 0$. A mapping $T : X \to Y$ is called $\epsilon-$preserving approximation if and only if for all subspaces G of X and all $x \in X$,*

$$T(P_{G,\epsilon}(x)) = P_{T(G),\epsilon}(T(x)). \tag{4.5}$$

where $P_{G,\epsilon}(x) = \{g_0 \in G : \|x - g_0\| \leq \|x - g\| \quad \text{for all } g \in G\}$

Theorem 4.1.5. *[45] Let $T : X \to X$ be preserving approximation, where X is a normed space, then $T = kU$ such that $k \in \mathbb{R}$ and U is an isometry.*

Lemma 4.1.6. *[45] Let X be a normed linear space and $\epsilon > 0$. Then every isometry operator $T : X \to X$ is $\epsilon-$preserving approximation.*

Theorem 4.1.6. *[45] Suppose $(X, \|.\|)$ and $(Y, \|.\|)$ be two normed linear spaces, $\epsilon > 0$ and $T : X \to Y$ is onto preserving $\epsilon-$approximation.*

(i) If T is linear, then

$$\forall x, y \in X, x \perp_\epsilon y \Rightarrow T(x) \perp_\epsilon T(y)$$

(ii) For a subspace G of X, $T(\widehat{G_\epsilon}) = \widehat{T(G)_\epsilon}$.

4.2 Some results on best approximation

Theorem 4.2.1. *Let $(X, \|.\|)$ be a normed linear space and if $\forall f \in X \exists g_0 \in G : f - g_0 \perp_P G$, then $g_0 \in P_G(f)$.*

Proof. Let $f \in X$ and $g \in G$ and $f - g_0$ is Pythagorean orthogonal to g. Then

$$\begin{aligned}
&\|f - g_0 - g\|^2 = \|f - g_0\|^2 + \|g\|^2 \\
\Rightarrow &\|f - g_0\|^2 \leq \|f - g_0 - g\|^2 \\
\Rightarrow &\|f - g_0\| \leq \|f - g_0 - g\|
\end{aligned}$$

Setting $g + g_0 = h$, we get

$$\|f - g_0\| \leq \|f - h\| \quad \Rightarrow g_0 \in P_G(f).$$

□

Theorem 4.2.2. *Let $(X, \|.\|)$ be a normed linear space and $T : X \to X$ is an isometry operator. Then $T(g_0) \in T(P_G(x))$ if and only if $T(x) - T(g_0) \perp_B T(G)$.*

Proof. The proof has a similar idea to that of Mazaheri and Zadeh [45]. Assume $T(g_0) \in T(P_G(x))$ and $T(g_1) \in T(G)$. Then

$$\|T(x) - T(g_0)\| \leq \|T(x) - T(g_1)\|$$

Put $T(g_1) = T(g_0) - \alpha T(g)$ for any fixed $T(g) \in T(G)$ and $\alpha \in \mathbb{R}$. Then we have,

$$\begin{aligned}\|T(x) - T(g_0)\| &\leq \|T(x) - (T(g_0) - \alpha T(g))\| \\ &= \|(T(x) - T(g_0) + \alpha T(g))\|\end{aligned}$$

This shows that $T(x) - T(g_0)$ is Birkhoff orthogonal to $T(G)$.

Conversely assume $T(x) - T(g_0) \perp_B T(G)$. Then for all $\alpha \in \mathbb{R}$ and $T(g_0) \in T(G)$, we have

$$\|t(x) - T(g_0)| \leq \|T(x) - T(g_0) + \alpha T(g_1)\| \tag{4.6}$$

Let $T(g) \in T(G)$ be arbitrary and fixed and taking $T(g_1) = T(g_0) - T(g)$ and $\alpha = 1$ in the inequality (4.6), we get

$$\begin{aligned} & \|T(x) - T(g_0)\| \leq \|T(x) - T(g)\| \\ \Rightarrow \quad & T(g_0) \in T(P_G(x))\end{aligned}$$

□

Theorem 4.2.3. *Let $(X, \|.\|)$ be a normed linear space and $T : X \to X$ is an isometry operator. Then $T(g_0) \in T(P_G(x, \epsilon))$ if and only if $T(x) - T(g_0) \perp_B^{\epsilon} T(G)$.*

Proof. The proof has a similar idea to that of Mazaheri and Zadeh [45]. Assume $T(g_0) \in T(P_G(x, \epsilon))$ and $T(g_1) \in T(G)$. Then

$$\|T(x) - T(g_0)\| \leq \|T(x) - T(g_1)\| + \epsilon$$

Put $T(g_1) = T(g_0) - \alpha T(g)$ for any fixed $T(g) \in T(G)$ and $\mid \alpha \mid \leq 1$. Then we have,

$$\begin{aligned}\|T(x) - T(g_0)\| &\leq \|T(x) - (T(g_0) - \alpha T(g))\| + \epsilon \\ &= \|(T(x) - T(g_0) + \alpha T(g))\| + \epsilon\end{aligned}$$

This shows that $T(x) - T(g_0)$ is approximate Birkhoff orthogonal to $T(G)$.

Conversely assume $T(x) - T(g_0) \perp_B^{\epsilon} T(G)$. Then for all α with $\mid \alpha \mid \leq 1$ and $T(g_0) \in T(G)$, we have

$$\|t(x) - T(g_0)| \leq \|T(x) - T(g_0) + \alpha T(g_1)\| + \epsilon \tag{4.7}$$

Let $T(g) \in T(G)$ be arbitrary and fixed and taking $T(g_1) = T(g_0) - T(g)$ and $\alpha = 1$ in the inequality (4.7), we get

$$\begin{aligned} & \|T(x) - T(g_0)\| \leq \|T(x) - T(g)\| + \epsilon \\ \Rightarrow \quad & T(g_0) \in T(P_G(x, \epsilon)) \end{aligned}$$

$\square$

Recall that

$$\begin{aligned} R_G(x) &= \{g_{\in} G : \|g_0 - g\| \leq \|x - g\| \quad \text{for all } g \in G\} \quad \text{and} \\ R_G(x, \epsilon) &= \{g_{\in} G : \|g_0 - g\| \leq \|x - g\| + \epsilon \quad \text{for all } g \in G\} \end{aligned}$$

are the set of all co-approximation and $\epsilon-$co-approximation of x in G. If $T : X \to X$ is an isometry operator, then for all subspace G of X and all $x \in X$, we have $\|T(x)\| = \|x\|$. Keeping this in mind and using the similar concepts of (Theorem 4.2.2 and Theorem4.2.3), we prove the following lemma.

Lemma 4.2.1. *Let $(X, \|.\|)$ be a normed linear space and $T : X \to X$ is an isometry operator.. Then $T(g_0) \in T(R_G(x, \epsilon))$ if and only if $T(g) \perp_{\epsilon} T(x) - T(g_0)$.*

Proof. Assume $T(g_0) \in T(R_G(x, \epsilon))$. Then

$$\|T(g_0) - T(g^{'})\| \leq \|T(x) - T(g^{'})\| + \epsilon \tag{4.8}$$

For $\mid \beta \mid \leq 1$ and $\beta \neq 0$, put $T(g^{'}) = T(g_0) - \frac{1}{\beta} T(g)$. Then the inequality (4.8) becomes

$$\begin{aligned} \|T(g_0) - T(g_0) + \frac{1}{\beta} T(g)\| &\leq \|T(x) - T(g_0) + \frac{1}{\beta} T(g)\| + \epsilon \\ \Rightarrow \quad \|\frac{1}{\beta} T(g)\| &\leq \|T(g) + \beta(T(x) - T(g_0))\| + \epsilon\beta \\ \Rightarrow \quad \|T(g)\| &\leq \|T(g) + \beta(T(x) - T(g_0))\| + \epsilon \\ \Rightarrow \quad T(g) &\perp_{\epsilon} T(x) - T(g_0) \end{aligned}$$

Conversely assume that $T(g) \perp_{\epsilon} T(x) - T(g_0)$, then for all $| \beta | \leq 1$ and $T(g^{'}) \in T(G)$, we have

$$\|T(g^{'})\| \leq \|\beta(T(x) - T(g_0)) + T(g^{'})\| + \epsilon \tag{4.9}$$

Letting $T(g) \in T(G)$ and putting $T(g^{'}) = T(g_0) - T(g)$ and $\beta = 1$ in (4.9) to get the desired result showing that $T(g_0) \in T(R_G(x, \epsilon))$.

$\square$

Chapter 5

SMMARY AND CONCLUSION

5.1 Summary

The main part of the thesis begins from chapter 2 and ends with chapter 4. In chapter 2, the classical definition of orthogonality by Roberts has been generalized in terms of p-HH norm by proving (Theorem:2.2.2) and (Theorem:2.2.3). Using the concept of Lemma 2.2.1, an important concept in relation to the Robert and Birkhoff-James orthogonality has been proved in Theorem 2.2.4 and Theorem 2.3.40. A relation in regard to the equivalency of homogeneity and additivity of the isosceles orthogonality have been proved in the Throrem 2.4.15. In the case of bounded linear operators on Hilbert space H, Theorem 2.4.16, Proposition 8 and Proposition 9 have been proved as a new result. As a generalized concept of the isosceles and Pythagorean orthogonality, Carlsson played the central role to introduce new concept of orthogonality. After that, Kikianty and Dragomir generalized the concept of Carlsson's orthogonality in terms of p-HH norm. Motivated by both concepts, an attempt was made to introduce the Carlsson's orthogonality in terms of bounded linear operators and proved Theorem 2.6.9, Theorem 2.6.10, Theorem 2.6.11 and Theorem 2.6.12 in relation to this orthogonality.

Chapter 3 is significant in this research work due to the fact that some new concepts of orthogonality have been introduced with the help of the Carlsson orthogonality; and different properties of inner-product space in relation to these orthogonalities are also verified. Motivated by this concept, two new particular cases of the Carlsson orthogonality were introduced by taking real and complex constants, and Proposition 12 with Corollary 3.1.1, Proposition 13 with corollary 3.1.2, Lemma 3.1.1 and Lemma 3.1.2 were proved.

Furthermore, the orthogonality relation 3.6 in terms of 2-HH norm were introduced and 2-HH-N orthogonality via 2-HH norm as unique in any normed space has been proved. The orthogonality relation 3.6 in terms of bounded linear operators was also introduced and it has been proved that such orthogonality implies the Birkhoff-James orthogonality, however, the converse may not be true in the Theorem 3.2.2. For the norm attainment set of T (i. e. for any $x \in M_T$), the Theorem 3.2.3 with Corollary 14, and Proposition 14 has been proved.

The application part of orthogonality, which can be linked with the best approximation in normed linear space is included in chapter 4.The concept of best approximation amounts to the problem of minimizing distance. The Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3, and Lemma 4.2.1 have been proved in the set of approximation (resp. co-approximation) and $\epsilon-$approximation (resp. $\epsilon-$co-approximation) .

5.2 Conclusion

It is concluded that the Pythagorean and the isosceles orthogonalities via $2 - HH$ norm on a normed linear space satisfies the non-degeneracy, continuity and symmetry property; moreover, the homogeneity and additivity of the isosceles orthogonality with respect to $2 - HH$ norm are equivalent. It is found that the Carlsson orthogonality for norm attaining bounded linear operators is continuous. In addition, the Pythagorean orthogonality implies the Birkhoff-James orthogonality. In the case of norm attaining bounded linear operators on a Hilbert space H with disjoint support, the Carlsson orthogonality is symmetric and the isosceles orthogonality is equivalent to the Pythagorean orthogonality. Finally, the orthogonality of bounded linear operators in a real Hilbert space H implies Birkhoff-James orthogonality, but the converse may not be true. Regarding the uniqueness property, 2-HH-N orthogonality through the medium of 2-HH norm is unique in any normed space. Furthermore, the 2-HH-N orthogonality in terms of bounded linear operators implies the Birkhoff-James orthogonality, but the converse may not be true. In the case of isometry operator $T : X \to X$, the best approximation (resp. $\epsilon-$best approximation) of an element $T(g_0)$ is equivalent to orthogonality (resp. $\epsilon-$orthogonality) of images.

5.3 Recommendation for the future work

First of all, there are some properties of orthogonality still remained to check in the case of new special case of the Carlssion orthogonality in terms of of 2-HH norm and bounded linear operators. Secondly, the properties like: existence, uniqueness, additivity etc. are also not studied for the generalized Carlsson orthogonality in terms of operators. Finally, on the basis of this research, there may have several chances of connecting orthogonality in the theory of best approximation. Some of the possibilities are as follows:

- Study of existence and uniqueness property of new orthogonality in normed linear space;
- Use of orthogonalities like: isosceles, Pythagorean, Carlsson and new orthogonalities in the theory of best approximation;
- Study of homogeneity property of new new orthogonality through the medium of 2-HH norm and bounded linear operators;
- Study of existence and uniqueness property of Birkhoff-James orthogonality in terms bounded linear operators.

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APPENDIX

PUBLICATIONS

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2. Ojha, B.P., Bajracharya, P.M., Mishra.V N.: On uniqueness of new orthogonality via 2-HH norm in normed linear space, *Journal of Functions Spaces*, Volume 2020: https://doi.org/10.1155/2020/8835492.
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CONFERENCES/ WORKSHOPS/ SEMINARS

1. National Conference on Mathematics and Its Application, January 11-13, 2017, Chitwan, Nepal(presented paper entitled " Different Concepts of Derivatives".

2. 7^{th} National Conference on Mathematics and Its Applications, January 12-15, 2019, Butwal, Nepal (presented paper entitled " Some Particular Cases of Carlsson's Orthogonality on Normed Linear Spaces).

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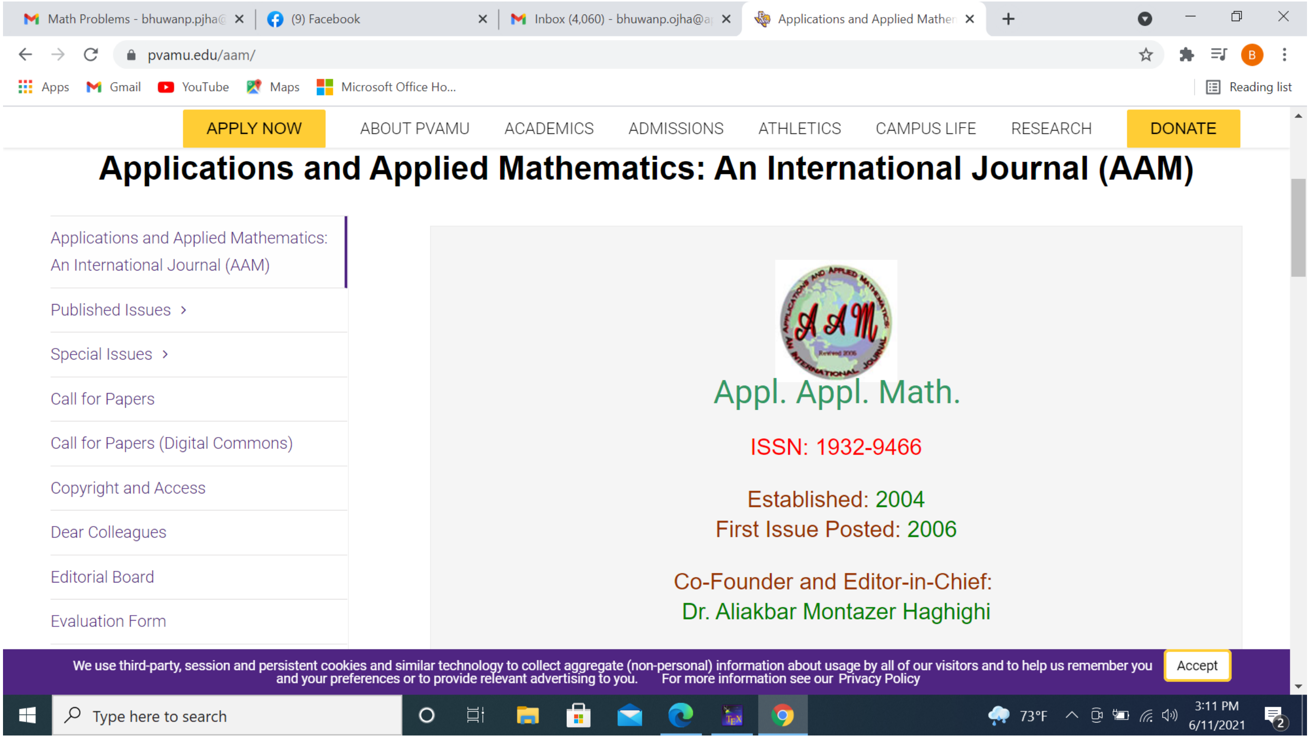

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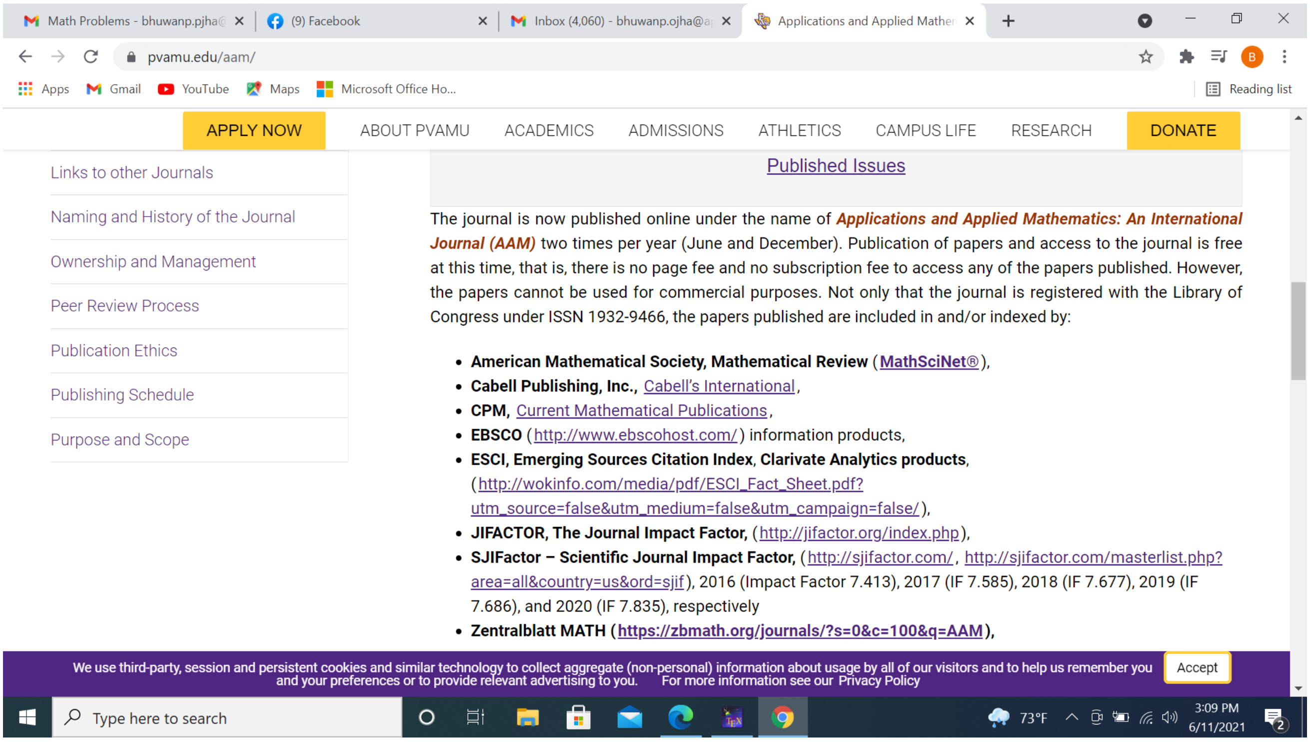

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Research Article

On Uniqueness of New Orthogonality via 2-HH Norm in Normed Linear Space

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This paper generalizes the special case of the Carlsson orthogonality in terms of the 2-HH norm in real normed linear space. Dragomir and Kikianty (2010) proved in their paper that the Pythagorean orthogonality is unique in any normed linear space, and isosceles orthogonality is unique if and only if the space is strictly convex. This paper deals with the complete proof of the uniqueness of the new orthogonality through the medium of the 2-HH norm. We also proved that the Birkhoff and Robert orthogonality via the 2-HH norm are equivalent, whenever the underlying space is a real inner-product space.

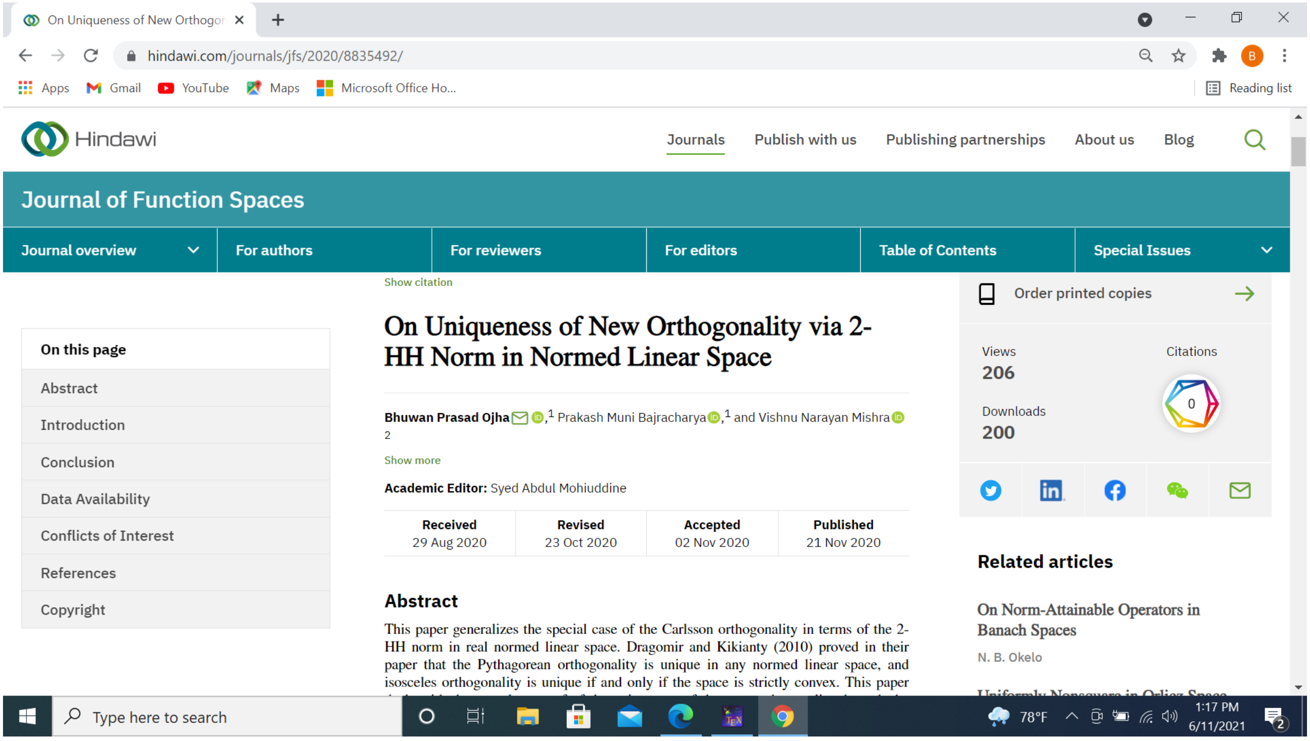
1. Introduction

Different notions of orthogonality in normed linear spaces have been developed by various mathematicians. As a generalization of orthogonality from inner product space to normed linear space "x is orthogonal to y if and only if $\|x+\lambda y\| = \|x-\lambda y\|$ identically in λ" was suggested by Robert ([1, 2]). However, it has the weakness that for some normed linear space, at least one of every pair of orthogonal elements would have to be zero, i.e., $\|x+\lambda y\| = \|x-\lambda y\|$ for all λ only if $x=0$ or $y=0$. This difficulty is not experienced in the isosceles, Pythagorean, and Birkhoff orthogonalities.

To study the difference of orthogonality in the complex case in comparison with the real case, Paul et al. in 2018 came with a new concept of Birkhoff-James orthogonality by introducing new definitions on complex reflexive Banach spaces and introduced more than one equivalent characterization of Birkhoff-James orthogonality of compact linear operators in the complex case [3]. In 1945, James came with the concept of the Pythagorean and isosceles orthogonalities, which characterize inner product space via their homogeneity and additivity [4]. James also discussed the existence property of isosceles orthogonality type. The property of the uniqueness of isosceles orthogonality was not discussed until Kapoor and Prasad's paper was published. They proved that the Pythagorean orthogonality is unique in any normed linear space; however, the isosceles orthogonality is unique if and only if the space is strictly convex [5].

Carlsson introduced a more general type of orthogonality treating the isosceles and Pythagorean orthogonalities are special cases [6]. Martini and Wu showed many interesting connections between the Birkhoff and isosceles orthogonality. They proved that if a linear map preserves the Birkhoff orthogonality, then it also preserves the isosceles orthogonality [7]. In 2007, Alsina and Tomas gave a different characterization of the inner product space with the help of weaker linearity axioms of the scalar product and Pythagoras/isosceles orthogonality [8].

Using the concept of the p-HH norm as described in the paper [9], Kikianty and Dragomir came up with a new notion of orthogonality with the help of the 2-HH norm, which is closely related to the Pythagorean and isosceles orthogonalities [10]. They proved that the Pythagorean orthogonality via 2-HH norm satisfies the nondegeneracy, continuity, and symmetry properties; however, it is neither additive nor



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Bhuwan Prasad Ojha,[1] Prakash Muni Bajracharya,[1] and Vishnu Narayan Mishra[2]

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Abstract

This paper generalizes the special case of the Carlsson orthogonality in terms of the 2-HH norm in real normed linear space. Dragomir and Kikianty (2010) proved in their paper that the Pythagorean orthogonality is unique in any normed linear space, and isosceles orthogonality is unique if and only if the space is strictly convex. This paper

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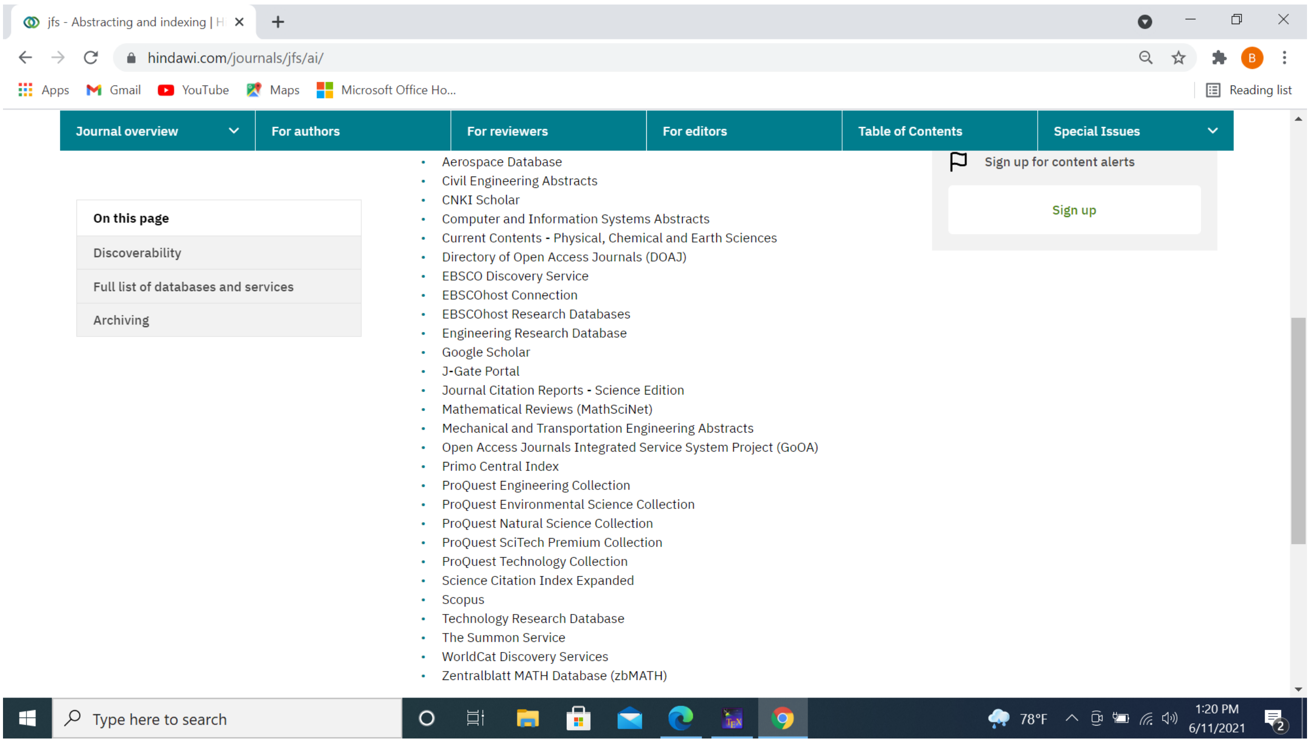

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A GLIMPSE ON BIRKHOFF-JAMES ORTHOGONALITY IN BANACH SPACES

B. P. OJHA AND P. M. BAJRACHARYA

Abstract. This paper is an overview of various results on Birkhoff-James orthogonality of operators in Hilbert space and Banach spaces. We mainly focus on Birkhoff orthogonality of linear(bounded and ompact) operators in terms of matrices, projection angles, Hilbert C^*-modules as well as on Banach modules. The article concludes with some open problems regarding possible correlation between Birkhoff-James orthogonality and Carlsson orthogonality, particularly in the case of Pythagorean orthogonality.

Дано огляд різноманітних результатів щодо ортогональності в сенсі Біркгофа-Джеймса операторів у гільбертових і банахових просторах. Переважно розглядається ортогональність за Біркгофом лінійних (обмежених і компактних) операторів у термінах матриць, кутів, гільбертових С*-модулів, а також банахових модулів. Наведені деякі відкриті питання стосовно співвідношень ортогональністю Біркгофа-Джеймса та ортогональністю Карлссона, зокрема для випадку піфагорової ортогональності.

1. Introduction

The concept of Birkhoff orthogonality began in 1935 [1]. In the literature of orthogonality this is known with some other names such as; Birkhoff- James orthogonality and Blaschke Birkhoff-James orthogonality (see [2]). In this paper [1, 3], an orthogonality which satisfies homogeneity but neither symmetric nor additive is defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all λ, is known as Birkhoff orthogonality or Birkhoff-James orthogonality. The geometrical meaning of Birkhoff orthogonality is that if x is an unit vector of a Banach space X and $y \in X$, then x is Birkhoff orthogonal to y means that the straight line $\{x + \lambda y : \lambda \in K\}$ is tangent to the unit ball of X at x. This concept is similar to the statement: suppose two lines l_1 and l_2 intersect at the point m, then $l_1 \perp l_2$ if and only if the distance from a point of l_2 to a given point n of l_1 is never less than the distance from m and n. [3] For any hyper-plane $H \subset X$, x is said to be orthogonal to H if $\forall x \in H, x \perp h$.

Bhatia and Semrl in [4] generalize the definition of Birkhoff orthogonality in terms of matrices. For any matrices A and B they denote the symbol $\|A\|$ for operator norm of A and A is orthogonal to B in the sense of Birkhoff-James iff for any complex number z, $\|A + zB\| \geq \|A\|$. A matrix A is orthogonal to B iff there exist a unit vector $x \in H$ such that $\|Ax\| = \|A\|$ and $\langle Ax, Bx \rangle = 0$ [4]. They also introduced Birkhoff- James orthogonality in [4] as $A \perp B$ if and only if $\|A + zB\|_p \geq \|A\|_p$, where $\|A\|_p$ denotes Schatten p-norm of A defined by $\|A\|_p = [\sum_{j=1}^{n} S_j(A)^p]^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $S_1(A) \geq$ ......$S_n(A)$ are singular values of A. Taking the special case for $p = 2$, Bhatia and Semrl in [4] also proved that the given orthogonality is equivalent to usual Hilbert space condition $\langle A, B \rangle = 0$, which defines an inner-product on the space of matrices as $\langle A, B \rangle = tr(A^*B)$. The norm associated to this inner product is $\|.\|_2$. In an infinite dimensional case [4], for

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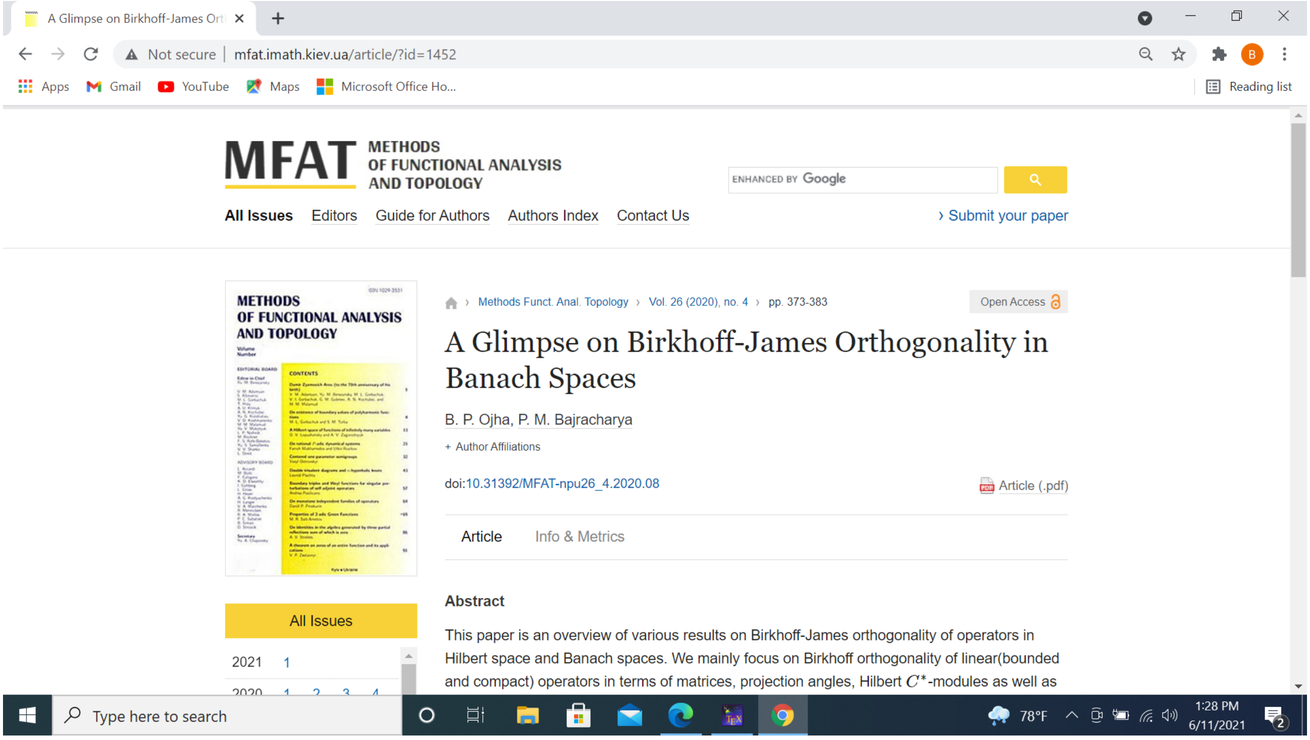
Keywords. Birkhoff-James orthogonality, Banach modules, Carlsson orthogonality, Robert orthogonality.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B. P. Ojha, P. M. Bajracharya
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Abstract
This paper is an overview of various results on Birkhoff-James orthogonality of operators in Hilbert space and Banach spaces. We mainly focus on Birkhoff orthogonality of linear(bounded and compact) operators in terms of matrices, projection angles, Hilbert C*-modules as well as
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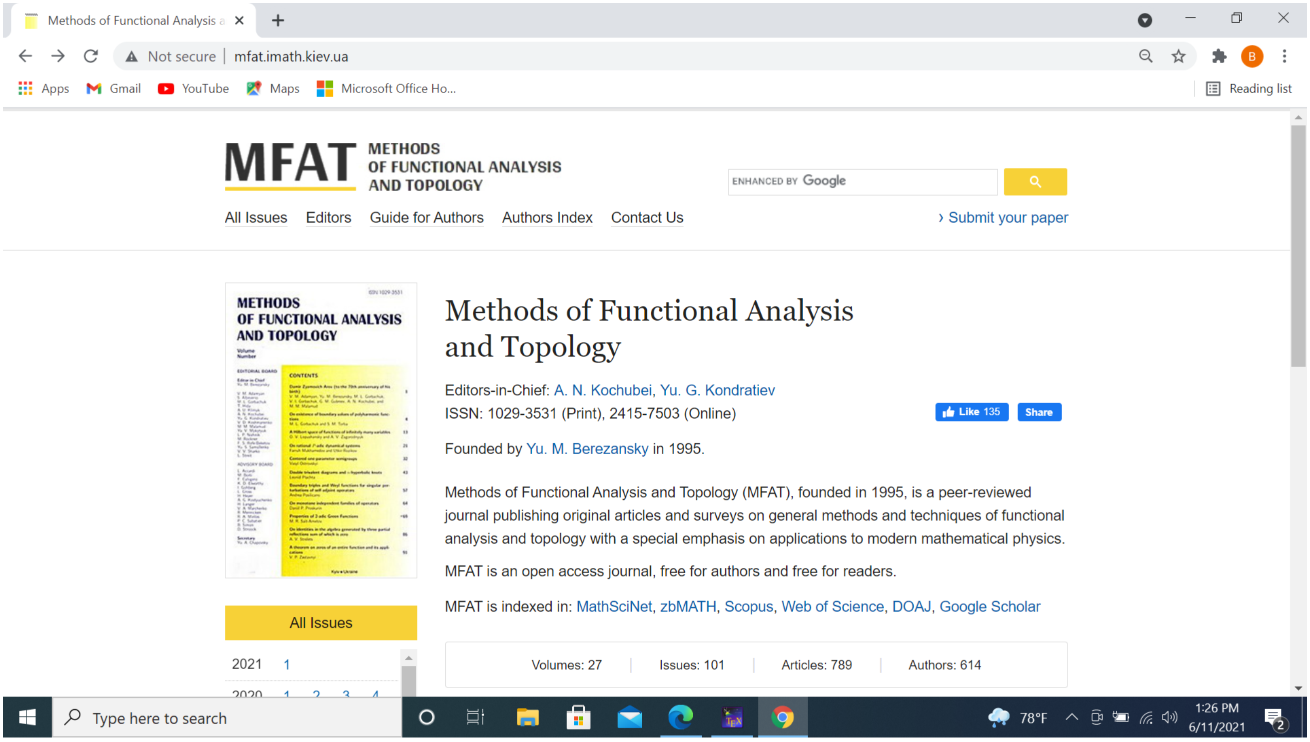
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Original Research Paper

Birkhoff Orthogonality and Different Particular Cases of Carlsson's Orthogonality on Normed Linear Spaces

Prakash Muni Bajracharya and Bhuwan Prasad Ojha

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Abstract: Let $x, y \in X$, where X is an inner-product space. We say x is orthogonal to y if $\langle x, y\rangle = 0$. When we move to general normed spaces there are many possibilities of extending the notion of orthogonality. Since 1934, different types of orthogonality relations in normed spaces have been introduced and studied. In this study, we enlist some properties of Birkhoff's orthogonality and Carlsson's orthogonality along with it we introduce two new particular cases of Carlsson's orthogonality and check some properties of othogonality in relation to these particular cases in normed spaces.

Keywords: Birkhoff Orthogonality, Carlsson Orthogonality, Minkowski Plane, Pythagorean Orthogonality, Robert Orthogonality

Introduction

Let X^* be dual space of a normed space X and S_X denotes the unit sphere of X given by $S_X = \{x \in X: \|x\| = 1\}$. The Hermite-Hadamard's inequality defined in (Mihai *et al*., 2018) is given by:

$$(b-a)\, f\left(\frac{a+b}{2}\right) \le \int_a^b f(t)\,dt \le (b-a)\frac{f(a)+f(b)}{2}.$$

We briefly describe the content of this paper: Section 1 contains the review of various properties and characterizations of Birkhoff orthogonality in normed linear spaces. Section 2 contains Carlsson orthogonality in classical sense and in HH-sense with the help of *p-HH* norm. Section 3 includes the main results of our research by introducing new cases of Carlsson orthogonality verifying some properties of the inner-product space in relation of these orthogonalities.

There are different orthogonality relations studied in the general normed linear space since from 1934. The Robert orthogonality condition which was introduced by Robert in 1934 is known as the first orthogonality defined in general normed spaces (Birkhoff, 1935). Robert orthogonality implies both Birkhoff orthogonality introduced by (Birkhoff, 1935) and isosceles orthogonality introduced by (James, 1945). Generalizing the Isosceles, Robert and Pythagorean orthogonalities in the normed space, (Carlsson, 1962) introduced a new type of orthogonality. After that numerous notions of orthogonality have been introduced. Kikianty and Dragomir (2010) introduced *p-HH*-norms ($1 \le p < \infty$) and some notions of orthogonality have been introduced by utilizing 2-*HH*-norm, which are closely related to the classical Pythagorean orthogonality and isosceles orthogonality. Kapoor and Prasad (1978) proved uniqueness property of isosceles orthogonality.

Mizuguchi (2017) let $(X, \|\cdot\|)$ be a real normed space. For any vector $x, y \in X$, we say that x is orthogonal to y ($x \perp_B y$) in the sense of Birkhoff if:

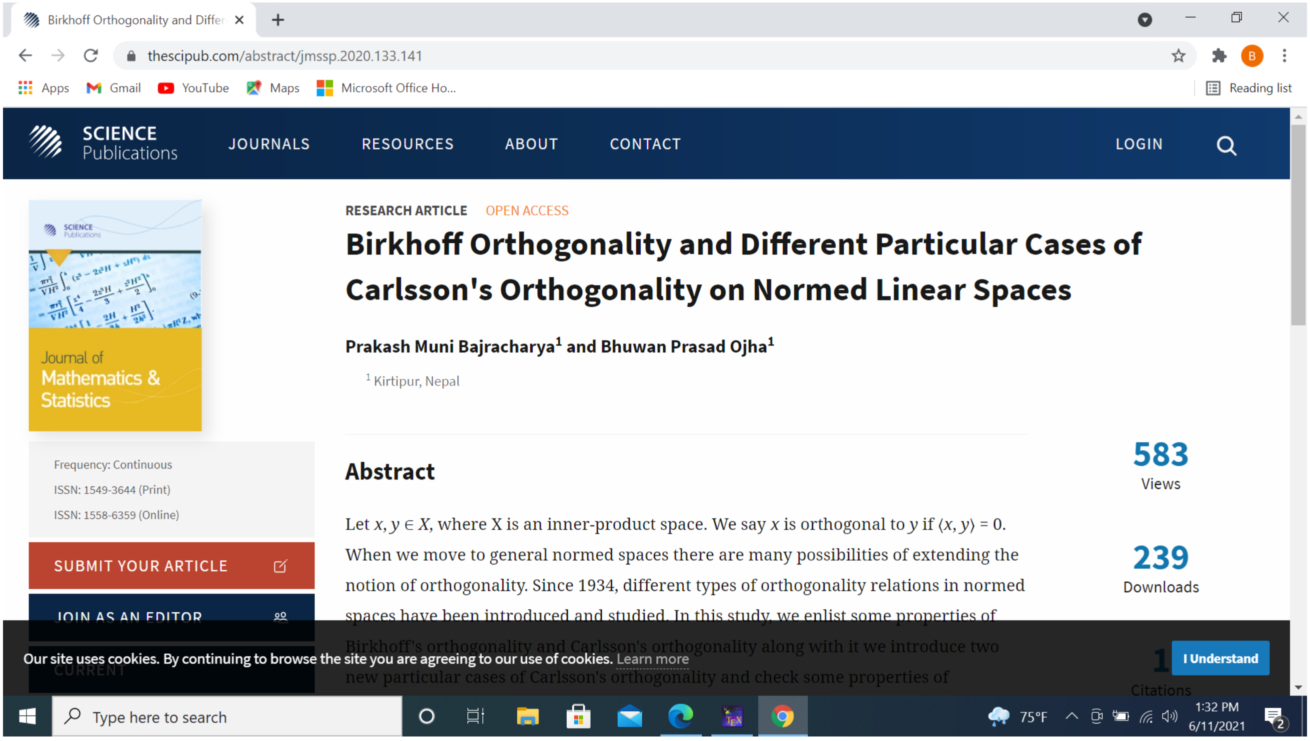
$$\|x\| \le \|x + \lambda y\| \ for\ all\ \lambda \in \mathbb{R}$$

James was the first who did a comprehensive study of the properties of Birkhoff orthogonality and therefore Birkhoff orthogonality is also known to as Birkhoff-James orthogonality. James (1945) introduced Isosceles and Pythagorean orthogonality proved that if for every x it is possible to find a vector y in a two dimensional subspace containing x such that $x \perp_B y$; then the space is necessarily an inner product space (Dragomir and Kikianty, 2010). If X is an inner product Space, then $\perp_B$ coincide with the standard orthogonality in inner product space. It is obvious that Birkhoff orthogonality is homogeneous; however, it is in general, neither symmetric nor additive.

Definition (Martini and Spirova, 2010)

A hyperplane of a normed linear space X is any proper closed linear subset H which is not properly contained in a proper linear subset of X, or any translation $x + H$ of such a linear subset H.

If $\{y_n\}$ is a sequence converging to y, x is orthogonal to $\{y_n\}$, then $x \perp y$. Hence for any x, the set



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Birkhoff Orthogonality and Different Particular Cases of Carlsson's Orthogonality on Normed Linear Spaces

Prakash Muni Bajracharya[1] and Bhuwan Prasad Ojha[1]

[1] Kirtipur, Nepal

Abstract

Let $x, y \in X$, where X is an inner-product space. We say x is orthogonal to y if $\langle x, y\rangle = 0$. When we move to general normed spaces there are many possibilities of extending the notion of orthogonality. Since 1934, different types of orthogonality relations in normed spaces have been introduced and studied. In this study, we enlist some properties of Birkhoff's orthogonality and Carlsson's orthogonality along with it we introduce two new particular cases of Carlsson's orthogonality and check some properties of

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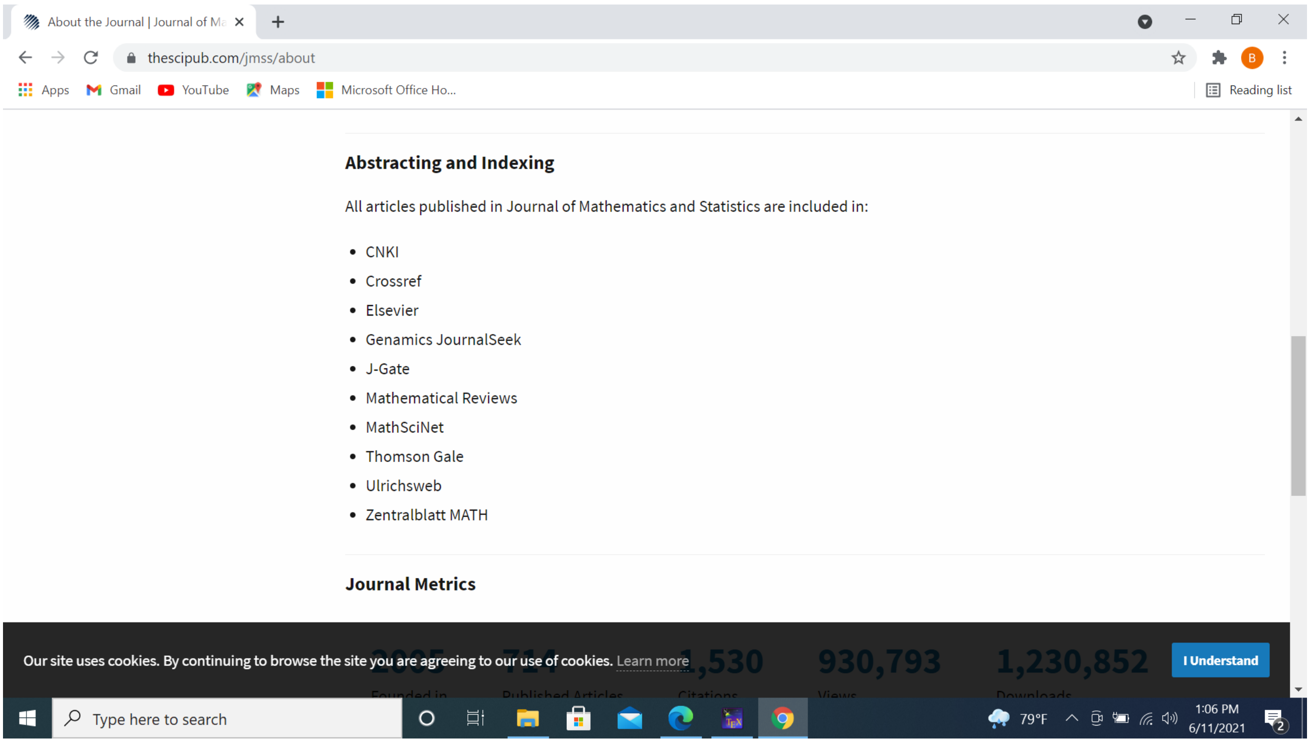

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RESEARCH PAPERS

2-HH NORM AND BIRKHOFF-JAMES ORTHOGONALITY IN NORMED SPACES

By

BHUWAN PRASAD OJHA *   PRAKASH MUNI BAJRACHARYA **

*-** *Department of Mathematics at Tribhuvan University, Kathmandu, Nepal.*

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ABSTRACT

For any normed space X, the p-HH norms X were introduced by Kikianty and Dragomir on $X^2 = X \times X$ of normed spaces. p-norms and p-HH norms induce the same topology, so they are equivalent, but are geometrically different. Besides that, E. Kikianty and S. S. Dragimor introduced HH-P orthogonality and HH-I orthogonality by using 2-HH norm and discussed main properties of these orthogonalities. The main purpose of this paper is to focus on the concept of 2-HH norm to Birkhoff and a new orthogonality in normed spaces, and we discuss some properties of these orthogonalities. It is proved that Robert orthogonality via 2-HH norm implies Birkhoff-James orthogonality via 2-HH norm; however, it is not necessary for the converse part.

Keywords: Birkhoff Orthogonality, Robert Orthogonality, p-HH Norm.

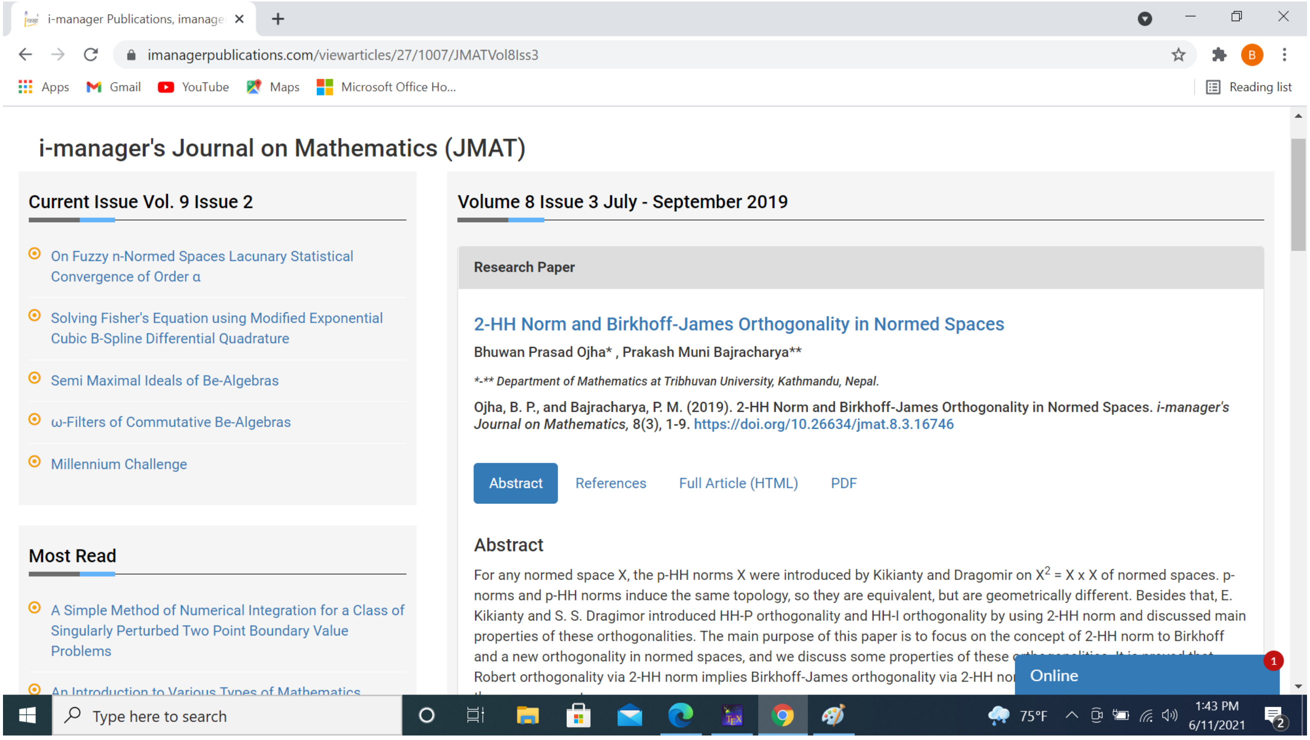
INTRODUCTION

Motivated by the Hermite-Hadamard inequality, Kikianty and Dragomir (2008) introduced p-HH norm by giving their equivalence to p-norms in X^2. Even though the equivalence of p-norms and p-HH norms are known, they are different due to the fact that p-HH norms not only depend on the size of vectors, but they depend on the relative position of the original vectors. Furthermore, these norms have been extended to X^n of a normed space X. To illustrate the quantitative comparison between p-norms and p-HH norms, Kikianty, in his thesis used the inequality $|h(x) - \frac{1}{b-a}\int_a^b h(t)\,dt| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\,M$ for a real valued function h(x) which is continuous and differentiable on (a, b) satisfying the condition $|h(x)| \le M$, where M is a real number.

Dragomir and Kikianty (2010), mentioned that the extension of sequence spaces also gives the fundamental difference of p-norms and p-HH norms. Kikianty and Dragomir (2008) mentioned in their paper that the classical norms can be extended from means on $(0, \infty)$ to normed space X in different ways; one calculates the norm of n vectors in X and then calculates the mean of resulting numbers and therefore these norms depend on the original vectors only through their norms which process is highly helpful to calculate the norm of X^n. The weighted arithmetic means are exponential in this case because of the fact that, one first computes a fixed linear combination of the original vectors and then gives the X-norm of the result which maintains the more of the structure of X^n, but a weighted arithmetic mean of non-zero vectors does not give the norm of X^n because its arithmetic mean may be zero. The p-HH norms keeps the responsiveness of the arithmetic means to the geometry of X^n due to the fact that their dependency not only depends on the size of vectors, also they depend on the relative position of the n original vectors in the space X.

In the paper of Kikianty and Sinnamon (2009), the p-norm is defined as follows: Let X be a normed space and for any positive integer n, $x = (x_1, \ldots, x_n) \in X^n$. Under the usual addition and scalar multiplication, it becomes a normed space when equipped with any of the following norms:

$$\|x\|_p = \begin{cases} (\|x_1\|^p + \|x_2\|^p + \ldots\ldots + \|x_n\|^p)^{\frac{1}{p}}, & 1 \le p < \infty \\ \max\{\|x_1\|, \|x_2\|, \ldots\ldots, \|x_n\|\}, & p = \infty \end{cases}$$



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2-HH Norm and Birkhoff-James Orthogonality in Normed Spaces

Bhuwan Prasad Ojha* , Prakash Muni Bajracharya**

*-** *Department of Mathematics at Tribhuvan University, Kathmandu, Nepal.*

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Abstract

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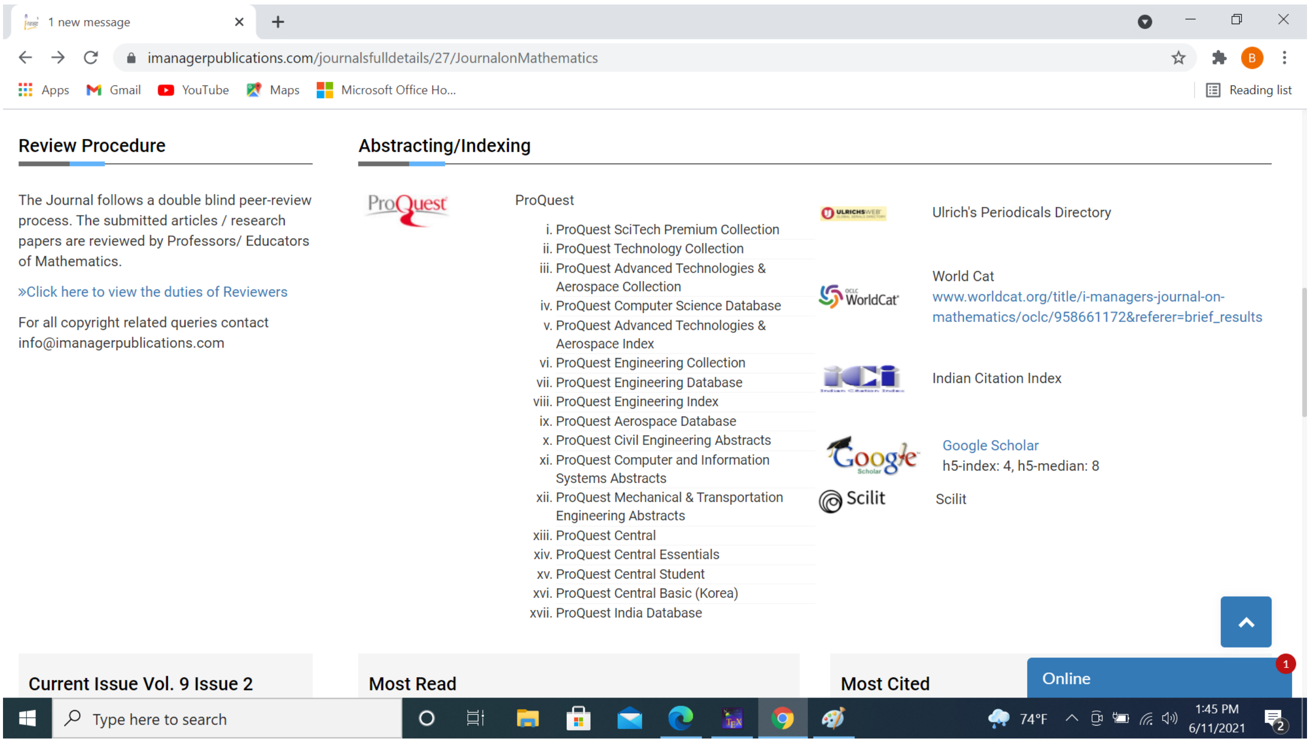

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Robert Orthogonality in Normed Linear Spaces Via 2-HH Norm

Bhuwan Prasad Ojha, Prakash Muni Bajracharya

Abstract

The p-HH norms on X^2 were introduced by Kikianty and Dragomir in 2008. Besides that, E. Kikianty and S.S. Dragimor introduced HH-P orthogonality and HH-I orthogonality by using 2-HH norm and discussed main properties of these orthogonalities. In this paper, we test the concept of 2-HH norm to Robert orthogonality in normed spaces and discuss some properties of this orthogonality.

Keywords: *Robert orthogonality, p-HH norm, Isosceles orthogonality, Pythagorean orthogonality, Hermite-Hadamards inequality* .

1 Introduction

The p-HH norms are equivalent to p-norms on X^2, as they induce the same topology, but geometrically they are different. The p-HH norm is an extension of the generalized logarithmic mean which is connected by the Hermite-Hadamards inequality to p-norm. The definition of the generalized logarithmic mean and Hermite-Hadamards inequality are as follows:

Definition. [6, 9] For any convex function $f : [a, b] \to \mathbb{R}([a, b] \subset \mathbb{R}$, the Hermite-Hadamard's inequality is defined as

$$(b-a)f(\frac{a+b}{2}) \leq \int_a^b f(t)dt \leq (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

. This inequality has been extended (see-12) for convex function $f : [x, y] \to \mathbb{R}$, where $[x, y] = \{(1-t)x + ty, t \in [0, 1]\}$. In that case Hermite-Hadamards integral inequality becomes

$$f(\frac{x+y}{2)} \leq \int_0^1 f\left[(1-t)x+ty\right]dt \leq \frac{f(x)+f(y)}{2} \qquad ......(1).$$

Using the convexity of $f(x) = \|x\|^p \quad (x \in X, p \geq 1)$ and relation (1) we have

$$\left\|\frac{x+y}{2}\right\| \leq \left[\int_0^1 \|(1-t)x+ty\|^p \, dt\right]^{\frac{1}{p}} \leq \frac{1}{2^{\frac{1}{p}}}(\|x\|^p + \|y\|^p)^{\frac{1}{p}}.$$

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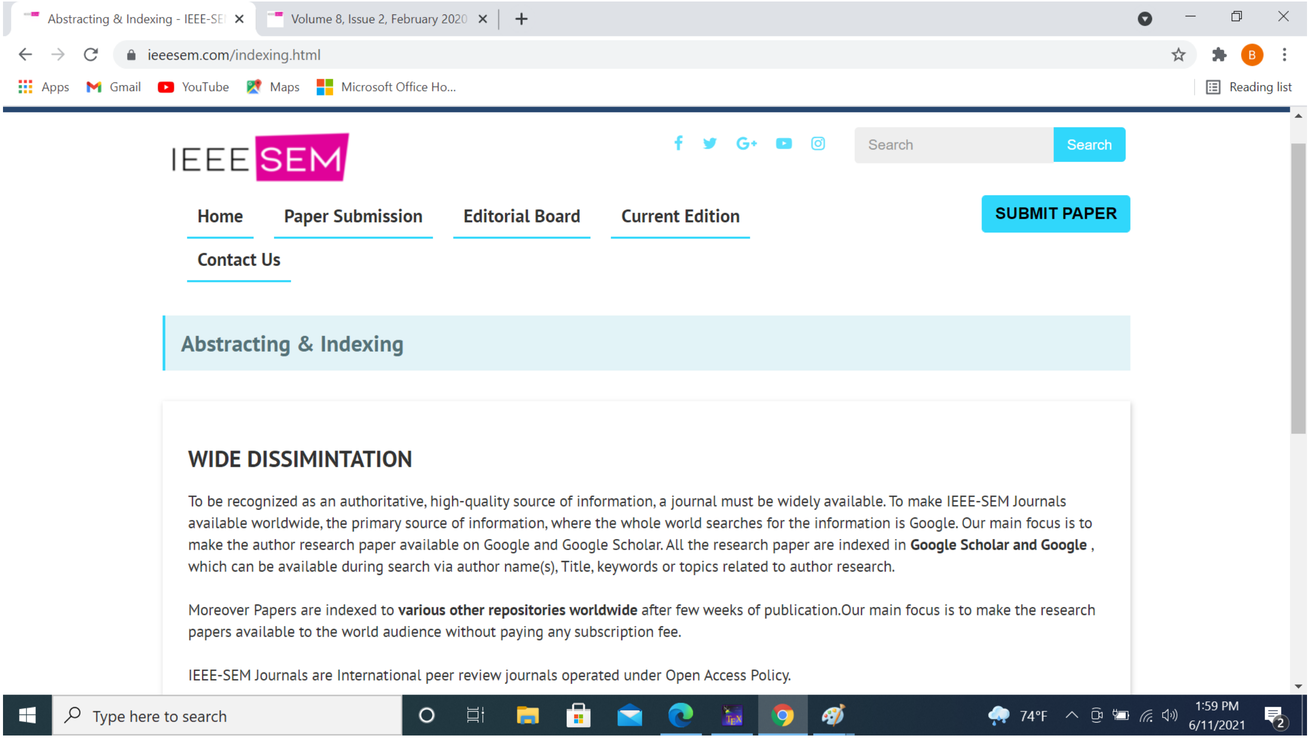

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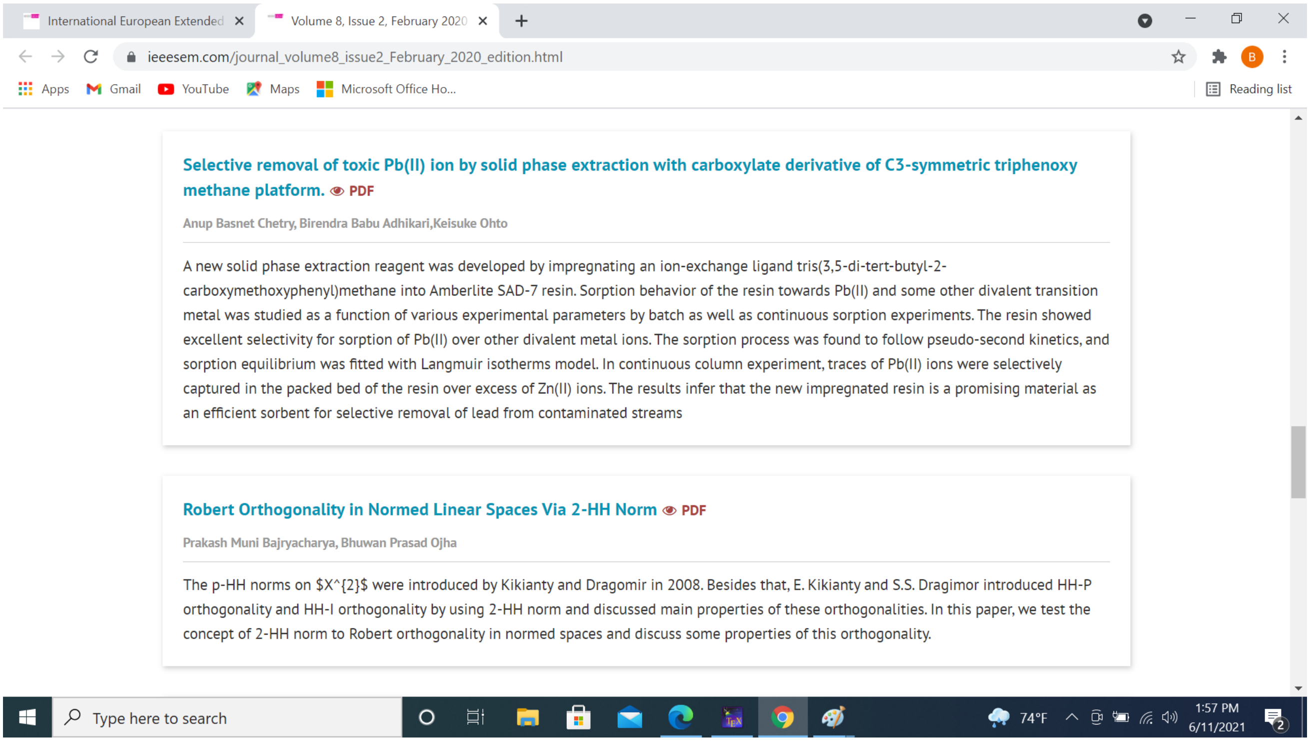

Selective removal of toxic Pb(II) ion by solid phase extraction with carboxylate derivative of C3-symmetric triphenoxy methane platform. PDF

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A new solid phase extraction reagent was developed by impregnating an ion-exchange ligand tris(3,5-di-tert-butyl-2-carboxymethoxyphenyl)methane into Amberlite SAD-7 resin. Sorption behavior of the resin towards Pb(II) and some other divalent transition metal was studied as a function of various experimental parameters by batch as well as continuous sorption experiments. The resin showed excellent selectivity for sorption of Pb(II) over other divalent metal ions. The sorption process was found to follow pseudo-second kinetics, and sorption equilibrium was fitted with Langmuir isotherms model. In continuous column experiment, traces of Pb(II) ions were selectively captured in the packed bed of the resin over excess of Zn(II) ions. The results infer that the new impregnated resin is a promising material as an efficient sorbent for selective removal of lead from contaminated streams

Robert Orthogonality in Normed Linear Spaces Via 2-HH Norm PDF

Prakash Muni Bajryacharya, Bhuwan Prasad Ojha

The p-HH norms on X^{2} were introduced by Kikianty and Dragomir in 2008. Besides that, E. Kikianty and S.S. Dragimor introduced HH-P orthogonality and HH-I orthogonality by using 2-HH norm and discussed main properties of these orthogonalities. In this paper, we test the concept of 2-HH norm to Robert orthogonality in normed spaces and discuss some properties of this orthogonality.

Birkhoff and New Orthogonality in Normed Linear Spaces Via 2-HH Norm

Bhuwan Prasad Ojha , Prakash Muni Bajracharya

Abstract

The p-HH norms were introduced by Kikianty and Dragomir on the Cartesian square of normed spaces . P-norms and p-HH norms induces the same topology, so they are equivalent, but geometrically they are different. Besides that, E. Kikianty and S.S. Dragimor introduced HH-P orthogonality and HH-I orthogonality by using 2-HH norm and discussed main properties of these orthogonalities. In this paper, we test the concept of 2-HH norm to Birkhoff and a new orthogonality in normed spaces and discuss some properties of these orthogonalities.

Keywords: *Birkhoff orthogonality,Hermite-Hadamard's inequality, Pythagorean orthogonality, p-HH norm, Logarithmic mean*

1 Introduction

An inner-product on X defines a norm on X by $\|x\|^2 = \langle x, x\rangle$. Every innerproduct spaces are normed spaces, but the converse may not be true. A best example of normed space which is not an inner-product space is $l^p = \{(x_n), x_n \in \mathbb{R} : \sum |x_n| < \infty\}$ for $p \neq 2$.

Definition. The $p - HH$ norm on $X^2 = X \times X$ is defined by

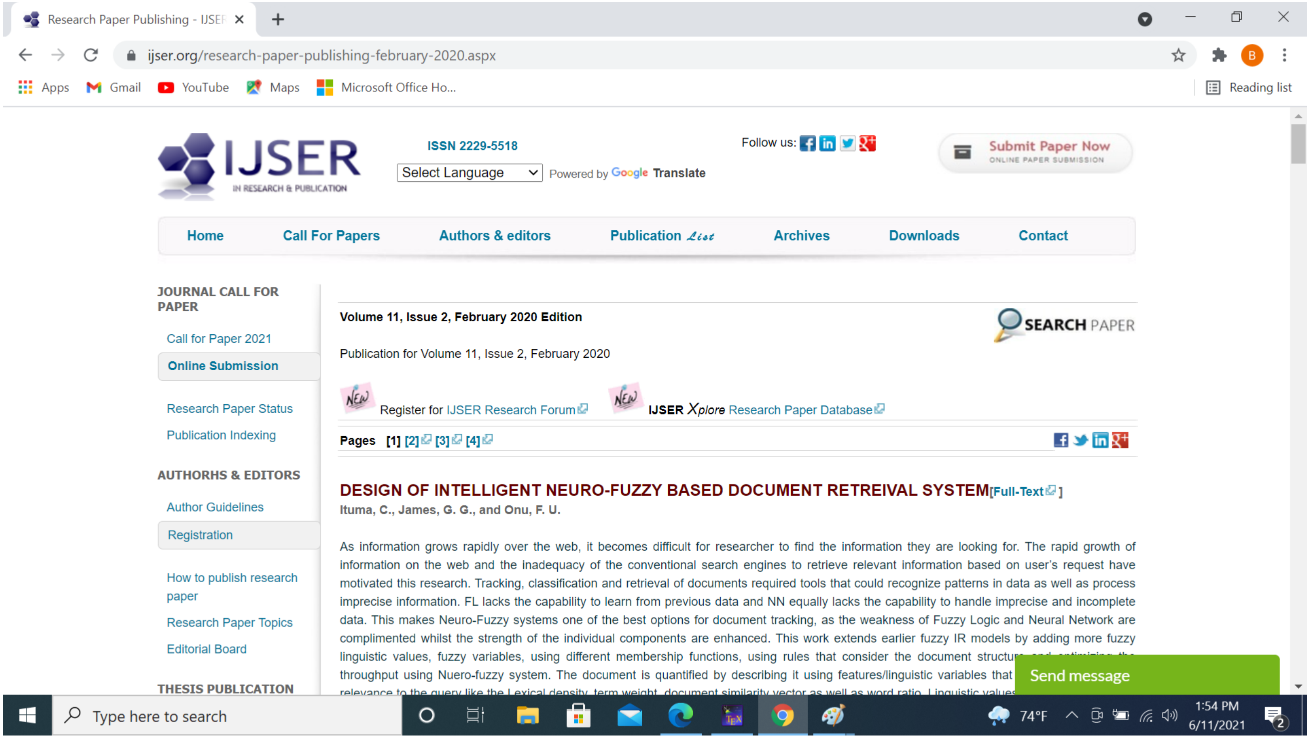
$$\|(x, y)\|_{p-HH} = (\int_0^1 \|(1-t)x + ty\|^p \, dt)^{\frac{1}{p}}$$

for any $x, y \in X^2$ and $1 \leq p < \infty$.

The 2-HH norm is defined as follows:

$$\begin{aligned}\|(x, y)\|_{2-HH}^2 &= \int_0^1 \|(1-t)x + ty\|^2 \, dt \\ &= \frac{1}{3}[\|x\|^2 + \langle x, y\rangle + \|y\|^2\end{aligned}$$

The p-HH norms are equivalent to p-norms on X^2, as they induce the same topology, but geometrically they are different. The p-HH norm is an extension of the generalized logarithmic mean which is connected by the Hermite-Hadamards inequality to p-norm. The definition of the generalized logarithmic mean and Hermite-Hadamards inequality are as follows:



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DESIGN OF INTELLIGENT NEURO-FUZZY BASED DOCUMENT RETREIVAL SYSTEM[Full-Text]

Ituma, C., James, G. G., and Onu, F. U.

As information grows rapidly over the web, it becomes difficult for researcher to find the information they are looking for. The rapid growth of information on the web and the inadequacy of the conventional search engines to retrieve relevant information based on user's request have motivated this research. Tracking, classification and retrieval of documents required tools that could recognize patterns in data as well as process imprecise information. FL lacks the capability to learn from previous data and NN equally lacks the capability to handle imprecise and incomplete data. This makes Neuro-Fuzzy systems one of the best options for document tracking, as the weakness of Fuzzy Logic and Neural Network are complimented whilst the strength of the individual components are enhanced. This work extends earlier fuzzy IR models by adding more fuzzy linguistic values, fuzzy variables, using different membership functions, using rules that consider the document structu... throughput using Nuero-fuzzy system. The document is quantified by describing it using features/linguistic variables that ...

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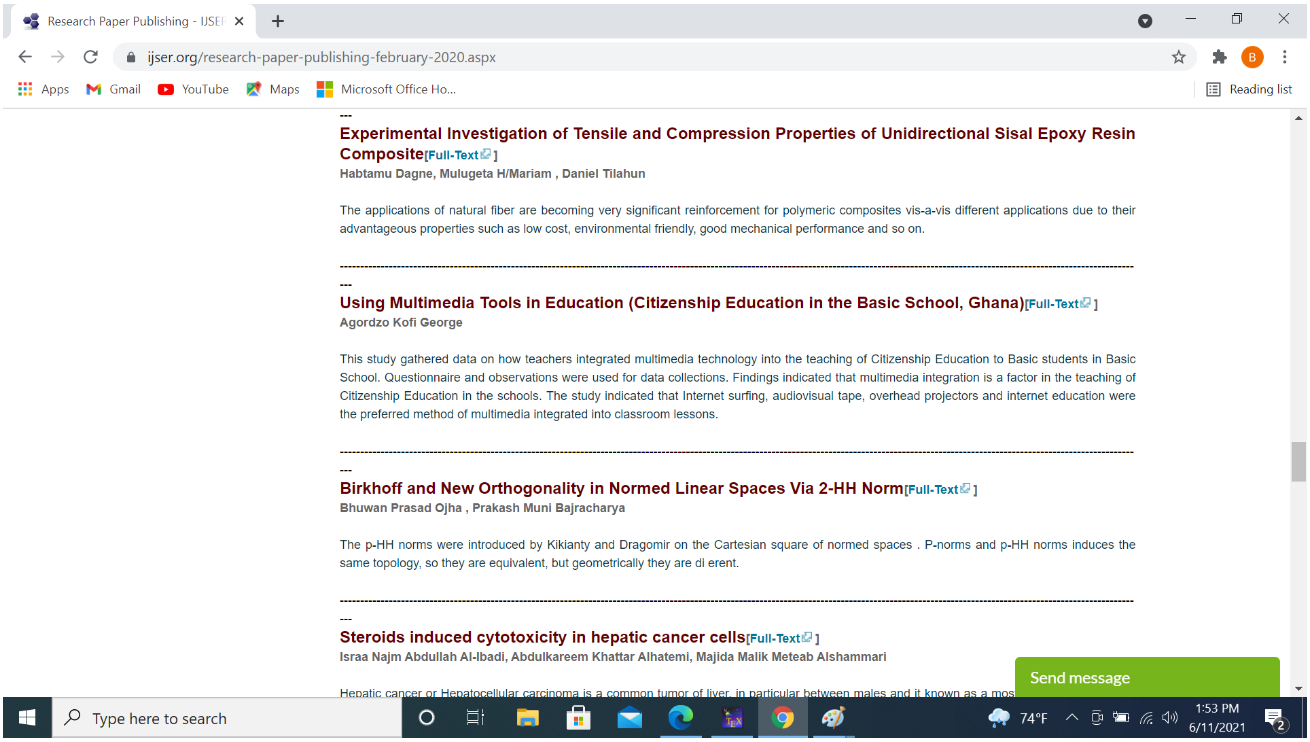

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Experimental Investigation of Tensile and Compression Properties of Unidirectional Sisal Epoxy Resin Composite[Full-Text]

Habtamu Dagne, Mulugeta H/Mariam , Daniel Tilahun

The applications of natural fiber are becoming very significant reinforcement for polymeric composites vis-a-vis different applications due to their advantageous properties such as low cost, environmental friendly, good mechanical performance and so on.

Using Multimedia Tools in Education (Citizenship Education in the Basic School, Ghana)[Full-Text]

Agordzo Kofi George

This study gathered data on how teachers integrated multimedia technology into the teaching of Citizenship Education to Basic students in Basic School. Questionnaire and observations were used for data collections. Findings indicated that multimedia integration is a factor in the teaching of Citizenship Education in the schools. The study indicated that Internet surfing, audiovisual tape, overhead projectors and internet education were the preferred method of multimedia integrated into classroom lessons.

Birkhoff and New Orthogonality in Normed Linear Spaces Via 2-HH Norm[Full-Text]

Bhuwan Prasad Ojha , Prakash Muni Bajracharya

The p-HH norms were introduced by Kikianty and Dragomir on the Cartesian square of normed spaces . P-norms and p-HH norms induces the same topology, so they are equivalent, but geometrically they are di erent.

Steroids induced cytotoxicity in hepatic cancer cells[Full-Text]

Israa Najm Abdullah Al-Ibadi, Abdulkareem Khattar Alhatemi, Majida Malik Meteab Alshammari

Hepatic cancer or Hepatocellular carcinoma is a common tumor of liver, in particular between males and it known as a mos...

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Relation of Pythagorean and Isosceles Orthogonality with Best approximation in Normed Linear Space

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Abstract

In an arbitrary normed space, though the norm not necessarily coming from the inner product space, the notion of orthogonality may be introduced in various ways as suggested by the mathematicians like R.C. James, B. D. Roberts, G. Birkhoff and S.O. Carlsson. We aim to explore the application of orthogonality in normed linear spaces in the best approximation. Hence it has already been proved that Birkhoff orthogonality implies best approximation and best approximation implies Birkhoff orthogonality. Additionally, it has also been proved that in the case of ε −orthogonality, ε −best approximation implies ε −orthogonality and vice-versa. In this article we established relation between Pythagorean orthogonality and best approximation as well as isosceles orthogonality and ε −best approximation in norned space.

Key words: Best approximation, Birkhoff orthogonality, Pythagorean orthogonality, ε −best approximation, Isosceles orthogonality.

Introduction

For any non-empty subset M of X, where X is a normed space, an element $m_0 \in$ M is called best best approximation to x $\in$ X from M if $\forall$ m $\in$ M, ||x-m_0||≤||x-m||. The collection of all such elements $m_0 \in$ M which are best approximation to x$\in$ X is denoted by P_M(x). If P_M(x) contains at least one element, then the subset M is called a proximal set. If for each x $\in$ X has a unique best approximation in M, in that case the set M is called Chebychev set of X. In another word the set M is called Chebychev if P_M(x) is singleton (Akramm, 2010).

Theorem 1.1. Let M be a subspace of a normed space X,
(i) If x $\in$ M, then P_M(x) ={x} (ii) If x $\in$ cl(M)\ M, then P_M(x) = $\emptyset$(Akramm 2010) (Singer 1974)

Proof. (i) Let x $\in$ M, then d(x, x) = 0 which implies that d(x, M) = 0.
Therefore P_M(x) = { x $\in$ M : ||x-y||= d(x, M)} = { x $\in$ M : ||x-y||= 0}={x}

Relation of Pythagorean and Isos ×

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Relation of Pythagorean and Isosceles Orthogonality with Best approximations in Normed Linear Space

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Keywords: Best approximation, Birkhoff orthogonality, Pythagorean

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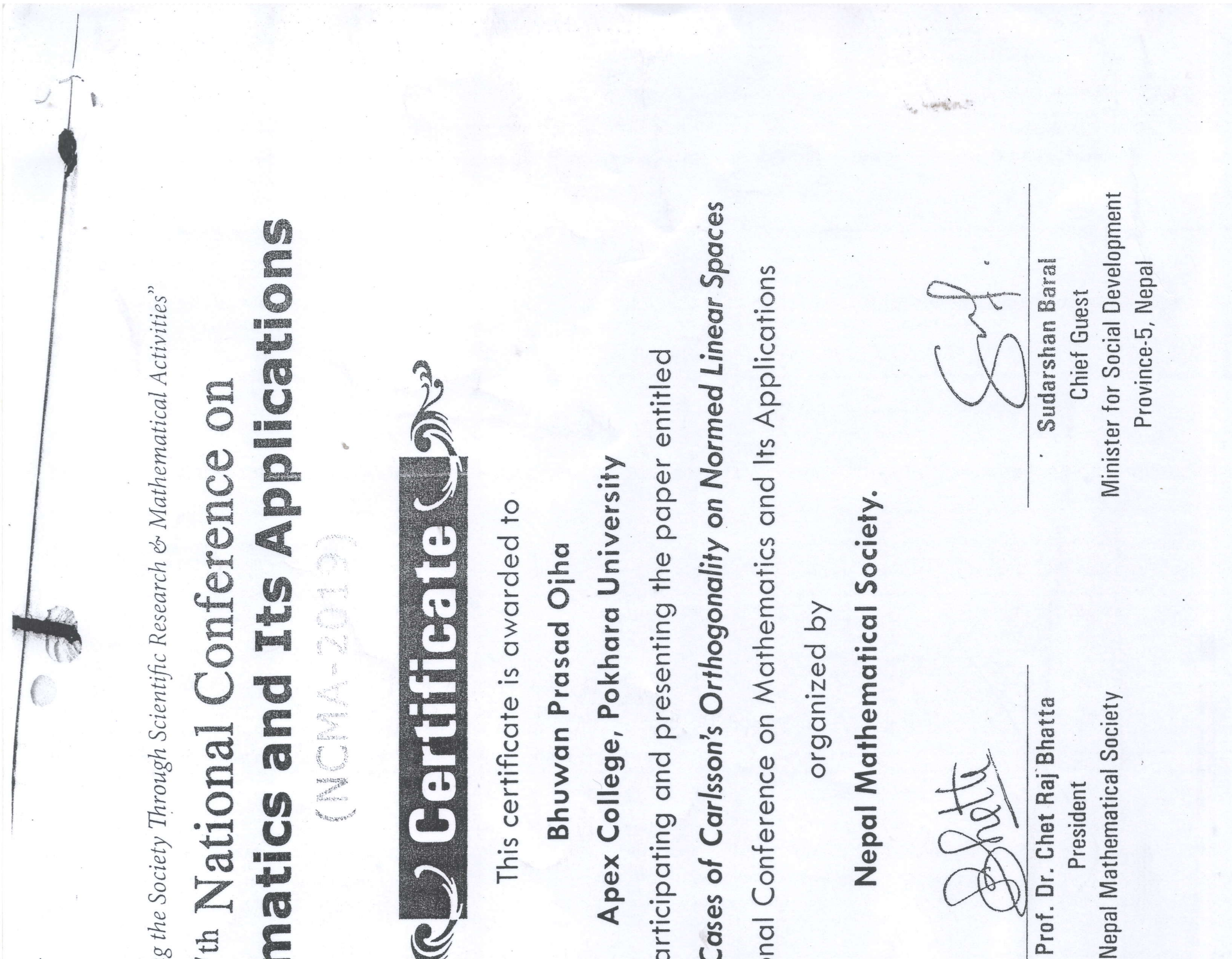

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