Numerical Modeling of Influence of Source in Heat Transformation: An Application in Blacksmithing Metal Heating

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THE CENTRAL DEPARTMENT OF MATHEMATICS
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BY

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SUBMITTED FOR THE
PARTIAL FULFILLMENT OF THE REQUIREMENT FOR
THE MASTER OF PHILOSOPHY (M.PHIL.) DEGREE
IN MATHEMATICS

AUGUST, 2022

DEDICATION

To

My Father

Shree Bhakta Kandel

My Mother

Shree Devi Kandel

Along with

My Beloved Wife

Sujata Poudel Kandel

STUDENT’S DECLARATION

This thesis entitled “**Numerical Modeling of Influence of Source in Heat Transformation: An Application in Blacksmithing Metal Heating**”, which has been submitted to the Central Department of Mathematics, Institute of Science and Technology (IOST), Tribhuvan University, Nepal for the partial fulfillment of the Master of Philosophy (M.Phil.) Degree in Mathematics, is a genuine work that I carried out under my supervisor Dr. Jeevan Kafle and that no sources other than those listed in the Bibliography have been used in this work. Moreover, this work has not been published or submitted elsewhere for the requirement of any degree programme.

—————————-

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RECOMMENDATION

This is to recommend that Mr. **Hari Prapanna Kandel** has prepared this thesis entitled "**Numerical Modeling of Influence of Source in Heat Transformation: An Application in Blacksmithing Metal Heating**" for the partial fulfillment of the Master of Philosophy (M.Phil.) in Mathematics under my supervision. To my knowledge, this work has not been submitted for any other degree. He has fulfilled all the requirements laid down by the Central Department of Mathematics, Institute of Science and Technology (IOST), Tribhuvan University (TU), Kirtipur for the submission of the thesis for the partial fulfillment of MPhil Degree in Mathematics.

——————-

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Date: August, 2022

LETTER OF APPROVAL

We certify that the Research Evaluation Committee of the Central Department of Mathematics, Tribhuvan University, Kirtipur approved this research work entitled "**Numerical Modeling of the Influence of Source in Heat Transformation: An Application in Blacksmithing Metal Heating**" carried out by Mr. **Hari Parapanna Kandel** in the scope and generality as a thesis in the partial fulfillment for the requirement of the M.Phil. degree in Mathematics.

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ABSTRACT

Partial differential equations (PDEs) are used to mimic a variety of real-world physical issues. A standard parabolic PDE of the form $u_t = \alpha u_{xx},\ (\alpha > 0)$ is an 1D heat equation. In a regular form of domain, the heat equation has an analytical solution. Computing an analytical solution becomes challenging, if not impossible, any time the domain of such modeled issues has an uneven shape. In this case, numerical methods can be used to find the numerical solution of these PDEs. Through the domain's discretization into a limited number of areas. One of the numerical techniques used to determine the numerical solutions of PDEs is the finite difference method (FDM). Here, the FTCSSfor the one-dimensional heat equation and the numerical computation of its solution using FTCSS are discussed. Furthermore, numerical solution and analytic solution of heat equation has been compared and analyzed. Additionally, the 1D heat equation with variable starting conditions (ICs) and numerous initial conditions (ICs)has been solved numerically using FDMs. Blacksmiths heated the parts at various temperatures and locations to mold different metals into the necessary shapes. The numerical solution method for the 1D heat problem given here can be used to solve heat equations used in engineering and scientific disciplines.

LIST OF ACRONYMS AND ABBREVIATIONS

IVP: Initial Value Problem

BC: Boundary Condition

PDE: Partial Differential Equation

FTCSS: Forward Time Central Space Scheme

FDM: Finite Different Method

PDE: Partial Differential Equation

LIST OF SYMBOLS

α	Diffussivity.
ρ	Density.
C_v	Specific heat capacity.
x	Position.
t	time.
A	Cross-sectional area.
$u(x,t)$	Temperature in the rod at position x and at instant t.

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Chapter 1

INTRODUCTI ON

1.1 Background

When there is a temperature difference between two places, an essential method of transferring energy is via heat. The term "heat" on the other hand, refers to the energy that is transferred throughout the procedure. In contrast, 'temperature' is a physical attribute of matter that expresses how cold or hot a thing or environment [13, 11]. The physical system between heat and/or the production, exploitation, conversion, and exchange of thermal energy is the topic of heat transfer. Heat transfer is divided into several modalities (See, in Fig. 1.1), including con-

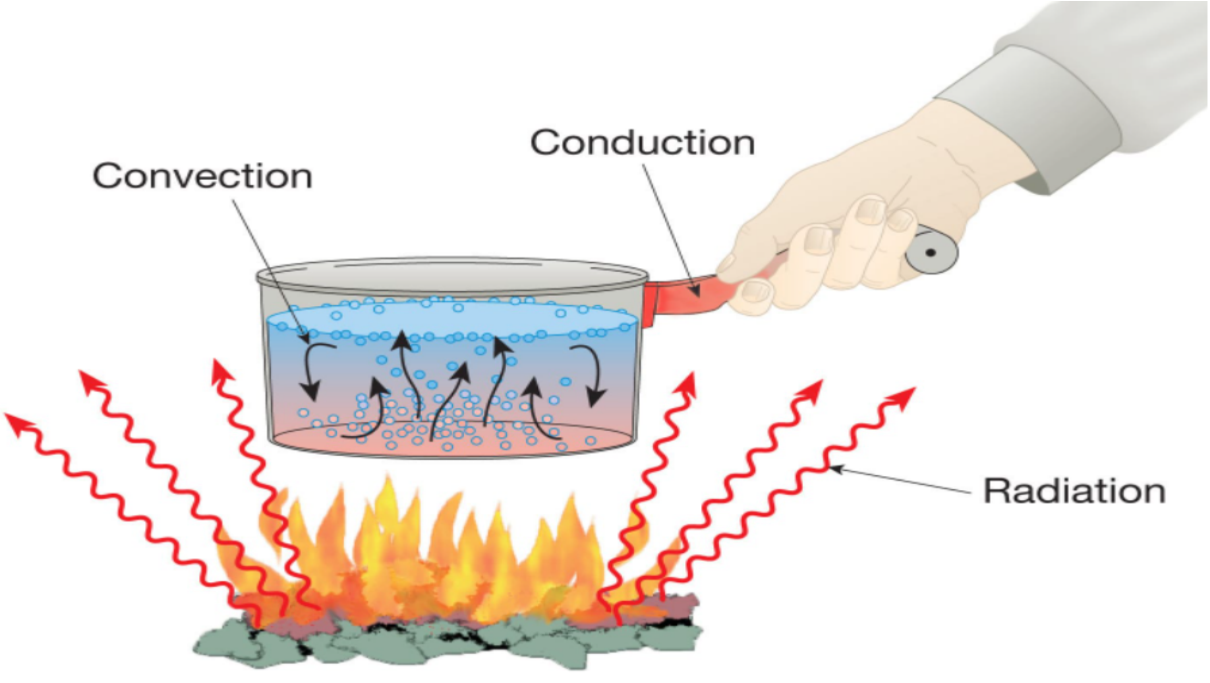

Figure 1.1: Various modes (i.e, conduction, convection, radiation) of Heat transfer [1].

duction, convection, thermal radiation, and energy transfer via phase shifts [6, 11]. Either as a result of an energy transfer from one molecule to another (without the molecules actually moving) or as a result of free electron motion if they are present occurs the conduction methods of heat transmission. As a result, this form of heat transmission is influenced by material qualities such as the medium's diffusivity [6, 11].

A differential equation can contain a derivative of an example function of one or more variables. Because there are always multiple independent variables in a physical setting, a partial differential equation (PDE) is made up of partial differential coefficients or the partial derivative of a dependent variable with many independent variables [6, 9, 13, 15]. So, PDEs are important in many fields of science and engineering. There are very few partial differential equations mainly linear and few number of nonlinear equations are solved [6, 9, 13, 15].. Only a small number of PDEs are difficult for us to solve; the majority of our work is dealing with linear and some nonlinear equations. [6, 9, 13, 15]. The diffusion equation and the heat equation both explain how the distribution of heat changes over time in a solid medium [3, 9, 13, 15]. In a specific area over time, the distribution of heat is represented by a type of the parabolic partial differential equation (PDE) known as the heat equation (or change in temperature) [4]. So, heat equation is significantly used in the diverse scientific fields [9, 10, 13, 14, 15]. The heat equation, which is pertinent to the investigation of chemical diffusion and other associated phenomena, is the source of the diffusion equation. Electricity can flow through geothermal gases with a constant heat or mass flux as well as through walls with a constant temperature or concentration. [9, 10, 13, 14, 15].

According to Chamkha and Khaled [7], The magnetic field affects heat transmission via mixed convection when internal heat generation or absorption is present. With the aid of analytical modeling and accurate solutions, an oscillatory rotating Burgers fluid flow that is limited by a plate has been studied [10].

Heat or diffusion equation describes how heat changes over time in a solid medium in mathematics and physics [9, 13, 15]. As an important PDE equation that illustrates the temporal dynamics of the heat dispersion in a certain area (Fig. 1.2). The heat equation is widely used to analyze Brownian motion, the Schrodinger equation for a free particle, thermal diffusivity in polymers, and particle diffusion. It is also used in the outer surfaces of rockets, bridges, trains, and freezers, as well as in the cancer model, picture analysis, and spatial ecology model. The heat equation is therefore essential in a wide range of scientific disciplines. [8, 9, 10, 13, 14, 15].

Suppose that $u(x, t)$ represents the temperature distribution at time t and position x. Then, one

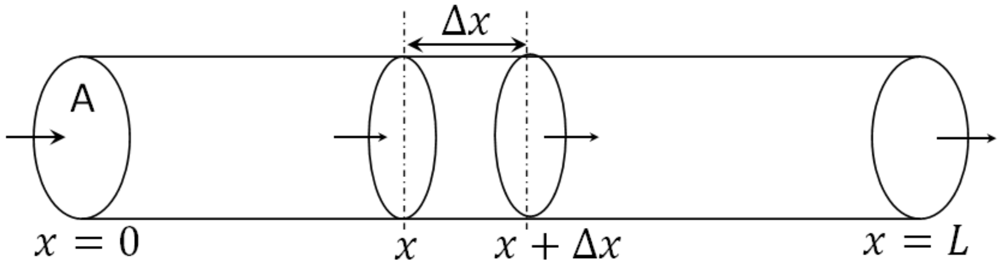

Figure 1.2: Heat flow in a rod.

dimensional heat equation is

$$u_t(x,t) = \alpha u_{xx}(x,t), \qquad c > 0 \tag{1.1}$$

with BCs $\quad u(0,t) = T_1, \quad u(L,t) = T_2; \quad t > 0$

and IC $\quad u(x,0) = f(x) \quad 0 \le x \le L$

where $f(x)$ is function of length which is the initial quantity of the heat delivered at length x of the rod, α is called thermal diffusivity, and T_1, T_2 are boundary temperatures. This is an illustration of a standard parabolic PDE [6, 8].

Taking the second order linear partial differential equations with regard to the aforementioned equation (1.1)

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = G \tag{1.2}$$

we get, $A = \alpha$ and $B = C = 0$.

$$\therefore B^2 - 4AC = 0$$

Here is an idea of a parabolic PDE [6]. The heat equation can be solved analytically in the world of regular shapes [17, 25]. Computing an analytical solution is highly challenging if the domain has an irregular form. Instead, we compute the solutions of partial differential equations using numerical methods. The domain is discretized into a finite number of areas, and the numerical solutions of PDEs are calculated using the finite difference method. The grid points of the domain are where the solutions are computed [17].

Numerical methods are employed to resolve the modeled partial differential equations. To find numerical solutions for heat transfor problems, the finite difference method (FDM) is used. Brook Taylor first proposed this approach in 1715, and George Boole (1860), C.M. Milne Thomson (1933), and Karaly Jordan (1939) have investigated it as abstract, self-standing mathematical objects[17, 25].

Using a finite number of regions to discretize the domain, finding the numerical solution to partial differential equations is easier. At the grid points of the domain, the solutions are calculated [17, 25].The numerical solutions of a 1D heat equation with IC and BCs do not always converge to the exact solutions when utilizing finite difference techniques [17, 25]. In finite difference approaches, it indicates numerical instability.

A blacksmith is a metalworker who creates different items out of steel or iron. Blacksmiths forge metals into tools, agricultural machinery, ornamental and religious objects, cooking utensils, gates, grilles, railings, light fixtures, furniture, sculpture, and munitions. Blacksmiths heat the metals as they are at work so that they are malleable and may be molded with hand tools like hammers, chisels, anvils, etc. They can determine the temperature distribution throughout the metal parts by looking at the bright hue of the metal pieces. One point on the metal may receive heat, or there may be numerous points. The location where a blacksmith works is referred to as a smithy, a forge, or a Blacksmith's shop, among other titles.

Heat can move from a hotter body to a cooler body, as we are all aware. Heat transfer has a major effect on temperature change; cooling lowers temperature while heating raises it. We assume throughout our research that there won't be any phase shifts and that the system won't change in any way. The results of experiments demonstrate that three variables-temperature change, system mass, and substance phasehave an impact on transmitted heat. One needs to heat various portions of the metal to various temperatures in order to twist it into the required shape. [2, 12, 22]. Well-known illustration is a blacksmith using heated iron shown in Fig 1.3.

Figure 1.3: Blacksmith working on hot iron [2].

1.2 Literature Review

In a linearly stratified stagnation flow, Chamkha and Khaled [7] investigated how its magnetic field affected mass and coupled convectional heat transmission when internal heat production or absorption was present. Incompressible Burgers fluid hydro magnetic oscillatory rotational flows have no proven analytical solutions or theories [10, 21, 22]. The heat equation was created in 1987 by Jean Baptise Joseph Fourier (17681830) as a manuscript for the Institute de France. In 1822, he also released his monograph, "Analytic Theory of Heat" [5, 18]. In the 1600s, scientists changed their minds about the relationship between heat and the mobility of tiny pieces of matter. But in the 1700s , People used to think of heat as a fluid-like substance[5].

One-dimensional (1D) heat transfer problems have been investigated utilizing finite difference method. Brook Taylor first presented FDM in 1715, and George Boole, C.M. Milne Thomson, and Karaly Jordan examined it as an abstraction of separate mathematical concepts [17, 25]. Makhtoumi [16] derived the investigation of heat diffusion in a 1D thin rod using analytical and numerical solutions. He applied the homotopy perturbation method (HPM) and the finite difference method(FDM) to the rod PDE system . The outcomes show the efficiency of homotopy perturbation as a numerical method for resolving partial differential equations.

Olaiju et al. [20] explored the explicit finite difference method (FDM), which was developed and used by computational software to solve a straightforward 1D heat equation problem. They discovered that the time step and mesh spacing have an impact on forecast accuracy. Additionally, the 1D heat equation solutions exhibit smooth and bounded temporary performance as they transition from an initial condition to a non-varying fixed state situation. Olaiju et al. [19] compared FEMs and FDMs for the air pollution problem simulation. According to their simulation results, FEMs and FDMs are effective at resolving the diffusion problem and are acceptable for reducing air pollution emissions for a healthier environment. Mebrate [17] presented numerical solutions using FEMs and FDMs to a 1D heat equation with IC and Dirichlet BCs. When the B-spline basis functions are taken into consideration, Dabral et al. [8] investigated numerical solution of the 1D heat equation by B-Spline FEM. Both quadratic and cubic B-splines are used to create the solution. With the Dirichlet boundary condition (BC), the simulation process was geometrically represented in three dimensions, taken in a distinct temporal space and different region of interest.

Wilson and Nickell [29] pointed out the error that estimates the discrete Galerkin FDMs for a nonlinear 1D heat equation. The numerical schemes studied were based on a classical transformation of the dependent variables by means of the enthalpy as well as the Kirchhoff transformation. Thomee [28] gives a very thorough overview of numerical analysis of PDEs with the development of FEMs and subsequently FEMs with a focus on mathematical elements like stability and convergence analysis.

Using finite difference methods, the one-dimensional heat equation with initial and boundary conditions has numerical solutions (FDMs).The impact of applying heat to the material at various sites will first be contrasted and examined. Next, the efficacy of heat applied in various locations will be studied and evaluated.

1.3 Objectives

The following are the major objectives of this research work:

Objective-I: To develop an understanding of the concept and applications of the partial differential equation to derive the heat equation.

Objective-II: To gain the insight on the proof of finite difference method (FDM) to compute the numerical solution of one-dimensional heat equation.

Objective-III: Utilization of FDMs to combine the numerical solutions of the 1D heat equation for a variety of beginning conditions and varied initial condition positions.

Objective-IV: In the field of science and engineering, to clarify need and the application of numerical methods for finding the solution of the heat equation.

1.4 Structure of Thesis

The first chapter provides a background of the work as well as general introduction of the work. In Chapter 2, derivation of one dimensional heat equations and its analytical solution by using variable separation method are presented. We also discuss the formulation of numerical method's Forward Time Central Space Scheme (FTCSS) along with consistency and Stability of the FTCSS. Chapter 3 focuses on the comparison between analytic and numerical solution along with, detailed discussion on the numerical simulation of 1D heat equation with variation of positions of the initial conditions and multiple initial conditions. Chapter 4 contains the summary of the work.

Chapter 2

Solution of One Dimensional Heat Equation

2.1 Derivation of One Dimensional Heat Equation

Consider the heat transfer in a one-dimensional homogeneous rod of length L and radius r, where A = πr^2 is the cross-sectional area. Assume $u = u(x,t)$ is the temperature distribution at time (t) and at position (x) along the rod which is to be determined [9, 13, 15, 26].

For which we make the following assumptions are made:

- Heat can enter and leave the rod only through its ends, that is, the rod is laterally insulated.
- The temperature is constant in all cross sections and heat only flows in the x-direction.
- Initially rod is at zero degree and both ends are held at constant temperature T_1 and T_2 degrees respectively [9, 13, 15, 26].

Let C_v be the specific heat capacity of the rod at constant volume. Consider the small segment $[x, x + \triangle x]$ of the rod [9, 13, 15, 26]. Then the amount of heat supplied in the segment $[x, x + \triangle x]$ is approximately given by

$$C_v \rho u(\xi, t) A \triangle x, \quad x \leq \xi \leq x + \triangle x.$$

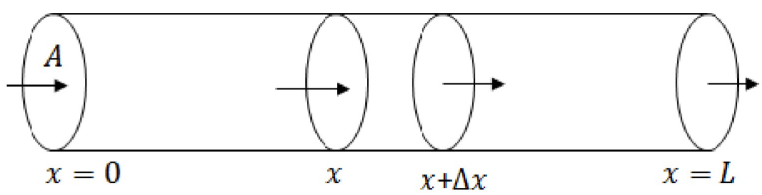

Figure 2.1: Heat flow in a rod.

where, mass=volume $\times$ density= $\rho A \triangle x$.

Let $\phi(x,t)$ denote the heat flux flowing through the face at x, then the energy balanced equation for the segment $[x, x+\triangle x]$ is,

$$\frac{\partial}{\partial t}(C_v \rho u(\xi,t) A \triangle x) = \phi(x,t) - \phi(x+\triangle x, t)$$

$$\therefore C_v \rho A u_t(\xi,t) = \frac{\phi(x,t) - \phi(x+\triangle x,t)}{\triangle x} \tag{2.1}$$

The product of the rate of heat flow into the segment at x minus the rate of heat flow out at $x+\triangle x$ must match the temporal rate of change of energy in the segment. Therefore from (2.1), as $\triangle x \to 0, \xi \to x$, we have

$$C_v \rho A u_t(x,t) = -\phi_x(x,t) \tag{2.2}$$

This is the fundamental rule governing energy conservation. By the Fourier law of hea, the negative of the temperature gradient $u_x(x,t)$ in the rod at a particular instant and the cross-sectional area A determine the heat flux at any location. This is,

$$\phi_x(x,t) = -k u_x(x,t) A \tag{2.3}$$

where, k is proportionality constant called thermal conductivity of the material.

$$C_v \rho u_t(x,t) = k u_{xx}(x,t)$$

$$\text{or,} \qquad u_t(x,t) = \alpha u_{xx}(x,t) \tag{2.4}$$

where,

$$\alpha = \frac{k}{C_v \rho}$$

. $(\alpha > 0)$ is called the thermal diffusivity of rod. The equation (2.4) is known as heat or diffusion equation. The IC is

$$u(x,0) = f(x), \qquad 0 \le x \le L$$

and the BCs are

$$u(0,t) = T_1 \qquad u(L,t) = T_2, \qquad t > 0$$

2.2 Analytic Solution of Heat Equation Using Separation of Variable Method

Let us consider the 1D heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}; \qquad 0 \le x \le L, \qquad t \ge 0 \tag{2.5}$$

with BCs $u(0,t) = T_1, \qquad u(L,t) = T_2; \qquad t > 0$
and IC $u(x,0) = f(x) \qquad 0 \le x \le L$
Let

$$u(x,t) = X(x).T(t) \tag{2.6}$$

be the solution [9, 13, 15] of equation (2.5). Substituting (2.6) into (2.5), we get

$$X\dot{T} = \alpha X''T$$

Therefore,

$$\frac{X''}{X} = \frac{\dot{T}}{\alpha T} = k$$

where k is called separation constant. This gives,

$$\frac{d^2X}{dx^2} - kX = 0 \tag{2.7}$$

And

$$\frac{dT}{dt} - \alpha kT = 0 \tag{2.8}$$

Case I: If $k = \lambda^2 > 0$, then (2.5) has solution

$$u(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})e^{\alpha\lambda^2 t} \tag{2.9}$$

where A and B are constants.

Case II: If $k = -\lambda^2 < 0$,then (2.5) has the solution

$$u(x,t) = (Ccos\lambda x + Dsin\lambda x)e^{-\alpha\lambda^2 t} \tag{2.10}$$

where C and D are constants.

Case III: If $k = 0$,then (2.7) has the solution,

$$u(x,t) = Ex + F. \tag{2.11}$$

where E and F are constants. We reject the solution (2.9), because as $t \to \infty$ the solution tends to ∞, and also reject the solution (2.11), because it is independent of t. Thus the equation (2.10) gives the solution of (2.5). To use the initial and boundary conditions, the constants in the solution are obtained [9, 13, 15].

2.3 Numerical Methods

Consider the IVP obtained above

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0 \tag{2.12}$$

$$\text{BC:} \quad u(0,t) = T_1, \quad u(1,t) = T_2$$

$$\text{IC:} \quad u(x,0) = f(x)$$

. To discover the solution of time-dependent PDEs like the heat equation, a number of approximations have been investigated. Numerical analysis is the study of various approximations in terms of accuracy, convergence, and stability. Making time as a constant while discretizing space is one method of numerically solving time-dependent partial differential equations [14, 23]. Here, we take into account the Forward Time Central Space Scheme (FTCSS), This is helpful for finding the approximate numerical solution of 1D heat equation.

2.3.1 Forward Time Central Space Scheme

In FTCSS, it is taken forward difference for time where as central difference for space.

Forward difference in time:

$$u(x, t + \triangle t) = u(x,t) + \triangle t \frac{\partial u(x,t)}{\partial t} + \frac{(\triangle t)^2}{2!} \frac{\partial^2 u(x,t)}{\partial t^2} + ...$$

After simplification

$$\frac{u(x, t + \triangle t) - u(x,t)}{\triangle t} = \frac{\partial u(x,t)}{\partial t} + O(\triangle t) \tag{2.13}$$

Central difference in space:

$$u(x + \triangle x, t) = u(x,t) + \triangle x \frac{\partial u(x,t)}{\partial x} + \frac{(\triangle x)^2}{2!} \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{(\triangle x)^3}{3!} \frac{\partial^3 u(x,t)}{\partial x^3} + ...$$

$$u(x - \triangle x, t) = u(x,t) - \triangle x \frac{\partial u(x,t)}{\partial x} + \frac{(\triangle x)^2}{2!} \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{(\triangle x)^3}{3!} \frac{\partial^3 u(x,t)}{\partial x^3} + ...$$

Adding above expressions, we get

$$\frac{u(x + \triangle x, t) - 2u(x,t) + u(x - \triangle x, t)}{(\triangle x)^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + O(\triangle x)^2 \tag{2.14}$$

Substituting the expression from the of equation (2.13) and equation (2.14) in equation (2.12), we have

$$\frac{u(x, t + \triangle t) - u(x,t)}{\triangle t} = \alpha \left(\frac{u(x + \triangle x, t) - 2u(x,t) + u(x - \triangle x, t)}{(\triangle x)^2} \right) + O(\triangle t, (\triangle x)^2)$$

The above equation can be written as

$$u(x, t + \triangle t) \approx u(x, t) + \alpha \left(\frac{\triangle t}{(\triangle x)^2} \right) \{u(x + \triangle x, t) - 2u(x, t) + u(x - \triangle x, t)\} \quad (2.15)$$

The spatial interval $[0, L]$ has been sub-divided into $M + 1$ equally spaced sample points

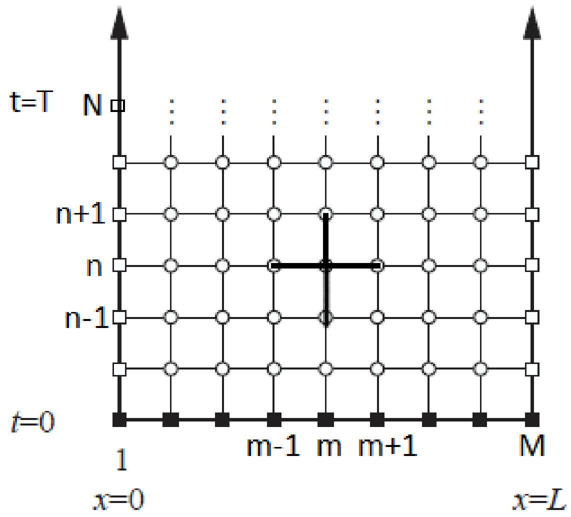

Figure 2.2: Grid points of space-time plane [24].

$x_m = m.\triangle x = m.h$. The time interval $[0, T]$ has been sub-divided into $N + 1$ equal time level $t_n = n.\triangle t = n.k$. We introduce approximations $u(x_m, t_n) \approx v_m^n$ for at each of these space-time points. Therefore,the equation (2.15) can be reducedin to

$$v_m^{n+1} = v_m^n + c(v_{m+1}^n - 2v_m^n + v_{m-1}^n) \quad (2.16)$$

where

$$c = \alpha \left(\frac{\triangle t}{(\triangle x)^2} \right) = \alpha, \frac{k}{h^2}$$

which is the required FTCSS [23, 24] for (2.12).

2.3.2 Consistency of FTCS Scheme

Let ϕ be a smooth function [27], then we have

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} - \alpha \frac{\phi_{m+1}^n - 2\phi_m^n + \phi_{m-1}^n}{h^2}$$

And,

$$P\phi = \phi_t - \alpha\phi_{xx}$$

Now,

$$\begin{aligned}
\phi_m^{n+1} &= \phi(t_n + k, x_m)\\
&= \phi_m^n + k\phi_t + O(k^2)\\
\therefore \tfrac{\phi_m^{n+1} - \phi_m^n}{k} &= \phi_t + O(k)
\end{aligned}$$

Again,

$$\phi_{m+1}^n = \phi_m^n + h\phi_x + \frac{h^2}{2!}\phi_{xx} + \frac{h^3}{3!}\phi_{xxx} + O(h^4)$$

And,

$$\begin{aligned}
\phi_{m-1}^n &= \phi_m^n - h\phi_x + \tfrac{h^2}{2!}\phi_{xx} - \tfrac{h^3}{3!}\phi_{xxx} + O(h^4)\\
\therefore \phi_{m+1}^n + \phi_{m-1}^n &= 2\phi_m^n + 2\tfrac{h^2}{2!}\phi_{xx} + O(h^4)\\
\therefore \frac{\phi_{m+1}^n - 2\phi_m^n + \phi_{m-1}^n}{h^2} &= \phi_{xx} + O(h^2)\\
\therefore P_{k,h}\phi &= \phi_t - \alpha\phi_{xx} + O(k) + O(h^2)\\
&= P\phi + O(K) + O(h^2)\\
\therefore P_{k,h}\phi - P\phi &= O(k) + O(h^2) \to 0 \qquad \text{as } h, k \to 0
\end{aligned}$$

Therefore, the FTCSS of equation (2.16) is in accordance with the degree of accuracy $(1, 2)$.

2.3.3 Stability of the FTCS Scheme

Putting $v_m^n = g^n e^{im\theta}$, we get

$$\begin{aligned}
g(\theta) &= 1 + c(e^{i\theta} - 2 + e^{-i\theta})\\
&= 1 + c(2cos\theta - 2)\\
&= 1 + 2c(cos\theta - 1)\\
\therefore g(\theta) &= 1 - 4c\ \ sin^2\tfrac{\theta}{2}
\end{aligned}$$

Now the scheme is stable iff,

$$\begin{aligned}
|g(\theta)| &\leq 1\\
\implies -1 &\leq g(\theta) \leq 1\\
\implies -1 &\leq 1 - 4c\ \ sin^2\tfrac{\theta}{2} \leq 1\\
\implies -2 &\leq -4c\ \ sin^2\tfrac{\theta}{2} \leq 0\\
\implies 0 &\leq 4c\ \ sin^2\tfrac{\theta}{2} \leq 2
\end{aligned}$$

Since this holds for all values of θ, in particular, take $\theta = \pi$ to get,

$$\begin{aligned}
4c.1 &\leq 2\\
\implies c &\leq \tfrac{1}{2}
\end{aligned}$$

Again, if $c \leq \frac{1}{2}$, then,

$$4c\ \ sin^2\frac{\theta}{2} \leq 4.\frac{1}{2}.1 = 2$$

$$\therefore 4c \;\; sin^2\frac{\theta}{2} \leq 2$$

Thus, the scheme (2.16) is stable [23, 27] iff $c \leq \frac{1}{2}$. Therefore, the numerical solution of 1D heat equation (2.12) with IC and BCs can be found to the FTCS scheme (2.16). By using FTCSS, the value of $c = \alpha\frac{k}{h^2}$ which is less than or equal to 0.5. By adjusting the time and space interval sizes, we can keep this state in place. To find the approximation that is more accurate, we must double the number of space and time divides.

Chapter 3

Comparison Between Analytical and Numerical Solution

3.1 Material having diffusivity 0.05

Let's use the heat equation as an example [11].

$$u_t = 0.05u_{xx} \qquad 0 \leq x \leq 1, \quad t \geq 0 \tag{3.1}$$

BCs: $u(0,t) = u(1,t) = 0; \quad t > 0$ IC: $u(x,0) = sin\pi x; \quad 0 \leq x \leq 1$

3.1.1 Analytical Solution

From equation (2.10) the 1D heat equation 3.1 has the following solution [9, 13, 15]:

$$u(x,t) = (Acos\lambda x + Bsin\lambda x)e^{-0.05\lambda^2 t}$$

Using the superposition principle and boundary conditions,we get

$$u(x,t) = \sum_{n=1}^{\infty} B_n sin(n\pi x)e^{-0.05(n\pi)^2 t}$$

Again, by using the IC,

$$u(x,t) = sin(\pi x)e^{-0.05\pi^2 t}$$

Now, we are eager to determine the rod's distant temperature at distance $x = 0.8$ m from the rod's starting position over time $t = 0.8$ hr. That is, we are interested to find $u(0.8, 0.8)$. From above solution we get,

$$u(0.8, 0.8) = 0.3961$$

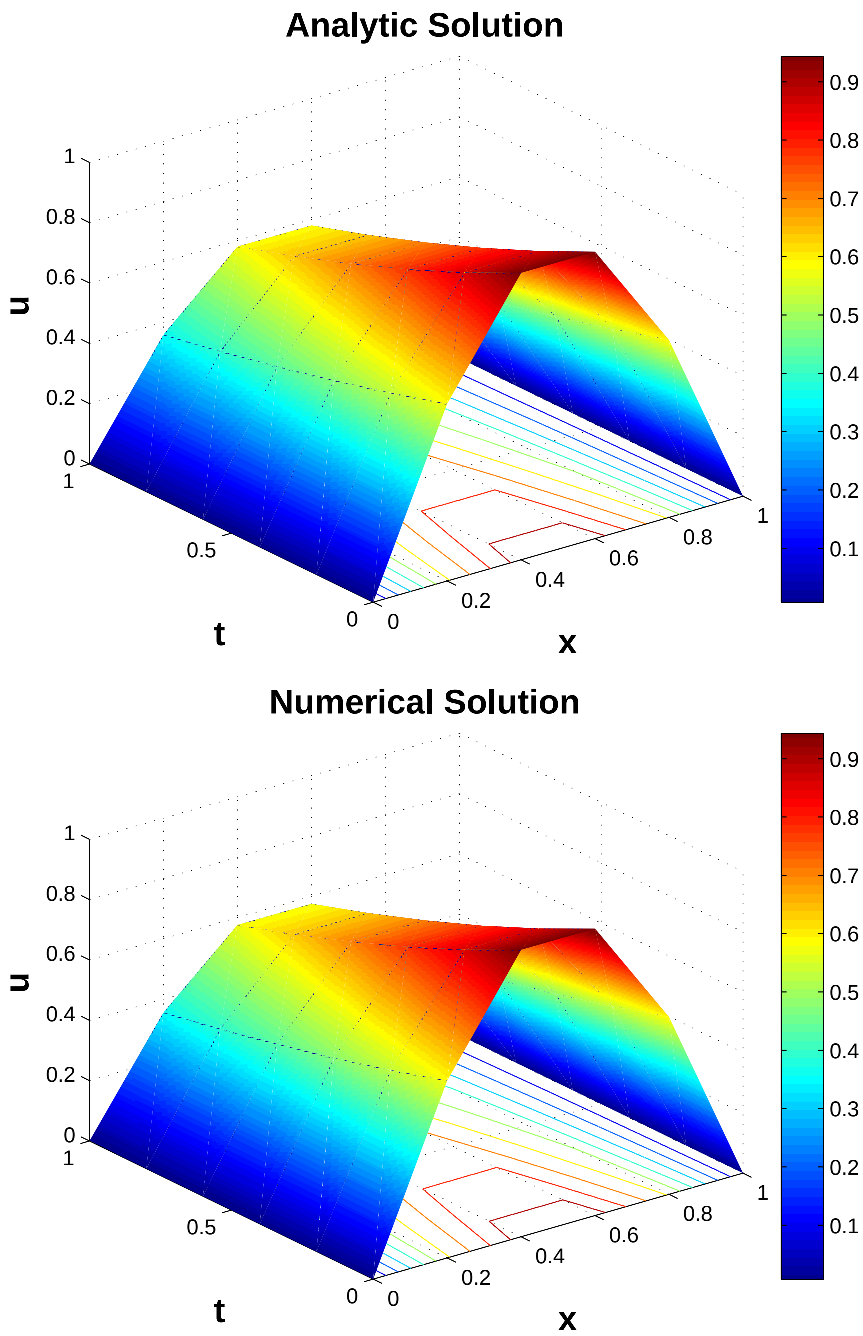

Figure 3.1: **Top:** analytic approach, **Bottom:** numerical approach of one dimensional heat equation 3.2.

3.1.2 Heat Equation is Numerically Solved Using the Finite Difference Method

The 1D heat equation 3.1has the following FTCS scheme:

$$v_m^{n+1} = v_m^n + c(v_{m+1}^n - 2v_m^n + v_{m-1}^n)$$

with $v_0^n = v_M^n = 0$, $v_m^0 = sin\pi x$ and $c = \frac{0.05k}{h^2}$.
Consider the length of time be $k = 0.2$ and space intervals be $h = 0.2$. Then,

$$c = \frac{0.05 \times 0.2}{0.2^2} = 0.25$$

Since FTCS scheme is stable if and only if $c \leq 0.5$. Therefore, For the aforementioned issue, FTCS is stable. Also, we have $v_m^0 = sin\pi x$

Then,

$v_0^0 = 0 \quad v_1^0 = 0.5878 \quad v_2^0 = 0.9511 \quad v_3^0 = 0.9511 \quad v_4^0 = 0.5878 \quad v_5^0 = 0$

For $m = 1$ and $n = 0$, we get $v_1^1 = v_1^0 + 0.25(v_2^0 - 2 \times v_1^0 + v_0^0)$

$v_1^1 = 0.5878 + 0.25(0.9511 - 2 \times 0.5878 + 0) = 0.5317$

Similarly, we have

$v_0^0 = 0 \quad v_1^0 = 0.5878 \quad v_2^0 = 0.9511 \quad v_3^0 = 0.9511 \quad v_4^0 = 0.5878 \quad v_5^0 = 0$
$v_0^1 = 0 \quad v_1^1 = 0.5317 \quad v_2^1 = 0.8602 \quad v_3^1 = 0.8602 \quad v_4^1 = 0.5317 \quad v_5^1 = 0$
$v_0^2 = 0 \quad v_1^2 = 0.4809 \quad v_2^2 = 0.7781 \quad v_3^2 = 0.7781 \quad v_4^2 = 0.4809 \quad v_5^2 = 0$
$v_0^3 = 0 \quad v_1^3 = 0.4350 \quad v_2^3 = 0.7038 \quad v_3^3 = 0.7038 \quad v_4^3 = 0.4350 \quad v_5^3 = 0$
$v_0^4 = 0 \quad v_1^4 = 0.3934 \quad v_2^4 = 0.6366 \quad v_3^4 = 0.6366 \quad v_4^4 = 0.3934 \quad v_5^4 = 0$
$v_0^5 = 0 \quad v_1^5 = 0.3559 \quad v_2^5 = 0.5758 \quad v_3^5 = 0.5758 \quad v_4^5 = 0.3559 \quad v_5^5 = 0$

Now, we find the error at particular case $x = 0.8\,m$ and $t = 0.8\,hr$
From analytic solution we get $u_{exact} = u(0.8, 0.8) = 0.3961$
Form FTCS approximate solution is $u_{approx} = v_4^4 = 0.3934$
Thus,

Error = $|u_{exact} - u_{approx}| = 0.0027.$

Therefore,

$$\%Error = \frac{0.0027}{0.3961} \times 100\% = 0.68\%$$

3.2 Various Initial Conditions with Numerical Solutions

Let's have a look at an illustration of a one-dimensional heat equation using an iron rod whose thermal diffusivity is $0.23cm^2/s = 0.000023m^2/s$ at $26.85°C$ temperature [29] as follows

$$u_t = 0.000023u_{xx} \qquad 0 \leq x \leq 1, \quad t \geq 0 \tag{3.2}$$

BCs: $u(0,t) = u(1,t) = 0; \quad t > 0$

And for the various trials, we use the following three initial conditions:

The above heat equation's FTCS scheme is [11]

$$v_m^{n+1} = v_m^n + c(v_{m+1}^n - 2v_m^n + v_{m-1}^n)$$

with $v_0^n = v_m^n = 0$. This FTCSS corresponds to the degree of accuracy (1, 2) which is stable if and only if $c \leq \frac{1}{2}$ [11, 27]. By shortening the gaps of time and space, we can keep the stability.The number of time and space divisions should be increased in order to discover the approximate value that is more accurate [11, 27]. Let the length space of the and time intervals' lengths be $h = 0.1$ and $k = 0.2$. So,

$$c = \frac{0.000023 \times k}{h^2} = \frac{0.000023 \times 0.2}{(0.1)^2} = 0.00046 \leq 0.5$$

The FTCS scheme is stable if and only if $c \leq 0.5$, In the foregoing, our FTCS scheme is stable.

3.2.1 Positional variation of the initial condition:

Blacksmiths in Nepali culture use to heat to iron in various places to make it repairable. First, using the aforementioned heat equation 3.1 and the left end, center as well as the rod's right ends, we explore the variation position of the beginning condition. Here, we first apply temperatures of $1500°C$ progressively to the rod's left, center, and right ends. The following initial circumstances are as a result:

IC1: $u(0.1, 0) = 1500$
IC2: $u(0.5, 0) = 1500$
IC3: $u(0.9, 0) = 1500$

A description of heat transport for the rod is then studied. The three's temperature distribution scenarios mentioned above is depicted in Fig. 3.2.

According to 3.2, the graph showing IC1 and IC3 only permits heat to move in right and left sides of one direction, respectively. In contrast, heat follows IC2 in either direction. Below in Fig. 3.3 is an area of x against distribution of temperatures.

The iron rod as displayed to be approximately 600 degree Celsius after 250 seconds of heating, while during the first and third situations, the rod is the sole about 500 $°C$. The BC, with the intention of 0 degrees Celsius from both ends, are the cause of this variation. The identical effect is produced by applying heat both close to the left and close to the right. However, if the content wasn't homogeneous or if the left and right ends have BC were different, the result would be more intriguing.

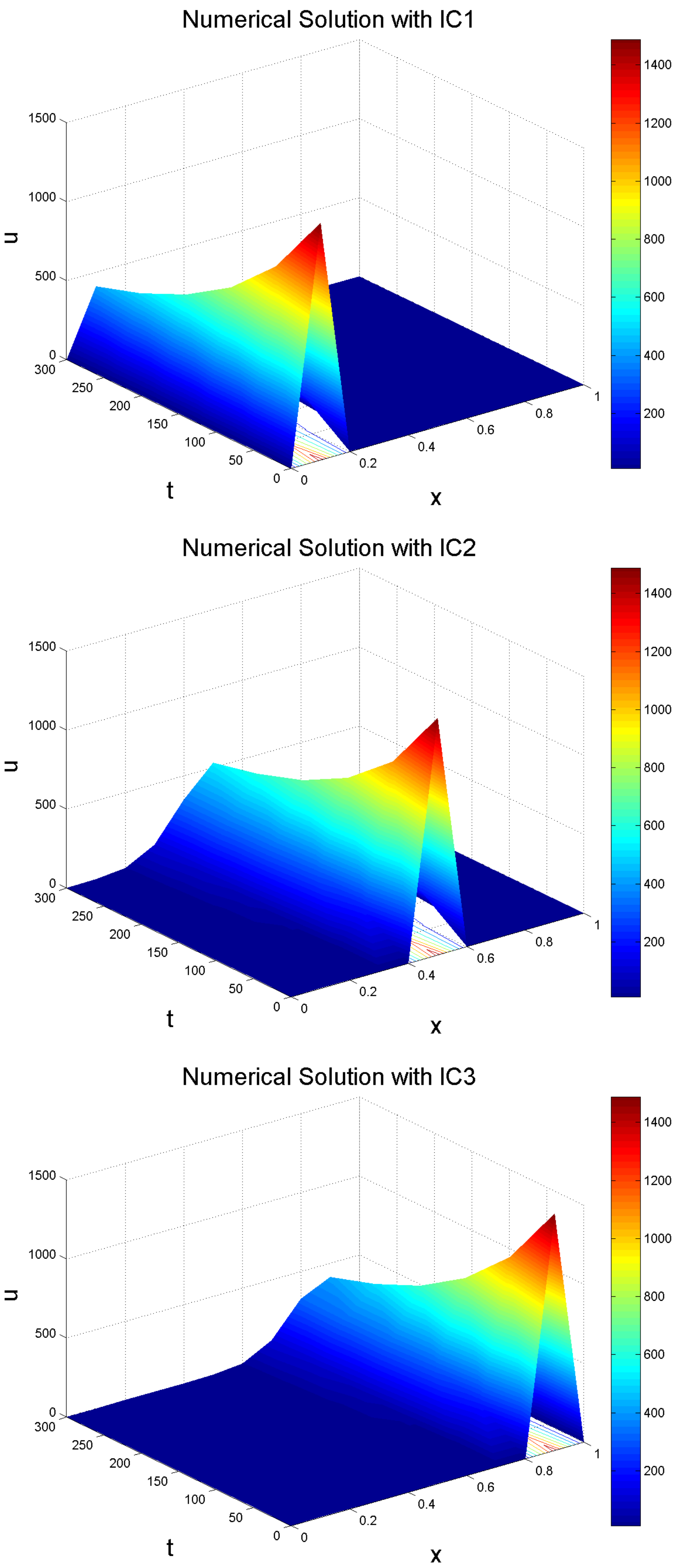

Figure 3.2: Temperature range that corresponds to **Top:** IC1, **Middle:** IC2 and **Botton:** IC3

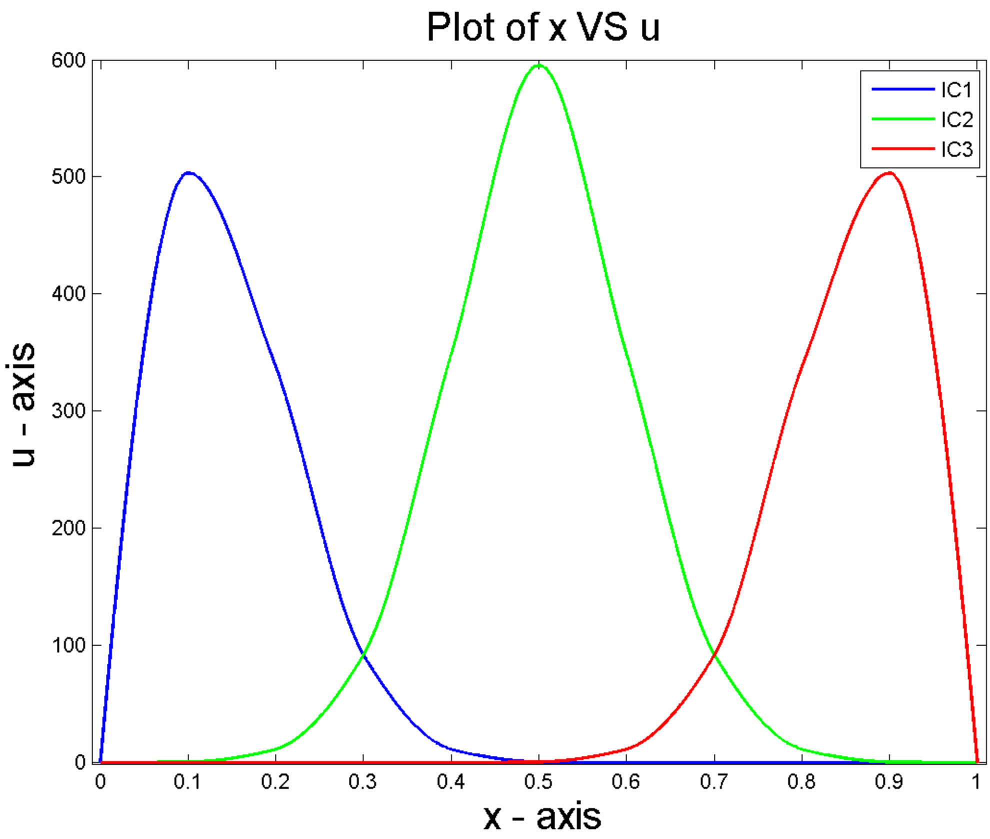

Figure 3.3: x vs u plot following 250 seconds

3.2.2 Initial situation at multiple positions

The aforementioned idea has been expanded in this case to account for various initial conditions. The first initial condition (IC1) is now available.

$$v_m^0 = \begin{cases} 1200 & \text{for } m = 5 \\ 0 & \text{otherwise} \end{cases}$$

Similar to this, we have the initial conditions IC2 and IC3 respectively.

$$v_m^0 = \begin{cases} 1200 & \text{for } m = 3, 7 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad v_m^0 = \begin{cases} 1200 & \text{for } m = 2, 5, 8 \\ 0 & \text{otherwise} \end{cases}$$

Figure 3.4 depicts the temperature distribution under various beginning conditions, including ICs 1, 2, and 3, maximum 300 seconds after the material was heated. The temperature of the material, in this case an iron rod, progressively decreases after it has been heated up, as seen in

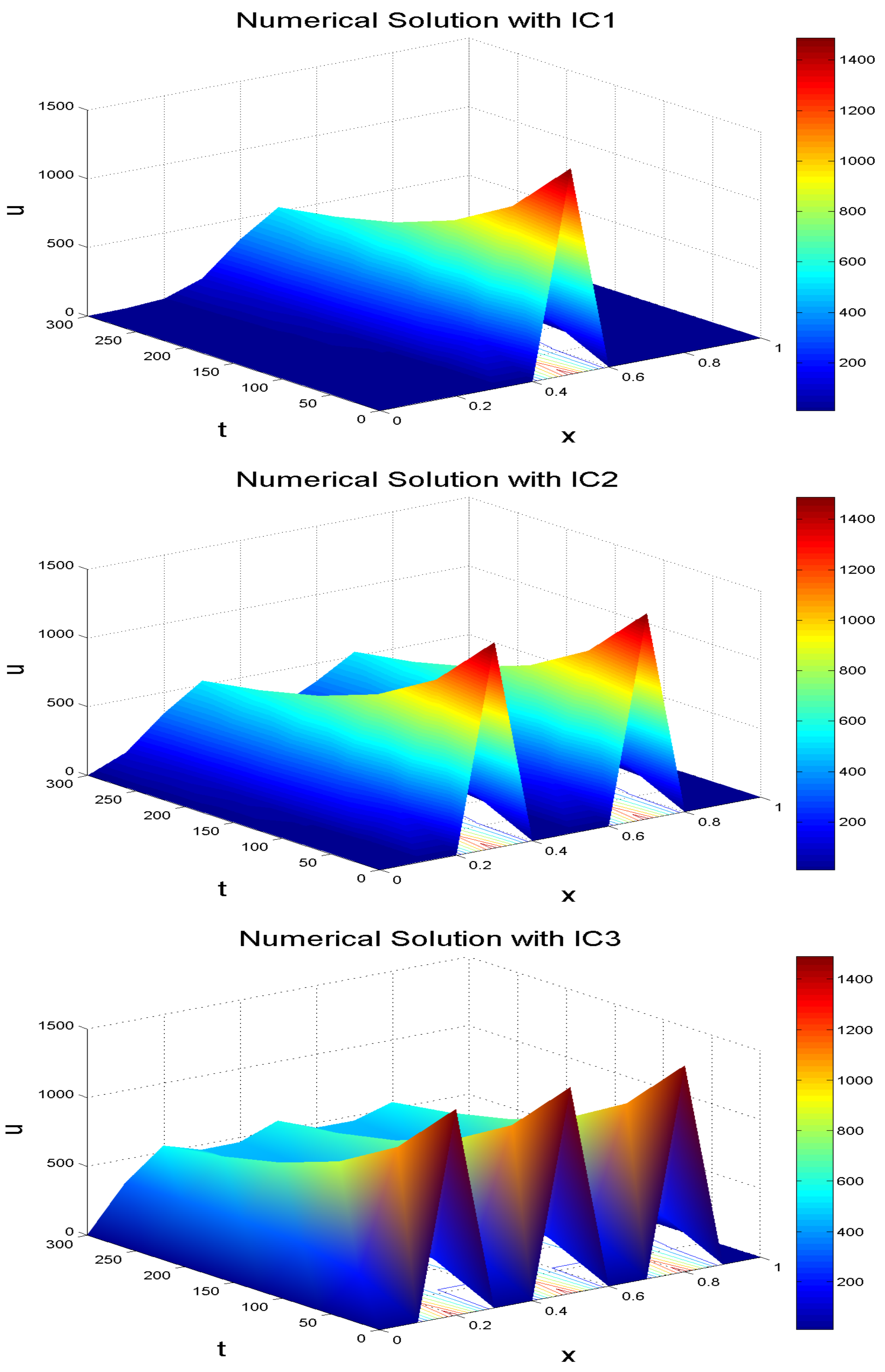

Figure 3.4: Different initial conditions for the numerical solution.

3.2

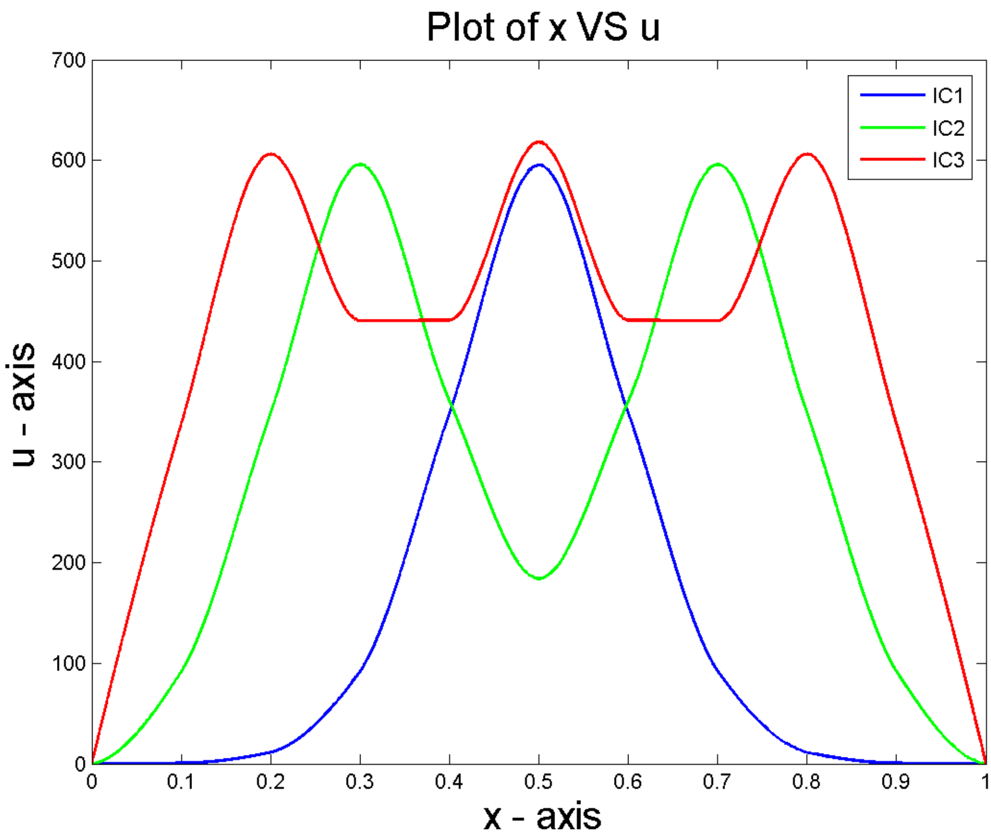

Figure 3.5: x vs u plot following 250 seconds

Fig. 3.4. The rod went from 1500 degrees Celsius to about 500 degrees Celsius in a short period of time.

As mentioned above, we heated the road at three distinct spots with IC1, IC2, and IC3: one position in IC1, two points in IC2, and three points in IC3. The numerical solution with IC3 in Fig. 3.4has a darker and browner front edge. This reveals that the rod's temperature distribution is more uniform in comparison to ICs 1 and 2. This happens as an outcome of the increase in the number of areas receiving heat. One of the iron road heating methods used by blacksmiths is seen in Figure 1.3. Blacksmiths and goldsmiths among other metal trades used this technique to heat iron rods and other materials.

The average temperature of the iron rod after 250 seconds of heating is around 590 degrees Celsius, as shown in Fig. 3.5.As a result of providing greater heat locations than ICs 1 and 2, the figure demonstrates that the average rod's internal temperature is higher more IC3 than it is with ICs 2 and 1. The result, we draw the conclusion that different metal businesses, such as blacksmiths and goldsmiths, have found that the iron rod's normal range is related the quantity of heat applied sites.

Chapter 4

SUMMARY

Partial differential equations (PDEs) utilized to simulate a number of physical issues, including wave and heat transmission, in the real world. Here, we first introduced the one-dimensional heat equation and provided a little history before modeling the equation and utilizing the approach of variable separation to find the analytic answer. Then, we discussed the FTCSS for the $1D$ heat equation, talked about its consistency and stability, and used it to get a numerical solution. Finally, we compared the analytical and numerical methods using an example.

Finally we discuss the corresponding computational software to solve heat equation for FTCSS and also analytic and numerical solution were plotted. Then, we discuss the temperature distribution with variation of positions of initial conditions, that is, the rod's left, middle, and right ends each are heated. Initially, we apply the $1500°C$. The rod of 1 m length's temperature at its far left, far center, and far right ends. A description of heat transport then in the rod studied. The real life situation of iron rod heating technique used by one of the blacksmith is shown. This type of iron rod and other resources heating technique also made use of different mental industry, used by blacksmiths and by goldsmiths too. Thus, we draw the conclusion that the typical temperature of iron rod is directly correlate with the locations where heat is apply.

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PRESENTATIONS

1. Kandel HP (2020) Comparison of Numerical Solutions of Parabolic Equation by Using Finite Difference Method and Finite Element Method. In Seminar cum Workshop on Mathematics and Its Applications organized by M. Phil. Scholars (2020) in Central Department of Mathematics, Tribhuvan University, Bhadra 13-14, 2077.

2. Kandel HP, Bagale LP, Kafle J (2021) Numerical Solution of Heat Equation by Variational Heat Supplies. In: International Conference on Analysis and Its Applications, April 9-11, 2021.

PUBLICATIONS

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Numerical Modelling on the Influence of Source in the Heat Transformation: An Application in the Metal Heating for Blacksmithing

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ABSTRACT

Many physical problems, such as heat transfer and wave transfer, are modeled in the real world using partial differential equations (PDEs). When the domain of such modeled problems is irregular in shape, computing analytic solution becomes difficult, if not impossible. In such a case, numerical methods can be used to compute the solution of such PDEs. The Finite difference method (FDM) is one of the numerical methods used to compute the solutions of PDEs by discretizing the domain into a finite number of regions. We used FDMs to compute the numerical solutions of the one dimensional heat equation with different position initial conditions and multiple initial conditions. Blacksmiths fashioned different metals into the desired shape by heating the objects with different temperatures and at different position. The numerical technique applied here can be used to solve heat equations observed in the field of science and engineering.

Keywords: Partial differential equation, Heat equations, Parabolic equations, Finite difference methods, Numerical solutions, Metal Heating, Blacksmithing.

1. INTRODUCTION

A partial differential equation (PDE) involves partial differential coefficients, i.e., the partial derivative of a dependent variable with more than one independent variable. As there is always more than one independent variable in a physical problem. PDEs are important in many branches of science and engineering. We can solve only a few PDEs, mostly linear equations and some nonlinear equations [1, 2, 8, 10]. The heat equation, also known as the diffusion equation in mathematics and physics, is a partial differential equation (PDE) that describes the distribution of heat evolution over time in a solid medium [2, 8, 10]. The heat equation is an important PDE which describes the variation in temperature (or distribution of heat) in a given region over time (Fig. 1). The heat equation (Diffusion equation) is widely used in particle diffusion, Brownian motion, Schrodinger equation for a free particle and thermal diffusivity in polymers. It's also used in metal processing industry, the outer surface of rockets, railway tracks and bridges, refrigerators, image analysis, cancer modeling and spatial ecological modeling. As a result, the heat equation is extremely important in a variety of scientific fields [2, 4, 5, 8, 9, 10, 24].

Let $u = u(x,t)$, then the partial differential equation of the form

$$u_t(x,t) = cu_{xx}(x,t), \; c > 0 \; ... (1.1)$$

where c is called thermal diffusivity, is one dimensional heat equation (diffusion equation). It is an example of a prototypical parabolic partial differential equation [1, 4].

In the regular shape domain, the heat equation has an analytic solution, whereas in the irregular shape domain, computing analytic solution of such equations is very difficult [12, 19]. As a result, we use numerical methods to compute the solution of the modeled partial differential equations. We use the finite difference method (FDM) to find numerical solutions to heat transfer problems, which was introduced by Brook Taylor in 1715 and has been studied as abstract self-standing mathematical objects in works by George Boole (1860), C.M. Milne Thomson (1933), and Karaly Jordan (1939) [12, 19].

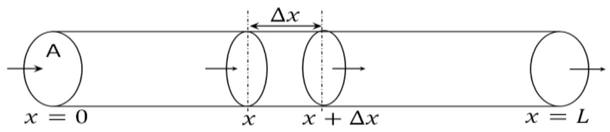

Fig. 1: Heat flow in a rod. Source….

By discretizing the domain into a finite number of regions, we can compute the numerical solution of partial differential equations. The solutions are computed at the domain's grid points [12, 19]. The numerical solutions of a one dimensional heat equation together with an initial condition and boundary conditions using finite difference methods does not always converge to the exact solutions [12, 19]. It denotes numerical instability in finite difference methods.

A blacksmith is a metalworker who fashions various objects out of iron or steel. By heating, hitting, twisting, and cutting metals, blacksmiths create objects such as gates, grilles, railings, light fixtures, furniture, sculpture, tools, agricultural implements, decorative and religious items, cooking utensils, and weapons. During their work, blacksmiths heat the metals to soften them so that they can be shaped with hand tools such as hammers, chisels, anvils, and so on. They estimate the temperature distribution on their metal pieces by looking at the glowing color on them. Heat may be applied to single or multiple locations on the metal [22].

We are all aware that heat transfers from a hotter body to a colder body. Temperature change is a major effect of heat transfer: heating raises the temperature while cooling lowers it. We assume that there is no phase change and that no work is done on or by the system throughout our work. Experiments show that the transferred heat is affected by three factors: temperature change, system mass, and substance phase [7, 17, 23]. In practice, we must apply different amounts of heat to different parts of the material in order to change the metals into the desired shape. One of the prominent examples is a blacksmith working on hot iron which is shown in Fig. 2.

Chamkha and Khaled [3] investigated the effect of magnetic field on coupled heat and mass transfer by mixed convection in a linearly stratified stagnation flow in the presence of internal heat generation or absorption. Exact analytic solutions and modeling for hydro magnetic oscillatory rotating flows of an incompressible Burgers fluid bounded by a plate [5, 16, 18]. Jean Baptise Joseph Fourier (1768-1830) invented the heat equation, which he presented as a manuscript to the Institute de France in 1807 AD and published in his monograph, Analytic Theory of Heat, in 1822 AD [13]. Scientists in the 1600s appeared to have been correct in their belief that heat is related to the motion of microscopic constituents of matter. However, in the 1700s, it was thought that heat was a separate fluid-like substance [25].

Fig. 2: Blacksmith working on hot iron [23].

We use the finite difference method (FDM) to find numerical solutions to heat transfer problems, which was introduced by Brook Taylor in 1715 and had been studied as abstract self-standing mathematical objects in works by George Boole (1860), C.M. Milne Thomson (1933) and Karaly Jordan (1939) [12, 19]. Makhtoumi [11] developed analytical and numerical solutions for studying heat diffusion investigating in a 1D thin rod. He used the rod PDE system to apply the homotopy perturbation method (HPM) and the finite difference method (FDM). Olaiju et al. [14, 15] investigated the explicit finite difference scheme and applied it to a simple 1D heat equation problem.

Finite difference methods are used to compute the numerical solutions of a one dimensional heat equation with initial and boundary conditions. To begin, the effect of heat supplied to the material at various positions will be investigated and compared. Finally, the effect of heat supplied in various positions will be investigated and compared.

2. Numerical Solutions for Variation of Initial Conditions

2.1 Heat Equation for 1D Iron Rod

Let us consider an example of a one dimensional heat equation in the case of iron rod, whose thermal diffusivity is $0.23cm^2/s = 0.000023m^2/s$ at 26.85^oC temperature [21] as follows

$u_t = 0.000023\, u_{xx}, \quad 0 \le x \le 1, \ \ t \ge 0$ ... (2.1)

BCs: $u(0,t) = u(1,t) = 0; \quad t > 0$

And for the different experiments we apply three different initial conditions.

2.1.1 Numerical Solution by Using FDM

The FTCS scheme of the above heat equation is [6]

$$v_m^{n+1} = v_m^n + c(v_{m+1}^n - 2v_m^n + v_{m-1}^n)$$

where, $v_0^n = v_m^n = 0$

This FTCSS is consistent with the order of accuracy (1, 2) and is stable iff $c \le \frac{1}{2}$ [6, 20]. We can maintain the condition of stability by resizing the lengths of space and time intervals. To find the more accurate approximation we have to increase the number of space and time partitions [6, 20]. Let the length of space and time intervals be $h = 0.1$ and $k = 0.2$ respectively. Then,

$$c = \frac{0.000023 \times k}{h^2} = \frac{0.000023 \times 0.2}{0.1^2} = 0.00046$$

We know that, the FTCS scheme is stable iff $c \le 0.5$, so our FTCS is stable for above problem.

A: Variation of position of initial condition:

In Nepali culture, blacksmith supplied heat at different position on the iron to make it fixable. Firstly, we discuss the variation position of initial condition, i.e., left end, middle and right end of the rod with heat equation (2.1) as above. Consider, a blacksmith applies the $1500C^o$ temperature near left end, at middle and near right end of the rod of 1m length successively. Thus, the corresponding initial conditions are:

IC1: $u(0.1, 0) = 1500$

IC2: $u(0.5, 0) = 1500$

IC3: $u(0.9, 0) = 1500$

After then we study the nature of heat transfer in the rod. The temperature distribution corresponding to the above three cases are shown in Fig. 3.

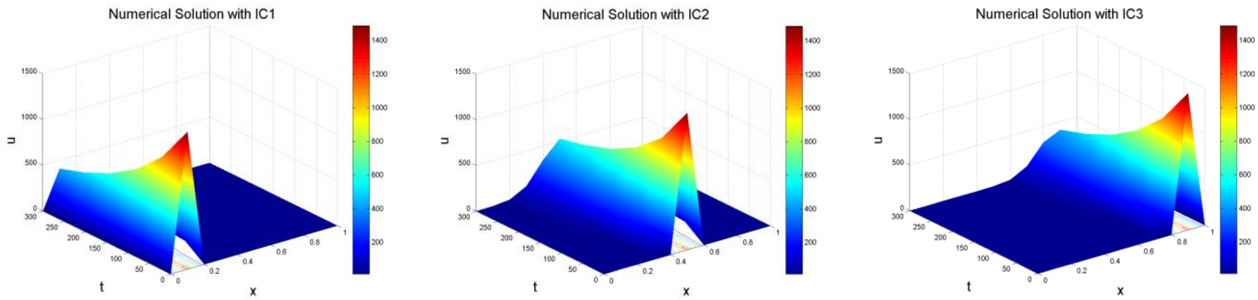

Fig. 3: Temperature distribution corresponding to **Top:** IC1, **Middle:** IC2 and **Bottom:** IC3.

From Fig. 3 we observe that, the figure with IC1 and IC3 allows heat to transfer in only one direction, to the right and left sides respectively. Heat, on the other hand, follows in either direction with IC2. A plot of the x versus temperature distribution is also shown below in Fig. 4.

This plot shows that after 250 sec. of heat, the iron rod is around 600 degrees Celsius, but in the first and third cases, the rod is only around 500 degrees Celsius. This variation is caused by the boundary conditions, which are set to 0°C at both ends. Heat applied to both the near left and near right produces the same result. However, the outcome would be more interesting if the material was non-homogeneous or if the boundary conditions on the left and right ends differed.

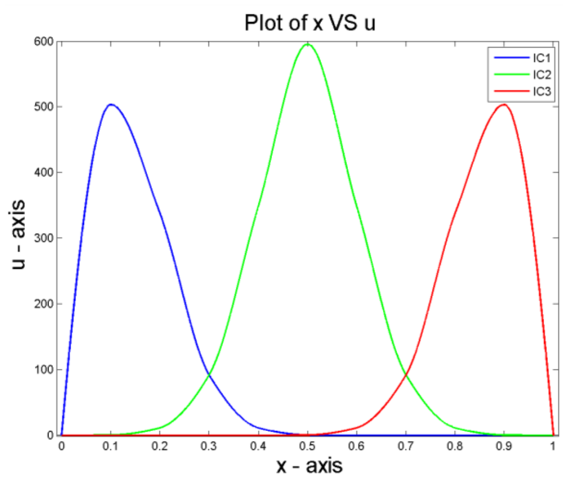

Fig. 4: Plot of x versus u after 250 sec. of heat supplied

B: Initial condition in Multiple Position: Here, the above concept has been generalized for the multiple initial conditions. Now for the first initial condition (IC1), we have

$$v_m^0 = \begin{cases} 1500 & \text{for } m = 5 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for the initial conditions IC2 and IC3 respectively, we have

$$v_m^0 = \begin{cases} 1500 & \text{for } m = 3,7 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$v_m^0 = \begin{cases} 1500 & \text{for } m = 2,5,8 \\ 0 & \text{otherwise} \end{cases}$$

Fig. 5 depicts the temperature distribution with different initial conditions, namely IC1, IC2 and IC3, up to 300 sec. from the instant of heating the material. According to Fig. 5, the temperature of the material, i.e., iron rod, gradually goes on decreasing from the time of heating. Over time, the rod cooled down to around $500^{\circ}C$ from $1500^{\circ}C$ in just 300 sec.

In IC1, we heated the road at one point, in IC2, we heated it at two points and in IC3, we heated at three different points, as stated above. The front edge of the numerical solution with IC3 in Fig. 5 is browner and darker red. This demonstrates that the rod has a more even temperature distribution than IC1 and IC2. This is because the number of heat-supplied locations has increased. Figure 2 depicts a real-world application of one of the blacksmiths' iron road heating techniques. This method of heating iron rods and other materials was also used by various metal industries, including blacksmiths and goldsmiths.

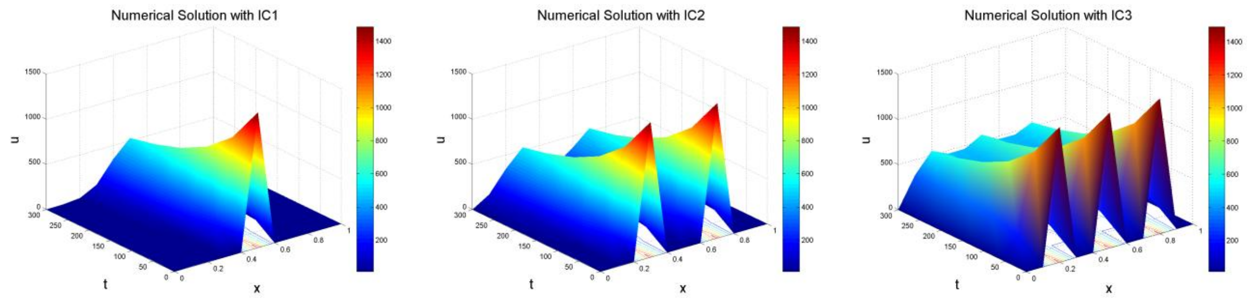

Fig. 5: Numerical solution of (2.1) with different initial conditions.

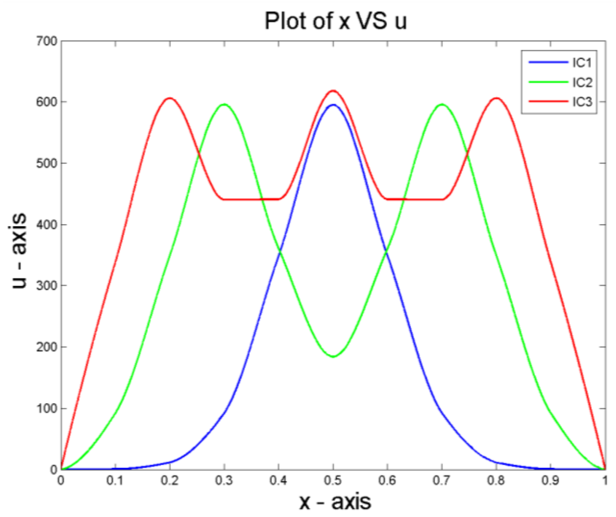

Fig. 6: Plot of x versus u after 250 sec. of heat supplied.

According to Fig. 6, the average temperature after 250 sec. of heating the iron rod is around 590°*C*. The figure shows that the amount of average temperature on the rod is greater with IC3 than that of IC2 and IC1 as a consequence of applying heat in more places than that of IC1 and IC2. As a result, we conclude that, the average temperature of the iron rod is proportional to the number of heat applied locations.

3. CONCLUSION

Using FDMs, we compute the numerical solution of one dimensional heat equation. To begin, we consider a one dimensional heat equation and consider the variety of positions of initial conditions, that is, heat is applied at the left, middle, and right ends of the rod. Initially, we applied the $1500^{\circ}C$ temperature to the aforementioned positions of the 1m long rod in succession. The nature of the heat transfer in the rod is then investigated. We discovered that when the temperature is applied to the center of the rod, it lasts longer. The outcome is due to the boundary conditions we used. Similarly, we applied 1500°C temperature to one position, then two positions, and finally three positions of the one dimensional iron rod. We conclude from this experiment that the average temperature of the iron rod is directly proportional to the number of heat applied locations. This method of heating iron rods and other materials was used by various metal industries, including blacksmiths and goldsmiths.

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