Remark Theorem[section] Corollary[section] [theorem]Lemma Proposition[section] Example[section]

TRIBHUVAN UNIVERSITY

INSTITUTE OF SCIENCE AND TECHNOLOGY

On the Study of Distribution of Primes and Twin Prime Conjecture

A Thesis
submitted in partial fulfillment of the requirements
for
the degree of MASTER OF PHILOSOPHY (Math)
by

Khagendra Adhikari

to the
CENTRAL DEPARTMENT OF MATHEMATICS
T.U., Kirtipur

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Student's Declaration

I hereby declare that I am the only author of this work and that no sources other than that listed here have been used in this work.

....

(**Khagendra Adhikari**)

Date: June 17, 2016

Supervisor's Recommendation

I hereby recommend that this thesis prepared under my supervision by **Khagendra Adhikari** entitled **Study on Distribution of Primes and Twin Prime Conjecture** during period prescribed by the rules and regulation of this institute in partial fulfillment of the requirements for the degree of M.Phil. in Mathematics be processed for the evaluation.

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We certify that the **Research Committee of the Central Department of Mathematics**, TU, Kirtipur approved this research work entitled **Study on Distribution of Primes and Twin Prime Conjecture** done by Mr. Khagendra Adhikari in the scope and generality as a thesis in the partial fulfillment for the requirement of the degree of M. Phil. in Mathematics.

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Dedication

To my family who always inspires me for further study.

Abstract

The distribution of primes, mainly focusing on the Tchebycheff estimates of prime counting function, Mertens Theorem which are most significant results for distribution of primes have beeen studied in this thesis. Distribution of Twin Primes, Twin Prime Conjecture and some developments towards the Twin Prime Conjecture is also studied. The alternative approaches for the Twin Prime Conjecture has also been studied in this thesis.

Contents

List of Symbols

We will use the following notations throughout the thesis.
For functions f and g we write:
1.$f(x) \sim g(x)$ which we read as f is asymptotic to g as $x \to \infty$ which mean that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$
2. If $g(x) > 0$ for all $x \geq a$, we write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ to be read f is big-O of g which mean that there is a positive constant C such that for all x, $\mid f(x) \mid \leq C \mid g(x) \mid$ for all $x \geq a$.
3. $f(x) = o(g(x))$ which we read as $f(x)$ is small oh to $g(x)$ as $x \to \infty$ which mean that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

For the most part, we use the standard notations for common number theoritic functions. These are usually defined at their first apperence, but for convenience we also list them here.

$\mathbb{R}$	:	The set of Real Numbers.
$\mathbb{C}$	:	The set of point in complex plane.
$\pi(x)$	:	Prime Counting Function: the number of Primes $p \leq x$
$\pi_2(x)$	:	Twin Prime Counting Function: the number of Primes $p \leq x; p+2$ also be a prime
$[x]$	:	The integral part of the real number x.
$\{x\}$	:	The fractional part of the real number x.
$\log x$	:	The natural logarithm of the real number x.
$d(n)$	:	The number of positive divisor of n.
$\phi(n)$	:	Euler's totient function: the number of reduced residue classes modulo n.
$\mu(n)$	:	Mobious function
$\Lambda(n)$	:	von Mangoldrt's function
$\rho(x)$	:	Correlation Coefficient

Chapter 1

Introduction

1.1 Introduction and Motivation

The positive integers are undoubtedly man's first mathematical creation. It is hard to say about the antiquity of the natural numbers but historical record shows that as early as 5700BC, the ancient Sumerian kept calender, so they must had some form of arithmetic.
Numbers were used for keeping records and for commercial transactions for over 5000 years before anyone thought studying numbers themselves in a systematic way. The first scientific approach to study of integers, that is, the true origin of the theory of numbers, is generally attributed to Greeks. Around 600BC Pythagoras and his disciples studied the integers and classified them as Even-Odd, Prime-composite [22].

A prime number is a natural number greater than 1 whose factors are 1 and itself. The prime numbers derive their peculiar importance from the fundamental theorem of arithmetic that every number can be constructed by multiplying together these numbers so that's why the primes are called building blocks of numbers. Despite of the simplicity in arithmetic, prime numbers have remained one of the most mysterious elements known in mathematics for nearly two thousands years. According to the nature of distribution of primes, they are classified in many types but most important in the history are, Fermat primes, Mersenne Primes, Sophie Germain Primes, Twin Primes, Cousin primes etc[17].

The problems in number theory seems to be very simple as like, How many primes are there?
How many primes are there less than certain number?
Is there exist certain natural number n which satisfies $x^n + y^n = z^n$ for positive integers x, y, z?
Are there infinitely many twin primes?

Although these questions seem to be very simple but their mathematical evidence are very complicated. Euclid $(300BC)$ and many others mathematicians gave the answer of the first question as ”There are infinitely many prime numbers”. But the second question is known as the prime number theorem conjectured by Gauss and Legendre at 1792 as

$$\pi(x) = \lim_{x\to\infty} \frac{x}{\log x}$$

and after 100 years J. Hadamard and C. J. de la Valle Poussion proved it. For the fourth question we have the famous conjecture “There are infinitely many Twin primes” known as Twin Prime Conjecture, mathematically

$$H_1 = \lim_{n\to\infty} (p_{n+1} - p_n) = 2$$

The Twin Prime Conjecture is possibly the most basic question one may ask, after they are satisfied with the Prime Number Theorem. The Twin Prime Conjecture already spawned a modern tool of combinatorial and analytic number theory the sieve theory [19].
The sieve theory was established at the beginning of the 20th century as a simple method to count prime numbers in intervals. Today it is a powerful tool to approach problems related to the Twin Prime Conjecture, e.g., Goldbach Conjecture. It was

already used to prove countless partial results supporting many conjectures and apparently there is much more for sieves to do. Why is that so, that the Twin Prime Conjecture resists any attempts to prove it? There is a fundamental difference between the question about the infinitude of prime numbers and the infinitude of twin prime pairs. The latter one involves not only multiplicative properties of numbers, but also additive properties. These two branches of number theory have numerous books dedicated to each of them separately. The history shows that the most difficult problems are those, which involve both domains.

It is a shame that the truth about such basic facts is hidden from us. Hopefully, one day we will understand the primes or, as Paul Erdos once said "It will be another million years, at least, before we understand the primes" [21].

Although, it has no proof yet, but remarkable progress in the study of distribution of primes and twin prime conjecture has been made. There are probably no direct, practical conclusions that can be drawn from the Twin Prime Conjecture. But, just as was in the case of Fermat's Last Theorem, research toward the unproven conjecture usually yields some additional understanding and tools that can be used in other situations. Like as, until very recently, the number theory was considered only as a pure branch of mathematics with virtually no practical application in the real life. Hardy said in his A mathematician's Apology[21] "No one has yet discovered any war like purposed to be served by the theory of numbers or relativity, and it seems unlikely that anyone will do so for many years". But due to unpredictability of prime numbers, they have become a deterministic part of computer and data security. Now, everyday millions of people use RSA encryption scheme and complicated ciphers without even knowing about it. In this chapter we will discuss the some historical development of prime numbers, distribution of primes and number Theorem, Twin Prime numbers its distribution and Twin Prime Conjecture.

1.2 Historical Background

1.2.1 Development of Prime Numbers and Prime Number Theorem

Although the Pythagoras and his disciples classified the integers as prime and composite around 600BC, the prime numbers and their properties were first studied extensively by ancient Greek mathematicians. Before Greeks, there are some evidences of Babylonians and Egyptians having and under standing of prime integers through their sexigesimal system of division. The most valued source available on the ancient Greek's expertise involving the prime integers is the ancient text elements. Elements has remained one of the most influential works of mathematics ever produced. The Elements by Euclid (300BC) has thirteen volumes among them Volume VII (Propositions 30,31,32) and volume IX (propositions 14 and 20) are related to the properties of primes. The proposition 20 of volume IX is the first result dealing with distribution of primes which stated as *Prime numbers are more than any assigned multitude of prime numbers* [23]. Later on many mathematicians ; T. J. Stiltjes, J. Brun, Euler, L. Kroneker, J. J. Sylvesters etc prove the infinitude of the prime number in different approaches.The propositions found in Elements are the groundwork of prime number theory. Despite the sophisticated principles presented in Elements, Euclid left very little instruction as to how one would go about finding the precise location of a prime integer, or even how to know whether or not a given integer is prime.

Eratosthenes (around 200 BC)of studied a method to produce a list of prime numbers known as the sieve of Eratosthenes, which is a convenient way to produce list of prime numbers below some bound. Extended tables of primes were constructed in the 17th century. In 1770 Johnn Henrich Lambert (1728-1777) had compiled a table of all primes up to $102,000$. At the beginning of the 17th century Pierre de Fermat devised a method to factor large numbers, known as the Fermat's little theorem which is stated as *if p is prime, then for any integer a,* $a^p = a(mod\ p)$.It would be nearly 100 years later would Fermat's Little theorem be proved by Euler. The theorem serves as a test to determine whether or not the input value is prime. The application of Fermat's Little Theorem is widely used today in mathematics and computer security [14].

Fermat tried to expand on Euclid's theorem on the relationship between prime numbers and perfect numbers and ended up producing a theorem that tests for primality. All of the innovations listed were stepping stones in widening our knowledge about the true nature of prime integers.

In 1737 Leonard Euler [26] made a direct connection between the prime numbers and tools of analyzing by proving that

$$\frac{1}{1-2^{-n}}.\frac{1}{1-3^{-n}}.\frac{1}{1-5^{-n}}... = \sum_{k=1}^{\infty}\frac{1}{k^n} = \zeta(n)$$

which displays the connection between the zeta function on the right to the prime number to the left. Mathematicians have studied this function in order to prove the properties of the prime numbers ever since. Using this formula Euler proved that reciprocal series of the prime number diverges which prove that prime numbers are infinitely many.

At the last of 18th century, a new perspective of prime counting function was emerged especially by a young mathematician Carl Fredric Gauss. At the fifteen, while he was studying the logarithm tables, conjectured a new relationship between the prime number and natural logarithm of the natural number which was denoted by Gauss in the formula $\pi(x) \sim \dfrac{x}{\log x}$ where $\pi(x)$ denotes the number of prime number from 1 to x[19]. But this result in some how give the quite estimation of prime numbers on the lower value of x, but goes farther and farther away away from the true value.

Six years later than Gauss, in 1798 French mathematician Legendre devised a similar formula for computing the number of primes in a set of number from 1 to x and conjectured that [2]

$$\pi(x) \sim \frac{x}{A\log x - B}$$

where A and B are constant. In his second paper in 1808 he made this approximation more precise by proving the values $A = 1$ and $B = 1.08366$.

Gauss first formula was not more accurate to count the prime numbers in certain block. In 1849 he conjectured in a letter to the astronomer Johann Franz Encke(1791-1865) [28] that

$$\pi(x) \sim Li(x) = \int_2^x \frac{dt}{\log t}$$

In this letter Gauss confessed that he was not aware of the Legendre's work. By using the calculus it can be shown that $Li(x) \approx \dfrac{x}{\log x}$.

In 1851 Pafunty Lvovich Chebyshev (1821-18194) made a significant step in the direction of the prime number theorem by giving the precise order of magnetuide of $\pi(x)$ by proving [23]

$$C_1 \frac{x}{\log x} \leq \pi(x) \leq C_2 \frac{x}{\log x}$$

for sufficiently large x. He also proved that if $\lim\limits_{x\to\infty} \dfrac{\pi(x)}{\frac{x}{\log x}}$ exists, it is necessarily equal to 1. In fact Chebyshev proved with $C_1 = 0.92129$ and $C_2 = 1.1056$. This enabled him to proved the Bertrand's Postulate, namely that for all $x \geq 1$ there exist prime p such that $x \leq p \leq 2x$.

A few years after the appearance of Chebychev's paper, a path to the proof of PNT was laid out by Riemann in his only published paper at 1859 in number theory. Riemann observed that $\zeta(x)$ is holomorphic in the half plane $Re(s) > 1$ and that it can be continued analytically to a meromorphic function whose only singularity is a simple pole at $s = 1$. Riemann was interested in $\zeta(x)$ because the Euler identity provide a connection between the analyticity properties of $\zeta(x)$ and PNT. He conjectured several properties of these zeros, all but one of which were proved around the end of the 19th century by Hadamard and H. Van Mangoldt. The one conjecture that remains to this day which has been selected by the Clay mathematics institute as one of the seven millenium problems so-called Riemann Hypothesis stated as "All zeros of $\zeta(x)$ with $0 \leq Re(s) \leq 1$ lies on the real line $Re(s) = \frac{1}{2}$". A proof of these conjectures was achieved independently in 1896 by Hadamard and de la Vallaee Poussin [7] and is now known as the Prime Number Theorem.

In fact, we can state stronger forms of the Prime Number Theorem, giving error estimates for the above asymptotic relationship. We find the best error estimates when we assume the Riemann Hypothesis (RH). Assuming RH, Koch showed in 1901 that[5]

$$\pi(x) = Li(x) + O(\sqrt{x} logx)$$

The best known unconditional bound is

$$\pi(x) = Li(x) + O(x \exp(\frac{-A(\log x)^{\frac{3}{5}}}{(\log \log x)^{\frac{1}{5}}}))$$

Although the Prime Number Theorem is a very satisfying result, its proof naturally did not end the study of the distribution of primes. One outstanding question (to which we shall later return) concerns bounds on the difference Another question concerns twin primes, to which we next turn our attention.

1.2.2 Development of the Twin Prime Conjecture

The existence of the Twin Primes would have been easily understood by ancient Greeks but there is no evident that they were considered by them,or indeed by any mathematicians until the nineteen century. The term Twin Prime was coined by Paul Stackel in the late nineteen century[19]. de Poligance's was the person who stated the general conjecture about twin prime in 1849 as "Every even number is the difference of two consecutive primes in infinitely many ways". Taking the even number in the conjecture to be 2 immediately gives what we now call the Twin Prime Conjecture [5].

About 30 years after de Poligance, Glaisher inaugurated a project which continues to the present day by enumerating the twin primes up to 10^5. Glaisher used published tables of primes to show that $\pi_2(10^5) = 1224$ [16].

In 1919, Brun adapted and improved earlier work of Merlin [5] on the sieve of Eratosthenes to find the first non-trivial result concerning twin primes. Brun showed that

$$\pi_2(x) = O(\frac{x(\log \log x)^2}{\log^2 x}) \tag{1.1}$$

Brun obtained an effective version of this bound. In particular, he showed that for some x_0 all $x > x_0$

$$\pi_2(x) < 7200\frac{x}{\log^2 x}(\log\log x)^2 + \frac{x}{\log^6 x} + x^{\frac{3}{4}} \tag{1.2}$$

Brun immediately followed this work with the announcement of a stronger bound, in which he showed that

$$\pi_2(x) = O(\frac{x}{\log^2 x}) \tag{1.3}$$

and once again, he found an effective version of his bound with

$$\pi_2(x) < \frac{100x}{\log^2 x} \tag{1.4}$$

for some $x > x_0$ where x_0 is an effective computable constant. Brun's these works are important because from (1), it can be concluded that the reciprocal series of the twin primes is convergent in contrast with the prime numbers. The convergent value is known as the Brun's constant 'B'. But Brun did not find an explicit value of B. First by Selmer in 1942 and later on by Froberg [1961], Bowman [1973], Shank's,wrener [1974], Brent [1974], Nicely [1996, 2004, 2007], Sebah [2002], Klyve [2007] gave the bounds for B [5].

In 1923 Hardy and Littlewood (1923) made a more precise conjecture on the distribution of twin primes in the interval $[1, x]$ which gives the first asymtotic formula for the Twin Counting Function[8].

Yaminov(1986) formulated a new conjecture of distribution of Twin Primes as the twin primes are distributed among the primes in the same way as the primes among the natural numbers. On the basis of Tchebycheff's theorem and on the asymptotic distribution law of prime numbers, he constructed lower and upper estimates of the numbers of twin primes. He considered their empirical distribution function, having a high degree of accuracy. Under the assumption of the validity of the introduced conjecture, he gave a simple proof of the fact that the number of pairs of twin primes is infinite on the distribution law of twin primes, based on the superposition of the function $\pi_(x)$ with itself [3].

The most significance result about the Twin Prime Conjecture announced on April 17, 2013, by Yitang Zhang. He proved that for some integer N that is less than 70 million, there are infinitely many pairs of primes that differ by N[29]. Terence Tao subsequently proposed a Polymath Project collaborative effort to optimize Zhang's bound . As of April 14, 2014, one year after Zhang's announcement, according to the Polymath project wiki, the bound has been reduced to 246. Further, assuming the Elliotte Halberstam conjecture and its generalized form, the Polymath project wiki states that the bound has been reduced to 12 and 6, respectively [13]. These improved bounds were discovered using a different approach that was simpler than Zhang's and was discovered independently by James Maynard and Terence Tao.

1.3 The goal and structure of Thesis

The main goal of this work is to discuss about the elementary results on distribution of primes and Twin Prime Conjecture. Here we discuss some elementary results on distribution of primes focussing on Chebychev's bounds for prime counting function and Merten's theorems. We also try to give a brief account of the development of Twin Prime Conjecture and its alternative approaches.

In the chapter 2, we state some fundamental definitions and theorems they will be used throughout this work.

The chaptpter 3 will be for the some arithmetical functions and their important results which are useful for the study of the distribution of primes.

In the chapter 4, we will discuss about the distribution of primes and some theorem about it. The martens theorem and Tschebycheff estimates for prime counting function will be discuss in this chapter.

The chapter 5 will be for the Twin prime numbers, their distribution and about the Twin prime conjecture and its altenative approaches.

In chapter 6, we will summarise our entire work and way for future research.

Chapter 2

Some Preliminaries

In this chapter, we will discuss some basic concepts and results on Number Theory which are fundamentals for the study of prime numbers and their distribution. We based on [27][22] [11] [10] for this chapter.

2.1 Some Definitions

Definition 2.1.1. If an integer m, not zero, divides the difference $a - b$, we say that a is congruent to b modulo m and write $a \equiv b(\text{mod } m)$. If $a - b$ is not divisible by m, we say that a is not congruent to b modulo m, and in this case we write $a \not\equiv b(\text{mod } m)$.

Definition 2.1.2. We say d divides n and we write $d|n$ whenever $n = cd$, for some c. We also say that n is a multiple of d, that d is a divisor of n, or that d is a factor of n. If d does not divide n we write $d \nmid n$.

Definition 2.1.3. For given integers a and b, we say d is the greatest common divisor(gcd) of a and b and denoted by $(a, b) = d$ if
(a) $d \geq 0$ (d is non negative)
(b) $d \mid a$ and $d \mid b$ (d is a common divisor of a and b)
(c) $e \mid a$ and $e \mid b$ implies $e|d$ (ever common divisor divides d)

Definition 2.1.4. If gcd of a and b is 1 i.e $(a, b) = 1$ then the integers a and b are said to be relatively prime.

Definition 2.1.5. If $x \equiv y(\text{mod } m)$ then y is called a residue of x modulo m. A set $x_1, x_2, , ..., x_m$ is called a complete residue system modulo m if for every integer y there is one and only one x_j such that $y = x_j(\text{mod } m)$.

2.2 Some Theorems

Theorem 2.2.1. *Every integer $n > 1$ is either a prime number or a product of prime numbers.*

Proof. We use induction on n. The theorem is clearly true for $n = 2$. Assume it is true for every integer $< n$. Then if n is not prime it has a positive divisor $d \neq 1, d \neq n$. Hence $n = cd$,where $c \neq n$. But both c and d are $< n$ and > 1 so each of c, d is a product of prime numbers, hence so is n. □

Theorem 2.2.2. *Euclid There are infinitely many prime numbers.*

Proof. Euclid's Proof:
Suppose there are only a finite number, say $p_1, p_2, ..., p_n$. Let $N = 1 + p_1, p_2, ..., p_n$. Now $N > 1$ so either N is prime or N is a product of primes. Of course N is not prime since it exceeds each p_i. Moreover, no p_i divides N (if $p_i|N$ then p_i divides the difference $N - p_1.p_2...p_n = 1$). This contradicts Theorem 2.2.1. □

Theorem 2.2.3. *(Fundamental theorem of arithmetic)*
Every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.

Theorem 2.2.4. *(The division algorithm):*
Given integers a and b with $b > 0$, there exists a unique pair of integers q and r such that $a = bq + r$, with $0 \leq r < b$.
Moreover, $r = 0$ if, and only if, $b|a$.

Note: We say that q is the quotient and r the remainder obtained when b is divided into a.

Theorem 2.2.5. *There are arbitrarily large gaps in the series of primes. Stated otherwise, given any positive integer k there exist k consecutive composite integers.*

Proof. Consider the integers

$$(k+1)!+2, (k+1)!+3, ..., (k+1)!+k, (k+1)!+k+1$$

Every one of these is composite because j divides $(k+1)!+j$ if $2 \leq j \leq (k+1)$ $\square$

Now we turn our concern to a slight difference view of divisibility known as congruences. A congruence is nothing more than a statement about divisibility. However, it is more than just a convenient notation. The theory of congruence was introduced by Carl Friedrich Gauss (1777-1855). In his book *Disquisitiones Arithmeticae*, written at age 24, Gauss introduced the theory of congruences, which gained ready acceptance as a fundamental tool for the study of number theory.

Theorem 2.2.6. *Let $(a, m) = 1$. Let $r_1, r_2, ..., r_n$ be a complete or reduced residue system modulo m. Then $ar_1, ar_2, ..., ar_n$ is complete, or a reduced, residue system, respectively, modulo m.*

Theorem 2.2.7. *(Fermat's Theorem) Let p denotes a prime. If $p \nmid a$ then $a^{p-1} \equiv 1(mod\ p)$. For every integer a, $a^p \equiv a(mod\ p)$.*

We will prove this theorem as a corollary of the following theorem.

Theorem 2.2.8. *(Euler's generalization of Fermat's Theorem) If $(a, m) = 1$, then*

$$a^{\phi(m)} \equiv 1(modm)$$

Proof. Let $r_1, r_2, ..., r_{\phi(m)}$ be a reduced residue system modulo m. Then by theorem (2.2.6),

$ar_1, ar_2, ..., ar_{\phi(m)}$ is also a reduced residue system modulo m. Hence corresponding to each r_i there is one and only one ar_i such that $r_i \equiv ar_j(\text{mod } m)$. Further more, different r_i will have different corresponding ar_j. This means that the numbers $ar_1, ar_2, ..., ar_{\phi(m)}$ are just residues modulo m of $r_1, r_2, ..., r_{\phi(m)}$, but not necessarily using in the same order. Multiplying and using the properties of congruence, we obtain

$$\prod_{j=1}^{\phi(m)} (ar_j) \equiv \prod_{i=1}^{\phi(m)} r_i(\text{mod } m)$$

and hence

$$a^{\phi(m)} \prod_{j=1}^{\phi(m)} r_j \equiv \prod_{j=1}^{\phi(m)} r_j(\text{mod } m)$$

Now $(r_j, m) = 1$. Hence cancel the r_i and we obtained $a^{\phi(m)} \equiv 1(\text{mod } m$ □

proof of Fermat's Theorem.

If $p \nmid a$, then $(a, p) = 1$ and $a^{\phi(m)} \equiv 1(\text{mod } p)$. Then we have $\phi(p) = p - 1$, Then the first part of the Fermat's Theorem follows and the second part is obvious.

Theorem 2.2.9. *If $(a, m) = 1$ then there is an x such that $ax \equiv$ (mod m). any two such x are congruent (mod$m$). If $(a, m) > 1$ then there is no such x.*

Theorem 2.2.10. *Wilson's Theorem*

If p is a prime, then $(p - 1)! \equiv -1$(mod p).

Chapter 3

Arithmetical Functions and Some Basic Results

In this chapter we will discuss some arithmetical functions and preliminary results of them. These functions play vital role in the study of the distribution of prime numbers. We based on [27][22] [11] [10] for this chapter.

Definition 3.0.1. A real or complex-valued function defined on the positive integers is called an arithmetical function or number-theoretic function.

An arithmetical function f is said to be multiplicative function if f is not identically zero and if

$$f(mn) = f(m)f(n), \text{whenever} \quad (m, n) = 1$$

A multiplicative function f is called completely multiplicative if we also have

$$f(mn) = f(m)f(n), \text{ for all } m, n$$

Example: Let $f_\alpha(n) = n^\alpha$, where α is a fixed real or complex number. This is a completely multiplicative arithmetic function.

Theorem 3.0.11. *Suppose that the function $f : \mathbb{N} \to \mathbb{C}$ is multiplicative. Then the* *function* $g : \mathbb{N} \to \mathbb{C}$ *, defined by*

$$g(n) = \sum_{m \mid n} f(m)$$

for every $n \in \mathbb{N}$*, is multiplicative.*

Definition 3.0.2. A divisor function $d : \mathbb{N} \to \mathbb{C}$ is defined as

$$d(n) = \sum_{m \mid n} 1 \tag{3.1}$$

for every $n \in \mathbb{N}$, here the sum is taken over all positive divisor m of n. In other word the value of $d(n)$ denotes the number of positive divisors of the natural number n. On the other hand we define the function $\sigma : \mathbb{N} \to \mathbb{C}$ by writing

$$\sigma(n) = \sum_{m \mid n} m \tag{3.2}$$

for every $n \in \mathbb{N}$. Clearly the value of $\sigma(n)$ denotes the sum of the all the positive divisors of the natural number n.

Theorem 3.0.12. *Suppose that* $n \in \mathbb{N}$ *and that* $n = p_1^{u_1} ... p_r^{u_r}$ *is the canonical decomposition of* n*. Then,*

$$d(n) = (1 + u_1)...(1 + u_r) \quad \textit{and} \quad \sigma(n) = \frac{p_1^{u_1+1} - 1}{p_1 - 1} ... \frac{p_r^{u_r+1} - 1}{p_r - 1}$$

Proof. Every positive divisor m is of the form $m = p_1^{v_1} ... p_r^{v_r}$, where for every $j = 1, ..., r$, the integer v_j satisfies $0 \leq v_j \leq u_j$. It follows from (3.1) that $d(n)$ is the number of choices for the r-tuple $(v_1, ..., v_r)$. Hence

$$d(n) = \sum_{v_1}^{u_1} ... \sum_{v_r}^{u_r} 1 = (1 + u_1)...(1 + u_r)$$

. On the other hand, it follows from (3.2) that

$$\sigma(n) = \sum_{v_1=0}^{u_1} ... \sum_{v_r=0}^{u_r} p_1^{v_1} ... p_r^{v_r} = (\sum_{v_1=0}^{u_1} p_1^{v_1})...(\sum_{v_r=0}^{u_r} p_1^{v_1}(\sum_{v_r=0}^{u_r} p_r^{v_r})$$

Note now that for every $j = 1, ..., r$, we have

$$\sum_{v_j=0}^{u_j} p_j^{v_j} = 1 + p_j + p_j^2 + ... + p_j^{u_j}$$

The second assertion follows. $\square$

The result below is the simple deduction of the above result.

Theorem 3.0.13. *The arithmetic functions $d : \mathbb{N} \to \mathbb{C}$ and $\sigma : \mathbb{N} \to \mathbb{C}$ are both multiplicative.*

Theorem 3.0.14. *As $x \to \infty$, we have*

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(x^{\frac{1}{2}})$$

Here γ is Euler's constant and is defined by

$$\gamma = \lim_{x \to \infty} (\sum_{n \leq x} \frac{1}{n} - \log x) = 0.5772156649.....$$

The proof depends on the following results.

Theorem 3.0.15. *As $x \to \infty$, we have*

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x})$$

Proof. As $x \to \infty$, we have

$$\sum_{n\leq x}\frac{1}{n}=\sum_{n\leq x}(\frac{1}{x}+\int_{n}^{x}\frac{1}{u^2}du)=\frac{[x]}{x}+\sum_{n\leq x}\int_{n}^{x}\frac{1}{u^2}du=\frac{[x]}{x}+\int_{1}^{x}\frac{1}{u^2}(\sum_{n\leq u}1)du$$
$$=\frac{[x]}{x}+\int_{1}^{x}\frac{[u]}{u^2}du=\frac{[x]}{x}+\int_{1}^{x}\frac{1}{u}du-\int_{1}^{x}\frac{u-[u]}{u^2}du$$
$$=\log x+1+O(\frac{1}{x})-\int_{1}^{\infty}\frac{u-[u]}{u^2}du+\int_{x}^{\infty}\frac{u-[u]}{u^2}du$$
$$=\log x+(1-\int_{1}^{\infty}\frac{u-[u]}{u^2}du)+O(\frac{1}{x})$$

Letting $x\to\infty$ we find that

$$\gamma=\lim_{x\to\infty}(\sum_{n\leq x}\frac{1}{n}-\log x)=1-\int_{1}^{\infty}\frac{u-[u]}{u^2}du$$

This completes the theorem. $\square$

Proof of Theorem 3.0.14:

As $x\to\infty$, we have

$$\sum_{n\leq x}d(n)=\sum_{\substack{y;z\\ yz\leq x}}1=\sum_{y\leq x^{\frac{1}{2}}}\sum_{z\leq\frac{x}{y}}1+\sum_{z\leq x^{\frac{1}{2}}}\sum_{y\leq\frac{x}{z}}1-\sum_{y\leq x^{\frac{1}{2}}}\sum_{z\leq x^{\frac{1}{2}}}1$$
$$=2\sum_{y\leq x^{\frac{1}{2}}}[\frac{x}{y}]-[x^{\frac{1}{2}}]^2=2\sum_{y\leq x^{\frac{1}{2}}}\frac{x}{y}+O(\frac{1}{x^{\frac{1}{2}}})-(x^{\frac{1}{2}}+O(1))^2$$
$$=2x(\log x^{\frac{1}{2}}+\gamma+O(\frac{1}{x^{\frac{1}{2}}}))+O(x^{\frac{1}{2}})-x$$
$$=x\log x+(2\gamma-1)x+O(x^{\frac{1}{2}})$$

This completes the theorem.

Every number $n \in \mathbb{N}$ has divisor 1 and n, so we must have $\sigma(1) = 1$ and $\sigma(n) > 1$. The next theorem shows the behavior of $\sigma(n)$ an $n \to \infty$.

Theorem 3.0.16. *we have $\sigma(n) \ll n \log n$ as $n \to \infty$*

As in the nature of $d(n)$, the magnitude of $\sigma(n)$ fluctuates a great deal as $n \to \infty$. So it is fruitful to study its average. The following result deals the average behavior of $\sigma(n)$

Theorem 3.0.17. *As $x \to \infty$, we have*

$$\sum_{n \le x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log x)$$

Proof. As $x \to \infty$, we have

$$\begin{aligned}
\sum_{n \le x} \sigma(n) &= \sum_{n \le x} \sum_{m|n} \frac{n}{m} = \sum_{m \le x} \sum_{\substack{n \le x \\ m|n}} \frac{n}{m} \\
&= \sum_{m \le x} \sum_{r \le \frac{x}{m}} r = \sum_{m \le x} \frac{1}{2} [\frac{x}{m}](1 + [\frac{x}{m}]) \\
&= \frac{1}{2} \sum_{m \le x} (\frac{x}{m} + O(1))^2 = \frac{x^2}{2} \sum_{m \le x} \frac{1}{m^2} + O(x \sum_{m \le x} \frac{1}{m}) + O(\sum_{m \le x} 1) \\
&= \frac{x^2}{2} \sum_{m=1}^{\infty} \frac{1}{m^2} + O(x^2 \sum_{m > x} \frac{1}{m^2}) + O(x \log x) \\
&= \frac{\pi^2}{12} x^2 + O(x \log x)
\end{aligned}$$

This completes the proof. □

Definition 3.0.3. The Mobious function $\mu : \mathbb{N} \to \mathbb{C}$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } \ n = 1 \\ (-1)^r & \text{if } \ n = p_1 ... p_r \quad \text{a product of distinct primes,} \\ 0 & \text{otherwise} \end{cases}$$

Remark: A natural number which is not divisible by square of any prime number is known as square free number. Thus by definition of Mobious function $n \in \mathbb{N}$ is square free iff $\mu(n) = \pm 1$. By definition, it is not difficult task task to show that the Mobious function is multiplicative. Although the Moubious function does not fluctuate to $n \in \mathbb{N}$ as $n \to \infty$, its average is more stable which is shown in the following theorem.

Theorem 3.0.18. *Suppose that* $n \in \mathbb{N}$. *Then*

$$\sum_{m/n} \mu(n) = \begin{cases} 1 & \textit{if } \ n = 1 \\ 0 & \textit{if } \ n > 1 \end{cases}$$

Proof. Consider the function $f : \mathbb{N} \to \mathbb{C}$ defined by writing

$$f(n) = \sum_{m \mid n} \mu(n)$$

for every $n \in \mathbb{N}$.Since μ is multiplicative and by theorem 2.1 f is multiplicative. For $n = 1$ the result is trivial. To complete the theorem it is suffices to show that $f(p^k) = 0$ for every prime p and $k \in \mathbb{N}$. For

$$f(p^k) = \sum_{m \mid p^k} \mu(m) = \mu(1) + \mu(p) + \mu(p^2) + ... + \mu(p^k) = 1 - 1 + 0 + 0 + ... + 0 = 0$$

This completes the proof. □

This theorem is useful to prove the following two theorems.

Theorem 3.0.19. ***Mobious Inversion Formula***
For any function $f : \mathbb{N} \to \mathbb{C}$,*if the function* $g : \mathbb{N} \to \mathbb{C}$ *is defined by writing*

$$g(n) = \sum_{m \mid n} f(m)$$

for every $n \in \mathbb{N}$,*we have*

$$f(n) = \sum_{m \mid n} \mu(m) g(\frac{n}{m}) = \sum_{m \mid n} \mu(\frac{n}{m}) g(m).$$

Proof. The second equality is obvious. Also

$$\sum_{m \mid n} \mu(m) g(\frac{n}{m}) = \sum_{m \mid n} \mu(m) \left(\sum_{k \mid \frac{n}{m}} f(k) \right) = \sum_{\substack{k,m \\ km \mid n}} \mu(m) f(k) = \sum_{k \mid n} f(k) \left(\sum_{m \mid \frac{n}{k}} /mu(m) \right) = f(n)$$

This completes the proof. $\square$

Theorem 3.0.20. *For any function $g : \mathbb{N} \to \mathbb{C}$, if the function $f : \mathbb{N} \to \mathbb{C}$ is defined by writing*

$$f(n) = \sum_{m \mid n} \mu(\frac{n}{m}) g(m)$$

for every $n \in \mathbb{N}$,we have

$$g(n) = \sum_{m \mid n} f(m) = \sum_{m \mid n} f(\frac{n}{m})$$

Definition 3.0.4. If $n \geq 1$, the Euler (some times called Euler totient function) is defined to be the number of positive integers not exceeding n which are relatively prime to n; thus,

$$\phi(n) = \sum_{k=1}^{n} 1$$

Theorem 3.0.21. *For every number $n \in \mathbb{N}$, we have*

$$\sum_{m \mid n} \phi(m) = n$$

Applying the Mobious inversion formula to the above theorem (3.0.21), we get the following result.

Theorem 3.0.22. *For every $n \in \mathbb{N}$, we have*

$$\phi(n) = \sum_{m \mid n} \mu(m) \frac{n}{m} = n \sum_{m \mid n} \frac{\mu(m)}{m}$$

Theorem 3.0.23. *The Euler function $\phi : \mathbb{N} \to \mathbb{C}$ is multiplicative.*

Theorem 3.0.24. *For every $n \in \mathbb{N}$, we have*

$$\frac{1}{2} < \frac{\sigma(n)\phi(n)}{n^2} \leq 1$$

Theorem 3.0.25 (MERTENS). *As $x \to \infty$, we have*

$$\sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$$

Proof. As $x \to \infty$, we have, by Theorem (3.0.22), that

$$\begin{aligned}
\sum_{n \leq x} \phi(n) &= \sum_{n \leq x} \sum_{m \mid n} \mu(m) \frac{n}{m} = \sum_{m \leq x} \mu(m) \sum_{\substack{n \leq x \\ m \mid n}} \frac{n}{m} = \sum_{m \leq x} \mu(m) \sum_{r \leq \frac{x}{m}} r \\
&= \sum_{m \leq x} \mu(m) \frac{1}{2} \left[\frac{x}{m}\right] (1 + \left[\frac{x}{m}\right]) = \frac{1}{2} \sum_{m \leq x} \mu(m) (\frac{x}{m} + O(1))^2 \\
&= \frac{x^2}{2} \sum_{m \leq x} \frac{\mu(m)}{m^2} + O(x^2 \sum_{m \leq x} \frac{1}{m}) + O(\sum_{m \leq x} 1) \\
&= \frac{x^2}{2} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} + O(x^2 \sum_{m > x} \frac{1}{m^2}) + O(x \log x) \\
&= \frac{x^2}{2} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} + O(x \log x)
\end{aligned}$$

But $\sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = \frac{6}{\pi^2}$. Hence the result follows. □

Definition 3.0.5. Given arithmetic functions $f, g \in A$, we define the function $f, g : \mathbb{N} \to \mathbb{C}$ by writing

$$(f * g)(n) = \sum_{m \mid n} f(m) g(\frac{n}{m})$$

for every $n \in \mathbb{N}$. This function is called the Dirichlet Convolution(Dirichlet multiplication) of f and g.

The next theorem describes the algebraic properties of Dirichlet multiplication.

Theorem 3.0.26. *Dirichlet multiplication is commutative and associative. That is for any arithmetical functions f, g, k we have*

$$f * g = g * f \quad \textit{(Commutative Law)}$$

$$(f * g) * k = f * (g * k) \quad \textit{(Associativity)}$$

Definition 3.0.6. : For every integer $n \geq 1$ we define Mangoldt Function as

$$\Lambda(n) = \begin{cases} \log p & \text{if } \ n = p^m \text{ for some prime p and some } m \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Here is a short Table of values of $\Lambda(n)$:

n :	1	2	3	4	5	6	7	8	9	10
$\Lambda(n)$:	0	$\log 2$	$\log 3$	$\log 2$	$\log 5$	0	$\log 7$	$\log 2$	$\log 3$	0

Theorem 3.0.27. *If $n \geq 1$ we have*

$$\log n = \sum_{d \mid n} \Lambda(d)$$

Proof. The theorem is true if $n = 1$ since both numbers are 0. Therefore, assume that $n > 1$ and write

$$n = \prod_{k=1}^{r} {p_k}^{a_k}$$

Taking logarithms we have

$$\log n = \sum_{k=1}^{r} a_k \log p_k$$

Now consider the sum on the right side of the equality in theorem. The only non zero terms in the sum come from those divisors d of the form ${p_k}^m$ for $m = 1, 2, 3, ..., a_k$ and $k = 1, 2, 3, ..., r$. Hence

$$\sum_{d \mid n} \Lambda(d) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \Lambda({p_k}^m) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \log p_k = \sum_{k=1}^{r} a_k \log p_k = \log n$$

This completes the proof. $\square$

Chapter 4

Distribution of Prime Numbers

In this chapter we will discuss some asymptotic results about the distribution of prime numbers.

4.1 Approximation of Primes

The prime numbers are distributed randomly over the set of positive integers. So it is not still possible to predict the number of primes in certain interval, we called this as prime counting function, which is formally defined as

$$\pi(x) = \sharp\{p : p \text{ is a prime } \leq x\}$$

However, the behavior of $\pi(x)$ as a function of x has been the object of intense study by many celebrated mathematicians ever since the eighteenth century, there have been attempts to search their patterns, actually the explicit formula to describe their distribution. We have a famous theorem "Prime number theorem "

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1$$

posed on around 1792 and after around 100 years its first proof came to the world of mathematics. But the proof is analytical use of complex analysis of Riemann Zeta function, further its elementary proof was came at 1949. But these both proof are not easy and simple, takes several arguments and followed by sub results. The proof of this "crowing achievement" of analytical number theory is beyond the scope of this thesis. By the Prime number theorem posed by Legendre and Gauss independently, we have two approximations of prime numbers as

$$\pi(x) \sim \frac{x}{\log x} \approx Li(x)$$

Where

$$Li(x) = \int_2^x \frac{dt}{\log t}$$

The following table shows the approximation of prime numbers .

x	$\pi(x)$	Li(x)	$\frac{Li(x)}{\pi(x)}$	$\frac{x}{\log x}$	$\frac{\pi(x)}{x/\log(x)}$
10	4	5.1204	1.28	4.3429	0.9210
10^2	25	29.0809	1.1632	21.7147	1.1552
10^3	168	176.56	1.0509	144.7682	1.1319
10^4	1229	1245.092	1.012	1085.7362	1.1043
10^5	9592	9628.25	1.0038	8685.8896	1.0844
10^6	78498	78626.56	1.0016	72382.4136	1.0711
10^7	664579	664917.359	1.005091	620420.6884	1.0612
10^8	5761455	5762208.330	1.00013	5428681.0237	1.0572
10^9	50847534	50849233.91	1.0000033	48254942.43	1.0477

Table 4.1: Approximation of Primes

Here is a graph of x vs $\pi(x)$, $\text{Li}(x)$, $\dfrac{x}{\log x}$

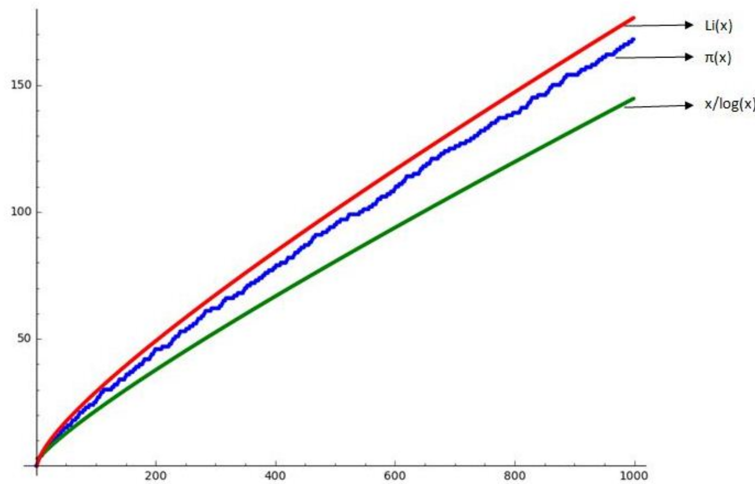

Figure 4.1: Graph of distribution of primes

Thus by observing the table and the graph shown in above we can conclude that $Li(x)$ is better approximation than $\dfrac{x}{\log x}$.

4.2 Some Analytical Results About Distribution on Primes

In many cases the value of a partial sum can be obtained by comparing it with an integral. Euler summation formula gives an exact expression for the error made in such approximation.

Theorem 4.2.1. *(Euler Summation Formula)[22]*
If f has continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt + f(x)([x]-x) - f(y)([y]-y)$$

The next theorem gives a number of asymptotic formulation which are the consequences of Euler summation formula. Here we use the Riemann Zeta function $\zeta(s)$ defined by the equation

$$\zeta(s) = \sum \frac{1}{n^s} \quad \text{if } \ s > 1$$

and by equation

$$\zeta(s) = \lim_{n\to\infty} \left(\sum_{n\le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right) \quad \text{if} \quad 0 < s < 1$$

Theorem 4.2.2. *[22] If $x \geq 1$, we have*

(a) $\sum\limits_{n\le x} \frac{1}{n} = \log x + C + O(\frac{1}{x})$

(b) $\sum\limits_{n\le x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$ *if* $s > 0, s \neq 0$

(c) $\sum\limits_{n> x} \frac{1}{n^s} = O(x^{1-s})$ *if* $s > 1$

(d) $\sum\limits_{n\le x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1}$

Proof. For (a) we take $f(t) = \frac{1}{t}$ in Euler Summation Formula to obtain

$$\begin{aligned}\sum_{n\le x} \frac{1}{n} &= \int_1^x \frac{dt}{t} - \int_1^x \frac{t-[t]}{t^2} dt + 1 - \frac{x-[x]}{x} \\ &= \log x - \int_1^x \frac{t-[t]}{t^2} dt + 1 + O(\frac{1}{x}) \\ &= \log x + 1 - \int_1^\infty \frac{t-[t]}{t^2} dt + \int_x^\infty \frac{t-[t]}{t^2} dt + O(\frac{1}{x})\end{aligned}$$

The improper integral $\int\limits_1^\infty (t-[t])t^{-2} dt$ exists, since it is dominated by $\int\limits_1^\infty t^{-2} dt$. Also

$$0 \geq \int_x^\infty \frac{t-[t]}{t^2} dt \geq \int_x^\infty \frac{1}{t^2} dt = \frac{1}{x}$$

so the last equation becomes

$$\sum_{n\le x}\frac{1}{n} = \log x + 1 - \int_1^\infty \frac{t-[t]}{t^2}dt + O(\frac{1}{x})$$

This proves (a) with $1 - \int_1^\infty \frac{t-[t]}{t^2}dt$

Letting $x \to \infty$ in (a) we find that

$$\lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n} - \log x\right) = 1 - \int_1^\infty \frac{t-[t]}{t^2}dt$$

$\square$

For the proof of remaining see [22].

Now we obtain some elementary identities involving $\mu(n)$ and $\Lambda(n)$ which will be used to studying the distribution of primes. These will be derived from the following partial sum of a Dirichlet Product $f * g$.

Theorem 4.2.3. *[27] If $h = f * g$, let*

$$H(x) = \sum_{n\le x} h(n), \quad F(x) = \sum_{n\le x} f(n), \quad \textit{and} \quad G(x) = \sum_{n\le x} g(n)$$

then we have

$$H(x) = \sum_{n\le x} f(n)G(\frac{x}{n}) = \sum_{n\le x} g(n)F(\frac{x}{n})$$

If $g(n) = 1$ for all n then $G(x) = [x]$, and the above theorem gives us the following corollary:

Theorem 4.2.4. *If $F(x) = \sum\limits_{n\le x} f(n)$ we have*

$$\sum_{n\le x}\sum_{d/n} f(d) = \sum_{n\le x} f(n)[\frac{x}{n}] = \sum_{n\le x} F(\frac{x}{n})$$

Now we take $f(n) = \mu(n)$ and $\Lambda(n)$ in theorem(4.2.4) to obtain the following identities which will be used in later in studying the distribution of primes.

Theorem 4.2.5. *[27] For $x \geq 1$ we have*

$$\sum_{n\leq x} \mu(n) \left[\frac{x}{n}\right] = 1$$

and

$$\sum_{n\leq x} \Lambda(n) \left[\frac{x}{n}\right] = \log[x]!$$

Proof. From theorem(3.0.18), we have

$$\sum_{n\leq x} \mu(n) \left[\frac{x}{n}\right] = \sum_{n\leq x} \sum_{d/n} \mu(d) = \sum_{n\leq x} \left[\frac{1}{n}\right] = 1$$

and

$$\sum_{n\leq x} \Lambda(n) \left[\frac{x}{n}\right] = \sum_{n\leq x} \sum_{d/n} \Lambda(n) = \log[x]!$$

□

Next we use Euler's summation formula to determine an asymptotic formula for $\log[x]!$.

Theorem 4.2.6.

$$\log[x]! = x\log x - x + O(\log x),$$

and hence

$$\sum_{n\leq x} \Lambda(n) \left[\frac{x}{n}\right] = x\log x - x + O(\log x),$$

Proof. Taking $f(t) = \log t$ in Euler's Summation Formula, we obtain

$$\sum_{n\leq x} \log n = \int_1^x \log t dt + \int_1^x \frac{t-[t]}{t} dt - (x-[x])\log x$$

$$= x\log x - x + 1 + \int_1^x \frac{t-[t]}{t} dt + O(\log x)$$

Since,

$$\int_{1}^{x} \frac{t-[t]}{t} dt = O\left(\int_{1}^{x} \frac{1}{t} dt\right) = O(\log x)$$

This proves the first part of the theorem. The second part of the theorem follows from the second part of the previous theorem.

□

Theorem 4.2.7. *[27] For $x \geq 2$, we have*

$$\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x)$$

Having the sum is extended over all primes $\leq x$

Proof. Since $\Lambda(n) = 0$ unless n is a prime power, so we have

$$\sum_{p \leq x} \left[\frac{x}{n}\right] \Lambda(n) = \sum_{p} \sum_{\substack{m=1^{\infty} \\ p^m \leq x}} \left[\frac{x}{p^m}\right] \Lambda(p^m)$$

Now, $p^m \leq x$ implies $p \leq x$. Also $[\frac{x}{p^m}] = 0$ if $p > 0$, so we can write the last sum as

$$\sum_{p \leq x} \sum_{m=1}^{\infty} \left[\frac{x}{p^m}\right] \log p = \sum_{p \leq x} \left[\frac{x}{p^m}\right] \log p + \sum_{p \leq x} \sum_{m=2}^{\infty} \left[\frac{x}{p^m}\right] \log p$$

Next we prove the last sum is $O(x)$, We have

$$\begin{aligned}
\sum_{p \leq x} \log p \sum_{m=2}^{\infty} \left[\frac{x}{p^m}\right] &\leq \sum_{p \leq x} \sum_{m=2}^{\infty} \frac{x}{p^m} \log p = x \sum_{p \leq x} \log p \sum_{m=2}^{\infty} (\frac{1}{p})^m \\
&= x \sum_{p \leq x} \log p \frac{1}{p^2} \frac{1}{1-\frac{1}{p}} = x \sum_{p \leq x} \frac{\log p}{p(p-1)} \\
&\leq \sum_{m=2}^{\infty} \frac{\log n}{n(n-1)} = O(x)
\end{aligned}$$

Hence we have shown that

$$\sum_{p \leq x} \left[\frac{x}{n}\right] \Lambda(n) = \sum_{p \leq x} \left[\frac{x}{p}\right] \log p + O(x)$$

Hence by using above theorem we get

$$\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x)$$

This completes the proof. $\square$

Shapiro's Tauberian Theorem[22]

Theorems relating different weighted averages of the same function are called Tauberian theorems. We discuss next a Tauberian theorem proved in 1950 by H. N. Shapiro. It relates sums of the form $\sum_{n \leq x} a(n)$ with those of the form $\sum_{n \leq x} a(n)[x/n]$ for non negative $a(n)$.

Theorem 4.2.8. *Let $\{a(n)\}$ be a non negative sequence such that*

$$\sum_{n \leq x} a(n)[\frac{x}{n}] = x \log x + O(x) \quad \textit{for all} \quad x \geq 1.$$

Then

(a) For $x \geq 1$ we have

$$\sum_{n \leq x} a(n) = \log x + O(1)$$

(b) There is a constant $B > 0$ such that

$$\sum_{n \leq x} a(n) \leq Bx \quad \textit{for all} x \geq 1$$

(c) There is a constant $A > 0$ and an $x_0 > 0$ such that

$$\sum_{n \leq x} a(n) \geq Ax \quad \textit{for all} \quad x \geq x_0$$

Now we apply the Shapiro's Theorem to prove the following Mertens's Theorem.

Theorem 4.2.9. *As $x \to \infty$ we have*

$$(a) \qquad \sum_{m \leq x} \frac{\Lambda(m)}{m} = \log x + O(x)$$

$$(b) \qquad \sum_{p\le x} \frac{\log p}{p} = \log x + O(1)$$

$$(c) \qquad \sum_{p\le x} \frac{1}{p} = \log\log x + O(1)$$

Proof. (a)

We have from Theorem(4.2.6)

$$\sum_{n\le x} \Lambda(n)\left[\frac{x}{n}\right] = x\log x - x + O(\log x)$$

This can be written as

$$\sum_{n\le x} \Lambda(n)\left[\frac{x}{n}\right] = x\log x + O(x)$$

Since $\Lambda(n) \ge 0$ we can apply the Shapiro's Theorem with $a(n) = \Lambda(n)$, we get the required result.

(b)

We have from Theorem (0) For $x \ge 2$

$$\sum_{p\le x} \left[\frac{x}{p}\right] \log p = x\log x + O(x)$$

This can be written as

$$\sum_{n\le x} \Lambda_1(n)\left[\frac{x}{n}\right] = x\log x + O(x)$$

Where Λ_1 is a function defined as follows:

$$\Lambda_1(n) = \begin{cases} \log p & \text{if } n \text{ is a prime } p \\ 0 & \text{if otherwise} \end{cases}$$

Since $\Lambda_1(n) \ge 0$, so we can use the Shapiro's Theorem with $a(n) = \Lambda_1(n)$, we get the required result.

(c)

Finally for every real number $x \geq 2$,let

$$T(x) = \sum_{p \leq x} \frac{\log p}{p}$$

Then it follows from (b) that there exists a positive absolute constant c such that $|T(x) - \log x| < c$ whenever $x \geq 2$. On the other hand,

$$\sum_{p \leq x} \frac{1}{p} = \sum_{p \leq x} \frac{\log p}{p} \left(\frac{1}{\log x} + \int_{p}^{x} \frac{dy}{y \log^2 y} \right) = \frac{T(x)}{\log x} + \int_{2}^{x} \frac{T(y)dy}{y \log^2 y}$$
$$= \frac{T(x) - \log x}{\log x} + \int_{2}^{x} \frac{T(y) - \log y}{y \log^2 y} + 1 + \int_{2}^{x} \frac{dy}{y \log y}$$

It follows that as $x \to \infty$, we have

$$\left| \sum_{p \leq x} \frac{1}{p} - \log \log x \right| < \frac{c_6}{\log x} + \int_{2}^{x} \frac{c_6 dy}{y \log^2 y} + 1 - \log \log 2 = O(1)$$

This inequality follows the (c) This completes the proof $\square$

4.2.1 Chebychev's type estimates

Getting upper and lower bounds for the prime counting function $\pi(x)$ is surprisingly difficult. Euclid's result that there are infinitely many primes shows that $\pi(x)$ tends to infinity, but the standard proofs of the infinitude of prime are indirect and do not give an explicit lower bound for $\pi(x)$, or give only a very weak bound. For example, Euclid's argument shows that the nth prime p_n satisfies the bound $p_n \leq p_1.p_2....p_{n+1} + 1$. By induction, this implies that $p_n \leq e^{e^{n-1}}$ for all n, from which one can deduce the bound $\pi(x) \geq \log \log x$ for sufficiently large x. This bound is far from the true order of $\pi(x)$, but it is essentially the best one can derive from Euclid's argument.

Euler's proof of the infinitude of primes proceeds by showing that $\sum_{n \leq x} \frac{1}{p} \leq \log \log x - c$ for some constant c and sufficiently large x. Although this gives the correct order for the partials sum of the reciprocals of primes (as we will see below, the estimate is accurate to within an error $O(1)$, one cannot deduce from this a lower bound for $\pi(x)$ of comparable quality. In fact, one can show that the most one can deduce from the above bound for $\sum_{n\le x}\frac{1}{p}$ is a lower bound of the form $\pi(x) \gg \log x$. While this is better than the bound obtained from Euclid's argument, it is still far from the true order of magnitude [23].

In the other direction, getting non-trivial upper bounds for $\pi(x)$ is not easy either. in the middle of the 19th century, the Russian mathematician P. L. Chebychev was able to determine the precise order of magnitude of the prime counting function $\pi(x)$, by showing that there exist positive constants C_1 and C_2 such that

$$C_1\frac{x}{\log x} \le \pi(x) \le C_2\frac{x}{\log x}$$

for all sufficiently large x. In fact, Chebychev proved such an inequality with constants $C_1 = 0.92...$, and $C_2 = 1.10....$ This enabled him to conclude that, for sufficiently large x (and, in fact, for all $x \ge 1$) there exists a prime p with $x < p \le 2x$, an assertion known as **Bertrand's postulate**[1] .
Here we show the Chebychev's Theorem from [10]. For this, first we establish a useful result for Chebychev theorem and then we state and prove the Chebychev theorem.

Theorem 4.2.10. *There exist positive absolute constants c_3 and c_4 such that*

$$\sum_{m\le x}\Lambda(m) \ge \frac{1}{2}x\log 2 \quad \textit{if} \quad x \ge c_3$$

and

$$\sum_{\frac{x}{2}<m\le x}\Lambda(m) \le c_4 x \quad \textit{if} \quad x \ge 0$$

Proof. If $m \in \mathbb{N}$ satisfies $\frac{x}{2} < m \le x$, then clearly $[\frac{x}{2m}] = 0$. It follows from this and 2nd part of Theorem 4.2.6, that as $x \to \infty$, we have

$$\sum_{m\le x}\Lambda(m)\left([\frac{x}{m}]-2[\frac{x}{2m}]\right)=\sum_{m\le x}\Lambda(m)[\frac{x}{m}]-2\sum_{m\le x/2}\Lambda(m)[\frac{x}{2m}]$$
$$=(x\log x-x+O(\log x))-2\left(\frac{x}{2}\log(\frac{x}{2})-\frac{x}{2}+O(\log x)\right)$$
$$=x\log 2+O(\log x)$$

Now, we consider a function $[\alpha]-2[\frac{\alpha}{2}]$. Clearly $[\alpha]-2[\frac{\alpha}{2}]<\alpha-2(\frac{\alpha}{2}-1)=2$. Note that the left hand side is an integer, so we most have $[\alpha]-2[\frac{\alpha}{2}]\ge 1$. It follows for all sufficiently large x,we have

$$\frac{1}{2}x\log 2<\sum_{m\le x}\Lambda(m)$$

Which proves the first part of the theorem.

On the other hand, if $\frac{x}{2}<m\le x$, then $[\frac{x}{m}]=1$ and $[\frac{x}{2m}]=0$, so that for all sufficiently large x, we have

$$\sum_{\frac{x}{2}<m\le\frac{x}{2}}\Lambda(m)\le c_5x$$

Hence the 2nd part of the theorem follows. □

Now we state and prove the Chebychev Theorem.

Theorem 4.2.11. *(Chebychev) There exist a positive absolute constants c_1 and c_2 such that for every real number $x\ge 2$ we have*

$$C_1\frac{x}{\log x}\le\pi(x)\le C_2\frac{x}{\log x}$$

Proof. To prove the lower bound, note that

$$\sum_{m\le x}\Lambda(m)=\sum_{\substack{p,n\\ p^n\le x}}\log p=\sum_{p\le x}(\log p)\sum_{1\le n\le[\frac{\log x}{\log p}x]}1=\sum_{p\le x(\log p)\left[\frac{\log x}{\log p}\right]}\le\pi(x)\log x$$

Then first part of the theorem 4.2.10, we have

$$\pi(x) \geq \frac{x \log 2}{2 \log x} \quad \text{if} \quad x \geq c_3$$

Since $\pi_2(x) = 1$, we get the lower bound for a suitable choice of c_1

To prove the upper bound note that in view of the second part of the Theorem ( ) and the definition of the von Mangoldt function, the inequality

$$\sum_{\frac{x}{2^{j+1}} < p < \frac{x}{2^j}} \log p \leq c_4 \frac{x}{2^j}$$

holds for every integer $j \geq 0$ and every real number $x \geq 0$. Suppose that $x \geq 2$. Let the integer $k \geq 0$ be defined such that $2^k < x^{\frac{1}{2}} \leq 2^{k+1}$. Then

$$\sum_{x^{\frac{1}{2}} < p \leq x} \log p \leq \sum_{j=0}^{\infty} \sum_{\frac{x}{2^{j+1}} < p < \frac{x}{2^j}} \log p \leq c_4 x \sum_{j=0}^{k} 2^{-j} < 2c_4 x$$

so that

$$\sum_{x^{\frac{1}{2}} < p \leq x} 1 \leq \sum_{x^{\frac{1}{2}} < p \leq x} \frac{\log p}{\log x^{\frac{1}{2}}} < \frac{4c_4 x}{\log x}$$

whence

$$\pi(x) \leq x^{\frac{1}{2}} + \frac{4c_4 x}{\log x} < \frac{C_2 x}{\log x}$$

for suitable C_2

Chapter 5

Twin Prime Numbers

5.1 Introduction:

One charm of the study of the integers is that they can be easily stated, which often sound simple, are often very difficult and some times even hopeless given the state of our knowledge. In 1912, at the lecture on International Mathematical congress, Edmund Landaur mentioned mentioned four old conjectures that are appeared hopeless at that time.
1. Goldbach's Conjecture: Every positive even integer is a sum of two primes.
2. Twin Prime Conjecture: 2 can be written as a difference of two primes in infinitely many ways.
3. Legendre's Conjecture: There is always a prime between n^2 and $(n+1)^2$.
4. There are infinitely many primes of the form $n^2 + 1$.

During the past 90 years, much intensive research has been conducted on all of these conjectures and due to of which , now we have many results concerning to these conjectures, but unfortunately, the current methods still cannot prove the above conjectures and these conjectures can still be called hopeless [19].

The first three of above conjectures are related, they concern the primes in some intervals. Here we will concentrate on the second one i.e

Conjecture 5.1.1. ***Twin Prime Conjecture:***. *There are infinitely many twin primes, i.e., numbers p and $p+2$, such that both of them are primes.*

The first pairs of twin primes are: $(3,5),(5,7),(11,13),(17,19),...$ with 5 being the only prime being in two pairs. Let's define the Twin Prime Counting Function as $\pi_2(x)$ a number of primes p, not bigger than x such that $p+2$ is also a prime, i.e.

$$\pi_2(x) = \sharp\{p : p,\ p+2 \text{ both are prime and } p \le x\}$$

We therefore have:

$$\pi_2(10) = 2$$
$$\pi_2(11) = 3$$
$$\pi_2(17) = 4$$
$$.........$$

5.2 Heuristic Approach to Twin Prime Numbers

To estimate the number of twin primes up to a natural number x one can use the distribution of the prime numbers and the prime number theory. This states that a number smaller than x has at least probability $\frac{1}{\log x}$ of being a prime number. This means if we pick two numbers smaller than x the probability of both of them being a prime numbers is at least $\frac{1}{(\log x)^2}$ but only when the event of "p is prime" is independent of the event "$p+2$ is prime". This isn't true if $p \equiv 1 \pmod 3$ and prime then $p+2 \equiv 0(\bmod 3)$ thus $p+2$ isn't prime.

One needs to correct for this dependence by some correction factor. The probability for an arbitrary number to be divisible by a number q is $\frac{1}{q}$. So the probability for two arbitrary numbers not to be divisible by a number q is $(1-\frac{1}{q})^2$. For two numbers p and

$p+2$ this is different because we $p \not\equiv 0(\mod q)$ and $p \not\equiv -2(\text{mod } q)$ which is in $\frac{2}{q}$ of the cases. The ratio between these factors is the correction factor. Thus the correction factor for any number $q > 2$ becomes:

$$\frac{1-\frac{2}{q}}{(1-\frac{1}{q})^2}$$

For $q = 2$ one has 1 modulo class which is restricted for p. This correction factor becomes:

$$\frac{1-\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$$

Being divisible by a small prime number is independent of the other small primes. Thus one may multiply the correction factors of the small prime numbers. In fact one may multiply over all prime numbers because when q is large the correction factor converges to 1. This suggests a definition of a twin prime constant of:

$$C_2 = 2 \prod_{\substack{q \text{ is a prime} \\ q \geq 3}} \frac{1-\frac{2}{q}}{(1-\frac{1}{q})^2} \approx 1.3203236316$$

This is the total correction factor over all primes q. One may guess the estimate of the number of twin prime pairs smaller than an integer x is:

$$C_2 \frac{x}{\log^2 x}$$

This would mean by a heuristic approach there would be infinitely many pairs of primes which differ 2. Because the formula for the estimate of the number of pairs under x goes to infinity when x goes to infinity.

Where as there is no proof that there are infinitely many twin primes,the empirical data on the function π_2 strongly suggests that this is indeed true. By the heuristic approach, Hardy and Littlewood (1923) conjectured that [8]

$$\pi_2(x) \sim 2 \prod_{p>2}^{\infty} (1 - \frac{1}{(1-p)^2}) \frac{x}{\log^2 x}$$

This fact would imply the infinitude of twin primes and also a simple asymptotic formula for $\pi_2(x)$, namely

The asymptotic representations by Hardy and Littlewood of the function $\pi_2(x)$, having great theoretical importance, do not possess sufficient simplicity for their practical use.

In 1986 Yaminov [3] formulated a new conjecture on the distribution law of twin primes, based on the superposition of the function $\pi(x)$ with itself. On the basis of Chebyshev's theorem on the asymptotic distribution law of prime numbers, he constructed lower and upper estimates of the numbers of twin primes, Under the assumption of the validity of the introduced conjecture, he also gave a simple proof of the fact that the number of pairs of twin primes is infinite (the problem of twin primes).

Conjecture 5.2.1. ***(Yaminov Conjecture:)***
The number of pairs af twin primes in the interval $[2, x]$ is approximately equal to the number of prime indices i in the set of prime numbers $p_i \in [2, x]$, i.e. the function $\pi_2(x)$ has the form

$$\pi_2(x) \approx \pi(\pi(x))$$

In other words, the twin primes are distributed among the primes in the same way as the primes among the natural numbers.

The following table shows the approximation of Twin Prime counting function .

x	$\pi_2(x)$	LW	$\pi(\pi(x))$	$\frac{\pi_2(x)}{\text{LW}}$	$\frac{\pi_2(x)}{\pi(\pi(x))}$
10	2	2.5786	2	0.7755	1
10^2	8	6.2371	9	1.2826	0.8888
10^3	35	27.6733	39	1.2647	0.8974
10^4	205	155.6443	201	1.31710	1.0337
10^5	1224	996.1147	1148	1.22877	1.0606
10^6	8169	6917.7582	7702	1.18092	1.0940

Table 5.1: Approximation of Twin Primes

Here is a graph of x vs. $\pi_2(x)$, LW(x) (i.e.Hardy Littlewood Approximation), $YA(x)$ (i.e. Yaminov's Approximation)

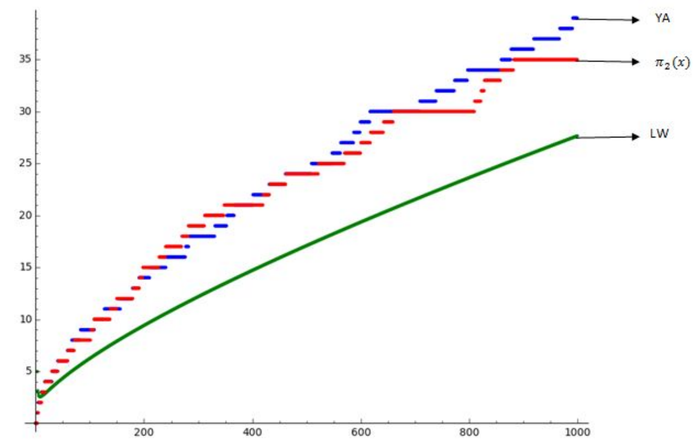

Figure 5.1: Graph of distribution of Twin primes

In the same paper [3],Yaminov constructed another empirical distribution function $\pi_2^*(x)$ for the Twin Primes lying in the interval $[2, x]$ as follows

$$\pi_2^*(x) = \frac{h_c \pi^2(x)}{x}$$

Where $h_c = 1.325067....$ is the ratio between the $\eta_{pp} = \dfrac{\pi_2(x)}{\pi(x)}$ and $\eta_p(x) = \dfrac{\pi(x)}{x}$ for $x \leq 10^6$. The right hand side of the equality is rounded off to integers.

The following table give the data regarding the function $\pi_2^*(x)$ and the degree of accuracy for $x \in [50, 1,000,000]$.However it can be applied also for $x > 1,000,000$. For example for $x = 37 \times 10^6$,there are $183,728$ twin primes. According to the above formula $\pi_2^*(x) = 183,463$, giving a relative error of $\delta = 0.0014$.

x	h	$\pi_2(x)$	$\pi_2^*(x)$	$\|\Delta\|$	$\frac{\Delta}{\pi_2(x)}$
50	1.333333	6	6	0	0
150	1.346938	11	11	0	0
500	1.329639	24	24	0	0
1500	1.3286742	49	50	1	0.0204
2000	1.307061	60	61	1	0.0167
3000	1.314223	81	82	1	0.0123
4000	1.348760	102	100	2	0.0196
5000	1.374114	123	119	4	0.0325
10000	1.330737	201	200	1	0.0050
15000	1.306672	268	274	4	0.0149
20000	1.321178	338	339	1	0.0030
25000	1.320680	403	404	1	0.0025
30000	1.316236	462	465	3	0.0065
40000	1.324637	585	585	0	0
50000	1.322696	697	698	1	0.0014
100000	1.330341	1224	1219	5	0.0041
200000	1.335088	2159	2143	16	0.0074
500000	1.302302	4494	4573	79	0.0176
1000000	1.342908	8164	8165	1	0.0001

Table 5.2: Approximation of Twin Primes by empirical distribution function π_2^*

Thus by this table it is shown that the function $\pi_2^*(x)$ counts the Twin Primes more accurately than other functions discussed above.

5.3 Results towards the twin prime conjecture

Theorem 5.3.1. ***Euclid(300 BC)****:There are infinitely many primes.[11]*

Theorem 5.3.2 (Polignac's Conjecture(1849):)**.** *[15] Let k be any positive even integer and let p_n be the nth prime number. Then, for infinitely many $n \in N$, we have $p_{n+1} - p_n = k$.*

Any k which satisfies Polignac's conjecture is called a Polignac number and the twin prime conjecture simply states that 2 is a Polignac number.

Theorem 5.3.3 (Burn (1916))**.** *[25]*

$$\sum_{p,p+2prime} \frac{1}{p} < \infty$$

By contrast, the sum of the reciprocals of the primes diverges and so this result shows that if there are infinitely many Twin Primes they have very large gaps
till around 1920 when Viggo Brun showed the following theorems [21]

Theorem 5.3.4. *Every sufficiently large even integer can be represented as a sum of two numbers each of which has at most nine prime factors.*

Theorem 5.3.5. *If n is large enough, then the interval $(n, n+\sqrt{n})$ contains a number with at most eleven prime factors.*

Theorem 5.3.6. *There are infinitely many pairs of numbers of difference 2, such that both of them have at most nine prime factors.*

He also showed

Theorem 5.3.7. *For sufficiently large x, the number of prime twins not exceeding x, denoted $\pi_2(x)$, is*

$$\pi_2(x) \leq \frac{100x}{\log^2 x}$$

Quite unjustly, Brun's methods were not recognized immediately. It seems that mathematicians did not believe that such elementary methods (Brun's sieve is basically a combinatorial tool) could be used to approach such difficult conjectures like those given above. There is an anecdote that E. Landau did not read Brun's paper for a decade because of this superstition. This skepticism was partially overcome when in 1933 L. G. Shnirelman proved his weak statement of Goldbach's conjecture [21]:

Theorem 5.3.8. *There exists a positive integer s, such that every sufficiently large integer is the sum of at most s primes.*

Another major milestone was set in 1947 by A. Selberg. Selberg's sieve method is simpler to understand and quite often leads to better results. This again is the example of the upper bound sieve. The methods of Brun and his successors work with numbers smaller than N, which are then sieved using primes not exceeding a certain threshold N^c . If we could set $c = \frac{1}{2}$, then the remaining numbers would be primes, of course, and we could estimate and bound precisely the number of primes in this range. But this is in general beyond the reach. One can see that all theorems of Brun above refer to numbers with a bounded number of prime factors. Some work was done to overcome this limitation. For example P. Kuhn in 1941 realized that one can obtain better bounds for the number of prime factors by "'weighting" the sieve in a certain way, relaxing the restriction. These ideas were used by J. R. Chen who in 1975 established

Theorem 5.3.9. *[12]*

(Chen's Theorem I). If n is large enough, then the interval $(n, n + \sqrt{n}$ contains an integer with at most two prime factors.

He also showed

Theorem 5.3.10. *(Chen's Theorem II). Every sufficiently large even number can be written as the sum of either two primes, or a prime and an integer that is a product of at most 2 primes,*

and

Theorem 5.3.11. *(Chen's Theorem III). There are infinitely many pairs of numbers of difference 2, such that the smaller number in the pair is a prime and the larger is a product of at most two primes.*

These results are proven using basically the same approach, it seems that all these problems are deeply connected.
On the basis of Chebyshev's theorem on the asymptotic distribution law of prime numbers, Yaminov constructed lower and upper estimates of the numbers of twin primes,for this he first proved the following theorem.

Theorem 5.3.12. *[3] If $x \geq 2y, \;\; y \geq 1$, then we have the following inequality.*

$$\pi(\frac{x}{y}) \geq \frac{\pi(x)}{y} \tag{5.1}$$

Proof. We have that

$$\pi(x_1 + x_2 + ... + x_k) \leq \pi(x_1) + \pi(x_2) + ...\pi(x_k)$$

if $x_1 = x_2 = ... = x_k = x$ i.e.

$$\pi(kx) \leq k\pi(x), \;\; k \in \mathbb{N} \tag{5.2}$$

We show the inequality (5.2) holds $\forall \;\; k \geq 1$. We consider the function of the continuous argument $k : z(k) = \frac{k\pi(x)}{\pi(kx)} \sim \frac{Li(x)}{Li(kx)}$. .it is increasing ,since

$$\frac{dz}{dk} = \frac{Li(x)}{Li^2(kx)} \left((Li(kx) - \frac{kx}{logx} \right) > 0$$

and since $z(1) = 1$, then $z(k) \geq 1 \quad \forall k \geq 1$, i.e., (6) holds.
Now we have for $y \geq 1$

$$\pi(x) = \pi(y\frac{x}{y}) \leq y\pi(\frac{x}{y})$$

from where the inequality (5.1) holds. The equality sign in (5) holds for $y = 1$. The theorem is proved. □

An estimate of the function $\pi_2(x)$ is given by the following theorem.

Theorem 5.3.13. *[3] For the values $x \geq 1000$ one has the inequalities*

$$\frac{C_1 x}{\log x(\log x - \log\log x)} < \pi_2(x) < \frac{1.159 C_2 x}{\log x(\log x - \log\log x)}$$

Where $C_1, C_2 (C_1 \leq 1 \leq C_2)$ are Chebyshev's coefficient.

Proof. There exist a constant number $\alpha, 0 < \alpha < 1$, such that one has inequality

$$\frac{\pi(x)}{1+\alpha} < \frac{x}{\log x} < \pi(x)$$

, whence
$$\pi(\frac{\pi(x)}{1+\alpha}) \le \pi(\frac{x}{\log x}) \le \pi(\pi(x))$$
. On the basis of Theorem (5.3.12) (inequality (5.1)) we have
$$\frac{\pi_2(x)}{1+\alpha} \le \pi(\frac{x}{\log x}) \le \pi_2(x) \tag{5.3}$$
By Chebyshev's theorem,
$$C_1\frac{x}{\log x} < \pi(x) < C_2\frac{x}{\log x}$$
Replacing here x by $\frac{x}{\log x}$, we obtain
$$\frac{C_1 x}{\log x(\log x - \log\log x)} < \pi(\frac{x}{\log x}) < \frac{C_2 x}{\log x(\log x - \log\log x)} \tag{5.4}$$
From the left hand inequality in (5.3), taking into account the right-hand inequality in (5.4), we find
$$\frac{\pi_2(x)}{1+\alpha} \le \pi(\frac{x}{\log x} < \frac{C_2 x}{\log x(\log x - \log\log x)}$$
whence
$$\pi_2(x) < \frac{(1+\alpha)C_2 x}{\log x(\log x - \log\log x)}$$
From the right hand equality in (5.3), taking account the left hand inequality in (5.4). we have
$$\frac{C_1 x}{\log x(\log x - \log\log x)} < \pi(\frac{x}{\log x}) < \pi_2(x)$$
i.e.
$$\frac{C_1 x}{\log x(\log x - \log\log x)}) < \pi(\frac{x}{\log x}) < \pi_2(x) < \frac{(1+\alpha)C_2 x}{\log x(\log x - \log\log x)} \tag{5.5}$$
For $x \ge 1000$ we have [9] $\pi(x)/\frac{x}{\log x} < 1.159$ or $\frac{\pi(x)}{1.159} < \frac{x}{\log x}$, i.e. $\alpha = 0.159$, and from the inequalities (5.5) there follows the assertion of the theorem.

$\square$

Yaminov also proof the following results in

Theorem 5.3.14. *Almost all the primes are not Twin primes i.e*

$$\pi_2(x) = o(\pi(x))$$

Corollary: Since $\pi(x) = o(x)$, from above theorem, we can have $\pi_2(x) = o(x)$ Indeed we have

$$\lim_{x\to\infty} \pi_2(x)/x = \lim_{x\to\infty} \pi_2(x)/\pi(x) \lim_{x\to\infty} \pi(x)/x = 0$$

Theorem 5.3.15. *if $y \geq 2, x \geq 4, x \geq y$, then we have the inequality*

$$\pi(x/y) \geq \pi(x) - \pi(y) \tag{5.6}$$

With the validity of conjecture (Yaminov) and with the help of inequality (5.6) Yaminov Proved the infinitude of Twin prime in the following theorem.

Theorem 5.3.16. *For $\pi(x) \geq 3 (x \geq 5)$, in the interval $(\pi(x), 2\pi(x))$ one has at least one pair of Twin primes ,i.e.*

$$\pi(2\pi(x)) - \pi(\pi(x)) \geq 1$$

Proof. Since, on the basis of conjecture (5.2.1), the twin primes are distributed in the set of primes in the same way as the primes in the set $\mathbb{N}$ we can use Theorem 5.3.15 (inequality (5.6)). If we set $v = \pi(x), u = 2\pi(x)$, then $v > 3 > 2, u > 6 > 4, u > v$ and

$$\pi(\frac{u}{v}) \leq \pi(u) - \pi(v)$$

or

$$\pi(2\pi(x)) - \pi(\pi(x)) \geq \pi(\frac{2\pi(x)}{\pi(x)}) = \pi(2) = 1$$

□

Remark: In the same manner as the assumptions of Theorem (5.3.15) hold; therefore,

$$\pi(3\pi(x)) - \pi(\pi(x)) \geq \pi(\frac{3\pi(x)}{\pi(x))} = \pi(3) = 2$$

Similarly,

$$\pi(5\pi(x)) - \pi(\pi(x)) \geq 3$$

i.e., between the numbers $\pi(x)$ and $5\pi(x)$ one has at least three pairs of twin primes etc. In general, if $\pi(m) = k$, then in the interval $(\pi(x); m\pi(x))$ one has at least k pairs of twin primes,

$$\pi(m\pi(x)) - \pi(\pi(x)) \geq \pi(\frac{m\pi(x)}{\pi(x))} = \pi(m) = k$$

We consider now the problem of twin primes. We divide the set of real numbers into intervals, whose lengths are unboundedly doubling:

$$(2^{n-1}\pi(x), 2^n\pi(x)), n = 1, 2, ... \tag{5.7}$$

Each pair of twin primes lies in exactly one interval. By Theorem 5.3.15, in each interval of the form (5.7) one has at least one pair of twin primes. Since the intervals (5.7) form an infinite set, it follows that the number of pairs of twin primes is infinite.

Theorem 5.3.17 (k-Tuples Conjecture:)**.** *[21]*

Let $\mathcal{H} = \{h_1, h_2, ...h_k\}$ an admissible set or an admissible k-Tuples,then there are infinitely many integers n, such that all of $n + h_1, n + h_2, \ldots, n + h_k$ are primes.

We know that $\{0, 2\}$ is admissible set,so with the twin prime conjecture is stated as *There are infinitely many primes p with $p + 2$ be prime.*

If one wants to prove the Twin Prime Conjecture the Prime k-Tuples conjecture has to be proven for $k = 2$

The proof of the Twin Prime Conjecture cannot go ahead without the study of the primes gaps.There are many famous results about the primes gap.

For each natural m ,let H_m denote by the quantity

$$H_m := \lim_{n \to \infty} inf(p_{n+m} - p_n)$$

Where $m \geq 1$ be a fix natural number and p_n denotes the nth prime. In othere words H_m is the least quantity such that there are infinitely many intervals of length H_m that contains the $m+1$ or more primes. Thus for instant the twin prime conjecture is equivalent to the assertion that $H_1 = 2$

Theorem 5.3.18 (Pintz and Yildirim(2009))**.** *[4] If Elliot-Halberstam conjecture is true then*

$$H_1 \leq 16$$

Theorem 5.3.19 (Zhang (2013):)**.** *[29]*

Suppose that $\mathcal{H}$ is admissible with $k \geq 3.5 \times 10^6$ then there are infinitely many positive integers n such that the set $\{n + h_1, n + h_2, \ldots, n + h_k\}$ contains at least two primes. Consequently ,

$$H_1 = \lim_{n \to \infty} inf(p_{n+1} - p_n) < 7 \times 10^7$$

In other words $p_{n+1} - p_n$ is bounded by 7×10^7 for infinitely many n.

As well his paper states that this result is not optimal.....,to replace this upper bound by a value as small as possible is an open problem. By the weak partial version of Elliot-Halberstan conjecture Zhang obtained this result i.e $H_1 < 7 \times 10^7$ Zhang proved this result for a fairly large value of k, that is $k > 3500000$ which has been reduced to $k > 632$ by the polymath8 team.

5.3.1 Polymath8 project and Refinement of Zhang's result

Polymath project is a massively collaborative mathematical project. This is a blog to sharing the new and challenging mathematical problem. From all over the world every mathematician can post their results about the project. The first polymath project was started in 2009 and polymath11 is open now.The main objectives of the polymath8 project initiated by Terry Tao is about the the refinement of the Zhang's result.There are two projects polymath8a and polymath8b.Polymath8a, "Bounded gaps between primes", was a project to improve the bound H_1 on the least gap between consecutive primes that was attained infinitely often, by developing the techniques of Zhang. This project concluded with a bound of $4,680$ of H_1.Polymath8b, *"Bounded intervals with many primes"*,is project to improve the value ofH_1 further, as well as H_m(the least gap between primes with $m-1$ primes between them that is attained infinitely often), by combining the Polymath8a results with the techniques of Maynard. This project concluded with a bound of H_1 is 246, as well as additional bounds on H_m. The following table summaries the current refinements on Zhang's results on assuming Elliott-Halberstam Conjecture(EH), Generalized Elliott- Halberstam Conjecture (GEH) and Deligne Theorem [24].

m	Conjectural	Assuming EH	Without EH	Without EH or Deligne
1	2	6 (on GEH) and 12 (on EH only)	246	246
2	6	252(on GEH) and 270(on EH only)	395106	474266
3	8	52116	24462654	32285928
4	12	474266	1404556152	2031558336
5	16	4137854	78602310160	124840189042
m	$(1+o(1)m\log m)$	$O(me^{2m})$	$O(\exp(3.815))$	$O(m\exp(4-\frac{4}{43})m)$

Table 5.3: Refinement of Zhang's Theorem

In October 2013, a postdoctoral assistant at Montreal, J. Maynard, made a second breakthrough by proving the bound

Theorem 5.3.20 (Maynard's Theorem(2013):)**.** *[13] Unconditionally ,*

$$H_1 \leq 600, H_m \leq Cm^3e^{4m}$$

for an absolute constant C and $m \geq 1$
if one assumes the Elliot-Halberstan conjecture,we have the following bounds $H_1 \leq 12$, $H_2 \leq 600$, $H_m \leq Cm^3e^{2m}$ for absolute constant C and $m \geq 1$.

This striking bound was obtained by a significantly simpler method; this is this method which we are going to expose in this course. Maynard's method allows for much stronger results non accessible to previous techniques: the existence of infinitely many k-tuples of primes clustered in intervals of bounded length.Maynard joined the Polymath8 project and these bounds were further to improved (April 2014).

Theorem 5.3.21 (Maynard's Theorem(2014):)**.** *[13]*

$H_1 \leq 246$
and $H_m = O(m\, exp((4 - \frac{4}{43})m))$

5.4 An Arithmetical Approach to Twin Prime Conjecture:

In [6] E. Benedetto gave an arithmetical approach to the Twin Prime Conjecture. His approach is based on the Eratoshenes Sieve. Eratosthenes was an ancient Greek mathematician and he is remembered for his prime number sieve, the "Sieve of Eratosthenes" which, in modified form, is still an important tool in number theory research. The ancient sieve of Eratosthenes is a simple algorithm for finding all prime numbers up to a given limit, by making a list of all integers and repeatedly striking out multiples of already found primes.

5.4.1 Sieving primes

Since 2 is the only even prime number, prime numbers are necessarily odd numbers. Therefore we know that the possible prime numbers are of the form

$$2k + 1$$

For example

$$3; k = 1$$

$$5; k = 2$$

$$7; k = 3$$

$$9; k = 4$$

$$...$$

$$...$$

$$...$$

Obviously, not all the odd numbers are prime numbers. For example for $k = 4$ we obtain 9 and this odd number is not prime. We now look for an arithmetical relationship that tells us what the odd numbers are to discard. The numbers to be discarded,are the product of two odd numbers and therefore are of the form $(2k_1 + 1)(2k_2 + 1)$.By multiplying the previous relation we get

$$4k_1k_2 + 2k_1 + 2k_2 + 1$$

This sieving function gives us all the odd numbers that are not primes. For example

k_1	k_2	$4k_1k_2 + 2k1 + 2k2 + 1$
1	1	9
1	2	15
2	2	25
...	...	...

We can write $4k_1k_2 + 2k_1 + 2k_2 + 1 = 2(2k_1k_2 + k_1 + k_2) + 1$ and therefore, by considering the following set

$$A = \{2k + 1, k \neq 2xy + x + y; x, y \in \mathbb{N}\}$$

the set of prime numbers is the following

$$P = \{2\} \cup \{A\}$$

5.4.2 Sieving Twin primes:

First of all let us write the following partition of natural numbers.

$6k-4$	$6k-3$	$6k-2$	$6k-1$	$6k$	$6k+1$
2	3	4	5	6	7
8	9	10	11	12	13
14	15	16	17	18	19
20	21	22	23	24	25
26	27	28	29	30	31
32	33	34	35	36	37
38	39	40	41	42	43
44	45	46	47	48	49
50	51	52	53	54	55
...	...	...	...	...	...
...	...	...	...	...	...

Table 5.4: Partition of Natural Numbers

We have $N = 1 \cup 6k-4 \cup 6k-3 \cup 6k-2 \cup 6k-1 \cup 6k \cup 6k+1$ and it is clear that all the primes, except 2 and 3, are of the form $6k \pm 1$. Obviously the numbers $6k \pm 1$ contain also the multiples of $5, 7$ and so on. Since

$$(6k_1+1)(6k_2+1) = 6(6k_1k_2 + k_1 + k_2) + 1 \in 6k+1$$

$$(6k_1-1)(6k_2-1) = 6(6k_1k_2 - k_1 - k_2) + 1 \in 6k+1$$

$$(6k_1-1)(6k_2+1) = 6(6k_1k_2 + k_1 - k_2) - 1 \in 6k-1$$

we consider the following sets

$$B = \{6k-1, k \neq 6xy + x - y : x, y \in \mathbb{N}\}$$

$$C = \{6k+1, k \neq 6xy + x + y, k \neq 6xy - x - y : x, y \in \mathbb{N}\}$$

Then the set of prime numbers is the following

$$P = \{2\} \cup \{3\} \cup \{B\} \cup \{C\}$$

We can write the following sieving rules

$$f_1 = 6xy + x + y$$

$$f_2 = 6xy - x - y$$

$$f_3 = 6xy + x - y$$

These relations give us all the values of k for which correspond composite numbers. The first two relations are the sieving rules for $6k + 1$, the third is the sieving rule for $6k - 1$. From how much we have said above, the integer numbers k that don't belong to none of the three sieving rules, correspond to couples of twins. For example   Therefore, if we succeed in knowing how many are the not selected numbers

k	$6k-1$	$6k+1$
1	5	7
2	11	13
3	17	19
5	29	31
7	41	43
10	59	61
12	71	73
17	101	103
...	...	...

k, we know how many are the couples of twins.We can observe that the numbers of the sieving rules make constant jumps and precisely jumps of $5, 7, 11, 13$ and so on. In fact we have $5k \pm 1, 7k \pm 1, 11k \pm 2, 13k \pm 2$ and so on. Therefore we can write the three sieving rules in an only symmetric matrix.

$5k \pm 1$	$7k \pm 1$	$11k \pm 2$	$13k \pm 2$	$17k \pm 3$	$19k \pm 3$	$23k \pm 4$	$25k \pm 4$	$29k \pm 5$	...
4	6	9	11	14	16	19	21	24	...
6	8	13	15	20	22	27	29	34	...
9	13	20	24	31	35	42	46	53	...
11	15	24	28	37	41	50	54	63	...
14	20	31	37	48	54	65	71	82	...
16	22	35	41	54	60	73	79	92	...
19	27	42	50	65	73	88	96	111	...
21	29	46	54	71	79	96	104	121	...
...	...	...	...	...	...	...	...	...	...

We can write the previous matrix as a triangle

$$4$$

$$6 \quad 6$$

$$9 \quad 8 \quad 9$$

$$11 \quad 13 \quad 13 \quad 11$$

$$14 \quad 15 \quad 20 \quad 15 \quad 14$$

$$16 \quad 20 \quad 24 \quad 24 \quad 20 \quad 16$$

$$19 \quad 22 \quad 31 \quad 28 \quad 31 \quad 22 \quad 19$$

$$21 \quad 27 \quad 35 \quad 37 \quad 37 \quad 35 \quad 27 \quad 21$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Finally we can say that the sieve sequence is deterministic and it has infinitely holes. Therefore there are infinitely many twin primes.

5.5 A statistical Argument for the Twin Primes Conjecture

Bruckman [18] gave a simple statistical argument of the Twin Prime Conjecture(He mentioned Weak Twin Prime Conjecture) by using the correlation analysis and the Prime Number Theorem.

Let

$$\pi^*(n) = \sum_{k=1}^{n} \delta(k) = \pi(n) - 1, (n \geq 3)$$

and

$$\pi_2(x) = \sum_{k=1}^{n} \delta(k)\delta(k+1)$$

where δ is the characteristic function for prime i.e, $\forall n \in \mathbb{N}$

$$\delta(n) = \begin{cases} 1 & \text{if } n \text{ is odd prime} \\ 0 & \text{otherwise} \end{cases}$$

Let $x_k = \delta(k), y_k = \delta(k+2)$. Then we have the pair of values $(0,0),(1,0),(0,1)$ and $(1,1)$. So by taking these values. it is worthwhile to compute correlation coefficient. So, the mean of their $\bar{x}$,$\bar{y}$ are taken in account. Also,it can be observed that $\pi_2^*(n+2) = \pi_2^*(n) + \theta_n$ where $\theta_n = 1$ or 0.

Also,

$$\overline{x} = \frac{\pi^*(n)}{n}, \quad \overline{y} = \frac{\pi^*(n+2)}{n} \quad \text{and} \quad \overline{xy} = \frac{\pi_2(n)}{n}$$

Thus now we are able to assert that the following

$$\lim_{n\to\infty} \frac{\overline{x}}{\overline{y}} = 1 \tag{5.8}$$

Also we have the following statistical formulae

$$\rho = \rho_{xy} = \frac{\overline{xy} - \overline{x}\,\overline{y}}{\sigma_x \sigma_y}$$

is called the correlation coefficient between the variables x and y, Where $\bar{x} = \frac{\sum_{1}^{n} x_k}{n}$, mean of x; $\bar{y} = \frac{\sum_{1}^{n} y_k}{n}$, mean of y; $\bar{xy} = \frac{x_1y_1 + x_2y_2 + ... + x_ny_n}{n}$, mean of xy; $\sigma^2(x) = \frac{\sum_{1}^{n}(x - x_k)^2}{n}$ and $\sigma^2(y) = \frac{\sum_{1}^{n}(y - y_k)^2}{n}$ with $\sigma(x) = \sqrt{\sigma^2(x)}$ and $\sigma(y) = \sqrt{\sigma^2(x)}$ called the standard deviation of the sample.

From the definition of ρ in this case, we see that the numerator of ρ is equal to $\pi_2(n)/n - \pi^*(n)\pi^*(n+2)/n^2$. Our argument, essentially, reduces to the conjecture that $\lim_{n\to\infty} \rho = +1$ in light of (5.8). In turn, this implies that the sign of ρ will be non-negative from some point on. In other words, we argue that for all sufficiently large n,

$$\pi_2(n) \geq \pi^*(n)\pi^*(n+2)/n. \tag{5.9}$$

In [20], it was shown, using the prime number theorem, that for all $n \geq 11$,

$$\pi(n) > n/\log n \tag{5.10}$$

from (5.9) and (5.10) for all sufficiently large n,

$$\pi_2(n) > n/(\log n)^2 \tag{5.11}$$

Since the expression $n/(\log n)^2$ is unbounded,(5.11) would establish TPC.

Chapter 6

Summary

In this thesis we presented the distribution of primes and Twin prime conjecture. We discussed some theorems about the distribution of primes. We presented the Merten's theorems in details which gives some basic concepts about the distribution of primes. We also discussed the Tschebycheff's estimates which gives the bounds for prime counting function. We also observed the estimation of primes by natural logarithm function and integral logarithm function numerically and graphically. In which we saw that integral logarithm function gives the better approximation for the prime numbers.

About the Twin prime, we briefly described heuristic approaches of the distribution of Twin primes with the work of Hardy, Littlewood and Yaminov. We presented both numerically and graphically estimates of Twin primes by the formulae given by them and observed that yaminov's approximation is better than that of Hardy and Littlewood. We discussed about another empirical function given by Yaminov and observed that this is more accurate than the privious one, and also discussed some theorems by Yaminov. Lastly we presented two alternative approaches of Twin prime conjecture; arithmetical approach by E. Benedetto and statistical approach by Bruckman

Although there are series of researches about the Twin prime conjecture, but it remains unproven till the date. The work by Zhang is the great breakthrough in this field. In this thesis, we presented chronologically the developments towards the Twin prime conjecture.

There has been lot of successful researches about the distribution of Twin primes and Twin prime conjecture but the mystery of Twin prime is still hidden with the great open problem

”There are infinitely many Twin primes”.

In future, I am interested to research in this field.

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