NEWTON TYPE ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS

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BY

JIVANDHAR JNAWALI

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DECLARATION

Thesis entitled **"Newton Type Iterative Methods for Solving Nonlinear Equations"** which is being submitted to the Central Department of Mathematics, Institute of Science and Technology (IOST), Tribhuvan University, Nepal for the award of the degree of Doctor of Philosophy (Ph.D.) is a research work carried out by me under the supervision of Prof. Dr. Chet Raj Bhatta, Central Department of Mathematics, Tribhuvan University and co-supervised by Dr. Pankaj Jain, Department of Mathematics, South Asian University, New Delhi, India.

This research is original and has not been submitted either in part or full in this or any other form to any university or institute, here or elsewhere, for the award of any degree.

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Jivandhar Jnawali

RECOMMENDATION

This is to recommend that **Mr. Jivandhar Jnawali** has carried out research entitled **"Newton Type Iterative Methods for Solving Nonlinear Equations"** for the award of Doctor of Philosophy (Ph.D.) in **Mathematics** under our supervision. To our knowledge, this work has not been submitted for any other degree.

He has fulfilled all the requirements laid down by the Institute of Science and Technology (IOST), Tribhuvan University, Kirtipur for the submission of the thesis for the award of Ph.D. degree.

....

Chet Raj Bhatta, PhD
Supervisor
(Professor)
Central Department of Mathematics
Tribhuvan University
Kirtipur, Kathmandu
Nepal

....

Pankaj Jain, PhD
Co-Supervisor
(Associate Professor)
Department of Mathematics
South Asian University
Chanakyapuri, New Delhi
India

[March 2017]

LETTER OF APPROVAL

On the recommendation of Prof. Dr. Chet Raj Bhatta and Dr. Pankaj Jain, this Ph.D. thesis submitted by Mr. Jivandhar Jnawali, entitled **“Newton Type Iterative Methods for Solving Nonlinear Equations”** is forwarded by Central Department Research Committee (CDRC) to Dean, IOST, T.U..

....

Narayan Prasad Pahari, PhD
Professor,
Head,
Central Department of Mathematics
Tribhuvan University
Kirtipur, Kathmandu
Nepal

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ABSTRACT

Solving single variable nonlinear equations efficiently is an important consideration in numerical analysis and has wide range of applications in all fields of science and engineering. Finding the analytic solutions of such equations is not always possible. Newton's method is the most widely used numerical method for solving such equations. In this thesis, we have developed several new Newton type iterative methods for solving nonlinear equations of a single variable. To obtain these methods, we used different techniques such as:

(i) amalgamation of existing methods;

(ii) amalgamation of existing and our investigated methods with the secant method;

(iii) amalgamation of existing methods and modified secant method;

(iv) idea of integral approximation; and

(v) use of inverse function methods.

The work done in this thesis is inspired by the work of Potra and Pták, Kasturiarachi, Jain, Weerakon and Fernando, Özban, Dhegain and Hajarian, Ujević, Erceg and Lakić, Amit and Basqular, Hasanov, Ivanov and Nedzhibov as well as recent work of McDaugall and Wotherspoon. For each method obtained in this thesis, the order of convergence has been calculated and compared with that of the similar existing methods. Also, most of the methods are supported by numerical examples.

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Chapter 1

Introduction

In this chapter, we collect certain definitions, notions and basic results of calculus and real analysis which have been used throughout the thesis. These things can be found in any standard textbook of calculus and/or analysis. Also, in this chapter, we give brief history of the work done by various researchers from time to time which motivated us to carry out our own investigations.

1.1 Preliminaries

We begin with the following:

Definition 1.1.1. [19] A function f defined on the set $\mathbb{R}$ of real numbers has the limit L at x_0, written

$$\lim_{x\to x_0} f(x) = L$$

if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon, \quad x \in \mathbb{R}.$$

Definition 1.1.2. [19] A function f defined on the set $\mathbb{R}$ of real numbers is said to be continuous at $x_0 \in \mathbb{R}$ if

$$\lim_{x\to x_0} f(x) = f(x_0).$$

The function f is continuous on the set $\mathbb{R}$ if it is continuous at each point $x_0 \in \mathbb{R}$.

Definition 1.1.3. [19] Let f be a function defined in an open interval (a, b) containing x_0. The function f is differentiable at x_0 if

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The quantity $f'(x_0)$ is called the derivative of f at x_0. A function that has a derivative at each point of $\mathbb{R}$ is said to be differentiable on $\mathbb{R}$.

The following is the well known Mean Value Theorem.

Theorem 1.1.4. [17] If f is a function continuous on some interval $[a, b]$ and differentiable on (a, b), then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Next is the famous Intermediate Value Theorem.

Theorem 1.1.5. [51] Let f be a continuous function defined on some interval $[a, b]$ and $f(a)f(b) < 0$, then the equation $f(x) = 0$ has at least one real root in the interval (a, b), i.e., there exists at least one point $c \in (a, b)$ such that $f(c) = 0$.

The following Taylor's Theorem will be used tremendously.

Theorem 1.1.6. [17] Suppose f is a continuous function on $[a, b]$, has n continuous derivatives on (a, b) and $f^{(n+1)}$ exists on $[a, b]$. Let $\alpha \in [a, b]$. For every $x \in [a, b]$, there exists a number ξ between x and α such that

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(\alpha)}{k!}(x - \alpha)^k \text{ and } R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - \alpha)^{n+1}.$$

Moreover, if the function f has continuous derivative of all orders at $x = \alpha$, then the above expression becomes

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(\alpha)}{k!}(x - \alpha)^k$$

which is called the Taylor series expansion of f about α.

The following Binomial expansion for any index will also be used frequently throughout the thesis.

Theorem 1.1.7. [103] Let n be a rational number and x be any real number such that $|x| < 1$, then

$$(1+x)^n = 1 + C(n,1)x + C(n,2)x^2 + C(n,3)x^3 + \cdots + C(n,r)x^r + \cdots .$$

Definition 1.1.8. [51] A real sequence $\{x_n\}$ is said to converge to a real number L, written $\lim_{n\to\infty} x_n = L$, if for given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon, \;\; \forall \;\; n \geq N.$$

Definition 1.1.9. [51] If $x_n, x_{n-1}, \ldots, x_{n-m+1}$ are m approximations to the root of the equation $f(x) = 0$, then a multipoint iteration method is defined by

$$x_{n+1} = \phi_f(x_n, x_{n-1}, \ldots, x_{n-m+1}). \tag{1.1.1}$$

The function ϕ_f is called the multipoint iteration function. For $m = 1$, we get the one point iteration method

$$x_{n+1} = \phi_f(x_n). \tag{1.1.2}$$

Definition 1.1.10. [17] A number $x \in \mathbb{R}$ is called a fixed point of a real valued function f if

$$f(x) = x.$$

Remark 1.1.11. A function f may have only one fixed point, more than one fixed points or no fixed point. For example, the function

$$f : (0,2) \to \mathbb{R}$$

defined by

$$f(x) = x^2$$

has only one fixed point $x = 1$. The function

$$f : [0,2] \to \mathbb{R}$$

defined by

$$f(x) = x^2$$

has two fixed point $x = 0, 1$. The function

$$f : (1, \infty) \to \mathbb{R}$$

defined by

$$f(x) = \frac{1}{x}$$

has no fixed point. The identity function has all of its points fixed.

The existence and uniqueness of fixed points is given by the famous Banach Fixed Point Theorem. Although the theorem is known for more general metric spaces, we shall state it for only reals. First, we define following.

Definition 1.1.12. [69] A mapping $f : \mathbb{R} \to \mathbb{R}$ is called a contraction mapping if there is a positive real number $\alpha < 1$ such that

$$|f(x) - f(y)| \leq \alpha |x - y|, \qquad x, y \in \mathbb{R}.$$

Theorem 1.1.13. [69](**Banach Fixed Point Theorem**) Let f be a contraction mapping on $\mathbb{R}$. Then f has a unique fixed point.

Definition 1.1.14. [19] Let $\{x_n\}$ be a sequence of iterates of some numerical method to solve a nonlinear equation $f(x) = 0$ and e_n be the error in x_n, i.e., $x_n = \alpha + e_n$, α being an exact root of $f(x)$ =0. The numerical method is said to be of order p if p is the smallest positive number such that there exists a constant A such that

$$e_{n+1} = A e_n^p, \qquad n = 1, 2, 3, \cdots$$

This equation is generally known as the error equation for the corresponding numerical method.

Definition 1.1.15. [51] The efficiency index of a numerical method is defined by $p^{\frac{1}{\theta}}$, where p is the order of the method and θ is the number of functions evaluation per iteration.

Theorem 1.1.16. [51] Let α be a simple root of the equation $f(x) = 0$ and (1.1.2) be the one point iteration method to approximate α. Then the method (1.1.2) is of order p if and only if

$$\phi_f(\alpha) = \alpha, \;\; \phi'_f(\alpha) = \phi''_f(\alpha) = \cdots = \phi_f^{p-1}(\alpha) = 0, \;\; \phi_f^p(\alpha) \neq 0.$$

Definition 1.1.17. [19] A root α of the equation $f(x) = 0$ is said to be a root of multiplicity m if f can be written in the form

$$f(x) = (x - \alpha)^m q(x),$$

where $q(\alpha) \neq 0$. A root of multiplicity one is called a simple root.

Theorem 1.1.18. [15] Let f be a continuous function with m continuous derivatives. The equation $f(x) = 0$ has a root of multiplicity m at $x = \alpha$ if and only if $f(\alpha) = f'(\alpha) = f''(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0$ but $f^{(m)}(\alpha) \neq 0$.

1.2 Some Classical Methods

In this section, we collect some classical iterative methods for solving nonlinear equations and methods used to approximate definite integrals.

Let $f(x) = 0$ be the given nonlinear equation. There are several standard classical methods to approximate the root of this equation see, e.g., [15], [17], [19], [24], [36], [51]. The two commonly used methods are the Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.2.1}$$

and the secant method

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n). \tag{1.2.2}$$

The Newton method is one point iteration method. This method is of order 2 and requires two functions evaluation per iteration. Consequently the efficiency index of Newton's method is $2^{\frac{1}{2}} \approx 1.414$.

The secant method is known to be of order 1.618. It is also known that it requires only one function evaluation per iteration giving the efficiency index 1.618. Comparing Newton's and secant methods, it is observed that although Newton's method has better rate of convergence but secant method is more efficient.

Next, consider the definite integral

$$\int_a^b f(x)\, dx. \tag{1.2.3}$$

There are several methods to find an approximate value of the above integral. Some of them to mention are trapezoidal rule, midpoint rule and Simpson's rule (see, [15], [17], [19], [24], [36], [51]) given respectively by

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}[f(b)+f(a)], \tag{1.2.4}$$

$$\int_a^b f(x)\,dx \approx (b-a)f\left(\frac{a+b}{2}\right), \tag{1.2.5}$$

and

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]. \tag{1.2.6}$$

There are several other methods to approximate integral. Some of them may be found in [1], [29], [33], [34], [45], [48], [56], [67], [74], [81], [99] and [104]. Few of these type of methods will be discussed in the subsequent chapters and will be used to derive efficient methods for solving nonlinear equations.

1.3 Brief Historical Background

Nonlinear equations are encountered quite often in all fields of science and engineering but solving such equation analytically is not always possible. In those situations, when an analytic solution cannot be obtained or it is difficult to obtain, numerical methods are used. In the previous section, we mentioned two such classical methods, namely, the Newton method and secant method. Some historical developments of the Newton and Newton type methods can be found in [20], [27], [82] and [106]. Over the years tremendous methods have appeared for solving nonlinear equations, each one is better than other in some or the other aspects. Some of these methods can be found in the papers [16], [21], [22], [23], [31], [32], [35], [38], [39], [44], [47], [55], [56], [57], [58], [61], [65], [66], [67], [68], [70], [71], [72], [74], [80], [88], [89], [90], [91], [92], [93], [94], [100], [107], [110] and [111].

It is difficult to trace out the whole development of all methods. However, in this section, we will mention the contribution of those researchers whose work motivated us to carry out our own investigations.

To begin with, let us mention the work of Kasturiarachi [64] who suggested that if the iteration of Newton's method and secant method are used alternatively, then the resulting method will be of order 3. The corresponding method is the following:

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f'(x_n)[f(x_n) - f(x_n^*)]},$$

where

$$x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The mixing of methods with the secant method has been seen to be very effective in order to increase the order of convergence. Such technique has been used by other authors as well, e.g., one may refer to [48], [49], [50] and [52].

In [104], Weerakoon and Fernando used the technique of numerical integration to improve Newton's method. In fact, they used the Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\, dt \tag{1.3.1}$$

and approximated the integral by trapezoidal rule that is

$$\int_{x_n}^{x} f'(t)\, dt = \frac{(x - x_n)}{2}[f'(x) + f'(x_n)]. \tag{1.3.2}$$

Then they obtained the variant of Newton's method which is given by the formula

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n^*)}, \tag{1.3.3}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$. In [81], Özban suggested that, in (1.3.1), if the integral is approximated by the mid point rule instead of the trapezoidal rule, then the corresponding method becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(\frac{x_n + x_n^*}{2})}, \tag{1.3.4}$$

where x_n^* same as in(1.3.3). Also in the same paper, Özban suggested the following harmonic mean Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)[f'(x_n) + f'(x_n^*)]}{2f'(x_n)f'(x_n^*)}. \tag{1.3.5}$$

It has been proved that the methods (1.3.3), (1.3.4) and (1.3.5) are all of order 3. Based on the same technique, Hasanov, Ivanov and Nedzhibov [40] obtained the following method which is based on Simson's rule.

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'\left(\frac{x_n+x_n^*}{2}\right) + f'(x_n^*)}. \tag{1.3.6}$$

Recently in 2010, Dehghan and Hajarian [29] approximated the integral in (1.3.1) by the linear combination of trapezoidal integration rule, mid point integration rule and harmonic mean rule as follows:

$$\int_{x_n}^{x} f'(t)\,dt = (x - x_n)\left[-\frac{2f'(x_n)f'(x)}{f'(x_n) + f'(x)} + \frac{3}{2}f'\left(\frac{x_n + x}{2}\right) - \frac{3}{4}\left(f'(x_n) + f'(x)\right)\right].$$

As a result, the following method was obtained:

$$x_{n+1} = x_n - \left[\frac{f(x_n)(f'(x_n) + f'(x_n^*))}{2f(x_n)f'(x_n^*)} + \frac{2}{3}\frac{f(x_n)}{f'(\frac{x_n+x_n^*}{2})} - \frac{4}{3}\frac{f(x_n)}{f'(x_n) + f'(x_n^*)}\right], \tag{1.3.7}$$

where x_n^* is same as in (1.3.3).

In [97], Ujević, Erceg and Lekić obtained a family of methods as a conclusion of similar method obtained by Ujević in [95] and [96] using distinct quadrature rules. Their method is following:

$$x_{n+1} = x_n + (z_n - x_n)\frac{f(x_n)}{\beta f(x_n) + \gamma f(z_n)}, \tag{1.3.8}$$

where $z_n = x_n - \alpha\frac{f(x_n)}{f'(x_n)}$, $0 < \alpha \leq 1$.

In [83], Potra and Pták suggested a modifications of Newton's method with third order convergence defined by

$$x_{n+1} = x_n - \frac{f(x_n) + f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)}. \tag{1.3.9}$$

Secant method (1.1.2) can be consider as a simplification of Newton's method (1.1.1), where $f'(x_n)$ is replace by

$$\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}. \tag{1.3.10}$$

The main limitation of this method with respect to the Newton method is of order since the expression (1.3.10) is not an optimal solution of $f'(x_n)$. In order to avoid this problem, Amat and Basqular [2] provided following generalization of secant method:

$$x_{n+1} = x_n - A_n^{-1} f(x_n), \tag{1.3.11}$$

where $A_n = [y_n, x_n; f] = \dfrac{f(x_n) - f(y_n)}{x_n - y_n}$, $y_n = x_n + \delta_n(x_{n-1} - x_n)$, $\delta_n \leq |O(\varepsilon_n)^{\frac{3}{2}}|$.

Very recently in 2014, McDaugall and Wotherspoon [75] obtained a modified Newton's method by a different strategy. Their method is the following:
If x_0 is the initial approximation, then

$$\begin{aligned} x_0^* &= x_0 && (1.3.12)\\ x_1 &= x_0 - \frac{f(x_0)}{f'[\frac{1}{2}(x_0 + x_0^*)]}. && (1.3.13) \end{aligned}$$

Subsequently for $n \geq 1$, the iterations can be obtained as

$$\begin{aligned} x_n^* &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1} + x_{n-1}^*)]} && (1.3.14)\\ x_{n+1} &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]}. && (1.3.15) \end{aligned}$$

It was proved that the last method is of order $1 + \sqrt{2}$. The methods presented in this section motivated us to carry out our own investigations.

1.4 Objectives

In most of the scientific and engineering problems, we come across nonlinear equations which need to be solved. Finding the exact solutions of such nonlinear equations is not always possible. Moreover many times, we do not require exact solutions. In such situations numerical methods are used. Some of the classical methods for solving nonlinear equations are bisection method, secant method, Regula-Falsi method and Newton's method. During last two decades, a lot of new methods have been obtained by several authors. Our main objective in the present thesis is to contribute in this direction by obtaining some effective numerical methods.

Chapter 2

Methods Based on McDougall and Wotherspoon Scheme

2.1 Introduction

Very recently in [75], McDougall and Wortherspoon obtained a method (1.3.12)-(1.3.15) with a slight modification in the standard Newton method and achieved order of convergence $1+\sqrt{2}$. The method (1.3.12)-(1.3.15) is a predictor-corrector type method. The predictor step is obtained just as the Newton step whereas in the corrector step, an arithmetic average is obtained between the previous two points and derivative is calculated at the average value.

First, we provide two variants of the method (1.3.12)-(1.3.15) by replacing the arithmetic average with geometric average and harmonic average. The corresponding methods are shown to be of order $1 + \sqrt{2}$ each.

Next, we construct a hybrid method by combining the iterations of the method (1.3.12)-(1.3.15) with the secant method. We show that the corresponding method is of order 3.5615. The motivation of combining two methods comes from the previous works of [48], [49], [50], [52] and [64], where the authors successfully obtained higher order of convergence.

The contents of this chapter is based on the author's paper [53].

2.2 The Methods with Harmonic and Geometric Averages

To begin with, we suggest the following method as a variant of (1.3.12)-(1.3.15) by replacing the arithmetic average with the harmonic average:

$$x_0^* = x_0 \tag{2.2.1}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'\left(\frac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.2.2}$$

followed by (for $n \geq 1$)

$$x_n^* = x_n - \frac{f(x_n)}{f'\left(\frac{2x_{n-1}x_{n-1}^*}{x_{n-1}+x_{n-1}^*}\right)} \tag{2.2.3}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{2x_nx_n^*}{x_n+x_n^*}\right)}. \tag{2.2.4}$$

The convergence of the method has been discussed in the following:

Theorem 2.2.1. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighbourhood of α. Then for solving the nonlinear equation $f(x) = 0$, the method (2.2.1)-(2.2.4) is convergent with order of convergence $1 + \sqrt{2}$.*

Proof. Denote $c_j = \frac{1}{j!} \cdot \frac{f^j(\alpha)}{f'(\alpha)}$, $j = 2, 3, 4 \cdots$. It is standard to work out that the error equation in the Newton method (1.2.1) is given by

$$e_{n+1} = c_2 e_n^2, \tag{2.2.5}$$

where e_n denotes the error in the iterate x_n and the terms with higher powers of e_n are ignored.

Let us now proceed with convergence analysis of the method (2.2.1)-(2.2.4). Let e_n and e_n^* denote the errors in the iterates x_n and x_n^*, respectively. Then obviously $e_0^* = e_0$ and in the view of (2.2.5), the error equation for (2.2.2) is given by

$$e_1 = c_2 e_0^2 \tag{2.2.6}$$

using which, Taylor series expansion and binomial expansion, the error equation for (2.2.3) with $n = 1$, i.e., for x_1^* is given by

$$\begin{aligned}
e_1^* &= e_1 - \frac{f(\alpha + e_1)}{f'(\alpha + e_0)} \\
&= e_1 - \frac{e_1 + c_2 e_1^2 + c_3 e_1^3 + O(e_1^4)}{1 + 2c_2 e_0 + 3c_3 e_0^2 + O(e_0^3)} \\
&= e_1 - \left(e_1 + c_2 e_1^2 + c_3 e_1^3 + O(e_1^4)\right)\left(1 - 2c_2 e_0 - 3c_3 e_0^2 + 4c_2^2 e_0^2 + O(e_0^3)\right) \\
&= 2c_2 e_0 e_1 \\
&= 2c_2^2 e_0^3, \qquad (2.2.7)
\end{aligned}$$

neglecting the higher powers of e_0.

Next, we find that

$$\begin{aligned}
\frac{2x_1 x^*}{x_1 + x_1^*} &= \frac{2(\alpha + e_1)(\alpha + e_1^*)}{(\alpha + e_1) + (\alpha + e_1^*)} \\
&= \left(\alpha + (e_1 + e_1^*) + \frac{e_1 e_1^*}{\alpha}\right)\left(1 + \frac{e_1 + e_1^*}{2\alpha}\right)^{-1} \\
&= \alpha + \frac{e_1 + e_1^*}{2},
\end{aligned}$$

neglecting the higher powers of e_1 and e_1^*. Therefore, the error equation for (2.2.4) with $n = 1$, i.e., x_2 can be obtained as follows:

$$\begin{aligned}
e_2 &= e_1 - \frac{f(\alpha + e_1)}{f'\left(\alpha + \frac{e_1 + e_1^*}{2}\right)} \\
&= e_1 - (1 + c_2 e_1^2 + c_3 e_1^3)\left[1 + 2c_2\left(\frac{e_1 + e_1^*}{2}\right) + 3c_3\left(\frac{e_1 + e_1^*}{2}\right)^2\right]^{-1} \\
&= c_2 e_1 e_1^* \\
&= 2c_2^4 e_0^5
\end{aligned}$$

by using (2.2.6) and (2.2.7). It can be shown, in general, that for $n \geq 2$, the errors respectively in x_n^* and x_n can be obtained recursively by the relations

$$e_n^* = c_2 e_n e_{n-1}$$

and

$$e_{n+1} = c_2 e_n e_n^*.$$

Using the above information, the errors at each stage in x_n^* and x_{n+1} are obtained and are tabulated below:

Table 2.1: Successive errors.

n	e_n	e_n^*
0	e_0	e_0
1	$c_2 e_0^2$	$2c_2^2 e_0^3$
2	$2c_2^4 e_0^5$	$2c_2^6 e_0^7$
3	$2^2 c_2^{11} e_0^{12}$	$2^3 c_2^{16} e_0^{17}$
4	$2^5 c_2^{28} e_0^{29}$	$2^7 c_2^{40} e_0^{41}$
5	$2^{12} c_2^{69} e_0^{70}$	$2^{17} c_2^{98} e_0^{99}$
$\vdots$	$\vdots$	$\vdots$

Note that, we obtain the same sequences $\{e_n\}$ and $\{e_n^*\}$ as obtained in [75]. Consequently, the method (2.2.1)-(2.2.4) is convergent with order of convergence $1+\sqrt{2}$. □

Next, we propose the following method that involves geometric average:

$$x_0^* = x_0 \tag{2.2.8}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(\sqrt{x_0 x_0^*})} = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.2.9}$$

followed by (for $n \geq 1$)

$$x_n^* = x_n - \frac{f(x_n)}{f'(\sqrt{x_{n-1} x_{n-1}^*})} \tag{2.2.10}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(\sqrt{x_n x_n^*})}. \tag{2.2.11}$$

We prove the following:

Theorem 2.2.2. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighbourhood of α. Then for solving nonlinear equation $f(x) = 0$, the method (2.2.8)-(2.2.11) is convergent with order of convergence $1 + \sqrt{2}$.*

Proof. As in the proof of Theorem 2.2.1, $e_0^* = e_0$ and $e_1 = c_2 e_0^2$. For $n = 1$, (2.2.10) becomes

$$x_1^* = x_1 - \frac{f(x_1)}{f'(x_0)},$$

which is exactly the same as obtained from (2.2.3) for $n = 1$. Therefore, the error e_1^* in (2.2.10) is as given by (2.2.7), i.e.,

$$e_1^* = 2c_2^2 e_0^3.$$

We now calculate the error in (2.2.11) for $n = 1$. We have

$$\begin{aligned} f'\left(\sqrt{x_1 x_1^*}\right) &= f'\left(\sqrt{(\alpha + e_1)(\alpha + e_1^*)}\right) \\ &= f'\left[\alpha\left(1 + \frac{e_1 + e_1^*}{\alpha} + \frac{e_1 e_1^*}{\alpha^2}\right)^{\frac{1}{2}}\right] \\ &= f'\left(\alpha + \frac{e_1 + e_1^*}{2}\right) \\ &= f'(\alpha)[1 + c_2(e_1 + e_1^*)] \end{aligned}$$

using the binomial expansion for fractions, Taylor's expansion and neglecting higher power terms of e_1 and e_1^*. Using this, the error e_2 in (2.2.11) can be calculated as

$$\begin{aligned} e_2 &= e_1 - (e_1 + c_2 e_1^2 + c_3 e_1^3)[1 + c_2(e_1 + e_1^*)]^{-1} \\ &= c_2 e_1 e_1^* \\ &= 2c_2^4 e_0^5. \end{aligned}$$

It can be shown, in general, that for $n \geq 2$, the errors e_n and e_n^* can be calculated recursively by the relations

$$e_n^* = c_2 e_n e_{n-1}$$

$$e_{n+1} = c_2 e_n e_n^*.$$

These relations are exactly the same as obtained in Theorem 2.2.1. Consequently, the method (2.2.8)-(2.2.11) is convergent with order of convergence $1+\sqrt{2}$. □

Remark 2.2.3. For any $n = 0, 1, 2, \cdots$, if $x_n x_n^* < 0$, then in the method (2.2.8)-(2.2.11), $\sqrt{x_n x_n^*}$ will not be real and hence the method will not proceed further. To avoid such situation, one has to be a little cautious. Although, the exact root of the given nonlinear equation is not known, but it is not difficult to know the sign of the root, e.g., one can plot the corresponding curve. In the case of positive root, if we start with a positive initial approximation x_0, then since the method is convergent, all iterates will be positive and there will be no negative product. The case of negative root can be handled similarly.

2.3 Combining with Secant Method

In this section, we provide a method by combining the iterations of the method (1.3.12)-(1.3.15) with secant method and show that the order of convergence of the resulting method is more than $1+\sqrt{2}$. Precisely, we propose the following method:
If x_0 is the initial approximation, then

$$x_0^* = x_0 \tag{2.3.1}$$

$$x_0^{**} = x_0 - \frac{f(x_0)}{f'[\frac{1}{2}(x_0 + x_0^*)]} = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.3.2}$$

$$x_1 = x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**}) \tag{2.3.3}$$

followed by (for $n \geq 1$)

$$x_n^* = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1} + x_{n-1}^*)]} \tag{2.3.4}$$

$$x_n^{**} = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]} \tag{2.3.5}$$

$$x_{n+1} = x_n^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**}). \tag{2.3.6}$$

For convergence of this method, we prove the following:

Theorem 2.3.1. *Let f be a function having sufficient number of continuous derivatives in a neighbourhood of α which is a simple root of the equation $f(x) = 0$. Then the method (2.3.1)-(2.3.6) to approximate the root α is convergent with order of convergence* 3.5615.

Proof. On the lines of the proofs of Theorems 2.2.1 and 2.2.2 and also the error equation of the standard secant method, it can be shown that the errors e_0^*, e_0^{**} and e_1 respectively in x_0^*, x_0^{**} and x_1 in the method (2.3.1)-(2.3.6) are given by

$$e_0^* = e_0$$

$$e_0^{**} = c_2 e_0^2$$

$$e_1 = c_2 e_0^* e_0^{**} = c_2^2 e_0^3.$$

Also the errors e_1^* and e_1^{**} respectively in x_1^* and x_1^{**} are given by

$$e_1^* = 2c_2 e_0 e_1 = 2c_2^3 e_0^4$$

$$e_1^{**} = c_2 e_1 e_1^* = 2^2 c_2^4 e_0^7.$$

It can be shown, in general, that for $n \geq 2$, the errors e_n^*, e_n^{**} and e_n respectively in x_n^*, x_n^{**} and x_n in the method (2.3.1)-(2.3.6) can be calculated recursively by the relations

$$e_n^* = c_2 e_{n-1} e_n$$

$$e_n^{**} = c_2 e_n e_n^*$$

$$e_n = c_2 e_{n-1}^* e_{n-1}^{**}.$$

The corresponding errors at each stage in x_n^*, x_n^{**} and x_n are obtained and tabulated as follows:

Table 2.2: Successive errors.

n	e_n	e_n^*	e_n^{**}
0	e_0	e_0	$c_2e_0^2$
1	$c_2^2e_0^3$	$2c_2^3e_0^4$	$2c_2^6e_0^7$
2	$2^2c_2^{10}e_0^{11}$	$2^2c_2^{13}e_0^{14}$	$2^4c_2^{24}e_0^{25}$
3	$2^6c_2^{38}e_0^{39}$	$2^8c_2^{49}e_0^{50}$	$2^{14}c_2^{88}e_0^{89}$
4	$2^{22}c_2^{138}e_0^{139}$	$2^{28}c_2^{177}e_0^{178}$	$2^{50}c_2^{316}e_0^{317}$
5	$2^{78}c_2^{494}e_0^{495}$	$2^{100}c_2^{633}e_0^{634}$	$2^{178}c_2^{1128}e_0^{1129}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

We make the analysis of the table as done in [75]. Note that the powers of e_0 in the error at each iterate from the sequence

$$3,\ 11,\ 39,\ 139,\ 495,\ 1763,\ 6279,\ 22363,\ \cdots$$

and the sequence of their successive ratios is

$$\frac{11}{3},\ \frac{39}{11},\ \frac{139}{495},\ \frac{495}{139},\ \frac{1763}{495},\ \frac{6279}{1763},\ \frac{22363}{6279},\ \cdots$$

or

$$3.67,\ 3.5454,\ 3.5641,\ 3.5611,\ 3.5616,\ 3.5615,\ 3.5615,\ \cdots.$$

This sequence approaches to a fixed number which approximately can be taken as 3.5615 which is the order of convergence of the method (2.3.1)-(2.3.6). $\square$

Alternative Proof

Here, we present different approach to prove Theorem 2.3.1. On the lines of the proofs of Theorems 2.2.1 and 2.2.2 and also the error equation of the standard secant method, it can be shown that the errors e_n^*, e_n^{**} and e_n respectively in x_n^*, x_n^{**} and x_n for $n \geq 2$ in the method (2.3.1)-(2.3.6) satisfy the following recursion formula:

$$\begin{aligned} e_n^* &= c_2 e_{n-1} e_n \\ e_n^{**} &= c_2 e_n e_n^* = c_2^2 e_{n-1} e_n^2 \\ e_{n+1} &= c_2^4 e_n^* e_n^{**} = c_2^4 e_{n-1}^2 e_n^3. \end{aligned} \tag{2.3.7}$$

To find the order of convergence of the method, we need a relation of the form

$$e_{n+1} = A e_n^p. \tag{2.3.8}$$

Thus,

$$e_n = A e_{n-1}^p \qquad \text{or} \qquad e_{n-1} = A^{-\frac{1}{p}} e_n^{\frac{1}{p}}. \tag{2.3.9}$$

From (2.3.7), (2.3.8) and (2.3.9), we have

$$A e_n^p = c_2^4 e_n^3 A^{-2/p} e_n^{2/p}.$$

Equating the power of e_n, we obtain

$$p = 3 + \frac{2}{p}$$

$$\text{or,} \qquad p^2 - 3p - 2 = 0$$

$$\text{or,} \qquad p = \frac{3 \pm \sqrt{17}}{2}.$$

Taking positive value, $p = 3.5615$. Thus, the order of convergence of the method (2.3.1)-(2.3.6) is 3.5615 and Theorem 2.3.1 is completely proved. □

Remark 2.3.2. Amalgamation of methods already exists in literature. For example, Kasturiarachi [64] amalgamated standard Newton and Secant methods, Jain [52] mixed iterations of Steffensen and Secant methods, Jain in [48] and [50] also mixed several methods with secant as well as with modified secant methods. It is noticed that whenever a method is combined with secant method, the order of convergence of the method gets increased by 1. In the present situation, the method (1.3.12)-(1.3.15) of McDougall and Wortherspoon [75] is of order $1+\sqrt{2}$ but the increment in our method (2.3.1)-(2.3.6) is more than 1 when combines with the secant method.

It is natural to consider the variants of methods (2.3.1)-(2.3.6), where in (2.3.2) and (2.3.5), the arithmetic mean is replaced by harmonic mean as well as geometric mean as done in methods (2.2.1)-(2.2.4) and (2.2.8)-(2.2.11), respectively.

Precisely, with harmonic mean, we propose the following method:

$$x_0^* = x_0 \tag{2.3.10}$$

$$x_0^{**} = x_0 - \frac{f(x_0)}{f'\left(\dfrac{2x_0x_0^*}{x_0 + x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.3.11}$$

$$x_1 = x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**}) \tag{2.3.12}$$

followed by (for $n \geq 1$)

$$x_n^* = x_n - \frac{f(x_n)}{f'\left(\dfrac{2x_{n-1}x_{n-1}^*}{x_{n-1} + x_{n-1}^*}\right)} \tag{2.3.13}$$

$$x_n^{**} = x_n - \frac{f(x_n)}{f'\left(\dfrac{2x_nx_n^*}{x_n + x_n^*}\right)} \tag{2.3.14}$$

$$x_{n+1} = x_n^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**}). \tag{2.3.15}$$

and with the geometric mean, we propose the following:

$$x_0^* = x_0 \tag{2.3.16}$$

$$x_0^{**} = x_0 - \frac{f(x_0)}{f'(\sqrt{x_0x_0^*})} = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.3.17}$$

$$x_1 = x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**}) \tag{2.3.18}$$

followed by (for $n \geq 1$)

$$x_n^* = x_n - \frac{f(x_n)}{f'\left(\sqrt{x_{n-1}x_{n-1}^*}\right)} \tag{2.3.19}$$

$$x_n^{**} = x_n - \frac{f(x_n)}{f'\left(\sqrt{x_nx_n^*}\right)} \tag{2.3.20}$$

$$x_{n+1} = x_n^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**}). \tag{2.3.21}$$

Using the arguments as used in the proofs of Theorems (2.2.1), (2.2.2) and (2.3.1), the following result can be proved. We omit the details.

Theorem 2.3.3. *Let f be a function having sufficient number of continuous* *derivatives in a neighbourhood of α which is a simple root of the equation $f(x) = 0$. Then the methods (2.3.10)-(2.3.15) as well as (2.3.16)-(2.3.21) to approximate the root α are convergent with order of convergence 3.5615.*

2.4 Algorithms and Numerical Examples

We give below an algorithm in order to implement the method (2.3.1)-(2.3.6):

Algorithm 2.4.1. *Step 1: For the given tolerance $\varepsilon > 0$ and iteration N, choose the initial approximation x_0 and set $n = 0$.*

Step 2: Follow the following sequence of expressions:

$$\begin{aligned} x_0^* &= x_0 \\ x_0^{**} &= x_0 - \frac{f(x_0)}{f'[\frac{1}{2}(x_0 + x_0^*)]} = x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**}). \end{aligned}$$

Step 3: For $n = 1, 2, 3, \ldots,$ calculate $x_2, x_3, x_4, \ldots$ by the following sequence of expressions:

$$\begin{aligned} x_n^* &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1} + x_{n-1}^*)]} \\ x_n^{**} &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]} \\ x_{n+1} &= x_n^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**}). \end{aligned}$$

Step 4: Stop if either $|x_{n+1} - x_n| < \varepsilon$ or $n > N$.

Step 5: Set $n = n + 1$ and repeat Step 3.

Example 2.4.2. We apply Algorithm (2.4.1) on the nonlinear equation

$$\cos x - xe^x + x^2 = 0. \tag{2.4.1}$$

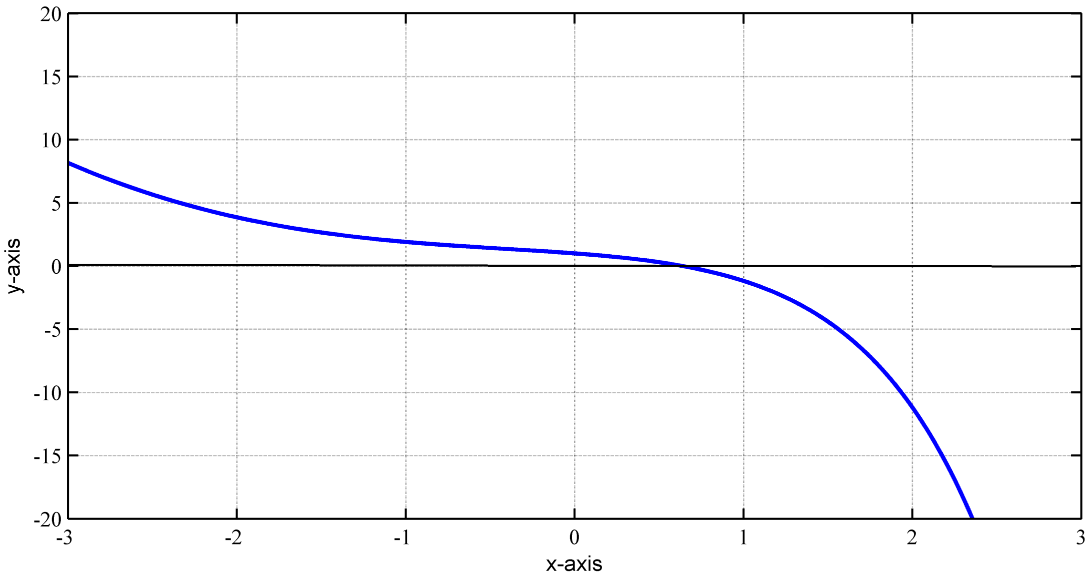

Figure 2.1: Graph of the function $f(x) = \cos x - xe^x + x^2$.

From Figure 2.1, it is clear that the equation (2.4.1) has a simple root in the interval $(0, 1)$. Table 2.3 shows the iterations of McDougall-Wortherspoon method (1.3.12)-(1.3.15), a third order method ([104]) and our method (2.3.1)-(2.3.6).

Table 2.3: Approximation of root of the equation (2.4.1) using different methods.

n	McDougall-Wotherspoon method	A third order method [104]	Present method (2.3.1)-(2.3.6)
1.	0.64132328499316349	0.64599588437664313	0.63915520442184104
2.	0.6391544362117092	0.63915411336536088	0.6391541004893474
3.	0.63915409833960735	0.63915408672427509	division by zero
4.	0.63915411809538092	0.63915409327226524	
5.	0.63915407824650872	0.63915409982025551	
6.	0.63915409800228429	0.63915410636824566	
7.	0.63915411775805786	0.63915411291623581	
8.	0.63915407790918577	0.63915408627515002	
9.	0.63915409766496134	division by zero	
10.	0.63915411742073491		
11.	0.63915407757186271		
12.	0.63915409732763828		
13.	0.63915411708341185		
14.	0.63915407723453976		
15.	0.63915409699031522		
16.	0.63915411674608891		
17.	0.63915407689721671		
18.	0.63915409665299228		
19.	0.63915411640876585		
20.	0.63915407655989365		

Example 2.4.3. We refer to the problem of "Solving a Crime" from [17]. The problem is of estimating the time of death of a person. It was noticed that the core temperatures of the corpse were 90°F and 85°F at 8 PM and 9 PM, respectively.

Also, it was noticed that due to the failure of air conditioner, the room temperature increased at the rate of 1°F per hour. Using the Newton's Law of Cooling, the problem reduces to solving the equations

$$\left(18 + \frac{1}{k}\right)e^{-k} - \frac{1}{k} - 12 = 0 \tag{2.4.2}$$

and

$$\left(18 + \frac{1}{k}\right)e^{-kt} + t - \frac{1}{k} - 26.6 = 0 \tag{2.4.3}$$

simultaneously, where k denotes the constant of proportionality and t denotes the time. The equations (2.4.2) and (2.4.3) are nonlinear and so precise values of k and t is difficult to find. The author in [17] used secant method to solve (2.4.2) with initial interval $(0.1, 1)$. After six iterations, the approximate value of k was obtained as 0.337114. Using $k = 0.337114$ in (2.4.3) and using secant method again with initial interval $(-2, 0)$, after six iterations, t was found to be -1.130939 which means that the man would have been dead approximately 1 hour 8 minutes before 8 PM. In Tables 2.4 and 2.5, we demonstrate that if instead of secant method, we apply our method (2.3.1)-(2.3.6), then we require much less than six iterations to reach the same conclusion.

Table 2.4: To calculate value of k in (2.4.2).

n	Mcdougall-Wortherspoon method	A third order method [104]	Present method (2.3.1)-(2.3.6)
1.	0.39424536527674747	0.28399501628622575	0.33729218050164789
2.	0.33712186735193811	0.33711307862711176	0.33711438414127259
3.	0.33711437423853269	0.33711439449543812	division by zero
4.	0.33711439130907683	division by zero	
5.	0.33711437857729748		
6.	0.33711439564784168		
7.	0.33711438291606233		
8.	0.33711437018428397		
9.	0.33711438725482684		
10.	0.33711437452304749		

Table 2.5: To calculate value of t in (2.4.3).

n	McDougall-Worthterspoon method	A third order method [104]	Present method (2.3.1)-(2.3.6)
1.	-0.43229389880795427	-1.4680241724237642	-1.1259217033754243
2.	-1.1245149717844931	-1.1310367658010889	-1.1309393994384249
3.	-1.1309384372848865	-1.13093937858347	division by zero
4.	-1.1309393943474491	division by zero	
5.	-1.1309393977356448		
6.	-1.1309394011238405		
7.	-1.1309394045120362		
8.	-1.130939407900232		
9.	-1.1309394112884277		
10.	-1.1309394146766234		

Remark 2.4.4. The examples in support of methods (2.3.10)-(2.3.15) as well as (2.3.16)-(2.3.21) have also been tested and verified. For conciseness, we avoid the details.

2.5 A Generalised Method

In this section, we propose the variant of Newton's method (1.2.1) whose iterative schemes are as follows:
If x_0 is the initial approximation, then

$$x_0^* = x_0 \tag{2.5.1}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(ax_0 + bx_0^*)} = x_0 - \frac{f(x_0)}{f'(x_0)}, \qquad \text{where } a + b = 1. \tag{2.5.2}$$

Subsequently for $n \geq 1$

$$x_n^* = x_n - \frac{f(x_n)}{f'(ax_{n-1} + bx_{n-1}^*)} \tag{2.5.3}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(ax_n + bx_n^*)]}. \tag{2.5.4}$$

For the convergence of this method we prove following result.

Theorem 2.5.1. *Let f be a function having sufficient number of continuous derivatives in a neighbourhood of α which is a simple root of the equation $f(x) = 0$. Then the method (2.5.1)-(2.5.4) to approximate the root α is convergent with order of convergence $1 + \sqrt{2}$ when $a = \frac{1}{2}$ and $b = \frac{1}{2}$ and for all other values of a and b, its order of convergence is 2.*

Proof. As in the lines of the proof of Theorem (2.2.1), denote $c_j = \frac{f^j(\alpha)}{j!f'(\alpha)}$. The errors e_0^*, e_1 and e_1^* respectively in x_0^*, x_1 and x_1^* are given by

$$e_0^* = e_0$$

$$e_1 = c_2 e_0^2$$

$$e_1^* = 2c_2^2 e_0^3.$$

From (2.5.4), the error e_2 in x_2 is given by

$$\begin{aligned} e_2 = e_1 - \frac{f(\alpha + e_1)}{f'[a(\alpha + e_1) + b(\alpha + e_1^*)} \\ = e_1 - \frac{f(\alpha + e_1)}{f'[\alpha + ae_1 + be_1^*)} \\ = e_1 - (e_1 + c_2 e_1^2 + c_3 e_1^3 + \cdots)[1 + 2c_2(ae_1 + be_1^*) + \cdots]^{-1} \\ = c_2(2a - 1)e_1^2 + 2c_2 b e_1 e_1^* - c_3 e_1^3 + \cdots . \end{aligned}$$

Thus, for $k > 1$, the error e_k in x_k is given by

$$e_k = c_2(2a - 1)e_{k-1}^2 + 2c_2 b e_{k-1} e_{k-1}^* - c_3 e_{k-1}^3 + \cdots .$$

From above it is clear that if a and b are other than $\frac{1}{2}$, the rate of convergence of method (2.5.1)-(2.5.4) is only 2 and when we take $a = \frac{1}{2}$ and $b = \frac{1}{2}$, this method becomes exactly the method (1.3.12)-(1.3.15) and hence order of convergence is $1 + \sqrt{2}$. This completes the proof. □

Chapter 3

Methods Based on Weerakoon and Fernando Scheme

3.1 Introduction

The method given by Weerakoon and Fernando [104] which is based on (1.3.1) and the integral involved is approximated by the trapezoidal rule (1.3.2). As a result, Weerakoon and Fernando obtained iterative method (1.3.3) for solving nonlinear equation $f(x) = 0$. The method so obtained is of third order. Here, the aim is to modify method (1.3.3). In fact, in (1.3.3), f' is a function of the previously calculated iterate. In our modification, f' would be a function of some other convenient points. It is proved that the corresponding method has order of convergence 5.1925. We follow the technique of McDougall and Wotherspoon [75] who modified Newton's method in a similar way yielding the order of convergence of their method as $1 + \sqrt{2}$.

Further, in [48] and [50], it was proved that if any method for solving nonlinear equation is used in conjunction with the standard secant method, then the order of the resulting method is increased by 1. We shall show, in this chapter (see Theorem 3.3.2), that this order can be increased by more than 1. In fact, we prove that if our own method (which is of order 5.1925) is combined with the secant method, then the new method is of order 7.275.

The contents of this chapter are based on the author's paper [54].

3.2 Method with Trapezoidal Rule

We propose the following method:

If x_0 is the initial approximation, then

$$\left.\begin{aligned} & x_0^* = x_0 \\ & x_1 = x_0^* - \frac{2f(x_0^*)}{f'(x_0^*) + f'(z_1)}, \\ \text{where} \quad & z_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \\ & x_1^* = x_1 - \frac{2f(x_1)}{f'(x_1) + f'(z_1^*)}, \\ \text{with} \quad & z_1^* = x_1 - \frac{f(x_1)}{f'[\frac{1}{2}(x_0 + x_0^*)]} = x_1 - \frac{f(x_1)}{f'(x_0)}. \end{aligned}\right\} \tag{3.2.1}$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$\left.\begin{aligned} & x_n^* = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n^*)}, \\ \text{where} \quad & z_n^* = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1} + x_{n-1}^*)]} \\ & x_{n+1} = x_n^* - \frac{2f(x_n^*)}{f'(x_n^*) + f'(z_{n+1})}, \\ \text{with} \quad & z_{n+1} = x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]}. \end{aligned}\right\} \tag{3.2.2}$$

Below, we prove the convergence result for the method (3.2.1)-(3.2.2).

Theorem 3.2.1. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then the method (3.2.1)-(3.2.2) is convergent and has the order of convergence* 5.1925.

Proof. Let e_n and e_n^* denote, respectively, the errors in the terms x_n and x_n^*. Also, we denote $c_j = \dfrac{f^j(\alpha)}{j!f'(\alpha)}$, $j = 2, 3, 4, \cdots$, which are constants. The error equation for the method (1.3.3) as obtained by Weerakoon and Fernando [104] is given by

$$e_{n+1} = ae_n^3,$$

where $a = c_2^2 + \frac{1}{2}c_3$ and we have neglected higher power terms of e_n. In particular, the error e_1 in x_1 in the equations (3.2.1) is given by

$$e_1 = ae_0^3. \tag{3.2.3}$$

We now proceed to calculate the error e_1^* in x_1^*. By using Taylor series expansion and binomial expansion, we get

$$\begin{aligned}\frac{f(x_1)}{f'(x_0)} &= \frac{f(\alpha+e_1)}{f'(\alpha+e_0)}\\ &= \big(e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)\big)\big(1 + 2c_2e_0 + 3c_3e_0^2 + O(e_0^3)\big)^{-1}\\ &= e_1 - 2c_2e_0e_1 + O(e_0^5)\end{aligned}$$

so that

$$x_1 - \frac{f(x_1)}{f'(x_0)} = \alpha + 2c_2e_0e_1 + O(e_0^5).$$

Consequently, by Taylor series expansion, it can be calculated that

$$\begin{aligned}f'(z_1^*) &= f'\left(x_1 - \frac{f(x_1)}{f'(x_0)}\right)\\ &= f'\big(\alpha + 2c_2e_0e_1 + O(e_0^5)\big)\\ &= f'(\alpha)\big(1 + 4c_2^2e_0e_1 + O(e_0^5)\big).\end{aligned}$$

Also

$$\begin{aligned}f'(x_1) &= f'(e_1+\alpha)\\ &= f'(\alpha)\big(1 + 2c_2e_1 + 3c_3e_1^2 + O(e_1^3)\big)\end{aligned}$$

so that

$$f'(x_1) + f'(z_1^*) = 2f'(\alpha)\big(1 + c_2e_1 + 2c_2^2e_0e_1 + O(e_0^5)\big). \tag{3.2.4}$$

Now, using (3.2.3) and (3.2.4), the error e_1^* in x_1^* in the equation (3.2.1) can be calculated as

$$\begin{aligned}e_1^* &= e_1 - \big(e_1 + c_2e_1^2 + O(e_1^3)\big)\big(1 + c_2e_1 + 2c_2^2e_0e_1 + O(e_0^5)\big)^{-1}\\ &= 2c_2^2e_0e_1^2\\ &= ba^2e_0^7,\end{aligned}$$

where $b = 2c_2^2$. Using e_1^*, we now compute the error e_2 in the term

$$x_2 = x_1^* - \frac{2f(x_1^*)}{f'(x_1^*) + f'(z_2)},$$

where

$$z_2 = x_1 - \frac{f(x_1)}{f'\big(\frac{x_1+x_1^*}{2}\big)}.$$

Now

$$\begin{aligned} f'\big(\frac{x_1 + x_1^*}{2}\big) &= f'\big(\alpha + \frac{e_1 + e_1^*}{2}\big) \\ &= f'(\alpha)\big(1 + c_2 e_1 + c_2 e_1^* + \frac{3}{4}c_3 e_1^2 + O(e_0^9)\big) \end{aligned}$$

so that

$$\begin{aligned} \frac{f(x_1)}{f'\big(\frac{x_1+x_1^*}{2}\big)} &= \big(e_1 + c_2 e_1^2 + O(e_1^3)\big)\big(1 + c_2 e_1 + c_2 e_1^* + \frac{3}{4}c_3 e_1^2 + O(e_0^9)\big)^{-1} \\ &= e_1 + \frac{1}{4}c_3 e_1^3 - c_2 e_1 e_1^* \end{aligned}$$

and therefore

$$z_2 = \alpha - \frac{1}{4}c_3 e_1^3 + c_2 e_1 e_1^*,$$

where the higher power terms are neglected. Thus

$$f'(z_2) = f'(\alpha)\big(1 - \frac{1}{2}c_2 c_3 e_1^3 + 2c_2^2 e_1 e_1^*\big)$$

and

$$f'(x_1^*) = f'(\alpha)\big(1 + 2c_2 e_1^* + 3c_3 e_1^{*2}\big).$$

Using the above considerations, the error e_2 in x_2 is given by

$$\begin{aligned} e_2 &= e_1^* - \big(e_1^* + c_2 e_1^{*2} + c_3 e_1^{*3}\big)\big(1 + c_2 e_1^* - \frac{1}{4}c_2 c_3 e_1^3\big)^{-1} \\ &= -\frac{1}{4}c_2 c_3 e_1^3 e_1^* \\ &= c e_1^3 e_1^*, \end{aligned}$$

where $c = -\frac{1}{4}c_2 c_3$. In fact, it can be worked out that for $n \geq 1$, the following relation holds:

$$e_{n+1} = ce_n^3 e_n^*. \tag{3.2.5}$$

In order to compute e_{n+1} explicitly, we need to compute e_n^*. We already know e_1^*. We now compute e_2^*. We have

$$x_2^* = x_2 - \frac{2f(x_2)}{f'(x_2) + f'(z_2^*)},$$

where

$$z_2^* = x_2 - \frac{f(x_2)}{f'\left(\frac{x_1+x_1^*}{2}\right)}.$$

Like above, it can be calculated that the error e_2^* is given by

$$e_2^* = de_1 e_2^2,$$

where $d = c_2^2$ and, again, it can be checked that in general, for $n \geq 2$, the following relation holds:

$$e_n^* = de_{n-1} e_n^2. \tag{3.2.6}$$

In the view of (3.2.5) and (3.2.6), the errors at each stage in x_n^* and x_{n+1} are calculated which are tabulated below:

Table 3.1: Successive errors.

n	e_n	e_n^*
0	e_0	e_0
1	ae_0^3	$a^2be_0^7$
2	$a^5bce_0^{16}$	$a^{11}b^2c^2de_0^{35}$
3	$a^{26}b^5c^6de_0^{83}$	$a^{57}b^{11}c^{13}d^3e_0^{182}$
4	$a^{135}b^{26}c^{32}d^6e_0^{431}$	$a^{296}b^{57}c^{70}d^{14}e_0^{945}$
5	$a^{701}b^{135}c^{167}d^{32}e_0^{2238}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$

It is observed that the powers of e_0 in the errors at each iterate form a sequence

$$3,\ 16,\ 83,\ 431,\ 2238,\ \cdots \tag{3.2.7}$$

and the sequence of their successive ratios is

$$\frac{16}{3}, \frac{83}{16}, \frac{431}{83}, \frac{2238}{431}, \cdots$$

or,

$$5.3334,\ 5.1875,\ 5.1927,\ 5.1925,\ \cdots .$$

This sequence seems to converge to the number 5.1925 approximately. Indeed, if the terms of the sequence (3.2.7) are denoted by $\{\, \alpha_i \,\}$, then it can be seen that

$$\alpha_i = 5\alpha_{i-1} + \alpha_{i-2}, \qquad i = 2, 3, 4 \cdots \tag{3.2.8}$$

If we set the limit

$$\frac{\alpha_i}{\alpha_{i-1}} = \frac{\alpha_{i-1}}{\alpha_{i-2}} = R,$$

Then dividing (3.2.8) by α_{i-1}, we obtain

$$R^2 - 5R - 1 = 0$$

which has its positive root as $R = \dfrac{5 + \sqrt{29}}{2} \approx 5.1925$. Hence the order of convergence of the method is at least 5.1925. □

Next, we give two variants of the method (3.2.1)-(3.2.2). Note that, in (3.2.1)-(3.2.2), the arithmetic average of the points $x_n, x_n^*,\ n = 0, 1, 2, \cdots$ has been used. We propose methods in which the arithmetic average is replaced by harmonic as well as geometric averages. With harmonic average, we propose the following method:
If x_0 is the initial approximation, then

$$\left.\begin{aligned}
& x_0^* = x_0 \\
& x_1 = x_0^* - \frac{2f(x_0^*)}{f'(x_0^*) + f'(z_1)}, \\
\text{where}\quad & z_1 = x_0 - \frac{f(x_0)}{f'\left(\frac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)} \\
& x_1^* = x_1 - \frac{2f(x_1)}{f'(x_1) + f'(z_1^*)}, \\
\text{with}\quad & z_1^* = x_1 - \frac{f(x_1)}{f'\left(\frac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_1 - \frac{f(x_1)}{f'(x_0)}.
\end{aligned}\right\} \tag{3.2.9}$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$\left.\begin{aligned} x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n^*)}, \\ \text{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_{n-1}x_{n-1}^*}{x_{n-1}+x_{n-1}^*}\right)} \\ x_{n+1} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*) + f'(z_{n+1})}, \\ \text{with} \qquad z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_nx_n^*}{x_n+x_n^*}\right)}. \end{aligned}\right\} \tag{3.2.10}$$

For the geometric average of the points $x_n, x_n^*,\ n = 0, 1, 2, \cdots$, the following method is proposed:

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_1 &= x_0^* - \frac{2f(x_0^*)}{f'(x_0^*) + f'(z_1)}, \\ \text{where} \qquad z_1 &= x_0 - \frac{f(x_0)}{f'\left(\sqrt{x_0x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1^* &= x_1 - \frac{2f(x_1)}{f'(x_1) + f'(z_1^*)}, \\ \text{with} \qquad z_1^* &= x_1 - \frac{f(x_1)}{f'\left(\sqrt{x_0x_0^*}\right)} = x_1 - \frac{f(x_1)}{f'(x_0)}. \end{aligned}\right\} \tag{3.2.11}$$

Subsequently, for $n \geq 1$, the iteration can be obtained as follows:

$$\left.\begin{aligned} x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n^*)}, \\ \text{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\sqrt{x_{n-1}x_{n-1}^*}\right)} \\ x_{n+1} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*) + f'(z_{n+1})}, \\ \text{with} \qquad z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\sqrt{x_nx_n^*}\right)}. \end{aligned}\right\} \tag{3.2.12}$$

The convergence of the methods (3.2.9)-(3.2.10) and (3.2.11)-(3.2.12) can be proved on the similar lines as those in Theorem 3.2.1. We only state the results below:

Theorem 3.2.2. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving nonlinear equation*

$f(x) = 0$, *the method (3.2.9)-(3.2.10) is convergent with order of convergence* 5.1925.

Theorem 3.2.3. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving nonlinear equation $f(x) = 0$, the method (3.2.11)-(3.2.12) is convergent with order of convergence* 5.1925.

3.3 Combining with Secant Method

In this section, we obtain a new iterative method by combining the iterations of method (3.2.1)-(3.2.2) with secant method and prove that the order of convergence is more than 5.1925. Precisely, we propose the following method:
If x_0 is the initial approximation, then

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_0^{**} &= x_0^* - \frac{2f(x_0^*)}{f'(x_0^*) + f'(z_1)}, \\ \text{where} \qquad z_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**}). \end{aligned}\right\} \tag{3.3.1}$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$\left.\begin{aligned} x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n^*)}, \\ \text{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)} \\ x_n^{**} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*) + f'(z_{n+1})}, \\ \text{where} \qquad z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)} \\ x_{n+1} &= x_1^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**}). \end{aligned}\right\} \tag{3.3.2}$$

Remark 3.3.1. In [48] and [50], it was proved that if the iterations of any method of order p for solving nonlinear equations are used alternatively with secant method, then the new method will be of order $p + 1$. Thus, in view of that result, the method (3.3.1)-(3.3.2) is certainly of order at least 6.1925. However, we prove below that the order is more.

Theorem 3.3.2. *Let f be a function having sufficient number of smooth derivatives in a neighborhood of α which is a simple root of the equation $f(x) = 0$. Then method (3.3.1)-(3.3.2) to approximate the root α is convergent with order of convergence* 7.275.

Proof. We argue on the lines of that of Theorem 3.2.1 and the error equation of the standard secant method. In particular, the errors e_0^*, e_0^{**} and e_1, respectively, in x_0^*, x_0^{**} and x_1 in (3.3.1) are given by

$$\begin{aligned}
e_0^* &= e_0 \\
e_0^{**} &= a e_0^3, \qquad \text{where } a = c_2^2 + \frac{1}{2}c_3 \\
e_1 &= c_2 e_0^* e_0^{**} = \lambda a e_0^4, \qquad \text{where } \lambda = c_2.
\end{aligned}$$

Also, the errors e_1^* in x_1^* in (3.3.2) is given by

$$\begin{aligned}
e_1^* &= 2c_2^2 e_0 e_1^2 \\
&= \lambda^2 a^2 b e_0^9, \qquad \text{where } b = 2c_2^2
\end{aligned}$$

and the error e_1^{**} in x_1^{**} in (3.3.2) is given by

$$\begin{aligned}
e_1^{**} &= -\frac{1}{4}c_2 c_3 e_1^3 e_1^* \\
&= c e_1^3 e_1^*,
\end{aligned}$$

where $c = -\frac{1}{4}c_2c_3$. In fact, it can be worked out that for $n \geq 1$, the following relation holds:

$$e_n^{**} = c e_n^3 e_n^*. \tag{3.3.3}$$

In order to compute e_n^{**} explicitly, we need to compute e_n and e_n^*. We have already computed e_1 and e_1^*. From the proof of Theorem 3.2.1

$$e_2^* = d e_1 e_2^2,$$

where $d = c_2^2$ and, again, it can be checked that the following relation holds:

$$e_n^* = d e_{n-1} e_n^2. \tag{3.3.4}$$

Also from (3.3.2), it can be shown that

$$e_2 = \lambda e_1^* e_2^{**}.$$

Thus, for $n \geq 1$, it can be shown that the error e_{n+1} in x_{n+1} in the method (3.3.1)-(3.3.2) satisfies the following recursion formula

$$e_{n+1} = \lambda e_n^* e_n^{**}. \tag{3.3.5}$$

Using the above information, the errors at each stage in x_n^*, x_n^{**} and x_n are obtained and tabulated as follows:

Table 3.2: Successive errors.

n	e_n	e_n^*	e_n^{**}
0	e_0	e_0	ae_0^3
1	$\lambda a e_0^4$	$\lambda^2 a^2 b e_0^9$	$\lambda^5 a^5 bce_0^{21}$
2	$\lambda^8 a^7 b^2 c e_0^{30}$	$\lambda^{17} a^{15} b^5 c^2 e_0^{64}$	$\lambda^{42} a^{36} b^{11} c^6 e_0^{154}$
3	$\lambda^{60} a^{51} b^{13} c^8 e_0^{218}$	$\lambda^{128} a^{109} b^{29} c^{17} e_0^{466}$	$\lambda^{308} a^{260} b^{68} c^{42} e_0^{1120}$
4	$\lambda^{437} a^{369} b^{97} c^{59} e_0^{1586}$	$\lambda^{934} a^{789} b^{208} c^{126} e_0^{3390}$	$\lambda^{2245} a^{1896} b^{499} c^{304} e_0^{8148}$
5	$\lambda^{3180} a^{2685} b^{707} c^{430} e_0^{11538}$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

We do the analysis of Table 3.2 as done in the proof of Theorem 3.2.1 for Table 3.1. Note that the powers of e_0 in the error at each iterate from the sequence

$$4,\ 30,\ 218,\ 1586,\ 11538,\ \cdots \tag{3.3.6}$$

and the sequence of their successive ratios is

$$\frac{30}{4}, \frac{218}{30}, \frac{1586}{218}, \frac{11538}{1586}, \cdots$$

or

$$7.5,\ 7.2667,\ 7.2752,\ 7.2749,\ \cdots .$$

If the terms of the sequence (3.3.6) are denoted by $\{N_i\}$, then it can be seen that

$$N_i = 7N_{i-1} + 2N_{i-2},\ i = 2,\ 3,\ 4,\ \cdots .$$

Thus, as in Theorem 3.2.1, the rate of convergence of method (3.3.1)-(3.3.2) is at least 7.275. □

Alternative Proof

Theorem 3.3.2 can also be proved using Definition 1.1.14 of the order of convergence. On the lines the proof of Theorem 3.3.2 and also the error equation of the standard secant method, it can be shown that the errors e_n^*, e_n^{**} and e_{n+1}, respectively, in x_n^*, x_n^{**} and x_{n+1} for $n \geq 1$ in the method (3.3.1)-(3.3.2) satisfy the following recursion formula:

$$\begin{aligned} e_n^* &= de_{n-1}e_n^2 \\ e_n^{**} &= ce_n^3 e_n^* = cde_{n-1}e_n^5 \\ e_{n+1} &= \lambda e_n^* e_n^{**} = \lambda cd^2 e_{n-1}^2 e_n^7. \end{aligned} \tag{3.3.7}$$

To find the order of convergence of the method, we need a relation of the form

$$e_{n+1} = Ae_n^p, \tag{3.3.8}$$

where A is some constant. Thus, we have

$$e_n = Ae_{n-1}^p \qquad \text{or} \qquad e_{n-1} = A^{-\frac{1}{p}} e_n^{\frac{1}{p}}. \tag{3.3.9}$$

From (3.3.7), (3.3.8) and (3.3.9), we obtain

$$Ae_n^p = \lambda cd^2 A^{-2/p} e_n^{2/p} e_n^7.$$

Equating the power of e_n,

$$
\begin{aligned}
p &= \frac{2}{p} + 7 \\
\text{or,} \qquad p^2 - 7p - 2 &= 0 \\
\text{or,} \qquad p &= \frac{7 \pm \sqrt{57}}{2}.
\end{aligned}
$$

Taking positive value, $p = 7.275$. Thus, the order of convergence of the method (3.3.1)-(3.3.2) is 7.275 and Theorem 3.3.2 is completely proved. □

It is natural to consider the variants of the method (3.3.1)-(3.3.2), where in the expression of z_n and z_n^*, the arithmetic mean is replaced by harmonic mean as well as geometric mean as done in methods (3.2.9)-(3.2.10) and (3.2.11)-(3.2.12), respectively. Precisely, with harmonic mean, we propose the following method:

$$
\left.
\begin{aligned}
x_0^* &= x_0 \\
x_0^{**} &= x_0^* - \frac{2f(x_0^*)}{f'(x_0^*) + f'(z_1)}, \\
\text{where} \qquad z_1 &= x_0 - \frac{f(x_0)}{f'\left(\frac{2x_0 x_0^*}{x_0 + x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)} \\
x_1 &= x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**})
\end{aligned}
\right\} \tag{3.3.10}
$$

followed by (for $n \geq 1$)

$$
\left.
\begin{aligned}
x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n^*)}, \\
\text{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_{n-1} x_{n-1}^*}{x_{n-1} + x_{n-1}^*}\right)} \\
x_n^{**} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*) + f'(z_{n+1})}, \\
\text{where} \qquad z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_n x_n^*}{x_n + x_n^*}\right)} \\
x_{n+1} &= x_1^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**})
\end{aligned}
\right\} \tag{3.3.11}
$$

and with the geometric mean, we propose the following :

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_0^{**} &= x_0^* - \frac{2f(x_0^*)}{f'(x_0^*) + f'(z_1)}, \\ \text{where} \qquad z_1 &= x_0 - \frac{f(x_0)}{f'\left(\sqrt{x_0 x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**}) \end{aligned}\right\} \tag{3.3.12}$$

followed by (for $n \geq 1$)

$$\left.\begin{aligned} x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n^*)}, \\ \text{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\sqrt{x_{n-1} x_{n-1}^*}\right)} \\ x_n^{**} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*) + f'(z_{n+1})}, \\ \text{where} \qquad z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\sqrt{x_n x_n^*}\right)} \\ x_{n+1} &= x_1^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**}). \end{aligned}\right\} \tag{3.3.13}$$

The convergence of the methods (3.3.10)-(3.3.11) and (3.3.12)-(3.3.13) can be proved by using the arguments as used in the proof of Theorem 3.3.2. We only state the results for conciseness.

Theorem 3.3.3. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving nonlinear equation $f(x) = 0$, the method (3.3.10)-(3.3.11) is convergent with order of convergence 7.275.*

Theorem 3.3.4. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving nonlinear equation $f(x) = 0$, the method (3.3.12)-(3.3.13) is convergent with order of convergence 7.275.*

3.4 Algorithms and Numerical Examples

We give below an algorithm to implement method (3.2.1)-(3.2.2):

Algorithm 3.4.1. *Step 1: For the given tolerance $\varepsilon > 0$ and iteration N, choose the initial approximation x_0 and set $n = 0$.*

Step 2: Follow the following sequence of expressions:

$$\begin{aligned}
x_0^* &= x_0 \\
x_1 &= x_0^* - \frac{2f(x_0^*)}{f'(x_0^*) + f'(z_1)}, \\
\textit{where} \qquad z_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
x_1^* &= x_1 - \frac{2f(x_1)}{f'(x_1) + f'(z_1^*)}, \\
\textit{where} \qquad z_1^* &= x_1 - \frac{f(x_1)}{f'\left(\frac{x_0+x_0^*}{2}\right)} = x_1 - \frac{f(x_1)}{f'(x_0)}.
\end{aligned}$$

Step 3: For $n = 1, 2, 3, \ldots$, calculate $x_2, x_3, x_4, \ldots$ by the following sequence of expressions:

$$\begin{aligned}
x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_n^*)}, \\
\textit{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)} \\
x_{n+1} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*) + f'(z_{n+1})}, \\
\textit{where} \qquad z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}
\end{aligned}$$

Step 4: Stop if either $|x_{n+1} - x_n| < \varepsilon$ or $n > N$.

Step 5: Set $n = n + 1$ and repeat Step 3.

Example 3.4.2. We apply method (3.2.1)-(3.2.2) on the nonlinear equation

$$\cos x - xe^x + x^2 = 0. \tag{3.4.1}$$

This equation has a simple root in the interval $(0, 1)$. Taking initial approximation as $x_0 = 1$, Table 3.3 shows the iterations of McDougall-Wotherspoon method, a third order method (1.3.3) and our method (3.2.1)-(3.2.2).

Table 3.3: Approximation of root of the equation (3.4.1) using different methods.

n	WF method (1.3.3)	MW method	Present method (3.2.1)-(3.2.2)
1.	1.1754860092539474	0.8903362174683696 6	0.644064524816892 69
2.	0.7117526001461193	0.66469560530044569	0.63915407608296659
3.	0.63945030188514695	0.63928150457301036	0.63915411559451774
4.	0.63915408656045591	0.63915408990276223	0.6391540955014231
5.	0.63915410631623149	0.63915410965853769	0.63915407540832936
6.	0.63915412607200606	0.6391540698096656	0.6391541149198805
7.	0.63915408622313585	0.63915408956544117	0.63915409482678587
8.	0.63915410597891142	0.63915410932121663	0.63915407473369212
9.	0.639154125734686	0.63915406947234454	0.63915411424524327
10.	0.63915408588581579	0.63915408922812	0.63915409415214863
11.	0.63915410564159136	0.63915410898389557	0.63915407405905489
12.	0.63915412539736594	0.63915406913502348	0.63915411357060603
13.	0.63915408554849573	0.63915408889079894	0.6391540934775114
14.	0.63915410530427119	0.63915410864657451	0.63915407338441765
15.	0.63915412506004576	0.63915406879770231	0.6391541128959688
16.	0.63915408521117556	0.63915408855347788	0.63915409280287416
17.	0.63915410496695113	0.63915410830925345	0.63915407270978042
18.	0.6391541247227257	0.63915406846038125	0.63915411222133156
19.	0.6391540848738555	0.63915408821615682	0.63915409212823693
20.	0.63915410462963107	0.63915410797193239	0.63915407203514318

Example 3.4.3. We consider the same equation as in Example 3.4.2 but now implement method (3.3.1)-(3.3.2) and compare with other methods. Table 3.4, shows the corresponding iterates. One can also compare the last columns of Table

3.3 and Table 3.4 which correspond to methods (3.2.1)-(3.2.2) and (3.3.1)-(3.3.2), respectively. This clearly indicates the fast convergence of (3.3.1)-(3.3.2) .

Table 3.4: Comparision of numerical solution of equation (3.4.1) using different methods.

n	WF method (1.3.3)	MW method	Present method (3.3.1)-(3.3.2)
1.	1.1754860092539474	0.8903362174683966	0.6391974712653039
2.	0.7117526001461193	0.6646956053004456	0.6391541058033836
3.	0.6394503018851469	0.6392815045730103	0.6391540989180736
4.	0.6391540865604559	0.6391540899027622	0.6391540920327637
5.	0.6391541063162314	0.6391541096585376	0.6391540851474537
6.	0.6391541260720060	0.639154069809665	0.6391541114512198
7.	0.6391540862231358	0.6391540895654411	division by zero
8.	0.6391541059789114	0.6391541093212166	
9.	0.639154125734686	0.6391540694723445	
10.	0.6391540858858157	0.63915408922812	
11.	0.6391541056415913	0.6391541089838955	
12.	0.6391541253973659	0.6391540691350234	
13.	0.6391540855484957	0.6391540888907989	
14.	0.6391541053042711	0.6391541086465745	
15.	0.6391541250600457	0.6391540687977023	
16.	0.6391540852111755	0.6391540885534778	
17.	0.6391541049669511	0.6391541083092534	
18.	0.639154124722725	0.6391540684603812	
19.	0.639154084873855	0.6391540882161568	
20.	0.6391541046296310	0.6391541079719323	

Chapter 4

Methods Based on Integral Approximation and Inverse Functions

4.1 Introduction

We consider the iterative methods to find the simple root of nonlinear equations $f(x) = 0$, where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval D is a scalar function. One of the most widely used numerical methods is Newton's method (1.2.1). In the recent years, tremendous variants of this method have appeared showing one or the other advantages over this method in some sense. Weerakoon and Fernando in [104] used the Newton's theorem (1.3.1) and approximated the integral in Newton's theorem by trapezoidal rule (1.2.4) and then they obtained the variant of Newton's method.

Let us recall from Chapter 1 that Weerakoon and Fernando [104] used the techniques of numerical integration in order to improve Newton's method and obtained method based on trapezoidal rule. Later this technique was used by other people and obtained some similar methods based on harmonic mean rule, midpoint rule, Simpson's rule etc.

In [29], Dehghan and Hajarian approximated the integral $\int_{x_n}^{x} f'(t)\,dt$ by a linear combination of trapezoidal rule, midpoint rule and harmonic mean rule and obtained the following method:

$$x_{n+1} = x_n - \left[\frac{f(x_n)(f'(x_n) + f'(x_n^*))}{2f(x_n)f'(x_n^*)} + \frac{2}{3}\frac{f(x_n)}{f'(\frac{x_n+x_n^*}{2})} - \frac{4}{3}\frac{f(x_n)}{f'(x_n) + f'(x_n^*)} \right]. \tag{4.1.1}$$

In this chapter, to begin with, we generalize the method (4.1.1) and we propose the method:

$$x_{n+1} = x_n - \left[a\frac{2f(x_n)}{f'(x_n)+f'(x_n^*)} + b\frac{f(x_n)(f'(x_n)+f'(x_n^*))}{2f'(x_n)f'(x_n^*)} + c\frac{f(x_n)}{f'(\frac{x_n+x_n^*}{2})} \right]. \quad (4.1.2)$$

We shall prove that, in general, for any values of a, b, c, the method (4.1.2) is of order 3 if $a+b+c=1$. However, the method can be made of order 4 for particular values of a, b and c. This is done in the Section 4.2. In Section 4.3, we study certain methods which are based on inverse function technique. This technique for certain other methods has been used by Jain [48], Homeir [45] etc.

Section 4.4 is devoted to the method that is composed of the Wang's method [99] and harmonic mean method. Some numerical examples based on these methods have been collected in Section 4.5. In Section 4.6 and 4.7, we study some hybrid methods which are combinations of some known methods. The order of convergence of these hybrid methods are more than the combining methods. Finally, in Section 4.8, we present more numerical examples to implement these methods.

Some contents of this chapter are based on author's papers [59] and [60]

4.2 Combination of Arithmetic Mean, Harmonic Mean and Midpoint Rule

As the main result of this section, we prove the following convergence result regarding the method (4.1.2):

Theorem 4.2.1. *Let the function f has sufficient number of continuous derivatives in a neighbourhood of α which is a simple zero of $f(x)=0$, that is, $f(\alpha)=0$ and $f'(\alpha)\neq 0$. Then for any values of a, b, c with $a+b+c=1$, the method (4.1.2) is of order 3. Moreover, $a=\frac{-2}{3}$, $b=1$ and $c=\frac{2}{3}$, then the method (4.1.2) is of order 4.*

Proof. Suppose that e_n is the error in nth iterate, that is, $x_n=\alpha+e_n$. Then using Taylor's expansions and after some calculation as in [81] and [104], we get

$$\frac{2f(x_n)}{f'(x_n)+f'(x_n^*)} = e_n - (c_2^2+\frac{1}{2}c_3)e_n^3 + O(e_n^4), \quad (4.2.1)$$

$$\frac{f(x_n)[f'(x_n)+f'(x_n^*)]}{2f'(x_n)f'(x_n^*)} = e_n - \frac{1}{2}c_3e_n^3 + O(e_n^4) \tag{4.2.2}$$

and

$$\frac{f(x_n)}{f'(\frac{x_n+x_n^*}{2})} = e_n + (\frac{1}{4}c_3 - c_2^2)e_n^3 + O(e_n^4), \tag{4.2.3}$$

where $c_j = \frac{1}{j!}\frac{f^j(\alpha)}{f'(\alpha)}, j = 2, 3, \cdots$. Substituting the values from (4.2.1), (4.2.2) and (4.2.3) in (4.1.2), we get

$$\begin{aligned} e_{n+1} &= e_n - (a+b+c)e_n + a(c_2^2 + \frac{1}{2}c_3)e_n^3 + b(\frac{1}{2}c_3e_n^3) - c(\frac{1}{4}c_3 - c_2^2)e_n^3 + O(e_n^4) \\ &= (\frac{a}{2} + \frac{b}{2} - \frac{c}{4})c_3e_n^3 + (a+c)c_2^2e_n^3 + O(e_n^4). \end{aligned}$$

Hence from above, the rate of convergence of method (4.1.2) is at least three and it become four for unique value of $a = \frac{-2}{3}$, $b = 1$ and $c = \frac{2}{3}$. Thus, fourth order convergent method is

$$x_{n+1} = x_n + \frac{2}{3}\frac{2f(x_n)}{f'(x_n)+f'(x_n^*)} - \frac{f(x_n)(f'(x_n)+f'(x_n^*))}{2f(x_n)f'(x_n^*)} - \frac{2}{3}\frac{f(x_n)}{f'(\frac{x_n+x_n^*}{2})}. \tag{4.2.4}$$

□

The method (4.2.4) is same as Dehghan and Hajarian method [29] but they approximated the indefinite integral in Newton's theorem by linear combination of trapezoidal integration rule, midpoint integral rule and harmonic mean rule and there is no idea how they choose constants. The cost of getting fourth order method from third order methods is only we have to evaluate one more function.

4.3 Inverse Function Method

Homeir [45] and Jain [48] used Newton's theorem (1.3.1) for the inverse function $x = f^{-1}(y) = g(y)$ instead of $y = f(x)$, that is,

$$g(y) = g(y_n) + \int_{y_n}^{y} g'(t)\,dt. \tag{4.3.1}$$

If we approximate the indefinite integral in (4.3.1) by harmonic mean rule, we get

$$\int_{y_n}^{y} g'(t)\,dt = (y - y_n)\frac{2g'(y_n)g'(y)}{g'(y_n)+g'(y)}. \tag{4.3.2}$$

Hence from (4.3.1) and (4.3.2), we obtain

$$g(y) = g(y_n) + (y - y_n)\frac{2g'(y_n)g'(y)}{g'(y_n) + g'(y)},$$

where $y_n = f(x_n)$. Now using the fact that $g'(y) = (f^{-1})'(y) = [f'(x)]^{-1}$ and that $y = f(x) = 0$, we obtain

$$\begin{aligned} x &= x_n + (0 - f(x_n))\frac{2\frac{1}{f'(x_n)}\frac{1}{f'(x)}}{\frac{1}{f'(x_n)} + \frac{1}{f'(x)}} \\ &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(x)}. \end{aligned}$$

Thus when $x \to x_{n+1}$ and in right side if we use $x_n^* = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, then, we get the iterative formula

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n^*)}. \tag{4.3.3}$$

This formula is exactly same as the formula (1.3.3) obtained by approximating the indefinite integral of Newton's theorem (1.3.1) using the trapezoidal rule for the function $y = f(x)$.

Again, if we approximate the indefinite integral in (4.3.1) by midpoint rule, we get

$$\int_{y_n}^{x} g'(t)\,dt = (y - y_n)g'\left(\frac{y + y_n}{2}\right).$$

Hence from equation(4.3.1), we get

$$\begin{aligned} & g(y) = g(y_n) + (y - y_n)g'\left(\frac{y + y_n}{2}\right) \\ \textit{or}, \quad & x = x_n + (0 - f(x_n))\frac{1}{f'(\frac{x + x_n}{2})}. \end{aligned}$$

Consequently, we obtain the following iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(\frac{x_n + x_n^*}{2})}, \tag{4.3.4}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$. This method is exactly same as the method given by

(1.3.4).

Finally if we approximate the indefinite integral in equation (4.3.1) by trapezoidal rule, we get

$$\int_{y_n}^{y} g'(t)\,dt = \frac{(y-y_n)}{2}[g'(y)+g'(y_n)].$$

Also from (4.3.1), we get

$$\begin{aligned} g(y) &= g(y_n) + \frac{(y-y_n)}{2}[g'(y)+g'(y_n)] \\ or, x &= x_n + \frac{(0-f(x_n))}{2}[\frac{1}{f'(x)}+\frac{1}{f'(x_n)}] \\ &= x_n - \frac{f(x_n)[f'(x_n)+f'(x)]}{2f(x_n)f'(x)}. \end{aligned}$$

Therefore the following iterative formula is obtained:

$$x_{n+1} = x_n - \frac{f(x_n)[f'(x)+f'(x_n^*)]}{2f(x_n)f'(x_n^*)}, \tag{4.3.5}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$. This formula is same as the (1.3.5) obtained by approximating the indefinite integral of equation (1.3.1) using the harmonic mean rule for the function $y = f(x)$. From above it is clear that the fourth-order convergence method based on inverse function obtained by combining the methods which are obtained respectively by approximating the indefinite integral of Newton's formula by harmonic mean rule, midpoint rule and trapezoidal rule is also given the same formula as (4.2.4).

4.4 Combination of Harmonic Mean Rule and Wang's Rule

Wang [99] approximated the integral in Newton's theorem (1.3.1) by using formula

$$\int_{x_n}^{x} f'(t)\,dt = (x-x_n)\left[(1-\beta)f'(x_n) + \beta f'\left(x_n - \frac{f(x_n)}{2\beta f'(x_n)}\right)\right],\ \beta \neq 0 \tag{4.4.1}$$

and in [81], Özban approximated the integral by using the harmonic mean rule, that is,

$$\int_{x_n}^{x} f'(t)\,dt = (x - x_n)\frac{2f'(x)f'(x_n)}{f'(x) + f'(x_n)}. \tag{4.4.2}$$

From Section 4.3, Newton's theorem (1.3.1) for the inverse function $x = f^{-1}(y) = g(y)$ instead of $y = f(x)$, takes the form (4.3.1). If we approximate the integral in (4.3.1) by harmonic mean rule, we obtain

$$\int_{y_n}^{y} g'(t)\,dt = (y - y_n)\frac{2g'(y)g'(y_n)}{g'(y) + g'(y_n)}, \tag{4.4.3}$$

and from Wang formula (4.4.1)

$$\int_{y_n}^{y} g'(t)\,dt = (y - y_n)\left[(1-\beta)g'(y_n) + \beta g'\left(y_n - \frac{g(y_n)}{2\beta g'(y_n)}\right)\right]. \tag{4.4.4}$$

If we approximate the integral of (4.3.1) by linear combination of harmonic and Wang rules, we get

$$\begin{aligned} g(y) = g(y_n) + (y - y_n)&\left[(1-\theta)\frac{2g'(y)g'(y_n)}{g'(y) + g'(y_n)}\right. \\ &\left. +\theta\left((1-\beta)g'(y_n) + \beta g'\left(y_n - \frac{g(y_n)}{2\beta g'(y_n)}\right)\right)\right], \end{aligned}$$

where $y_n = f(x_n)$ and θ is any real number. Now using the fact that $g'(y) = (f^{-1})'(y) = [f'(x)]^{-1}$ and that $y = f(x) = 0$ as in [48], we get

$$x_{n+1} = x_n - (1-\theta)\frac{2f(x_n)}{f'(x_n) + f'(x_n^*)} - \theta f(x_n)\left[\frac{(1-\beta)}{f'(x_n)} + \frac{\beta}{f'\left(x_n - f(x_n)/2\beta f'(x_n)\right)}\right], \tag{4.4.5}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$.

We shall prove below that the order of convergence of the method (4.4.5) is at least three in general and for particular values of θ and β, it can be more.

Theorem 4.4.1. *Let function f has sufficient number of derivative in the neighborhood of α, which is a simple zero of f, that is, $f(\alpha) = 0, f'(\alpha) \neq 0$. Then, the method (4.4.5) is of order at least 3 and for unique values of $\theta = \frac{5}{3}$ and $\beta = \frac{5}{8}$, it is of order 4.*

Proof. Suppose e_n is the error in the nth iterate x_n and denote $c_j = \frac{1}{j!}\frac{f^j(\alpha)}{f'(\alpha)}$, $j =$

$2, 3, 4, \cdots$. Then $x_n = \alpha + e_n$. From Taylor's series about α, we obtain

$$\begin{aligned} f(x_n) &= f(\alpha + e_n) \\ &= f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \frac{e_n^3}{3!} f'''(\alpha) + O(e_n^4) \\ &= f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)], \end{aligned} \tag{4.4.6}$$

and

$$\begin{aligned} f'(x_n) &= f'(\alpha + e_n) \\ &= f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) + \frac{e_n^3}{3!} f^{iv}(\alpha) + O(e_n^4) \\ &= f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)]. \end{aligned} \tag{4.4.7}$$

From equation(4.4.6) and (4.4.7)

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= \left(e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)\right)\left(1 + 2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)\right)^{-1} \\ &= e_n - c_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 + O(e_n^4) \end{aligned} \tag{4.4.8}$$

and hence

$$x_n^* = \alpha + c_2 e_n^2 + 2(c_3 - c_2^2) e_n^3 + O(e_n^4). \tag{4.4.9}$$

After some calculations, we get

$$\begin{aligned} f'(x_n^*) &= f'(\alpha)[1 + 2c_2^2 e_n^2 + 4(c_2 c_3 - c_2^3) e_n^3 + O(e_n^4)], \\ f'(x_n) + f'(x_n^*) &= 2f'(\alpha)\left[1 + 2c_2 e_n + \left(c_2^2 + \frac{3}{2}c_3\right) e_n^2 + 2(c_2 c_3 - c_2^3 + c_4) e_n^3 + O(e_n^4)\right], \end{aligned} \tag{4.4.10}$$

$$f'\left(x_n - \frac{f(x_n)}{2\beta f'(x_n)}\right) = f'(\alpha)\left[1 + 2c_2(1 - \frac{1}{2\beta})e_n + \left(\frac{c_2^2}{\beta} + 3c_3\left(1 - \frac{1}{2\beta}\right)^2\right) e_n^2 + O(e_n^3)\right], \tag{4.4.11}$$

$$\frac{2f(x_n)}{f(x_n) + f'(x_n^*)} = e_n - (c_2^2 + \frac{1}{2}c_3) e_n^3 + O(e_n)^4 \tag{4.4.12}$$

and

$$f(x_n)\left[\frac{(1-\beta)}{f'(x_n)}+\frac{\beta}{f'\left(x_n-\dfrac{f(x_n)}{2\beta f'(x_n)}\right)}\right]=e_n+\left[\frac{1}{\beta}c_2^2-(2c_2^2-c_3)-\frac{3}{4\beta}c_3\right]e_n^3+O(e_n)^4. \tag{4.4.13}$$

Consequently, from (4.4.5)

$$\begin{aligned} e_{n+1} &= e_n-(1-\theta)\left[e_n-\left(c_2^2+\frac{1}{2}c_3\right)e_n^3+O(e_n)^4\right] \\ &\quad -\theta\left[e_n+\left(\frac{1}{\beta}c_2^2-(2c_2^2-c_3)-\frac{3}{4\beta}c_3\right)e_n^3+O(e_n)^4\right] \\ &= (1-\theta)(c_2^2+\frac{1}{2}c_3)e_n^3+\theta[-\frac{1}{\beta}c_2^2+(2c_2^2-c_3)+\frac{3}{4\beta}c_3]e_n^3+O(e_n^4). \end{aligned} \tag{4.4.14}$$

For unique values of $\theta=\frac{5}{3}$ and $\beta=\frac{5}{8}$, we get from the (4.4.14) that $e_{n+1}=O(e_n^4)$. Thus, the order of convergence of family of method (4.4.5) is at least three and it gives the fourth order method for unique value of $\theta=\frac{5}{3}$ and $\beta=\frac{5}{8}$. The new fourth order method is, thus, given by

$$x_{n+1}=x_n+\frac{4}{5}\frac{f(x_n)}{f(x_n)+f'(x_n^*)}-\frac{5}{24}f(x_n)\left[\frac{3}{f'(x_n)}+\frac{5}{f'\left(x_n-\dfrac{4f(x_n)}{5f'(x_n)}\right)}\right], \tag{4.4.15}$$

where $x_n^*=x_n-\frac{f(x_n)}{f'(x_n)}$. Hence, theorem (4.4.1) is completely proved. □

4.5 Numerical Examples

In order to check the performance of the introduced fourth-order method (4.4.15), we give the numerical results on some functions. We also compare the results of this method with Newton's method (NM), Weerakoon and Fernando (WF) method and Wang Method. Numerical computations have been performed using the Matlab software rounding to 16 significant decimal digits. We use the stopping criteria $|x_{n+1}-x_n|<\varepsilon$ where $\varepsilon=(10)^{-8}$ for the iterative process of our results.

The functions and their roots α which are used as numerical examples are given below:

1. $f_1(x)=\cos x-xe^x+x^2$, $\alpha=0.639154069332008$
2. $f_2(x)=(x-1)^8-1$, $\alpha=2$

3. $f_3(x) = x^3 + 4x^2 - 10,$ $\alpha = 1.365230013414007$

Table 4.1: $f_1(x) = \cos x - xe^x + x^2$ and initial guess $x_0 = 1$.

n	Newton method	WF method	Wang method	Present method (4.4.15)
1	0.724644697567095	0.665881945014898	0.662938687425796	0.645000328957045
2	0.644658904870270	0.639169572742496	0.639163361765903	0.639154096912870
3	0.639177807467281	0.639154096332011	0.639154096332008	
4	0.639154096773051			

Table 4.2: $f_2(x) = (x-1)^8 - 1$, and initial guess $x_0 = 2.5$.

n	Newton method	WF method	Wang method	Present method (4.4.15)
1	2.319815957933242	2.244107083493465	2.236236164659438	2.177268585515064
2	2.172758071196855	2.068965943835277	2.060772262760990	2.014981732655870
3	2.067131900591505	2.003572340012848	2.002277983579724	2.000002165842810
4	2.013060520391305	2.000000704582805	2.000000159604880	2.000000000000000
5	2.000574370391506	2.000000000000000	2.000000000000000	
6	2.000001152667968			
7	2.000000000004650			

Table 4.3: $f_3(x) = x^3 + 4x^2 - 10$ and initial guess $x_0 = 1$.

n	Newton method	WF method	Wang method	Present method (4.4.15)
1	1.454545454545455	1.345024237239806	1.346506300114548	1.369968652351256
2	1.368900401069519	1.365227728691384	1.365228321128059	1.365230013487738
3	1.365236600202116	1.365230013414097	1.365230013414097	
4	1.365230013435367			

Table 4.4: Comparision of different methods.

Function	Newton method		W-F method		Wang method		Present method (4.4.15	
	TNFE	TNI	TNFE	TNI	TNFE	TNI	TNFE	TNI
f_1	8	4	9	3	9	3	8	2
f_2	14	7	15	5	15	5	16	4
f_3	8	4	9	3	9	3	8	2

TNI : Total number of iterations, TNFE : Total number of functions evaluation

From above discussion, it turns out that the method (4.4.5) is a family of at least third order methods which contains Weerakoon and Fernando method, inverse Wang method as well as many other third order methods. For the unique value of θ and β, it gives a fourth order method which is actually linear combination of two third order methods, inverse harmonic mean and inverse Wang. The numerical experiment results shows that new introduced fourth order method can easily compete with classical Newton method, Wang method as well as Weerakoon and Fernando method. Also this method does not require the computation of second or higher order derivatives.

4.6 Method Based on Simpson Rule

In [40], Hassanov, Ivnov and Nedhzhibov suggested a new iterative method (1.3.6) by approximating the integral in Newton's theorem (1.3.1) using the Simpson rule. Combining this method with the quadratically convergent method due to Kanwar, Kukreja, and Singh [63],

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + p\sqrt{f'^2(x_n) + 4f^2(x_n)}}, \tag{4.6.1}$$

Mir, Ayub and Rafiq [77] obtained a new third order method,

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'\left(\frac{x^*_{n+1}+x_n}{2}\right) + f'(x^*_{n+1})} \tag{4.6.2}$$

$$\text{where} \quad x^*_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + p\sqrt{f'^2(x_n) + 4f^2(x_n)}},$$

for solving nonlinear equation $f(x) = 0$, where p is chosen as positive or negative sign so as to make the denominator largest in magnitude to avoid numerical difficulties. Our aim, in this section, is to find a variant of method (4.6.2).

Using inverse function $x = f^{-1}(y) = g(y)$ instead of $y = f(x)$ in Newton's theorem (1.3.1), we get (4.3.1). If we approximate the integral in (4.3.1) by using Simpson rule, we get

$$g(y) = g(y_n) + \left(\frac{y - y_n}{6}\right)\left[g'(y_n) + 4g'\left(\frac{y + y_n}{2}\right) + g'(y)\right],$$

where $y_n = f(x_n)$. Now using the fact that $g'(y) = (f^{-1})'(y) = [f'(x)]^{-1}$ and that $y = f(x) = 0$, we obtain

$$x = x_n - \frac{1}{6}f(x_n)\left[\frac{1}{f'(x_n)} + \frac{4}{f'\left(\frac{x+x_n}{2}\right)} + \frac{1}{f'(x)}\right].$$

Thus, we obtain the iterative formula

$$x_{n+1} = x_n - \frac{1}{6}\left[\frac{f(x_n)}{f'(x_n)} + \frac{4f(x_n)}{f'\left(\frac{x^*_{n+1}+x_n}{2}\right)} + \frac{f(x_n)}{f'(x^*_{n+1})}\right] \tag{4.6.3}$$

$$\text{where} \quad x^*_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + p\sqrt{f'^2(x_n) + 4f^2(x_n)}}.$$

For the convergence analysis of method (4.6.3), we prove the following result.

Theorem 4.6.1. *Let f be a function having sufficient number of derivatives in a neighborhood of α which is a simple zero of equation $f(x) = 0$. Then the iterative method (4.6.3) is of order three for $p = 1$*

Proof. Suppose e_n is the error in the nth iterate x_n and denote $c_j = \frac{1}{j!}\frac{f^j(\alpha)}{f'(\alpha)}$, $j =$

$2, 3, 4, \cdots$. Then $x_n = \alpha + e_n$. On the line of the proof of Theorem 4.4.1, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)],$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)],$$

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4).$$

Also x^*_{n+1} can be written as

$$x^*_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)} \frac{1}{\left(1 + \sqrt{1 + 4\left(\frac{f(x_n)}{f'(x_n)}\right)^2}\right)}. \tag{4.6.4}$$

Substituting the value of $\frac{f(x_n)}{f'(x_n)}$ and using the binomial theorem for any rational number, after some calculations, we get

$$x^*_{n+1} = \alpha + c_2 e_n^2 + [2(c_3 - c_2^2) + 1]e_n^3 + O(e_n^4). \tag{4.6.5}$$

Now, by using Taylor's series, we have

$$f'(x^*_{n+1}) = f'(\alpha)[1 + 2c_2^2 e_n^2 + \{4c_2(c_3 - c_2^2) + 2\}(e_n^3) + O(e_n^4)]. \tag{4.6.6}$$

Also,

$$\frac{x^*_{n+1} + x_n}{2} = \alpha + \frac{1}{2}[e_n + c_2 e_n^2 + \{2(c_3 - 2c_2^2) + 1\}(e_n^3) + O(e_n^4)]. \tag{4.6.7}$$

Using Taylor's series again, we find that

$$f'\left(\frac{x^*_{n+1} + x_n}{2}\right) = f'(\alpha)[1 + c_2 e_n + (c_2^2 + \frac{3}{4}c_3)e_n^2 + O(e_n^3)]. \tag{4.6.8}$$

So that

$$\begin{aligned}\frac{f(x_n)}{f'\left(\frac{x^*_{n+1}+x_n}{2}\right)} &= e_n + (c_3 - 2c_2^2 - \frac{3}{4}c_3 + c_2^2)e_n^3 + O(e_n^4) \\ &= e_n + (\frac{1}{4}c_3 - c_2^2)e_n^3 + O(e_n^4),\end{aligned} \tag{4.6.9}$$

and

$$\frac{f(x_n)}{f'(x^*_{n+1})} = e_n + c_2e_n^2 + (c_3 - 2c_2^2)e_n^3 + O(e_n^4). \tag{4.6.10}$$

Thus, the error equation of (4.6.3) becomes

$$\begin{aligned} e_{n+1} &= e_n - \frac{1}{6}\Big[e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4) + 4\{e_n + (\frac{1}{4}c_3 - c_2^2)e_n^3 + O(e_n^4)\} \\ &\qquad + e_n + c_2e_n^2 + (c_3 - 2c_2^2)e_n^3 + O(e_n^4)\Big] \\ &= \frac{2}{3}c_2^2e_n^3 + O(e_n^4). \end{aligned} \tag{4.6.11}$$

Hence order of convergence of method (4.6.3) is 3. □

4.7 Increasing the Order of Convergence

In this section, we obtain two variants of Mir, Ayab and Rafiq method (4.6.2) by combining with secant and modified secant methods.

The order of convergence of Mir, Ayab and Rafiq method (4.6.2) for solving nonlinear equation can be raised to 4 if the iterations are performed alternately with secant method (1.2.2). The corresponding method becomes

$$x_{n+1} = \overline{x_n} - \frac{\overline{x_n} - x_n}{f(\overline{x_n}) - f(x_n)} f(\overline{x_n}), \tag{4.7.1}$$

$$\text{where} \quad \overline{x_n} = x_n - \frac{6f(x_n)}{f'(x_n) + 4f'\left(\frac{x^*_{n+1}+x_n}{2}\right) + f'(x^*_{n+1})} \tag{4.7.2}$$

$$\text{with} \quad x^*_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + p\sqrt{f'^2(x_n) + 4f^2(x_n)}}. \tag{4.7.3}$$

In this method, p is chosen positive or negative sign so as to make the denominator largest in magnitude to avoid numerical difficulties. For the order of convergence of the method (4.7.1)-(4.7.3), we prove the following.

Theorem 4.7.1. *Let f be a function having sufficient number of derivatives in a neighborhood of α which is a simple root of equation $f(x) = 0$. Then, the iterative method (4.7.1)-(4.7.3) is of order 4 for $p = 1$.*

Proof. Let e_n and $\overline{e_n}$ be the errors in x_n and $\overline{x_n}$, respectively, that is, $x_n = \alpha + e_n$ and $\overline{x_n} = \alpha + \overline{e_n}$. Denote $c_j = \frac{f^j(\alpha)}{j!f'(\alpha)}$, $j = 2, 3, 4\cdots$. It was shown by Mir, Ayub and Rafiq in [77] that the error equation of (4.7.2) is given by

$$\overline{e_n} = c_2^2 e_n^3 + O(e_n^4). \tag{4.7.4}$$

Here,

$$\begin{aligned}
\overline{x_n} - x_n &= (\alpha + \overline{e_n}) - (\alpha + e_n) \\
&= \overline{e_n} - e_n \\
&= c_2^2 e_n^3 - e_n + O(e_n^4)
\end{aligned}$$

By Taylor's expansion, we get

$$\begin{aligned}
f(x_n) &= f(\alpha + e_n) \\
&= f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)]
\end{aligned}$$

and using (4.7.4), we obtain

$$\begin{aligned}
f(\overline{x_n}) &= f(\alpha + \overline{e_n}) \\
&= f'(\alpha)[c_2^2 e_n^3 + O(e_n^6)].
\end{aligned}$$

Thus, we get

$$\begin{aligned}
f(\overline{x_n}) - f(x_n) &= f'(\alpha)[c_2^2 e_n^3 + O(e_n^6)] - f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)] \\
&= -f'(\alpha) e_n [1 + c_2 e_n + (c_3 - c_2^2) e_n^2 + O(e_n^3)]
\end{aligned}$$

and

$$\begin{aligned}
\frac{(\overline{x_n} - x_n) f(\overline{x_n})}{f(\overline{x_n}) - f(x_n)} &= \frac{[c_2^2 e_n^3 - e_n + O(e_n^4)] f'(\alpha)[c_2^2 e_n^3 + O(e_n^6)]}{-f'(\alpha) e_n [1 + c_2 e_n + (c_3 - c_2^2) e_n^2 + O(e_n^3)]} \\
&= [c_2^2 e_n^3 + O(e_n^5)][1 + c_2 e_n + (c_3 - c_2^2) e_n^2 + O(e_n^3)]^{-1} \\
&= c_2^2 e_n^3 - c_2^3 e_n^4 + O(e_n^5).
\end{aligned}$$

Thus, the error equation in (4.7.1) is given by

$$\begin{aligned} e_{n+1} &= \overline{e_n} - c_2^2 e_n^3 + c_2^3 e_n^4 + O(e_n^5) \\ &= c_2^2 e_n^3 + O(e_n^4) - c_2^2 e_n^3 + c_2^3 e_n^4 + O(e_n^5) \\ &= A e_n^4 + O(e_n^5), \end{aligned}$$

where A is some constant. The assertion is, therefore, proved. □

Next, the iterations of Mir, Ayab and Rafiq method (4.6.2) are performed alternately with modified secant method (1.3.11). Then the corresponding method becomes

$$\begin{aligned} x_{n+1}^* &= x_n - \frac{2f(x_n)}{f'(x_n) + p\sqrt{f'^2(x_n) + 4f^2(x_n)}} \\ \overline{x_n} &= x_n - \frac{6f(x_n)}{f'(x_n) + 4f'\left(\frac{x_{n+1}^* + x_n}{2}\right) + f'(x_{n+1}^*)} \\ x_{n+1} &= \overline{x_n} - A_n^{-1} f(\overline{x_n}), \end{aligned} \tag{4.7.5}$$

$$\text{where} \quad A_n = [y_n, \overline{x_n}; f] = \frac{f(\overline{x_n}) - f(y_n)}{\overline{x_n} - y_n}$$

$$\text{with} \quad y_n = \overline{x_n} + \delta_n(x_n - \overline{x_n}), \qquad \delta_n \leq |O(e_n)^{\frac{3}{2}}|.$$

For the convergence analysis of method (4.7.5), we prove following result:

Theorem 4.7.2. *Let f be a function having sufficient number of derivatives in a neighborhood of α which is a simple root of $f(x) = 0$. Then, the iterative method (4.7.5) is of order at least 6 for $p = 1$.*

Proof. We prove this theorem in the line of proof of Theorem 4.7.1. The error $\overline{e_n}$ in $\overline{x_n}$ is given by

$$\overline{e_n} = c_2^2 e_n^3 + O(e_n^4).$$

Since

$$y_n = \overline{x_n} + \delta_n(x_n - \overline{x_n}) = \overline{x_n} + a_n, \quad \text{where } a_n = \delta_n(x_n - \overline{x_n}),$$

by using Taylor expansion, we get

$$f(y_n) = f(\overline{x_n} + a_n) = f(\overline{x_n}) + a_n f'(\overline{x_n}) + \frac{a_n^2}{2} f''(\overline{x_n}) + \cdots .$$

Also

$$
\begin{aligned}
A_n &= \frac{f(\overline{x_n}) - f(y_n)}{\overline{x_n} - y_n} \\
&= \frac{a_n f'(\overline{x_n}) + \dfrac{a_n^2}{2} f''(\overline{x_n}) + \cdots}{a_n} \\
&= f'(\overline{x_n}) + \frac{a_n}{2} f''(\overline{x_n}) + \cdots \\
&= f'(\overline{x_n}) + O(a_n).
\end{aligned}
$$

Thus, from (4.7.5), we have

$$
\begin{aligned}
x_{n+1} &= \overline{x_n} - \frac{f(x_n)}{A_n} \\
&= \overline{x_n} - \frac{f(x_n)}{f'(\overline{x_n}) + O(a_n)} \\
&= \overline{x_n} - \frac{\dfrac{f(x_n)}{f'(\overline{x_n})}}{1 + O(a_n)} \\
&= \overline{x_n} - \frac{f(x_n)}{f'(\overline{x_n})}[1 + O(a_n)]^{-1} \\
&= \overline{x_n} - \frac{f(x_n)}{f'(\overline{x_n})}[1 - O(a_n) + O(a_n^2) + \cdots] \\
&= \overline{x_n} - \frac{f(x_n)}{f'(\overline{x_n})} + O(a_n).
\end{aligned}
$$

Thus, the method (4.7.5) can be written as

$$
x_{n+1} = \overline{x_n} - \frac{f(\overline{x_n})}{f'(\overline{x_n})} + O(\delta_n |x_n - \overline{x_n}|). \qquad (4.7.6)
$$

From (4.7.6), we have

$$
\begin{aligned}
e_{n+1} &= \overline{e_n} - \frac{f(\alpha+\overline{e_n})}{f'(\alpha+\overline{e_n})} + O(\delta_n|\alpha+e_n-\alpha-\overline{e_n}|) \\
&= \overline{e_n} - \left(\overline{e_n} + c_2\overline{e_n}^2 + c_3\overline{e_n}^3 + O(\overline{e_n}^4)\right)\left(1 + 2c_2\overline{e_n} + 3c_3\overline{e_n}^2 + O(\overline{e_n}^3)\right)^{-1} + O(\delta_n|e_n-\overline{e_n}|) \\
&= \overline{e_n} - (\overline{e_n} + c_2\overline{e_n}^2 - c_2\overline{e_n}^2 + O(\overline{e_n}^3) + O(\delta_n|e_n-\overline{e_n}|) \\
&= c_2\overline{e_n}^2 + O(\delta_n|e_n-\overline{e_n}|) + \cdots \\
&= c_2(c_2^2e_n^3)^2 + O(\delta_n|e_n - c_2^2e_n^3|) + \cdots \\
&= c_2^5e_n^6 + O(\delta_n|e_n - c_2^2e_n^3|) + \cdots
\end{aligned} \tag{4.7.7}
$$

Since $\delta_n \leq |O(\overline{e_n})^{\frac{3}{2}}|$, so if we assume that $\delta_n \leq |O(\overline{e_n})^2|$, that is, $\delta_n \leq |O(e_n)^6|$, then the order of the method (4.7.5) is at least 6. □

Remark 4.7.3. In practice, we do not have any information about the solution. In this situation, we use the same strategy as the Amat and Basquier indicated in [2] to obtain δ_n. The possible strategy to obtain $\{\delta_n\}$ can be

$$
\delta_0 = O(10^{-k}) \leq O(e_0)^{\frac{3}{2}}
$$
$$
\delta_n = O(\delta_0^{2^n}),
$$

where k is an integer such that

$$
O(10^{-k}) \leq |f(\alpha) - f(x_0)| = |f(x_0)| \leq O(|\alpha - x_0|).
$$

4.8 Further Numerical Examples

In this section, we shall compare the performance of newly introduced methods (4.6.3), (4.7.1) and (4.7.5) with the some existing methods. To avoid the numerical difficulties in newly introduced methods, we take p equal to 1 or -1 so as to make the denominator largest in magnitude. For the comparison, we use Matlab Software and stopping criteria $|x_{n+1} - x_n| < (10)^{-12}$ or $|f(x_{n+1})| < (10)^{-14}$ for the iterative process of our results.

Example 4.8.1. We apply methods (4.6.3), (4.7.1) and (4.7.5) on the nonlinear equation

$$
x^6 - x - 1 = 0 \tag{4.8.1}
$$

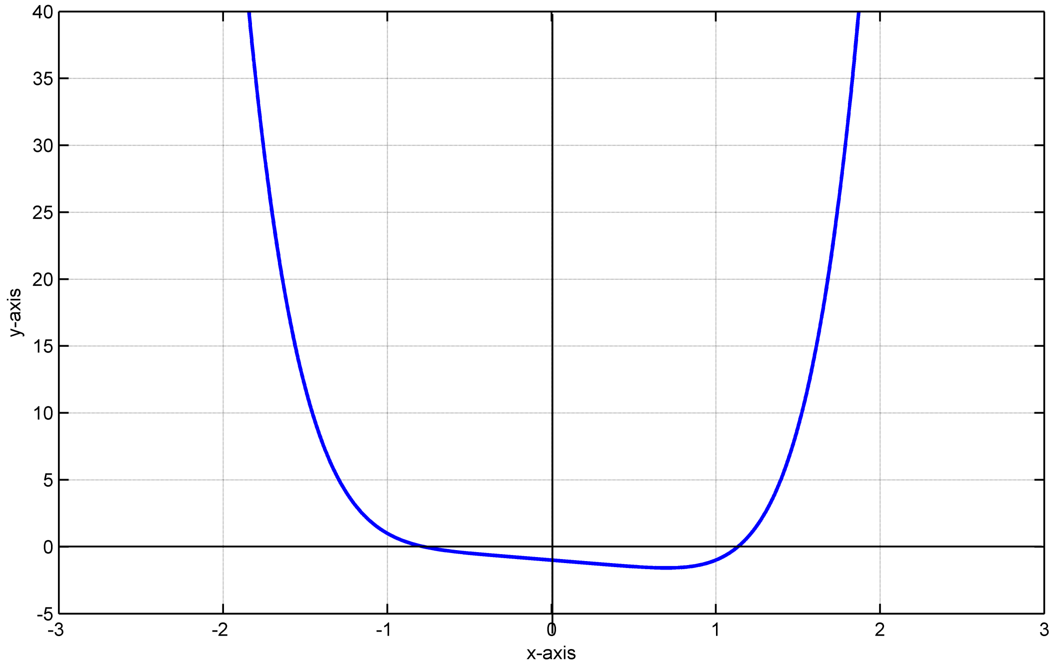

Figure 4.1: Graph of the function $f(x) = x^6 - x - 1$.

From the Figure 4.1, it is clear that one of the simple roots of (4.8.1) lies in the interval $(1, 2)$. Taking initial approximation as $x_0 = 1$, Table 4.5 shows the iterations of the Newton method, Mir, Ayab and Rafiq method (4.6.2), and introducing methods (4.6.3), (4.7.1) and (4.7.5).

Table 4.5: Comparision of different methods.

Method	n	x_n	$\|x_n - x_{n-1}\|$	$\|f(x_n)\|$
Newton's method	1	1.200000000000000	0.200000000000000	0.785983999999999
	2	1.143575842503044	0.056424157496956	0.093031957363097
	3	1.134909462242086	0.008666380260958	0.001907397172335
	4	1.134724221386558	0.000185240855528	0.000000853719439
	5	1.134724138401536	0.000000082985022	0.000000000000171
	6	1.134724138401519	0.000000000000017	0.000000000000001
Mir Ayub and Rafiq method(p=1)	1	1.114331079664285	0.114331079664285	0.199698426927704
	2	1.134671170083933	0.020340090419648	0.000544848619648
	3	1.134724138400651	0.000052968316718	0.000000000008935
	4	1.134724138401519	0.000000000000868	0.000000000000001
Present method (4.6.3) (p=1)	1	1.124122502547814	0.124122502547814	0.106305164737796
	2	1.134719404836294	0.010596902288480	0.000048696604577
	3	1.134724138401519	0.000004733565225	0.000000000000005
Present method (4.7.1)	1	1.142859996170415	0.142859996170415	0.085360616958645
	2	1.134724198032685	0.008135798137730	0.000000613463385
	3	1.134724138401519	0.000000059631166	0.000000000000001
Present method (4.7.5) ($\delta_0 = 0.15$)	1	1.134888830552744	0.134888830552744	0.001694966368938
	2	1.134724138401519	0.000164692151225	0.000000000000003

Example 4.8.2. We apply methods (4.6.3), (4.7.1)and (4.7.5) on the nonlinear

equation

$$\sin^2 x - x^2 + 1 = 0. \tag{4.8.2}$$

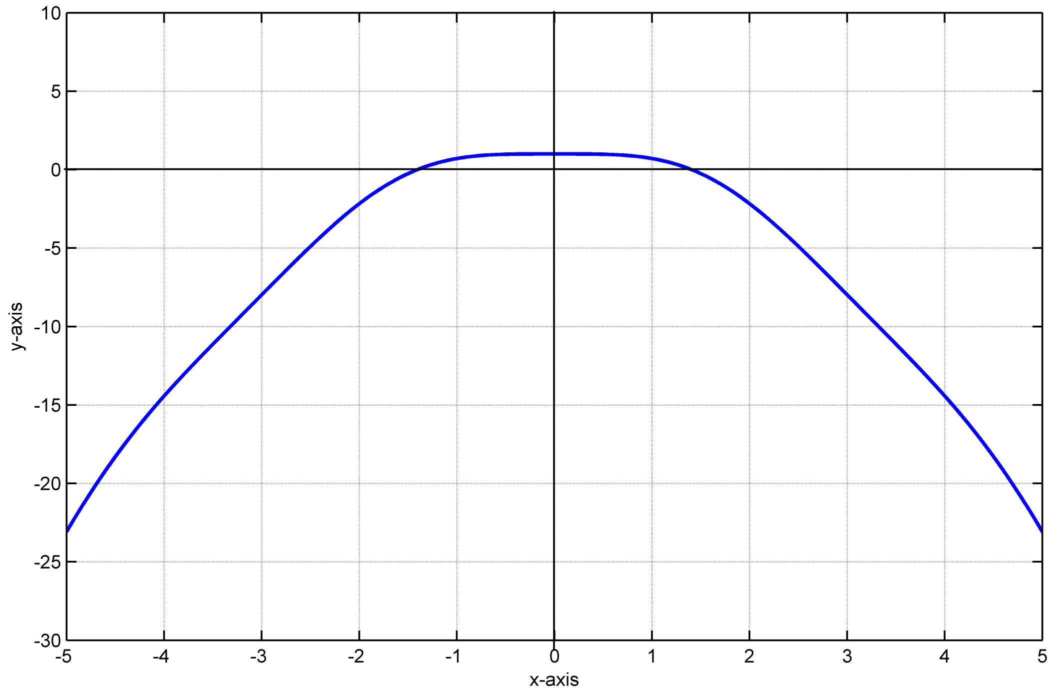

Figure 4.2: Graph of the function $f(x) = \sin^2 x - x^2 + 1$.

From the Figure 4.2, it is clear that one of the simple roots of (4.8.2) lies in the interval $(1, 2)$. Taking initial approximation as $x_0 = 2$, Table 4.6 shows the iterations of the Kanwar, Kukreja, and Singh method (4.6.1), Mir, Ayab and Rafiq method (4.6.2), and introducing methods (4.6.3), (4.7.1)and (4.7.5) when we apply on (4.8.2).

Table 4.6: Comparision of different methods.

Method	n	x_n	$\|x_n - x_{n-1}\|$	$\|f(x_n)\|$
Kanwar, Kukreja, and Singh method	1	1.611941042164389	0.388058957835611	0.600045855942974
	2	1.436057898886696	0.175883143277692	0.080307137216431
	3	1.405269547988669	0.030788350898027	0.001932292231027
	4	1.404492122350878	0.000777425637791	0.000001177029015
	5	1.404491648215517	0.000000474135360	0.000000000000437
	6	1.404491648215341	0.000000000000176	0.000000000000000
Mir, Ayab, and Rafiq method (p=-1)	1	1.463929732530918	0.536070267469082	0.154467321201545
	2	1.404614185557379	0.059315546973539	0.000304224828280
	3	1.404491648216471	0.000122537340908	0.000000000002804
	4	1.404491648215341	0.000000000001130	0.000000000000000
Present methods (4.6.3) (p=-1)	1	1.457909810256813	0.542090189743187	0.138190341228102
	2	1.404554376091742	0.053355434165071	0.000155727901671
	3	1.404491648215442	0.000062727876299	0.000000000000251
	4	1.404491648215341	0.000000000000101	0.000000000000000
Present method (4.7.1)	1	1.422910813234791	0.577089186765209	0.046386337604170
	2	1.404491702133259	0.018419111101532	0.000000133849771
	3	1.404491648215341	0.000000053917918	0.000000000000000
Present method (4.7.5)($\delta_0 = 0.1$)	1	1.404751654708388	0.595248345291612	0.000645590555316
	2	1.404491648215341	0.000260006493047	0.000000000000000

Remark 4.8.3. From the above comparison tables, we observe that newly introduced methods are easily compete with existing Newton's method, Kanwar, Kukreja, and Singh's method (4.6.1) and Mir, Ayab and Rafiq's method (4.6.2). For the suitable choice of the values of p and δ_0, these methods provide better results.

Chapter 5

Methods Based on Ujević, Erceg and Lekić Scheme

5.1 Introduction

In the process of finding efficient numerical method for solving nonlinear equations, Ujević, Erceg and Lekić [97] suggested a family of methods (1.3.8) as a conclusion of two methods suggested by Ujević in [95] and [96] using different quadrature rules. To find the method effective in practice, they seek the best possible parameters in the method (1.3.8) and the particular method becomes

$$\left.\begin{aligned} x_{n+1} &= x_n + (z_n - x_n)\frac{f(x_n)}{f(x_n) - f(z_n)}, \\ \text{where} \qquad z_n &= x_n - \frac{f(x_n)}{f'(x_n)}. \end{aligned}\right\} \tag{5.1.1}$$

The order of convergence of above method is 3. In Section 5.2, we modify this method by using modified Newton method given by McDougall and Wotherspoon instead of classical Newton method and in Section 5.3, we combine our method which we will obtain in Section 5.2 with secant method. In both cases, the order of convergence is increased. Section 5.4 contains certain numerical examples based on these methods.

5.2 McDaugall and Wortherspoon Type Method

We suggest the following method as a variant of Ujević, Erceg and Lekić method. If x_0 is the initial approximation, then

$$
\left.\begin{aligned}
x_0^* &= x_0 \\
x_1 &= x_0 + \frac{(z_0 - x_0)f(x_0)}{f(x_0) - f(z_0)}, \\
\text{where} \qquad z_0 &= x_0 - \frac{f(x_0)}{f'(\frac{x_0^*+x_0}{2})} = x_0 - \frac{f(x_0)}{f'(x_0)}.
\end{aligned}\right\} \tag{5.2.1}
$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$
\left.\begin{aligned}
x_n^* &= x_n + \frac{(z_n^* - x_n)f(x_n)}{f(x_n) - f(z_n^*)}, \\
\text{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)} \\
x_{n+1} &= x_n + \frac{(z_n - x_n)f(x_n)}{f(x_n) - f(z_n)}, \\
\text{where} \qquad z_n &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]}.
\end{aligned}\right\} \tag{5.2.2}
$$

Below we prove the convergent result for the method (5.2.1)-(5.2.2).

Theorem 5.2.1. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving nonlinear equation $f(x) = 0$, the method (5.2.1)-(5.2.2) is convergent with order of convergence $\frac{3+\sqrt{17}}{2} \approx 3.5615$.*

Proof Let e_n and e_n^* denote respectively the errors in the terms x_n and x_n^*. Also, we denote $c_j = \dfrac{f^j(\alpha)}{j!f'(\alpha)}$, $j = 2, 3, 4...$, which are constants. Then from (5.2.1) $x_0^* = x_0$ implies $e_0^* = e_0$. We now proceed to calculate the error e_1 in x_1. By using Taylor series expansion and binomial expansion, we get

$$
\begin{aligned}
z_0 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
&= \alpha + e_0 - \frac{f(\alpha + e_0)}{f'(\alpha + e_0)} \\
&= \alpha + e_0 - \frac{f'(\alpha)[e_0 + c_2e_0^2 + c_3e_0^3 + O(e_0^4)]}{f'(\alpha)[1 + 2c_2e_0 + 3c_3e_0^2 + 4c_4e_0^3 + O(e_0^4)]} \\
&= \alpha + e_0 - [e_0 + c_2e_0^2 + c_3e_0^3 + O(e_0^4)][1 + 2c_2e_0 + 3c_3e_0^2 + 4c_4e_0^3 + O(e_0^4)]^{-1} \\
&= \alpha + e_0 - [e_0 - c_2e_0^2 + 2(c_2^2 - c_3)e_0^3 + O(e_0^4)] \\
&= \alpha + c_2e_0^2 + (2c_3 - 2c_2^2)e_0^3 + O(e_0^4).
\end{aligned}
$$

So that after some calculation, we get

$$z_0 - x_0 = -e_0 + c_2 e_0^2 + (2c_3 - 2c_2^2)e_0^3 + O(e_0^4),$$

$$f(z_0) = f'(\alpha)[c_2 e_0^2 + (2c_3 - 2c_2^2)e_0^3 + O(e_0^4)],$$

$$f(x_0) - f(z_0) = e_0 f'(\alpha)[1 + 2c_2^2 e_0^2 - c_3 e_0^2 + O(e_0^3)],$$

and
$$\frac{(z_0 - x_0)f(x_0)}{f(x_0) - f(z_0)} = -e_0 + c_2^2 e_0^3 + O(e_0^4).$$

Hence from (5.2.1),

$$\alpha + e_1 = \alpha + e_0 - e_0 + c_2^2 e_0^3 + O(e_0^4)$$

so that

$$e_1 = a e_0^3, \tag{5.2.3}$$

where $a = c_2^2$ and we have neglected the higher power of e_n. Again,

$$x_1^* = x_1 + \frac{(z_1^* - x_1)f(x_1)}{f(x_1) - f(z_1^*)}. \tag{5.2.4}$$

Here

$$z_1^* - x_1 = x_1 - \frac{f(x_1)}{f'[\frac{1}{2}(x_0 + x_0^*)]} - x_1 = -\frac{f(x_1)}{f'(x_0)},$$

so that

$$(z_1^* - x_1)f(x_1) = -\frac{[f(x_1)]^2}{f'(x_0)}.$$

Since

$$\begin{aligned} f(x_1) &= f(\alpha + e_1) \\ &= f'(\alpha)[e_1 + c_2 e_1^2 + c_3 e_1^3 + O(e_1^4)], \end{aligned}$$

we have

$$\begin{aligned}
\frac{[f(x_1)]^2}{f'(x_0)} &= \frac{[f(\alpha+e_1)]^2}{f'(\alpha+e_0)} \\
&= [f'(\alpha)]^2 \frac{[e_1+c_2e_1^2+c_3e_1^3+O(e_1^4)]^2}{[1+2c_2e_0+3c_3e_0^2+4c_4e_0^3+O(e_0^4)]} \\
&= f'(\alpha)[e_1^2+2c_2e_1^3+\cdots][1+2c_2e_0+3c_3e_0^2+4c_4e_0^3+O(e_0^4)]^{-1} \\
&= f'(\alpha)[e_1^2-2c_2e_1^2e_0+\cdots] \\
&= e_1f'(\alpha)[e_1-2c_2e_0e_1+O(e_0^5).
\end{aligned}$$

Also

$$\begin{aligned}
f(z_1^*) &= f\left[x_1-\frac{f(x_1)}{f'(x_0)}\right] \\
&= f\left[\alpha+e_1-\frac{e_1+c_2e_1^2+c_3e_1^3+O(e_1^4)}{1+2c_2e_0+3c_3e_0^2+O(e_0^3)}\right]
\end{aligned}$$

After some calculation, we get

$$f(z_1^*) = f'(\alpha)[2c_2e_0e_1+3c_3{e_0}^2e_1-4c_2^2e_0^2e_1+O(e_0^6)]$$

so that

$$f(x_1)-f(z_1^*) = e_1f'(\alpha)[1-(2c_2e_0+3c_3e_0^2-4c_2^2e_0^2)+O(e_0^3)]$$

and

$$\begin{aligned}
(z_1^*-x_1)\frac{f(x_1)}{f(x_1)-f(z_1^*)} &= -[e_1-2c_2e_0e_1+O(e_0^5)][1-(2c_2e_0+3c_3e_0^2-4c_2^2e_0^2)+O(e_0^3)]^{-1} \\
&= -[e_1-2c_2e_0e_1+O(e_0^5)][1+2c_2e_0+3c_3e_0^2-4c_2^2e_0^2+4c_2^2e_0^2+\cdots] \\
&= -e_1+(4c_2^2-3c_3)e_0^2e_1+O(e_0^6). \qquad (5.2.5)
\end{aligned}$$

Now, using (5.2.5), the error e_1^* in x_1^* in (5.2.4) can be calculated as

$$\begin{aligned}
e_1^* &= e_1+[-e_1+(4c_2^2-3c_3)e_0^2e_1+0(e_0^6)] \\
&= (4c_2^2-3c_3)e_0^2e_1 \\
&= abe_0^5, \qquad (5.2.6)
\end{aligned}$$

where $b = 4c_2^2 - 3c_3$ and we have neglected the higher power terms of e_0. Now, we compute the error e_2 in the term

$$x_2 = x_1 + (z_1 - x_1)\frac{f(x_1)}{f(x_1) - f(z_1)},$$

where

$$z_1 = x_1 - \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}.$$

Now

$$\begin{aligned} f'\left(\frac{x_1+x_1^*}{2}\right) &= f'(\alpha + \frac{e_1+e_1^*}{2}) \\ &= f'(\alpha)(1 + c_2e_1 + c_2e_1^* + \frac{3}{4}c_3e_1^2 + O(e_1^3) \end{aligned}$$

so that

$$\begin{aligned} \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)} &= (e_1 + c_2e_1^2 + O(e_1^3))\left(1 + c_2e_1 + c_2e_1^* + \frac{3}{4}c_3e_1^2 + O(e_1^3)\right)^{-1} \\ &= e_1 + \frac{1}{4}c_3e_1^3 - c_2e_1e_1^* \end{aligned}$$

and therefore

$$z_1 = \alpha - \frac{1}{4}c_3e_1^3 + c_2e_1e_1^*,$$

where the higher power terms are neglected. Thus

$$f(z_1) = f'(\alpha)[c_2e_1e_1^* + c_2^2e_1^2e_1^* - \frac{1}{4}c_3e_1^3]$$

and

$$f(x_1) - f(z_1) = e_1f'(\alpha)(1 + c_2e_1 + c_3e_1^2 - c_2e_1^* - c_2^2e_1e_1^* + \frac{5}{4}c_3e_1^2).$$

Also

$$(z_1 - x_1)f(x_1) = -\frac{[f(x_1)]^2}{f'[\frac{1}{2}(x_1 + x_1^*)],}$$

so that

$$(z_1 - x_1)f(x_1) = -e_1f'(\alpha)(e_1 + c_2e_1^3 - c_2e_1e_1^* + \frac{5}{4}c_3e_1^3).$$

Using above considerations, the error e_2 in x_2 is given by

$$\begin{aligned} e_2 &= -3c_2^2 e_1^2 e_1^* + c_2^2 e_1 (e_1^*)^2 \\ &= -3c_2^2 e_1^2 e_1^* \\ &= c e_1^2 e_1^*, \end{aligned}$$

where $c = -3c_2^2$. In fact it can be worked out for $n \geq 1$, the following relation holds:

$$e_{n+1} = c e_n^2 e_n^*. \tag{5.2.7}$$

In order to compute e_{n+1} explicitly, we need e_n^*. We already find e_1^*. We now compute e_2^*. We have

$$\begin{aligned} x_2^* &= x_2 + (z_2^* - x_2)\frac{f(x_2)}{f(x_2) - f(z_2^*)}, \\ \text{where} \qquad z_2^* &= x_2 - \frac{f(x_2)}{f'(\frac{x_1 + x_1^*}{2}).} \end{aligned}$$

Similar as above, it can be calculated the error e_2^* is given by

$$e_2^* = d e_1^2 e_2,$$

where $d = \frac{4}{3}c_3$ and, again, it can be checked that in general for $n \geq 2$, the following relation holds:

$$e_n^* = d e_{n-1}^2 e_n. \tag{5.2.8}$$

In the view of (5.2.7) and (5.2.8), the error at each stage in x_n^* and x_{n+1} are calculated which are tabulated below:

Table 5.1: Successive errors.

n	e_n	e_n^*
0	e_0	e_0
1	ae_0^3	abe_0^5
2	$a^3bce_0^{11}$	$a^5bcde_0^{17}$
3	$a^{11}b^3c^4de_0^{39}$	$a^{17}b^5c^6d^2e_0^{61}$
4	$a^{39}b^{11}c^{15}d^4e_0^{139}$	$a^{61}b^{17}c^{23}d^7e_0^{217}$
5	$a^{139}b^{39}c^{44}d^{15}e_0^{495}$	...
$\vdots$	$\vdots$	$\vdots$

It is observed that the powers of e_0 in the errors at each iterate from a sequence

$$3,\ 11,\ 39,\ 139,\ 495,\ 1763,\ 6279,\ 22363,\ ... \tag{5.2.9}$$

and the sequence of their successive ratios is

$$\frac{11}{3},\ \frac{39}{11},\ \frac{139}{39},\ \frac{495}{139},\ \frac{1763}{495},\ \frac{6279}{1763},\ \frac{22363}{6279},\ \ldots$$

or,

$$3.67,\ 3.5454,\ 3.5641,\ 3.5611,\ 3.5616,\ 3.5615,\ 3.5615, \cdots .$$

This sequence seems to converge the number 3.5615 approximately. The numbers α_i in the sequence (5.2.9) are related by the relation

$$\alpha_i = 3\alpha_{i-1} + 2\alpha_{i-2},\ i = 2, 3, 4 \cdots . \tag{5.2.10}$$

If we set the limit

$$\frac{\alpha_i}{\alpha_{i-1}} = \frac{\alpha_{i-1}}{\alpha_{i-2}} = R,$$

Then dividing (5.2.10) by α_{i-1}, we obtain

$$R^2 - 3R - 2 = 0,$$

which has its positive root as $R = \frac{3+\sqrt{17}}{2} \approx 3.5615$.
Hence we conclude that the order of convergence of method is at least 3.5615. □

5.3 Increasing the Order of Convergence

In this section, we derive a new method by combining the iterations of method (5.2.1)-(5.2.2) with the secant method and show that order of convergence of resulting method is increased by more than one. Precisely, we propose the following method:
If x_0 is the initial approximation, then

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_0^{**} &= x_0 + \frac{(z_0 - x_0)f(x_0)}{f(x_0) - f(z_0)}, \\ \text{where} \qquad z_0 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0^{**} - \frac{x_0^{**} - x_0^*}{f(x_0^{**}) - f(x_0^*)} f(x_0^{**}). \end{aligned}\right\} \tag{5.3.1}$$

Subsequently, for $n \geq 1$, the iteration can be obtained as follows:

$$\left.\begin{aligned} x_n^* &= x_n + \frac{(z_n^* - x_n)f(x_n)}{f(x_n) - f(z_n^*)}, \\ \text{where} \qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1} + x_{n-1}^*}{2}\right)} \\ x_n^{**} &= x_n - \frac{(z_n - x_n)f(x_n)}{f(x_n) - f(z_n)}, \\ \text{where} \qquad z_n &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n + x_n^*)]} \\ x_{n+1} &= x_n^{**} - \frac{x_n^{**} - x_n^*}{f(x_n^{**}) - f(x_n^*)} f(x_n^{**}). \end{aligned}\right\} \tag{5.3.2}$$

For the convergence of this method, we prove the following:

Theorem 5.3.1. *Let f be a function having sufficient number of smooth deriva-*

tives in a neighborhood of α which is a simple root of $f(x) = 0$. Then the method (5.3.1)-(5.3.2) to approximate the root α is convergent with order of convergence $2 + 2\sqrt{2} \approx 4.828$.

Proof We prove this theorem on the line of the proof of Theorem 5.2.1 and error equation of standard secant method. In particular, the errors e_0^*, e_0^{**} and e_1, respectively, in x_0^*, x_0^{**} and x_1 in (5.3.1) are given by

$$e_0^* = e_0$$

$$e_0^{**} = ae_0^3, \qquad \text{where} \qquad a = c_2^2$$

$$e_1 = c_2 e_0^* e_0^{**} = \lambda a e_0^4, \qquad \text{where} \qquad \lambda = c_2.$$

Also the error e_1^* in x_1^* in (5.3.2) is given by

$$e_1^* = (4c_2^2 - 3c_3)e_0^2 e_1 = \lambda a b e_0^6, \qquad \text{where} \qquad b = 4c_2^2 - 3c_3$$

and the error e_1^{**} in x_1^{**} in (5.3.2) is given by

$$e_1^{**} = ce_1^2 e_1^* = c\lambda^3 a^3 b e_0^{14}, \qquad \text{where} \qquad c = -3c_2^2.$$

In fact, it can be worked out that for $n \geq 1$, the following relation holds:

$$e_n^{**} = ce_n^2 e_n^*. \tag{5.3.3}$$

In order to compute e_n^{**} explicitly, we need to compute e_n and e_n^*. We have already computed e_1 and e_1^*. From the proof of Theorem 5.2.1,

$$e_2^* = de_1^2 e_2,$$

where $d = \frac{4}{3}c_3$ and again it can be verified that the following relation holds:

$$e_n^* = de_{n-1}^2 e_n. \tag{5.3.4}$$

Also from (5.3.2), it can be shown that

$$e_2 = \lambda e_1^* e_1^{**} = \lambda^5 a^4 b^2 c e_0^{20}.$$

Thus, for $n \geq 1$, it can be shown that the error e_{n+1} in x_{n+1} in the method (5.3.1)-(5.3.2) satisfies the following recursion formula

$$e_{n+1} = \lambda e_n^* e_n^{**}. \tag{5.3.5}$$

Using the above information, the error at each stage in x_n^*, x_n^{**}, and x_n are obtained and calculated as follows:

Table 5.2: Successive errors.

n	e_n	e_n^*	e_n^{**}
0	e_0	e_0	ae_0^3
1	$\lambda a e_0^4$	$\lambda ab e_0^6$	$\lambda^3 a^3 b e_0^{14}$
2	$\lambda^5 a^4 b^2 c e_0^{20}$	$\lambda^7 a^6 b^2 c d e_0^{28}$	$\lambda^{17} a^{14} b^6 c^4 d e_0^{68}$
3	$\lambda^{25} a^{20} b^8 c^5 d^2 e_0^{96}$	$\lambda^{35} a^{28} b^{12} c^7 d^3 e_0^{136}$	$\lambda^{75} a^{68} b^{28} c^{18} d^7 e_0^{328}$
4	$\lambda^{111} a^{96} b^{40} c^{25} d^{10} e_0^{464}$	$\lambda^{161} a^{136} b^{56} c^{35} d^{15} e_0^{656}$	$\lambda^{383} a^{328} b^{136} c^{86} d^{25} e_0^{1584}$
5	$\lambda^{545} a^{464} b^{192} c^{121} d^{40} e_0^{2240}$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

We construct the analysis of the table as done in [75]. Note that powers of e_0 in the error at each iterate form the sequence

$$4,\ 20,\ 96,\ 464,\ 2240,\ \cdots \tag{5.3.6}$$

and sequence of their successive ratios is

$$\frac{20}{4},\ \frac{96}{20},\ \frac{464}{96},\ \frac{2240}{464},\ \cdots$$

or,

$$5,\ 4.8,\ 4.84,\ 4.82,\ \cdots .$$

If the terms of the sequence (5.3.6) are denoted by α_i, then it can be seen that

$$\alpha_i = 4\alpha_{i-1} + 4\alpha_{i-2}.$$

Thus, the rate of convergence of method (5.3.1)-(5.3.2) is at least $2 + 2\sqrt{2} \approx 4.82$.
□

5.4 Numerical Examples

In order to check the performance of the newly introduced methods (5.2.1)-(5.2.2) and (5.3.1)-(5.3.2), the test functions and their roots α which are used as numerical examples are as follows:

(i) $f_1(x) = (x-1)^8 - 1$, $\alpha = 2$

(ii) $f_2(x) = \sin^2 x - x^2 + 1$, $\alpha = 1.40449164821534$

(iii) $f_3(x) = \cos x - xe^x + x^2$, $\alpha = 0.639154069332008$

(iv) $f_4(x) = x^3 + 4x^2 - 10$, $\alpha = 1.365230013414097$

Numerical computations have been performed using Matlab software and stopping criteria $|x_{n+1} - x_n| < 10^{-12}$ and $|f(x_n)| < 10^{-14}$. We also compare the result of these methods between them as well as with Newton's method and Ujević, Erceg and Lekić (UEL)method

Table 5.3: $f_1(x) = (x-1)^8 - 1$ and initial guess $x_0 = 3$.

n	Newton's method	UEL method	Present method (5.2.1)-(5.2.2)	Present method (5.3.1)-(5.3.2)
1	2.750976562500000	2.621212292220119	2.621212292220119	2.536240556881722
2	2.534581615819526	2.321482528817460	2.240259790619724	2.116363645929317
3	2.348995976046720	2.106434089229419	2.029715679791304	2.000368108924736
4	2.195747198046065	2.009090545951117	2.000022929984292	2.000000000000000
5	2.082041836760382	2.000008831906093	2.000000000000000	
6	2.018764916659598	2.000000000000008		
7	2.001166173395949	2.000000000000000		
8	2.000004743257317			
9	2.000000000078744			
10	2.000000000000000			

Table 5.4: $f_2(x) = \sin^2 x - x^2 + 1$ and initial guess $x_0 = 1$.

n	Newton's method	UEL method	Present method (5.2.1)-(5.2.2)	Present method (5.3.1)-(5.3.2)
1	1.649190196932272	1.320546154049013	1.320546154049013	1.442226482471086
2	1.439042347687187	1.404061768716632	1.404460568207670	1.404491649744238
3	1.405385086160459	1.404491648166524	1.404491648215341	1.404491648215341
4	1.404492272936243	1.404491648215341		
5	1.404491648215647			
6	1.404491648215341			

Table 5.5: $f_3(x) = \cos x - xe^x + x^2$ and initial guess $x_0 = 1$.

n	Newton's method	UEL method	Present method (5.2.1)-(5.2.2)	Present method (5.3.1)-(5.3.2)
1	0.724644697567095	0.660764858475215	0.660764858475215	0.644691946674196
2	0.644658904870270	0.639160213376992	0.639154122061457	0.639154096332009
3	0.639177807467281	0.639154096332008	0.639154096332008	
4	0.639154096773051			
5	0.639154096332008			

Table 5.6: $f_4(x) = x^3 + 4x^2 - 10$ and initial guess $x_0 = 1.5$.

n	Newton's method	UEL method	Present method (5.2.1)-(5.2.2)	Present method) (5.3.1)-(5.3.2)
1	1.500000000000000	1.365738635364349	1.365738635364349	1.365262021937744
2	1.373333333333333	1.365230013445703	1.365230013414097	1.365230013414097
3	1.365262014874627	1.365230013414097		
4	1.365230013916147			
5	1.365230013414097			

In this chapter, we have obtained two new higher order Newton type iterative methods for solving nonlinear equations. The method (5.2.1)-(5.2.2) needs one more function evaluation than Ujević, Erceg and Lekić method and two more functions evaluation than Newton's method. However, numerical examples have shown that this method is easily compete with cited methods. Also we derived new hybrid method (5.3.1)-(5.3.2) by combining method (5.2.1)-(5.2.2) with secant method. It is shown that resulting method is of order 4.828 and the computational cost is comparable with that of the methods cited in the tables.

Chapter 6

Method Based on Potra and Pták Scheme

6.1 Introduction

In [83], Potra and Pták propose the third order iterative method (1.3.9) for solving nonlinear equations of single variables as a modification of Newton's method. In Section 6.2, we modify this method using McDougall and Wortherspoon scheme and obtain new method having order of convergence 3.5615. Again, in Section 6.3 and 6.4, we combine the iterations of this method with secant method and modified secant method given by Amat and Busquier in [2], respectively, and obtain two new methods having order of convergence 4 and 6. Finally, we observe some numerical examples to compare the performance of these modified methods with some existing methods.

6.2 McDougall and Wortherspoon Type Scheme

As the first aim of this chapter, we propose the following method by modifying the method (1.3.9) given by Potra and Pták using McDaugall and Wotherspoon Scheme:

If x_0 is the initial approximation, then

$$x_0^* = x_0 \tag{6.2.1}$$

$$x_1 = x_0 - \frac{f\left[x_0 - \frac{f(x_0)}{f'\left(\frac{x_0+x_0^*}{2}\right)}\right] + f(x_0)}{f'\left(\frac{x_0+x_0^*}{2}\right)}$$

$$= x_0 - \frac{f\left[x_0 - \frac{f(x_0)}{f'(x_0)}\right] + f(x_0)}{f'(x_0)}. \tag{6.2.2}$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$x_n^* = x_n - \frac{f\left[x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)}\right] + f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)}, \tag{6.2.3}$$

$$x_{n+1} = x_n - \frac{f\left[x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}\right] + f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}. \tag{6.2.4}$$

For the convergence of the method (6.2.1)-(6.2.4), we prove the following:

Theorem 6.2.1. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. If x_0 is sufficiently close to α, then the method (6.2.1)-(6.2.4) is convergent and has the order of convergence* 3.5615.

Proof. Let e_n and e_n^* denote, respectively, the errors in the terms x_n and x_n^*. Also, we denote $c_j = \dfrac{f^j(\alpha)}{j!f'(\alpha)}$, $j = 2, 3, 4...$, which are constants. Then from (6.2.1), $x_0^* = x_0$ implies $e_0^* = e_0$. We now proceed to calculate the error e_1 in x_1. By using Taylor series expansion and binomial expansion, we get

$$\begin{aligned}
x_0 - \frac{f(x_0)}{f'(x_0)} &= \alpha + e_0 - \frac{f(\alpha + e_0)}{f'(\alpha + e_0)} \\
&= \alpha + e_0 - \frac{f'(\alpha)[e_0 + c_2e_0^2 + c_3e_0^3 + O(e_0^4)]}{f'(\alpha)[1 + 2c_2e_0 + 3c_3e_0^2 + 4c_4e_0^3 + O(e_0^4)]} \\
&= \alpha + e_0 - [e_0 + c_2e_0^2 + c_3e_0^3 + O(e_0^4)][1 + 2c_2e_0 + 3c_3e_0^2 + 4c_4e_0^3 + O(e_0^4)]^{-1} \\
&= \alpha + c_2e_0^2 + (2c_3 - 2c_2^2)e_0^3 + O(e_0^4),
\end{aligned}$$

so that after some calculations, we get

$$f\left(x_0 - \frac{f(x_0)}{f'(x_0)}\right) = f'(\alpha)[c_2e_0^2 + (2c_3 - 2c_2^2)e_0^3 + c_2^3e_0^4 + O(e_0^5)],$$

$$f\left(x_0 - \frac{f(x_0)}{f'(x_0)}\right) + f(x_0) = f'(\alpha)[e_0 + 2c_2e_0^2 + 3c_3e_0^3 - 2c_2^2e_0^3 + c_2^3e_0^4 + O(e_0^5)]$$

and

$$\begin{aligned}\frac{f\left(x_0 - \frac{f(x_0)}{f'(x_0)}\right) + f(x_0)}{f'(x_0)} &= [e_0 + 2c_2e_0^2 + 3c_3e_0^3 - 2c_2^2e_0^3 + c_2^3e_0^4 + O(e_0^5)] \\ &\quad [1 + 2c_2e_0 + 3c_3e_0^2 + 4c_4e_0^3 + O(e_0^4)]^{-1} \\ &= e_0 - 2c_2^2e_0^3 + O(e_0^4).\end{aligned}$$

Hence from (6.2.2),

$$\alpha + e_1 = \alpha + e_0 - e_0 + 2c_2^2e_0^3 + O(e_0^4)$$

i.e.,

$$e_1 = ae_0^3, \tag{6.2.5}$$

where $a = 2c_2^2$ and we have neglected the higher power of e_n. Again, from (6.2.3)

$$x_1^* = x_1 - \frac{f\left[x_1 - \frac{f(x_1)}{f'(x_0)}\right] + f(x_1)}{f'(x_0)}. \tag{6.2.6}$$

Here

$$\begin{aligned}f(x_1) &= f(\alpha + e_1) \\ &= f'(\alpha)[e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)], \\ f'(x_0) &= f'(\alpha + e_0) \\ &= 1 + 2c_2e_0 + 3c_3e_0^2 + 4c_4e_0^3 + O(e_0^4).\end{aligned}$$

Also

$$f\left[x_1 - \frac{f(x_1)}{f'(x_0)}\right] = f\left[\alpha + e_1 - \frac{e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)}{1 + 2c_2e_0 + 3c_3e_0^2 + O(e_0^3)}\right].$$

After some calculation, we get

$$f\left[x_1 - \frac{f(x_1)}{f'(x_0)}\right] = f'(\alpha)[2c_2e_0e_1 + 3c_3{e_0}^2e_1 - 4c_2^2e_0^2e_1 + O(e_0^6)]$$

so that

$$f\left[x_1 - \frac{f(x_1)}{f'(x_0)}\right] + f(x_1) = f'(\alpha)[e_1 + 2c_2e_0e_1 + 3c_3e_0^2e_1 - 4c_2^2e_0^2e_1 + \cdots]$$

and

$$\begin{aligned}\frac{f\left[x_1 - \frac{f(x_1)}{f'(x_0)}\right] + f(x_1)}{f'(x_0)} &= [e_1 + 2c_2e_0e_1 + 3c_3e_0^2e_1 - 4c_2^2e_0^2e_1 + \cdots] \\ &\quad [1 + 2c_2e_0 + 3c_3e_0^2 + \cdots]^{-1} \\ &= e_1 - 4c_2^2e_0^2e_1 + O(e_0^4). \qquad (6.2.7)\end{aligned}$$

From (6.2.6), the error e_1^* in x_1^* can be calculated as

$$\begin{aligned}e_1^* &= e_1 - [e_1 - 4c_2^2e_0^2e_1 + O(e_0^4)] \\ &= 4c_2^2e_0^2e_1 + O(e_0^4) \\ &= abe_0^5, \qquad (6.2.8)\end{aligned}$$

where $b = 4c_2^2$ and we have neglected the higher power terms of e_0.

Next, we compute the error e_2 in x_2. Now,

$$\begin{aligned}\frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)} &= \frac{f'(\alpha)[e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)]}{f'\left(\alpha + \frac{e_1+e_2}{2}\right)} \\ &= \frac{e_1 + c_2e_1^2 + c_3e_1^3 + O(e_1^4)}{1 + c_2e_1 + c_2e_1^* + \frac{3}{4}c_3e_1^2 + O(e_1^3)} \\ &= e_1 + \frac{1}{4}c_3e_1^3 - c_2e_1e_1^* - c_2^2e_1^2e_1^* + \cdots\end{aligned}$$

so that

$$x_1 - \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)} = \alpha - \frac{1}{4}c_3e_1^3 + c_2e_1e_1^* + c_2^2e_1^2e_1^*,$$

where the higher power terms are neglected. Thus

$$f\left(x_1 - \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}\right) = f'(\alpha)[c_2e_1e_1^* + c_2^2e_1^2e_1^* - \frac{1}{4}c_3e_1^3]$$

and

$$f\left(x_1 - \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}\right) + f(x_1) = e_1f'(\alpha)(1+c_2e_1+c_3e_1^2+c_2e_1^*+c_2^2e_1e_1^*-\frac{1}{4}c_3e_1^2+\cdots).$$

Also

$$\frac{f\left(x_1 - \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}\right) + f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)} = e_1 - \frac{3}{2}c_3e_1^2e_1^* + \cdots .$$

From (6.2.4),

$$x_2 = x_1 - \frac{f\left(x_1 - \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}\right) + f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}$$

Thus, substituting the values, we get

$$\begin{aligned} \alpha + e_2 &= \alpha + e_1 - (e_1 - \frac{3}{2}c_3e_1^2e_1^* + \cdots) \\ \Rightarrow e_2 &= \frac{3}{2}c_3e_1^2e_1^* + O(e_0^{12}) \\ \therefore \qquad e_2 &= \frac{3}{2}c_3e_1^2e_1^* = a^3bce_0^{11}, \end{aligned}$$

where $c = \frac{3}{2}c_3$. In fact, it can be worked out for $n \geq 1$, that the following relation holds:

$$e_{n+1} = ce_n^2e_n^*. \tag{6.2.9}$$

In order to compute e_{n+1} explicitly, we need e_n^*. We already find e_1^*. We now compute e_2^*. From (6.2.3)

$$x_2^* = x_2 - \frac{f\left[x_2 - \dfrac{f(x_2)}{f'\left(\frac{x_1+x_1^*}{2}\right)}\right] + f(x_2)}{f'\left(\frac{x_1+x_1^*}{2}\right)}.$$

Using similar process as above, the error e_2^* in x_2^* can be calculated as

$$e_2^* = de_1^2 e_2,$$

where $d = c_2^2$ and, again, it can be checked that, in general, for $n \geq 2$, the following relation holds:

$$e_n^* = de_{n-1}^2 e_n. \tag{6.2.10}$$

From (6.2.9) and (6.2.10), it is clear that the errors e_n^* and e_{n+1}, respectively, in x_n^* and x_{n+1} for $n \geq 2$ in the method (6.2.1)-(6.2.4) satisfy the following recursion formula:

$$e_n^* = de_{n-1}^2 e_n \tag{6.2.11}$$

$$e_{n+1} = ce_n^2 e_n^*. \tag{6.2.12}$$

To find the order of convergence of the method, we need a relation of the form

$$e_{n+1} = Ae_n^p, \tag{6.2.13}$$

where A is some constant. Thus,

$$e_n = Ae_{n-1}^p \qquad \text{or} \qquad e_{n-1} = A^{-\frac{1}{p}} e_n^{\frac{1}{p}}. \tag{6.2.14}$$

From (6.2.11), (6.2.12), (6.2.13) and (6.2.14),

$$Ae_n^p = ce_n^2 e_n^* = ce_n^2 de_{n-1}^2 e_n = cde_n^2 A^{\frac{-2}{p}} e_n^{\frac{2}{p}} e_n = cdA^{\frac{-2}{p}} e_n^{(3+\frac{2}{p})}.$$

Equating the power of e_n,

$$p = 3 + \frac{2}{p}$$

$$\text{or,} \qquad p^2 - 3p - 2 = 0$$

$$\text{or,} \qquad p = \frac{3 \pm \sqrt{17}}{2}.$$

Taking positive value, $p = 3.5615$. Thus, the order of convergence of the method (6.2.1)-(6.2.4) is 3.5615. □

6.3 Combining With Secant Method

The aim of this section is to improve the rate of convergence of the method (1.3.9) given by Potra and Pták in [83]. For this, we propose the following method in which iterations are performed alternately from method (1.3.9) and secant method (1.2.2):

$$x_{n+1} = \overline{x_n} - \frac{\overline{x_n} - x_n}{f(\overline{x_n}) - f(x_n)} f(\overline{x_n}), \tag{6.3.1}$$

$$\text{where} \qquad \overline{x_n} = x_n - \frac{f(x_n) + f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)}. \tag{6.3.2}$$

Let us now discuss the convergence of the above mentioned method (6.3.1)-(6.3.2).

Theorem 6.3.1. *Let α be a simple zero of sufficiently differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval D. If x_0 is sufficiently close to α, then the proposed method (6.3.1)-(6.3.2) has convergence of order at least 4.*

Proof. Let e_n and $\overline{e_n}$ be the errors in x_n and $\overline{x_n}$, respectively, that is, $x_n = \alpha + e_n$ and $\overline{x_n} = \alpha + \overline{e_n}$. Denote $c_j = \dfrac{f^j(\alpha)}{j! f'(\alpha)}$. If we give a little attention on the proof of Theorem 6.2.1, it is clear that the error equation of (6.3.2) is given by

$$\begin{aligned} \overline{e_n} &= 2c_2^2 e_n^3 + O(e_n^4) \\ &= A e_n^3 + O(e_n^4), \qquad \text{where } A = 2c_2^2. \end{aligned} \tag{6.3.3}$$

Here

$$\begin{aligned}\overline{x_n} - x_n &= (\alpha + \overline{e_n}) - (\alpha + e_n)\\ &= \overline{e_n} - e_n\\ &= Ae_n^3 - e_n + O(e_n^4)\end{aligned}$$

By Taylor's expansion, we get

$$\begin{aligned}f(x_n) &= f(\alpha + e_n)\\ &= f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)]\end{aligned}$$

and using (6.3.3), we obtain

$$\begin{aligned}f(\overline{x_n}) &= f(\alpha + \overline{e_n})\\ &= f'(\alpha)[Ae_n^3 + O(e_n^6)].\end{aligned}$$

Thus, we get

$$\begin{aligned}f(\overline{x_n}) - f(x_n) &= f'(\alpha)[Ae_n^3 + O(e_n^6)] - f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)]\\ &= -f'(\alpha)e_n[1 + c_2e_n + (c_3 - A)e_n^2 + O(e_n^3)]\end{aligned}$$

and

$$\begin{aligned}\frac{(\overline{x_n} - x_n)f(\overline{x_n})}{f(\overline{x_n}) - f(x_n)} &= \frac{[Ae_n^3 - e_n + O(e_n^4)]f'(\alpha)[Ae_n^3 + O(e_n^6)]}{-f'(\alpha)e_n[1 + c_2e_n + (c_3 - A)e_n^2 + O(e_n^3)]}\\ &= [Ae_n^3 + O(e_n^5)][1 + c_2e_n + (c_3 - A)e_n^2 + O(e_n^3)]^{-1}\\ &= [Ae_n^3 - Ac_2e_n^4 + O(e_n^5)].\end{aligned}$$

Thus, the error equation in (6.3.1) is given by

$$\begin{aligned}e_{n+1} &= \overline{e_n} - Ae_n^3 + Ac_2e_n^4 + O(e_n^5)]\\ &= Ae_n^3 + O(e_n^4) - Ae_n^3 + Ac_2e_n^4 + O(e_n^5)\\ &= \lambda e_n^4 + O(e_n^5),\end{aligned}$$

where λ is some constant. Thus, the order of convergence of the method (6.3.1)-(6.3.2) is at least 4 and the theorem is proved. $\square$

6.4 Combining with Modified Secant Method

Again, we recall the following derivative free method presented by the Amat and Basquier in [2]:

$$x_{n+1} = x_n - A_n^{-1} f(x_n), \tag{6.4.1}$$

$$\text{where} \qquad A_n = [y_n, x_n; f] = \frac{f(x_n) - f(y_n)}{x_n - y_n},$$

$$y_n = x_n + \delta_n(x_{n-1} - x_n), \qquad \delta_n \leq |O(e_n)^{\frac{3}{2}}|.$$

This is the second order method. They obtained this method by modifying classical secant method. We shall prove that if we use the iterates alternatively from the method (1.3.9) and this method, the resulting method will be of order at least 6 for the suitable choice of δ. Thus, we propose following method:

$$x_{n+1} = \overline{x_n} - A_n^{-1} f(\overline{x_n}), \tag{6.4.2}$$

$$\text{where} \qquad A_n = [y_n, \overline{x_n}; f] = \frac{f(\overline{x_n}) - f(y_n)}{\overline{x_n} - y_n},$$

$$y_n = \overline{x_n} + \delta_n(x_n - \overline{x_n}), \qquad \delta_n \leq |O(e_n)^{\frac{3}{2}}|$$

$$\text{and} \qquad \overline{x_n} = x_n - \frac{f(x_n) + f\left(x_n - \dfrac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)}.$$

For the convergence of method (6.4.2), we prove the following:

Theorem 6.4.1. *Let α be a simple zero of a sufficiently differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval D. If x_0 is sufficiently close to α, then the method (6.4.2) has convergence of order at least 5.5 and it become 6 for the suitable choice of δ.*

Proof. We prove this theorem on the line of the proof of Theorem 4.7.2. From Theorem 6.3.1, the error $\overline{e_n}$ in $\overline{x_n}$ is given by

$$\overline{e_n} = 2c_2^2 e_n^3 + O(e_n^4).$$

Since

$$y_n = \overline{x_n} + \delta_n(x_n - \overline{x_n}) = \overline{x_n} + a_n, \qquad a_n = \delta_n(x_n - \overline{x_n}),$$

we have by Taylor expansion

$$f(y_n) = f(\overline{x_n} + a_n) = f(\overline{x_n}) + a_n f'(\overline{x_n}) + \frac{a_n^2}{2} f''(\overline{x_n}) + \cdots .$$

Also from the proof Theorem (4.7.2), the method (6.4.2) can be written as

$$x_{n+1} = \overline{x_n} - \frac{f(x_n)}{f'(\overline{x_n})} + O(a_n).$$

Thus, the method (6.4.2) can be written as

$$x_{n+1} = \overline{x_n} - \frac{f(\overline{x_n})}{f'(\overline{x_n})} + O(\delta_n |x_n - \overline{x_n}|), \tag{6.4.3}$$

From (6.4.3), we have

$$\begin{aligned}
e_{n+1} &= \overline{e_n} - \frac{f(\alpha + \overline{e_n})}{f'(\alpha + \overline{e_n})} + O(\delta_n |\alpha + e_n - \alpha - \overline{e_n}|) \\
&= \overline{e_n} - \left(\overline{e_n} + c_2 \overline{e_n}^2 + c_3 \overline{e_n}^3 + O(\overline{e_n}^4)\right)\left(1 + 2c_2\overline{e_n} + 3c_3\overline{e_n}^2 + O(\overline{e_n}^3)\right)^{-1} + O(\delta_n |e_n - \overline{e_n}|) \\
&= \overline{e_n} - (\overline{e_n} + c_2\overline{e_n}^2 - 2c_2\overline{e_n}^2 + O(\overline{e_n}^3) + O(\delta_n |e_n - \overline{e_n}|) \\
&= c_2\overline{e_n}^2 + O(\delta_n |e_n - \overline{e_n}|) + \cdots \\
&= c_2(2c_2^2 e_n^3)^2 + O(\delta_n |e_n - 2c_2^2 e_n^3|) + \cdots \\
&= 4c_2^5 e_n^6 + O(\delta_n |e_n - 2c_2^2 e_n^3|) + \cdots .
\end{aligned} \tag{6.4.4}$$

Since $\delta_n \leq |O(\overline{e_n})^{\frac{3}{2}}|$, so if we assume that $\delta_n \leq |O(\overline{e_n})^2|$, that is, $\delta_n \leq |O(e_n)^6|$, then the order of convergence of the method (6.4.2) is at least 6. But, if we take $\delta_n = |O(\overline{e_n})^{\frac{3}{2}}|$, then order of convergence of the method becomes 5.5. □

Remark 6.4.2. If the solution of nonlinear equation is unknown, we use the same idea as indicated in Remark 4.7.3 to determine the value of δ_n.

6.5 Numerical Examples

In this section, we present numerical examples which demonstrate the performance of methods obtained in the previous sections. We compare these methods with Newton's method, Weerakoon and Fernando (WF) method [104] and Potra and Pták (PP) method (1.3.9). To perform the numerical calculation, we use Matlab Software and stopping criteria $|x_{n+1} - x_n| < (10)^{-12}$ or $|f(x_{n+1}| < (10)^{-14}$.

Example 6.5.1. We apply methods (6.2.1)-(6.2.4) and (6.3.1)-(6.3.2) on the non-linear equation

$$3x + \sin x - e^x = 0. \tag{6.5.1}$$

To determine appropriate initial approximation of root, let us draw the graph of function $f(x) = 3x + \sin x - e^x$.

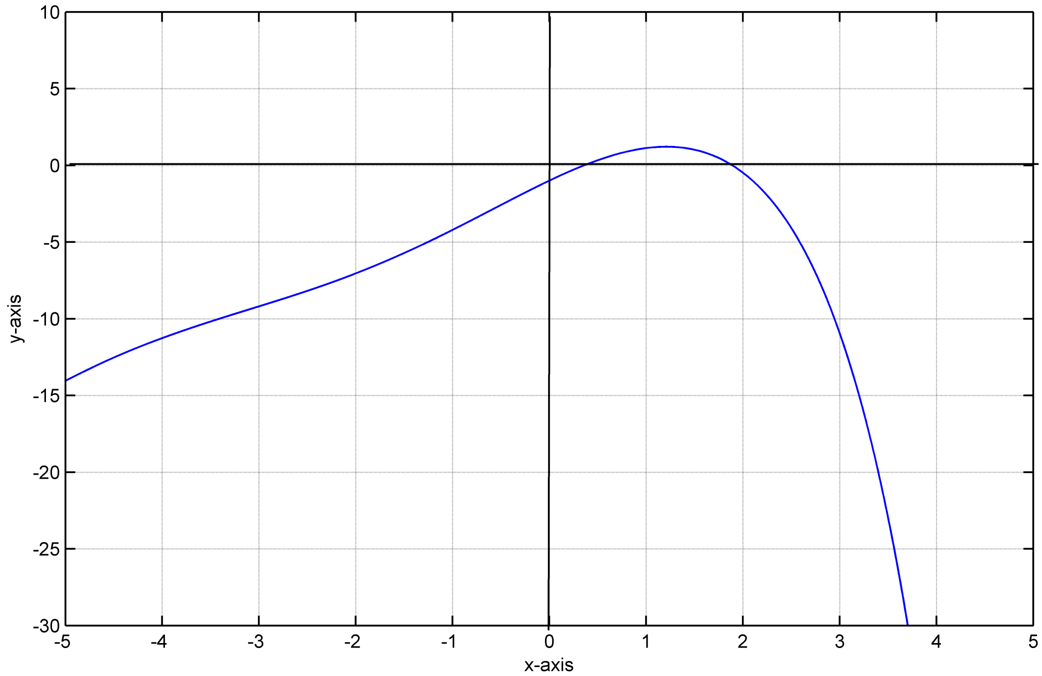

Figure 6.1: Graph of the function $f(x) = 3x + \sin x - e^x$.

From Figure 6.1, it is clear that the equation (6.5.1) has a simple root in $(1, 2)$ and another simple root in $(0, 1)$. Taking initial approximation as $x_0 = 3$. Table 6.1 shows the iterations of Newton's method, Potra and Pták method (1.3.9), Weerakoon and Fernando method [104] and our methods (6.2.1)-(6.2.4) and (6.3.1)-(6.3.2).

Example 6.5.2. Again, we apply methods (6.2.1)-(6.2.4) and (6.3.1)-(6.3.2) on the nonlinear equation

$$f(x) = x^3 + 2x^2 - 3x - 1 = 0. \tag{6.5.2}$$

From the Intermediate Value Theorem, it is clear that at least one of the roots of this equation lies in the interval $(0, 2)$ since $f(0)f(2) < 0$. Taking initial approximation as $x_0 = 2$, Table 6.2 shows the iterations of the same methods which we use in example 6.5.1

Example 6.5.3. Finally, we apply methods (6.3.1)-(6.3.2) and (6.4.2) on the nonlinear equation

$$(x - 2)^{23} - 1 = 0 \tag{6.5.3}$$

By inspection of above equation, it is clear that $x = 3$ is the root of this equation. Taking initial approximation as $x_0 = 4$. Table 6.3 shows the iterations of Newton's method, Potra and Pták method (1.3.9) and our methods (6.3.1)-(6.3.2) and (6.4.2).

Table 6.1: Comparision of different methods.

Method	n	x_n	$\|x_n - x_{n-1}\|$	$\|f(x_n)\|$
Newton's method	1	2.394517490417379	0.605482509582621	3.099858826809431
	2	2.038072024241837	0.356445466175543	0.668781324813446
	3	1.907609975098321	0.130462049143516	0.070325609970161
	4	1.890320120516808	0.017289854581513	0.001142545401351
	5	1.890029810354966	0.000290310161841	0.000003319010836
	6	1.890029729251992	0.000000081102975	0.000000000000025
	7	1.890029729251985	0.000000000000006	0.000000000000000
PP method	1	2.223022716747521	0.776977283252479	1.771401528396967
	2	1.918429998489068	0.304592718258453	0.114786469963655
	3	1.890068371118427	0.028361627370641	0.000151999733143
	4	1.890029729252092	0.000038641866335	0.000000000000419
	5	1.890029729251985	0.000000000000107	0.000000000000000
WF method	1	2.182401798156115	0.817598201843885	1.501646202703620
	2	1.905217119872699	0.277184678283415	0.060614766751669
	3	1.890033324087735	0.015183795784964	0.000014139991158
	4	1.890029729251985	0.000003594835750	0.000000000000000
Present method (6.2.1) -(6.2.4)	1	2.223022716747521	0.776977283252479	1.771401528396967
	2	1.903254578217390	0.319768138530131	0.052683015289491
	3	1.890030006989384	0.013224571228006	0.000001092453759
	4	1.890029729251985	0.000000277737399	0.000000000000000
Present method (6.3.1) -(6.3.2)	1	2.072980588934883	0.927019411065118	0.853003949652775
	2	1.891066539031407	0.181914049903475	0.004082261437855
	3	1.890029729254035	0.001036809777372	0.000000000008063
	4	1.890029729251985	0.000000000002050	0.000000000000000

Table 6.2: Comparision of different methods.

Method	n	x_n	$\|x_n - x_{n-1}\|$	$\|f(x_n)\|$
Newton's method	1	1.470588235294118	0.529411764705882	2.093832688784858
	2	1.247132678773134	0.223455556520984	0.308997039524917
	3	1.200698732423872	0.046433946349262	0.012278976846436
	4	1.198694926456477	0.002003805967394	0.000022485705760
	5	1.198691243528430	0.000003682928048	0.000000000075905
	6	1.198691243515997	0.000000000012432	0.000000000000000
PP method	1	1.347421606542067	0.652578393457933	1.035129693410847
	2	1.202167869692565	0.145253736849502	0.021293690797169
	3	1.198691313272115	0.003476556420450	0.000000425885336
	4	1.198691243515997	0.000000069756118	0.000000000000001
WF method	1	1.317412413069151	0.682587586930849	0.805382355879769
	2	1.199882716602041	0.117529696467110	0.007282302633090
	3	1.198691245071338	0.001191471530703	0.000000009495896
	4	1.198691243515997	0.000000001555341	0.000000000000000
Present method (6.2.1) -(6.2.4)	1	1.347421606542067	0.652578393457933	1.035129693410847
	2	1.199673166705735	0.147748439836332	0.006000378396200
	3	1.198691243536719	0.000981923169016	0.000000000126513
	4	1.198691243515997	0.000000000020722	0.000000000000000
Present method (6.3.1) -(6.3.2)	1	1.262611277391092	0.737388722608908	0.413379428130185
	2	1.198711091370871	0.063900186020221	0.000121180247031
	3	1.198691243515997	0.000019847854874	0.000000000000000

Table 6.3: Comparision of different methods.

n	PP method	Present method (6.3.1)-(6.3.2)	Present method (6.4.2) $\delta_n = (3 - x_n)^{3/2}$	Present method (6.4.2) $\delta_n = (3 - x_n)^2$
1	3.881762281757005	3.843143104308642	3.814658265096324	3.729238036112728
2	3.770514691012868	3.698588407251381	3.645089902139495	3.526770236835179
3	3.665844113356311	3.565371837885375	3.484739002121267	3.361258490695884
4	3.567362235341004	3.442608817238754	3.338026931042994	3.220194565600152
5	3.474705145072179	3.329507542251746	3.205117622709691	3.099435117528323
6	3.387535965688113	3.225479138393579	3.088517260188958	3.014759658594771
7	3.305558823234900	3.130819006660631	3.010352168613209	3.000003318880189
8	3.228575744377471	3.050435204423297	3.000000483378891	
9	3.156705026488715	3.004892673718495	3.000000000000000	
10	3.091180490016622	3.000001327332550		
11	3.036856948205285	3.000000000000000		
12	3.005553045154960			
13	3.000036207524484			
14	3.000000000011477			
15	3.000000000000000			

Chapter 7

Conclusion

7.1 Summary of the Work Done

In the present thesis entitled **"Newton Type Iterative Methods for Solving Nonlinear Equations"**, an attempt has been made to investigate and obtain new numerical methods for solving nonlinear equations. Chapter 1 contains basic preliminaries required in the subsequent chapters. Also brief historical background has been given in this chapter motivating the present study. Chapters 2-6 contain our own investigation given briefly below.

Chapter 2 is inspired by the method of McDaugall and Wotherspoon [75] which is of order $1+\sqrt{2}$. We have obtained new iterative methods for solving nonlinear equations. Also we derived certain hybrid methods by using these methods and the standard secant method. The resulting methods turned out to be of higher order of convergence and are more efficient than the existing ones.

Newton type iterative methods with higher order of convergence are obtained in Chapter 3 which are based on the work of Weerakoon and Fernando [104] and also the same techniques of Chapter 2 have been used.

Chapter 4 contains results which are based on integral approximation and inverse functions. We obtained a fourth order method by combining arithmetic mean Newton's method, Harmonic mean Newton's method and midpoint Newton's method. Again, a method has been obtained by using the idea of inverse function and approximating the integral in Newton's theorem by the linear combination of harmonic mean rule and a more general mean given by Wang [99]. Also a method based on Simpson's rule and inverse function has been made. Moreover, two variants of Mir, Ayub and Rafiq method [77] have been presented by amalgamating this method with the standard secant method and modified secant method [2].

Chapter 5 is motivated by the method obtained by Ujević, Erceg and Lekić [97] while Chapter 6 is based on the work of the Potra and Pták [83]. In these chapters, we used the McDaugall and Wotherspoon scheme [75] and the idea of combining the known methods with the secant method as well as the modified secant method [2].

For all the methods that have been obtained, we have computed the order of convergence and compared these methods with similar existing methods. Also all these methods are free from second and higher order derivatives. All methods are supported by numerical examples.

7.2 Recommendation for Further Work

There is a lot of scope of the work done in this thesis. The work on this thesis is rich with new idea and results which may motivate for further research on this area. In the literature, several Newton type methods are dealing with system of nonlinear equations. Also the Newton's method and its various generalization are known in the framework of Banach spaces. For such work one may refer to [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [18], [23], [25], [26], [28], [30], [37], [41], [42], [43], [46], [62], [73], [76], [78], [79], [84], [85], [86], [101], [102], [105], [108], [109] and [112]. It is of interest if the numerical methods obtained in this thesis could be generalized and studied for the system of nonlinear equations as well as in the framework of Banach space.

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APPENDIX
Research Papers

1. P. Jain, C. R. Bhatta, and J. Jnawali, *Modified Newton type methods with higher order convergence*, Jordan J. Math. and Stat., 8(2015), 327–341.

2. P. Jain, C. R. Bhatta, and J. Jnawali, *Newton type iterative methods with higher order of convergence*, J. Numer. Anal. Approx. Theory, 45(2016), 14–26.

3. J. Jnawali and C. R. Bhatta, *A new variant of Newton's method with fourth-order convergent*, Journal of Institute of Science and Technology, 21(2016), 86–89.

4. J. Jnawali and C. R. Bhatta, *Iterative methods for solving nonlinear equations with fourth-order convergence*, Tribhuvan University Journal, (accepted for publication).

Attended Seminars/Conferences

1. Seminar on Mathematics and Its Applications held at Central Department of Mathematics Tribhuvan University on September 19-20, 2014. Presented a paper entitled " A Variant of Newton's Method for Solving Nonlinear Equation with Forth-order Convergence".

2. International seminar on mathematics -2014 held at Biratnagar, Nepal on December 27-28, 2014. Presented a paper entitled "Combination of Modified Newton's and Secant Methods for Solving Nonlinear Equations with Order of Convergence $(3+\sqrt{17})/2$".

3. International Congress on Industrial and Applied Mathematics(ICIAM) held in Beijing, China on August 10-14, 2015. Presented a poster paper entitled "Combination of Modified Newton's and Secant Methods for Solving Nonlinear Equations with Order of Convergence $(3+\sqrt{17})/2$".

4. International Conference on Function Spaces and Inequalities during December 8-12, 2015 organized by Department of Mathematics, South Asian University, New Delhi, India. Presented a paper entitled " Newton's Type Iterative Methods with High Efficiency".

5. The Seventh National Conference on Science and Technology during March 29-31, 2016 organised by Nepal Academy of Science and Technology. Presented a paper entitled "A New Variant of Newton's Method with Fourth-order Convergence"

6. International Conference on Applications of Mathematics to Nonlinear Science during May 26-29, 2016, Kathmandu, Nepal. Presented a paper entitled "Some Higher Order Convergent Newton Type Iterative Methods"

7. National Conference on Mathematics and its Applications during January 11-13, 2017, Chitwan, Nepal. Presented a paper entitled "Some Iterative Methods for Solving Nonlinear Equations"

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NEWTON TYPE ITERATIVE METHODS WITH HIGHER ORDER OF CONVERGENCE

PANKAJ JAIN*, CHET RAJ BHATTA† and JIVANDHAR JNAWALI**

Abstract. Newton type iterative methods with higher order of convergence are obtained. The order of convergence is further increased by amalgamating these methods with the standard secant method. The methods are compared to the similar recent methods.

MSC 2010. 65H05.
Keywords. Newton method, secant method, iterative method, nonlinear equation, order of convergence.

1. INTRODUCTION

Quite often, we come across numerous nonlinear equations which need to be solved. If the equation is not a polynomial equation, then it is not always easy to deal with such equations. To this end, one or the other numerical iterative method is employed. One such classical standard method is the Newton method

$$x_{n+1} = x_n - \tfrac{f(x_n)}{f'(x_n)}$$

which is quadratically convergent. Over the years, a lot of methods have appeared, each one claims to be better than the other in some or the other aspect. We mention here the method given by Weerakoon and Fernando [8] which is based on the Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\lambda)\, d\lambda$$

and the integral involved is approximated by the trapezoidal rule, i.e.,

$$\int_{x_n}^{x} f'(\lambda)\, d\lambda = \tfrac{(x-x_n)}{2}\big(f'(x) + f'(x_n)\big).$$

*Department of Mathematics, South Asian University, Akbar Bhawan, Chanakya Puri, New Delhi-110021, India, e-mail: pankaj.jain@sau.ac.in and pankajkrjain@hotmail.com.
†Central Department of Mathematics, Tribhuvan University, Kirtipur Kathmandu, Nepal, e-mail: chetbhatta0@gmail.com.
**Central Department of Mathematics, Tribhuvan University, Kirtipur Kathmandu, Nepal, e-mail: jnawalij@gmail.com.

As a result, Weerakoon and Fernando obtained the following iterative method for solving the nonlinear equation $f(x) = 0$:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_{n+1})}, \tag{1}$$

where $z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

The method so obtained is of third order. In the present paper, the aim is to modify method (1). In fact, in (1), f' is a function of the previously calculated iterate. In our modification, f' would be a function of some other convenient point. It is proved that the corresponding method has order of convergence 5.1925. We follow the technique of McDougall and Wotherspoon [7] who modified Newton's method in a similar way yielding the order of convergence of their method as $1 + \sqrt{2}$.

Further, in [3], it was proved that if any method for solving nonlinear equation is used in conjunction with the standard secant method then the order of the resulting method is increased by 1. We shall show, in this paper (see Theorem 3.2), that this order can be increased by more than 1. In fact, we prove that if our own method (which is of order 5.1925) is combined with the secant method than the new method is of order 7.275.

2. THE METHOD AND THE CONVERGENCE

We propose the following method:

If x_0 is the initial approximation, then

$$\left.\begin{aligned} x_0^* &= x_0 \\ x_1 &= x_0^* - \frac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)}, \\ \text{where}\quad z_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1^* &= x_1 - \frac{2f(x_1)}{f'(x_1)+f'(z_1^*)}, \\ \text{with}\quad z_1^* &= x_1 - \frac{f(x_1)}{f'[\frac{1}{2}(x_0+x_0^*)]} = x_1 - \frac{f(x_1)}{f'(x_0)}. \end{aligned}\right\} \tag{2}$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$\left.\begin{aligned} x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_n^*)}, \\ \text{where}\quad z_n^* &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_{n-1}+x_{n-1}^*)]} \\ x_{n+1} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})}, \\ \text{with}\quad z_{n+1} &= x_n - \frac{f(x_n)}{f'[\frac{1}{2}(x_n+x_n^*)]}. \end{aligned}\right\} \tag{3}$$

Below, we prove the convergence result for the method (2)–(3).

Theorem 1. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then the method (2)–(3) is convergent and has the order of convergence* 5.1925*.*

Proof. Let e_n and e_n^* denote respectively the errors in the terms x_n and x_n^*. Also, we denote $c_j = \frac{f^j(\alpha)}{j!f'(\alpha)}$, $j = 2, 3, 4...$, which are constants. The error equation for the method (1) as obtained by Weerakoon and Fernando [8] is given by

$$e_{n+1} = ae_n^3,$$

where $a = c_2^2 + \frac{1}{2}c_3$ and we have neglected higher power terms of e_n. In particular, the error e_1 in x_1 in the equations (2) is given by

$$e_1 = ae_0^3. \tag{4}$$

We now proceed to calculate the error e_1^* in x_1^*. By using Taylor series expansion and binomial expansion, we get

$$\begin{aligned}\frac{f(x_1)}{f'(x_0)} &= \frac{f(\alpha+e_1)}{f'(\alpha+e_0)}\\ &= \left(e_1 + c_2e_1^2 + c_3e_1^3 + \mathcal{O}(e_1^4)\right)\left(1 + 2c_2e_0 + 3c_3e_0^2 + \mathcal{O}(e_0^3)\right)^{-1}\\ &= e_1 - 2c_2e_0e_1 + \mathcal{O}(e_0^5)\end{aligned}$$

so that

$$x_1 - \frac{f(x_1)}{f'(x_0)} = \alpha + 2c_2e_0e_1 + \mathcal{O}(e_0^5).$$

Consequently, by Taylor series expansion, it can be calculated that

$$f'(z_1^*) = f'(\alpha)\left(1 + 4c_2^2e_0e_1 + \mathcal{O}(e_0^5)\right).$$

Also

$$f'(x_1) = f'(\alpha)\left(1 + 2c_2^2e_1 + 3c_3e_1^2 + \mathcal{O}(e_1^3)\right)$$

so that

$$f'(x_1) + f'(z_1^*) = 2f'(\alpha)\left(1 + c_2e_1 + 2c_2^2e_0e_1 + \mathcal{O}(e_0^5)\right). \tag{5}$$

Now, using (4) and (5), the error e_1^* in x_1^* in the equation (2) can be calculated as

$$\begin{aligned}e_1^* &= e_1 - \left(e_1 + c_2e_1^2 + \mathcal{O}(e_1^3)\right)\left(1 + c_2e_1 + 2c_2^2e_0e_1 + \mathcal{O}(e_0^5)\right)^{-1}\\ &= 2c_2^2e_0e_1^2\\ &= ba^2e_0^7,\end{aligned}$$

where $b = 2c_2^2$. Using e_1^*, we now compute the error e_2 in the term

$$x_2 = x_1^* - \frac{2f(x_1^*)}{f'(x_1^*)+f'(z_2)},$$

where

$$z_2 = x_1 - \frac{f(x_1)}{f'\left(\frac{x_1+x_1^*}{2}\right)}.$$

Now

$$f'\big(\tfrac{x_1+x_1^*}{2}\big) = f'\big(\alpha + \tfrac{e_1+e_1^*}{2}\big)$$
$$= f'(\alpha)\big(1 + c_2e_1 + c_2e_1^* + \tfrac{3}{4}c_3e_1^2 + \mathcal{O}(e_0^9)\big)$$

so that

$$\frac{f(x_1)}{f'\big(\frac{x_1+x_1^*}{2}\big)} = \big(e_1 + c_2e_1^2 + \mathcal{O}(e_1^3)\big)\big(1 + c_2e_1 + c_2e_1^* + \tfrac{3}{4}c_3e_1^2 + \mathcal{O}(e_0^9)\big)^{-1}$$
$$= e_1 + \tfrac{1}{4}c_3e_1^3 - c_2e_1e_1^*$$

and therefore

$$z_2 = \alpha - \tfrac{1}{4}c_3e_1^3 + c_2e_1e_1^*,$$

where the higher power terms are neglected. Thus

$$f'(z_2) = f'(\alpha)(1 - \tfrac{1}{2}c_2c_3e_1^3 + 2c_2^2e_1e_1^*)$$

and

$$f'(x_1^*) = f'(\alpha)\big(1 + 2c_2e_1^* + 3c_3{e_1^*}^2\big).$$

Using the above considerations, the error e_2 in x_2 is given by

$$e_2 = e_1^* - (e_1^* + c_2{e_1^*}^2 + c_3{e_1^*}^3)\big(1 + c_2e_1^* - \tfrac{1}{4}c_2c_3e_1^3\big)^{-1}$$
$$= -\tfrac{1}{4}c_2c_3e_1^3e_1^*$$
$$= ce_1^3e_1^*,$$

where $c = -\frac{1}{4}c_2c_3$. In fact, it can be worked out that for $n \geq 1$, the following relation holds:

$$e_{n+1} = ce_n^3e_n^*. \tag{6}$$

In order to compute e_{n+1} explicitly, we need to compute e_n^*. We already know e_1^*. We now compute e_2^*. We have

$$x_2^* = x_2 - \tfrac{2f(x_2)}{f'(x_2)+f'(z_2^*)},$$

where

$$z_2^* = x_2 - \frac{f(x_2)}{f'\big(\frac{x_1+x_1^*}{2}\big)}.$$

Like above, it can be calculated that the error e_2^* is given by

$$e_2^* = de_1e_2^2,$$

where $d = c_2^2$ and, again, it can be checked that in general, for $n \geq 2$, the following relation holds:

$$e_n^* = de_{n-1}e_n^2. \tag{7}$$

In the view of (6) and (7), the error at each stage in x_n^* and x_{n+1} are calculated which are tabulated below:

n	e_n	e_n^*
0	e_0	e_0
1	ae_0^3	$a^2be_0^7$
2	$a^5bce_0^{16}$	$a^{11}b^2c^2de_0^{35}$
3	$a^{26}b^5c^6de_0^{83}$	$a^{57}b^{11}c^{13}d^3e_0^{182}$
4	$a^{135}b^{26}c^{32}d^6e_0^{431}$	$a^{296}b^{57}c^{70}d^{14}e_0^{945}$
5	$a^{701}b^{135}c^{167}d^{32}e_0^{2238}$	
$\vdots$	$\vdots$	$\vdots$

Table 1. Successive errors.

It is observed that the powers of e_0 in the errors at each iterate form a sequence

$$3,\ 16,\ 83,\ 431,\ 2238,\ \ldots \tag{8}$$

and the sequence of their successive ratios is

$$\tfrac{16}{3},\ \tfrac{83}{16},\ \tfrac{431}{83},\ \tfrac{2238}{431},\ \ldots$$

or,

$$5.3334,\ 5.1875,\ 5.1927,\ 5.1925,\ \ldots$$

This sequence seems to converge to the number 5.1925 approximately. Indeed, if the terms of the sequence (8) are denoted by $\{\,\alpha_i\,\}$, then it can be seen that

$$\alpha_i = 5\alpha_{i-1} + \alpha_{i-2}, \qquad i = 2, 3, 4... \tag{9}$$

If we set the limit

$$\tfrac{\alpha_i}{\alpha_{i-1}} = \tfrac{\alpha_{i-1}}{\alpha_{i-2}} = R,$$

Then dividing (9) by α_{i-1}, we obtain

$$R^2 - 5R - 1 = 0$$

which has its positive root as $R = \frac{5+\sqrt{29}}{2} \approx 5.1925$. Hence the order of convergence of the method is at least 5.1925. □

Next, we give two variants of the method (2)–(3). Note that, in (2)–(3), the arithmetic average of the points $x_n, x_n^*,\ n = 0, 1, 2...$ has been used. We propose methods in which the arithmetic average is replaced by harmonic as well as geometric averages. With harmonic average, we propose the following method: If x_0 is the initial approximation, then

$$(10)\qquad \left.\begin{aligned}
x_0^* &= x_0\\
x_1 &= x_0^* - \frac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)},\\
\text{where}\quad z_1 &= x_0 - \frac{f(x_0)}{f'\left(\frac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_0 - \frac{f(x_0)}{f'(x_0)}\\
x_1^* &= x_1 - \frac{2f(x_1)}{f'(x_1)+f'(z_1^*)},\\
\text{with}\quad z_1^* &= x_1 - \frac{f(x_1)}{f'\left(\frac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_1 - \frac{f(x_1)}{f'(x_0)}.
\end{aligned}\right\}$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$(11)\qquad \left.\begin{aligned}
x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_n^*)},\\
\text{where}\quad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_{n-1}x_{n-1}^*}{x_{n-1}+x_{n-1}^*}\right)}\\
x_{n+1} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})},\\
\text{with}\quad z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{2x_nx_n^*}{x_n+x_n^*}\right)}.
\end{aligned}\right\}$$

For the geometric average of the points $x_n, x_n^*,\ n = 0, 1, 2...$, the following method is proposed:

$$(12)\qquad \left.\begin{aligned}
x_0^* &= x_0\\
x_1 &= x_0^* - \frac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)},\\
\text{where}\quad z_1 &= x_0 - \frac{f(x_0)}{f'(\sqrt{x_0x_0^*})} = x_0 - \frac{f(x_0)}{f'(x_0)}\\
x_1^* &= x_1 - \frac{2f(x_1)}{f'(x_1)+f'(z_1^*)},\\
\text{with}\quad z_1^* &= x_1 - \frac{f(x_1)}{f'(\sqrt{x_0x_0^*})} = x_1 - \frac{f(x_1)}{f'(x_0)}.
\end{aligned}\right\}$$

Subsequently, for $n \geq 1$, the iteration can be obtained as follows:

$$(13)\qquad \left.\begin{aligned}
x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_n^*)},\\
\text{where}\quad z_n^* &= x_n - \frac{f(x_n)}{f'(\sqrt{x_{n-1}x_{n-1}^*})}\\
x_{n+1} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})},\\
\text{with}\quad z_{n+1} &= x_n - \frac{f(x_n)}{f'(\sqrt{x_nx_n^*})}.
\end{aligned}\right\}$$

The convergence of the methods (10)–(11) and (12)–(13) can be proved on the similar lines as those in Theorem 1. We only state the results below:

THEOREM 2. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving non-linear equation $f(x) = 0$, the method (10)–(11) is convergent with order of convergence* 5.1925.

THEOREM 3. *Let α be a simple zero of a function f which has sufficient number of smooth derivatives in a neighborhood of α. Then for solving non-linear equation $f(x) = 0$, the method (12)–(13) is convergent with order of convergence* 5.1925.

3. METHODS WITH HIGHER ORDER CONVERGENCE

In this section, we obtain a new iterative method by combining the iterations of method (2)–(3) with secant method and prove that the order of convergence is more than 5.1925. Precisely, we propose the following method: If x_0 is the initial approximation, then

$$(14)\qquad \left.\begin{aligned} x_0^* &= x_0 \\ x_0^{**} &= x_0^* - \frac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)}, \\ \text{where}\qquad z_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0^{**} - \frac{x_0^{**}-x_0^*}{f(x_0^{**})-f(x_0^*)} f(x_0^{**}). \end{aligned}\right\}$$

Subsequently, for $n \geq 1$, the iterations can be obtained as follows:

$$(15)\qquad \left.\begin{aligned} x_n^* &= x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_n^*)}, \\ \text{where}\qquad z_n^* &= x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)} \\ x_n^{**} &= x_n^* - \frac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})}, \\ \text{where}\qquad z_{n+1} &= x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)} \\ x_{n+1} &= x_1^{**} - \frac{x_n^{**}-x_n^*}{f(x_n^{**})-f(x_n^*)} f(x_n^{**}). \end{aligned}\right\}$$

REMARK 4. In [3], it was proved that if the iterations of any method of order p for solving nonlinear equations are used alternatively with secant method, then the new method will be of order $p+1$. Thus, in view of that result, the method (14)–(15) is certainly of order at least 6.1925. However, we prove below that the order is more.

THEOREM 5. *Let f be a function f having sufficient number of smooth derivatives in a neighborhood of α which is a simple root of the equation $f(x) = 0$. Then method (14)–(15) to approximate the root α is convergent with order of convergence* 7.275.

Proof. We argue on the lines of that of Theorem 1 and the error equation of the standard secant method. In particular, the errors e_0^*, e_0^{**} and e_1, respectively, in x_0^*, x_0^{**} and x_1 in equations (14) are given by

$$\begin{aligned} e_0^* &= e_0 \\ e_0^{**} &= ae_0^3, \qquad \text{where } a = c_2^2 + \tfrac{1}{2}c_3 \\ e_1 &= \lambda a e_0^4, \qquad \text{where } \lambda = c_2. \end{aligned}$$

Also, the errors e_1^* in x_1^* in equation (15) is given by

$$\begin{aligned} e_1^* &= 2c_2^2 e_0 e_1^2 \\ &= \lambda^2 a^2 b e_0^9, \qquad \text{where } b = 2c_2^2 \end{aligned}$$

and the error e_1^{**} in x_1^{**} in equation (15) is given by

$$\begin{aligned} e_1^{**} &= -\tfrac{1}{4}c_2 c_3 e_1^3 e_1^* \\ &= c e_1^3 e_1^*, \end{aligned}$$

where $c = -\frac{1}{4}c_2c_3$. In fact, it can be worked out that for $n \geq 1$, the following relation holds:

$$e_n^{**} = c e_n^3 e_n^*. \tag{16}$$

In order to compute e_n^{**} explicitly, we need to compute e_n and e_n^*. We have already computed e_1 and e_1^*. From the proof of Theorem 1

$$e_2^* = d e_1 e_2^2,$$

where $d = c_2^2$ and, again, it can be checked that the following relation holds:

$$e_n^* = d e_{n-1} e_n^2. \tag{17}$$

Also from (15), it can be shown that

$$e_2 = \lambda e_1^* e_2^{**}.$$

Thus, for $n \geq 1$, it can be shown that error e_{n+1} in x_{n+1} in the method (14)–(15) satisfies the following recursion formula

$$e_{n+1} = \lambda e_n^* e_n^{**} \tag{18}$$

Using the above information, the errors at each stage in x_n^*, x_n^{**} and x_n are obtained and tabulated as follows:

We do the analysis of Table 2 as done in the proof of Theorem 1 for Table 1. Note that the powers of e_0 in the error at each iterate from the sequence

$$4,\ 30,\ 218,\ 1586,\ 11538,\ \ldots. \tag{19}$$

and the sequence of their successive ratios is

$$\tfrac{30}{4},\ \tfrac{218}{30},\ \tfrac{1586}{218},\ \tfrac{11538}{1586},\ \ldots$$

or

$$7.5,\ 7.2667,\ 7.2752,\ 7.2749,\ \ldots.$$

n	e_n	e_n^*	e_n^{**}
0	e_0	e_0	ae_0^3
1	$\lambda a e_0^4$	$\lambda^2 a^2 b e_0^9$	$\lambda^5 a^5 bce_0^{21}$
2	$\lambda^8 a^7 b^2 ce_0^{30}$	$\lambda^{17} a^{15} b^5 c^2 e_0^{64}$	$\lambda^{42} a^{36} b^{11} c^6 e_0^{154}$
3	$\lambda^{60} a^{51} b^{13} c^8 e_0^{218}$	$\lambda^{128} a^{109} b^{29} c^{17} e_0^{466}$	$\lambda^{308} a^{260} b^{68} c^{42} e_0^{1120}$
4	$\lambda^{437} a^{369} b^{97} c^{59} e_0^{1586}$	$\lambda^{934} a^{789} b^{208} c^{126} e_0^{3390}$	$\lambda^{2245} a^{1896} b^{499} c^{304} e_0^{8148}$
5	$\lambda^{3180} a^{2685} b^{707} c^{430} e_0^{11538}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2. Successive errors.

If the terms of the sequence (19) are denoted by $\{N_i\}$, then it can be seen that

$$N_i = 7N_{i-1} + 2N_{i-2}, \quad i = 2, 3, 4, ...$$

Thus, as in Theorem 1, the rate of convergence of method (14)–(15) is at least 7.275. □

It is natural to consider the variants of the method (14)–(15), where in the expression of z_n and z_n^*, the arithmetic mean is replaced by harmonic mean as well as geometric mean as done in methods (10)–(11) and (12)–(13), respectively. Precisely, with harmonic mean, we propose the following method:

$$(20) \qquad \left.\begin{aligned} x_0^* &= x_0 \\ x_0^{**} &= x_0^* - \tfrac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)}, \\ \text{where} \quad z_1 &= x_0 - \tfrac{f(x_0)}{f'\left(\frac{2x_0x_0^*}{x_0+x_0^*}\right)} = x_0 - \tfrac{f(x_0)}{f'(x_0)} \\ x_1 &= x_0^{**} - \tfrac{x_0^{**}-x_0^*}{f(x_0^{**})-f(x_0^*)} f(x_0^{**}) \end{aligned}\right\}$$

followed by (for $n \geq 1$)

$$(21) \qquad \left.\begin{aligned} x_n^* &= x_n - \tfrac{2f(x_n)}{f'(x_n)+f'(z_n^*)}, \\ \text{where} \quad z_n^* &= x_n - \tfrac{f(x_n)}{f'\left(\frac{2x_{n-1}x_{n-1}^*}{x_{n-1}+x_{n-1}^*}\right)} \\ x_n^{**} &= x_n^* - \tfrac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})}, \\ \text{where} \quad z_{n+1} &= x_n - \tfrac{f(x_n)}{f'\left(\frac{2x_nx_n^*}{x_n+x_n^*}\right)} \\ x_{n+1} &= x_1^{**} - \tfrac{x_n^{**}-x_n^*}{f(x_n^{**})-f(x_n^*)} f(x_n^{**}) \end{aligned}\right\}$$

and with the geometric mean, we we propose the following :

$$
(22) \qquad \left.\begin{aligned}
x_0^* &= x_0\\
x_0^{**} &= x_0^* - \tfrac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)},\\
\text{where} \qquad z_1 &= x_0 - \tfrac{f(x_0)}{f'\left(\sqrt{x_0 x_0^*}\right)} = x_0 - \tfrac{f(x_0)}{f'(x_0)}\\
x_1 &= x_0^{**} - \tfrac{x_0^{**}-x_0^*}{f(x_0^{**})-f(x_0^*)} f(x_0^{**})
\end{aligned}\right\}
$$

followed by (for $n \geq 1$)

$$
(23) \qquad \left.\begin{aligned}
x_n^* &= x_n - \tfrac{2f(x_n)}{f'(x_n)+f'(z_n^*)},\\
\text{where} \qquad z_n^* &= x_n - \tfrac{f(x_n)}{f'\left(\sqrt{x_{n-1}x_{n-1}^*}\right)}\\
x_n^{**} &= x_n^* - \tfrac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})},\\
\text{where} \qquad z_{n+1} &= x_n - \tfrac{f(x_n)}{f'(\sqrt{x_n x_n^*})}\\
x_{n+1} &= x_1^{**} - \tfrac{x_n^{**}-x_n^*}{f(x_n^{**})-f(x_n^*)} f(x_n^{**}).
\end{aligned}\right\}
$$

The convergence of the methods (20)–(21) and (22)–(23) can be proved by using the arguments as used in the proof of Theorem 5. We skip the details for conciseness.

4. ALGORITHMS AND NUMERICAL EXAMPLES

We give below an algorithm to implement the method (2)–(3):

ALGORITHM 6. *Step 1 : For the given tolerance* $\varepsilon > 0$ *and iteration* N*, choose the initial approximation* x_0 *and set* $n = 0$*.*
Step 2 : Follow the following sequence of expressions:

$$
\begin{aligned}
x_0^* &= x_0\\
x_1 &= x_0^* - \tfrac{2f(x_0^*)}{f'(x_0^*)+f'(z_1)},\\
\textit{where} \qquad z_1 &= x_0 - \tfrac{f(x_0)}{f'(x_0)}\\
x_1^* &= x_1 - \tfrac{2f(x_1)}{f'(x_1)+f'(z_1^*)},\\
\textit{where} \qquad z_1^* &= x_1 - \tfrac{f(x_1)}{f'\left(\frac{x_0+x_0^*}{2}\right)} = x_1 - \tfrac{f(x_1)}{f'(x_0)}
\end{aligned}
$$

Step 3 : For $n = 1, 2, 3, \ldots,$ calculate $x_2, x_3, x_4, \ldots$ by the following sequence of expressions:

$$x_n^* = x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_n^*)},$$

$$\textit{where} \qquad z_n^* = x_n - \frac{f(x_n)}{f'\left(\frac{x_{n-1}+x_{n-1}^*}{2}\right)}$$

$$x_{n+1} = x_n^* - \frac{2f(x_n^*)}{f'(x_n^*)+f'(z_{n+1})},$$

$$\textit{where} \qquad z_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}$$

Step 4 : Stop if either $|x_{n+1} - x_n| < \varepsilon$ or $n > N$.
Step 5 : Set $n = n + 1$ and repeat Step 3.

Example 7. We apply method (2)–(3) on the nonlinear equation

$$\cos x - xe^x + x^2 = 0. \tag{24}$$

This equation has a simple root in the interval $(0, 1)$. Taking initial approximation as $x_0 = 1$, Table 3 shows the iterations of McDougall-Wotherspoon method, a third order method (1) and our method (2)–(3).

n	W-F Method (1)	M-W method	(2)–(3) method
1.	1.1754860092539474	0.89033621746836966	0.64406452481689269
2.	0.7117526001461193	0.66469560530044569	0.63915407608296659
3.	0.63945030188514695	0.63928150457301036	0.63915411559451774
4.	0.63915408656045591	0.63915408990276223	0.6391540955014231
5.	0.63915410631623149	0.63915410965853769	0.63915407540832936
6.	0.63915412607200606	0.6391540698096656	0.6391541149198805
7.	0.63915408622313585	0.63915408956544117	0.63915409482678587
8.	0.63915410597891142	0.63915410932121663	0.63915407473369212
9.	0.639154125734686	0.63915406947234454	0.63915411424524327
10.	0.63915408588581579	0.63915408922812	0.63915409415214863
11.	0.63915410564159136	0.63915410898389557	0.63915407405905489
12.	0.63915412539736594	0.63915406913502348	0.63915411357060603
13.	0.63915408554849573	0.63915408889079894	0.6391540934775114
14.	0.63915410530427119	0.63915410864657451	0.63915407338441765
15.	0.63915412506004576	0.63915406879770231	0.6391541128959688
16.	0.63915408521117556	0.63915408855347788	0.63915409280287416
17.	0.63915410496695113	0.63915410830925345	0.63915407270978042
18.	0.6391541247227257	0.63915406846038125	0.63915411222133156
19.	0.6391540848738555	0.63915408821615682	0.63915409212823693
20.	0.63915410462963107	0.63915410797193239	0.63915407203514318

Table 3. Numerical results for different methods.

EXAMPLE 8. We consider the same equation (24) but now implement method (14)–(15) and compare with other methods. Table 4, shows the corresponding iterates. One can also compare the last columns of Table 3 and Table 4 which correspond to methods (2)–(3) and (14)–(15), respectively. This clearly indicates the fast convergence of (14)–(15).

n	W-F Method (1)	M-W method	(14)–(15) method
1.	1.1754860092539474	0.89033621746836966	0.63919747126530391
2.	0.7117526001461193	0.66469560530044569	0.63915410580338361
3.	0.63945030188514695	0.63928150457301036	0.63915409891807362
4.	0.63915408656045591	0.63915408990276223	0.63915409203276374
5.	0.63915410631623149	0.63915410965853769	0.63915408514745375
6.	0.63915412607200606	0.6391540698096656	0.63915411145121981
7.	0.63915408622313585	0.63915408956544117	division by zero
8.	0.63915410597891142	0.63915410932121663	
9.	0.639154125734686	0.63915406947234454	
10.	0.63915408588581579	0.63915408922812	
11.	0.63915410564159136	0.63915410898389557	
12.	0.63915412539736594	0.63915406913502348	
13.	0.63915408554849573	0.63915408889079894	
14.	0.63915410530427119	0.63915410864657451	
15.	0.63915412506004576	0.63915406879770231	
16.	0.63915408521117556	0.63915408855347788	
17.	0.63915410496695113	0.63915410830925345	
18.	0.6391541247227257	0.63915406846038125	
19.	0.6391540848738555	0.63915408821615682	
20.	0.63915410462963107	0.63915410797193239	

Table 4. Numerical results for different methods.

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