Definition[section]
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TRIBHUVAN UNIVERSITY

INSTITUTE OF SCIENCE AND TECHNOLOGY

A Study on Advances of Hurwitz Theorem

A Thesis
submitted in partial fulfillment of the requirements
for
the degree of MASTER OF PHILOSOPHY (Math)
by

Ramesh Gautam

to the
CENTRAL DEPARTMENT OF MATHEMATICS
T.U., Kirtipur

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Student's Declaration

I, hereby declare that the research work entitled **A study on advances of Hurwitz theorem** submitted here for the partial fulfillment of the Master of Philosophy (M. Phil.) degree in Mathematics to the Central Department of Mathematics, Tribhuvan University in june 2016 carried out under the guidance of my supervisor and has not been published or submitted elsewhere for the requirement of any degree programme. Any literature, data or works done by others are cited within this dissertation has been given due acknowledgement and are listed in the reference.

Ramesh Gautam

Date: June 20, 2016

Supervisor's Recommendation

The dissertation entitled **A study on advances of Hurwitz theorem** presented by Ramesh Gautam under our supervision is hereby submitted for the partial fulfllment of the Master of Philosophy (M. Phil.) degree in Mathematics to the Central Department of Mathematics, Tribhuvan University, Kirtipur, Nepal.

Dr. Ajaya Singh
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We certify that the **Research Committee of the Central Department of Mathematics**, TU, Kirtipur approved this research work entitled **A study on advances of Hurwitz's theorem** done by Mr. Ramesh Gautam in the scope and generality as a thesis in the partial fulfillment for the requirement of the degree of M. Phil. in Mathematics.

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Dedication

my Mother and my Father

Abstract

Irrationals numbers are complicated but dealing with them is most. Approximation of irrational numbers by rational numbers is possible, because the rational numbers are dense on real line. Continued fraction expansion of real numbers is one of the most important tool in Diophantine approximation. **A study on advances of Hurwitz theorem** is a part of Diophantine approximation, approximation of irrationals by rationals. In general, there is no closest rational approximaion. In fact, there are infinitely many and that will depend upon the size of denominator. By increasing the size of denominator, will find the better approximation. Hence how much we can pay for the denominator deteremined the best approximation for that context. Continued fraction expansion is an important tool to solve Pell's Fermat equation $x^2 - Dy^2 = \pm 1$ and solution of Pell's Fermat equation are applicable to find the rational approximation of $\sqrt{D}$. Adolf Hurwitz established the theorem in 1891 and claimed that, for any irrational real numbers α there exist infinitely many rationals $\frac{p}{q}$ such that

$$|\alpha - \frac{p}{q}| \leq \frac{1}{\sqrt{5}q^2}$$

. He also proved that this bound cannot be improved, if we consider whole set of irrationals. In the past, many work had done for the refinement of this bound by making the restrictions on the set of Irrationals or number of rational approximation. In this context, in 1948 A.V Prasad improved this bound for the finite number of rational approximation in the place of infinitely many rational approximation. Adolf Hurwitz 1906 and Hailiza Kamarul Haili and Norhayati Rosli in 2005 made some contribution in this direction. On the other hand R.T Worley in 1981 established the general result of rational approximation. In this context Bernadin Imbrampasic gave an explicit version for k=13 in 2013.

Contents

List of Symbols

$\mathbb{R}$	:	The field of Real numbers.
$\mathbb{C}$	:	The field of complex numbers.
$\mathbb{N}$	:	The set of natural numbers.
$\mathbb{Z}$	:	The ring of rational integers.
$\mathbb{Z}+$	:	The set of nonnegative rational integers.
$\mathbb{Q}$	:	The field of rational numbers.
$GL(2, \mathbb{Z})$		The set of all 2 by 2 matrices with integers and determinant ± 1.
$F(k)$	:	The set of all real numbers α such that $0 \leq \alpha \leq 1$ has no partial quotient greater than k and $F(0) = \phi$.
$\lfloor \alpha \rfloor$	:	The lower integral part of α
$\{\alpha\}$	:	The fractional part of α

Chapter 1

Introduction

This Chapter covers some motivations on Rational approximation with brief history and some useful definitions of Rational approximation and Continued fraction.

1.1 Motivation

Every point on the real line corresponds to a real number. That is there is one to one correspondence between the points on real line and the set of real numbers. For every interval on the real line, no matter how small, there are points corresponding to rational numbers. Mathematically, this situation is described by saying that the rational points are dense on the real line. It follows that every real number α can be approximated by a rational number to any degree of accuracy whatsoever[10].

Irrational numbers are complicated, hard to understand and hard to work with in comparison with rational numbers. For this reason, number theorists have long worked in the field of Diophantine approximation, the process of approximating irrational numbers with rational ones. What we are specifically concerned with here is determining when a rational number is a best rational approximate of an irrational number[12]. Given a real number α, how closely can it be approximated by rational numbers, to make this more precise, for any given positive ϵ is there a rational number $\dfrac{p}{q}$ within ϵ of α, so that the inequality $|\alpha - \frac{p}{q}| < \epsilon$ is satisfied ? The answer is yes because the rational numbers are dense on real line[10].

In number theory, the field of Diophantine approximation, named after Diophantus of Alexandria, deals with the approximation of real numbers by rational numbers. The first problem was to know how well a real number can be approximated by rational numbers. For this problem, a rational number $\frac{a}{b}$ is a "good" approximation of a real number α if the absolute value of the difference between $\frac{a}{b}$ and α may not decrease if $\frac{a}{b}$ is replaced by another rational number with a smaller denominator[7].

Suppose we want to approximate the irrational number $e = 2.71828....$ Intuitively, one might use the rational $\frac{2718}{1000}$, and observe $|e - \frac{2178}{1000}| = .00028.....$ At a glance, this appears to be a good approximation of e, but how do we know if it is a best approximation? How do we know we can't do better? What about the (not so intuitive) rational approximation, $\frac{1650}{607}$? One may compute $|e - \frac{1650}{607}| = .0000048...$, so in some sense, this is a "better" approximation. It brings us closer to the irrational we are approximating, and it is a simpler fraction, in that it uses lower numbers. Yet how do you come up with the number $\frac{1650}{607}$, and once you have it, how do you determine if it really is a best approximate? The first question was answered in the 18th century by means of continued fraction, We focus our study on second problem.

Clearly, there is no closest rational approximations to any irrational number. You can always keep increasing the denominator of the rational approximation, and in doing so, keep getting closer and closer to the irrational number you are trying to approximate. However, since the purpose of a rational approximate is to give us a number that is easier to work with, often we do not want our approximations to have extremely large denominators. Also, although in theory denominators can be arbitrarily large, computers only have a finite amount of memory, and so even if only for this reason alone, it is necessary to restrict the size of the denominator[12].

Knowing the "best" approximations of a given number, the main problem of the field is to find sharp upper and lower bounds of the above difference, expressed as a function of the denominator. It appears that these bounds depend on the nature of the real numbers to be approximated: the lower bound for the approximation of a rational number by another rational number is larger than the lower bound for algebraic numbers, which is itself larger than the lower bound for all real numbers. Thus a real number that may be better approximated than the bound for algebraic numbers is certainly a transcendental number. This allowed Liouville, in 1844, to produce the first explicit transcendental number. Later, the proofs that π and e are transcendental were obtained with a similar method

1.2 History

1.2.1 Rational Approximation

The history of Diophantine approximation is quite old: it includes, for instance, early estimates for π computations related to astronomical studies, the theory of continued fraction expansion. There are positive results: any irrational number has good rational approximations. One of the simplest tools to see this is Dirichlet's box principle, other methods are continued fraction expansions, Farey series, geometry of numbers (Minkowski's Theorem). There are negative results: no number has too good (and at the same time too frequent) approximations. Some results are valid for all (irrational) numbers, others only for restricted classes of numbers, like the class of algebraic numbers.

One main goal of the theory of Diophantine approximation is to compare, on the one hand, the distance between a given real number α and a rational number $\dfrac{p}{q}$, on the other hand, the denominator q of the approximant. An approximation is considered as sharp if $|\alpha - \frac{p}{q}|$ is small compared to q[15].

For example, Suppose we are trying to square the circle, that is, find a square whose area equals the area of a given circle. The ratio of the side of the resulting square to the diameter of the original circle will be an irrational number, so an approximation will have to be used in computations.

In 1842, the German mathematician Dirichlet's proved that for any irrational real number α there exists infinitely many rationals $\dfrac{p}{q}$ within $\dfrac{1}{q^2}$. In 1904, French Mathematician Fatou and in 1918, an American mathematician Grace Andrews described the computational approach of rationals with the Dirichlets condition. In their results, the rational approximations are obtained from the list of convergents and semi convergents of the continued fraction expansion of the irrational number α.

In 1798, the French Mathematician Legender's proved the result to give the criteria to be the convergents of continued fraction expansion of given irrational real number for which we are going to find the best rational approximation. In 1895, Austrian mathematician Vahlen established the result for calculation of the best rational approximation with Legendre's condition. It was proved that from the two consecutive convergents of continued fraction expansion of irrational number at least one satisfied the Legender's condition. In 1891, another German mathematician Hurwitz established the best approximation theorem for general irrational numbers with more refined approximation then the approximations given by Legender's condition.

In 1903, French mathematician Emile Borel described the computational approach of best rational approximation within the Hurwitz condition. It was proved that among the three consecutive convergents at least one satisfied the Hurwitz condition. which at once implies the Hurwitz result[17].

In 1770, Lagranges proved that each convergents $\dfrac{p}{q}$ lies within $\dfrac{1}{2q^2}$ of α and in 1878, Serret a French mathematician proved that if two irrational real numbers are equivalent then their continued fraction after certain stages onwards coincides. Malayasian Mathematicians Hailiza Kamarul Haili, and Norhayati Rosli in 2005 did some work for the refinement of Hurwitz constant by taking specific set of irrationals[7].

For all irrational number α the inequality

$$|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$$

has infinitely many solutions. If α is equivalent with $\dfrac{\sqrt{5}-1}{2}$ i.e a root of the quadratic equation $\alpha^2 + \alpha - 1 = 0$, the constant $\dfrac{1}{\sqrt{5}}$ can not be improved. If not, there are infinitely many solutions of

$$|\alpha - \frac{p}{q}| < \frac{1}{2\sqrt{2}q^2},$$

where the constant $\dfrac{1}{\sqrt{8}}$ can not be improved if α is a equivalent to a root of $\alpha^2+2\alpha-1=0$, otherwise, there are infinitely many solutions of

$$|\alpha-\frac{p}{q}|<\frac{5}{\sqrt{221}q^2},$$

where the constant $\dfrac{5}{\sqrt{221}}$ can not be improved for α equivalent to root of $5\alpha^2+11\alpha-5=0$,otherwise there are infinitely many solutions of

$$|\alpha-\frac{p}{q}|<\frac{13}{\sqrt{1517}q^2},$$

where the constant $\dfrac{13}{\sqrt{1517}}$ can not be improved for α equivalent to a root of $13\alpha^2+29\alpha-15=0$ and so on indefinitely . The sequence of numbers $\frac{1}{\sqrt{5}}, \dfrac{1}{\sqrt{8}}, \dfrac{5}{\sqrt{221}}, \dfrac{13}{\sqrt{1517}}$....... tends to $\dfrac{1}{3}$[10].

The sequence of quadratic irrationalities $(\theta_i), i\geq 1$, such that for any $i\geq 1$, if α is not $GL(2,Z)$ equivalent to any of $\theta_1,...,\theta_{i-1}$, then there is infinitely many rational numbers that satisfy

$$|\alpha-\frac{p}{q}|<\frac{1}{L_i q^2}$$

and this is best possible when α is $GL(2,\mathbb{Z})$ equivalent to θ_i.

Denoting by

$$m_1, m_2, ... = 1, 2, 5, 13, 29, ...$$

the sequence of Markoff numbers, one has $L_i=\sqrt{9-\frac{4}{m_i^2}}$ and

$$\theta_i=\frac{-3m_i+2k_i\pm\sqrt{9m_i^2-4}}{2m_i},$$

where k_i is an integer satisfying $a_ik_i\equiv b_i(\text{mod} m_i)$ and (a_i,b_i,m_i) is a solution of Markoff's equation with $m_i\geq\max(a_i,b_i)$. Here one assumes Markoff's Conjecture to get unicity of (a_i,b_i) [14][9].

Scott 1940 gave the idea of restricted rational approximations $\dfrac{p}{q}$ having the property of different cases such as both p and q are odd, p is even, q is odd and p is odd, q is even.

In 1948 A.V Prasad initiated the study of finite Diophantine approximation. He proved that, for any given irrationals if at least one rational approximation is asked then the constant $\frac{1}{\sqrt{5}}$ in Hurwitz theorem can be improved by $\frac{2}{3+\sqrt{5}}$. Similary L.C Eggan proved that for the restricted set of irrationals which are not equivalent with the golden ratio, then the constant for at least one rational approximation can be improved by $\frac{2}{2\sqrt{2}+3}$. A.V Prasad in 1948 also proved that for any irrational α, there are at least m pairs of relatively prime integers p and q , $q > 0$ satisfying the conditions with the constant

$$c_m = \frac{\sqrt{5}+1}{2} + \frac{p_{2m-1}}{q_{2m-1}},$$

where $\frac{p_n}{q_n}$ is the nth convergent of $\frac{\sqrt{5}-1}{2}$[13][19].

Further if $\alpha = \frac{\sqrt{5}-1}{2}$, there are exactly m solutions. L.C Eggan gave the extension theorem on Prasad result and prove that, For any irrational number which are not equivalent with $\frac{\sqrt{5}-1}{2}$ and any positive integer m, there are at least m solutions in relatively prime integers p,q and $q > 0$ with the constant

$$c_m = \sqrt{2} + 1 + \frac{p_{2m-1}}{q_{2m-1}}.$$

Moreover if the irrational is $\sqrt{2}-1$, there are exactly m solutions. These results are the special case of the following theorem. If n be a positive integer and let

$$\alpha_n = [0; n, n, n, n...] = \frac{\sqrt{n^2+4}-n}{2}.$$

For any positive integer m, let

$$c_m = \alpha_n + n + \frac{p_{2m-1}}{q_{2m-1}},$$

where $\frac{p_n}{q_n}$ are the jth convergent of α_n. Then if $\alpha = [a_0, a_1, ...]$is an irrational number with $a_j \geq n$ for infinitely many j, there are at least m solutions in relatively prime integers[13][19][23].

In 1995 Jingcheng Tong generalized the constant

$$c_m = \sqrt{5} + \frac{\sqrt{5}}{\left(\frac{7+3\sqrt{5}}{2}\right)^m - 1}$$

by using the Fibonacci sequence. In particular $c_1 = \frac{3+\sqrt{5}}{2}$, $c_2 = \frac{7+3\sqrt{5}}{6}$ and $c_3 = \frac{9+4\sqrt{5}}{8}$[22].

In 1981 R.T worley, Dujella 2004 established the generalized result of approximation by setting the approximating constant k for any positive real number and gave the corresponding rational approximation for any irrational number. At the same time Worley gave the explicit version of his result for k=2. A. Dujella, Bernadin. Ibrahimpasic, 2008 extend the Worley's work and gave the explicit version for $k = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$. In 2013 Bernadin Ibrahimpasi extend the work for k=13[5].

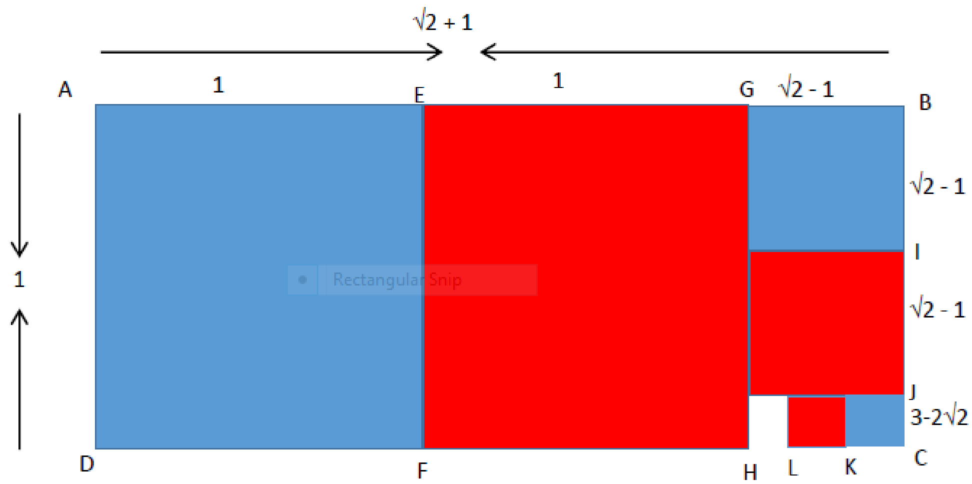
1.3 Geometric Interpretation of Irrational Numbers

Consider a real number $\sqrt{2}+1$. Start with a rectangle have side length 1 +$\sqrt{2}$ unit and breadth 1 unit. Decompose it into two squares with sides 1 and a smaller rectangle of sides of length 1 unit and breadth 1 +$\sqrt{2}-2 = \sqrt{2}-1$ unit. This second small rectangle has side lengths in the proportion $\frac{1}{\sqrt{2}-1} = 1+\sqrt{2}$, which is the same as for the large one. Hence the second small rectangle can be split into two squares and a third smaller rectangle, the sides of which are again in the same proportion. This process does not end. Hence $1+\sqrt{2}$ is an irrational number.

If we start with a rectangle having integer side lengths, then this process stops after finite number of steps (the side lengths of the successive rectangles produces a decreasing sequence of positive integers). Also, for a rectangle with side lengths in a rational proportion, this process stops after finitely may steps.

Set $t = \sqrt{2}+1 = 2.41421356...$ The continued fraction expansion of t is $[2; 2, ...] = [\bar{2}]$. Indeed, from $\sqrt{2}-1 = \frac{1}{\sqrt{2}+1}$, we deduce $t = 2 + \frac{1}{t}$

An interval of length $t = \sqrt{2}+1$ is decomposed into two intervals of length 1 and one of length $\frac{1}{t}$. Again $1 = \frac{2}{t} + \frac{1}{t^2}$, further the interval of length 1 is decomposed into two intervals of length $\frac{1}{t}$ and one of length $\frac{1}{t^2}$. At each step we get two large intervals and a small one. The process does not stop[23][14].



Geometrical Meaning of irrational number

1.4 Structure of Thesis

In this work, our plan of study is about the rational approximations of irrational real numbers. Continued fraction expansion of real irrationals and solutions of Pell's Fermat equation $x^2 - Dy^2 = \pm 1$ are important tool in rational approximation.

The Second chapter cover the basic results of continued fraction of real numbers.

The third chapter focus on the results of continued fraction expansion of $\sqrt{D}$, for non square positive integer D to solve Pell's Fermat equation, taking different types of D.

In the Fourth chapter, we discuss about the basic results of rational approximation by using Dirichlet Pigeonhole principle.It also covers the application of solution of Pell's Fermat equation in the rational approximation of $\sqrt{D}$.

In the fifth chapter, Development of Rational approximation of irrational numbers by using continued fraction and focus on the study of Hurwitz's theorem for irrationals.

The last chapter cover some more advanced results in Hurwitz's theorem to dicuss the two direction of approximation and elaborate the more recent results about the explicit version of L.T Worley's work.

Chapter seven includes Conclusion and some open problems.

Chapter 2

Some Preliminaries

In this chapter, we discuss about the basic results of continued fraction of real numbers and related definitions. Some properties of continued fraction of rational and irrational are throughly discussed from preliminary stage.

2.1 Some Definitions

Definition 2.1.1. *A rational number is a number that can be expressed in the form of* $\frac{p}{q}$*, where p and q are integers with* $q > 0$*.*

Definition 2.1.2. *A real number is a rational number if and only if it can be expressed as a terminating or repeating decimal, that is if a rationalnumber is not terminating decimal form then it is in the form*

$$\alpha = m.d_1 d_2 ... d_k d_{k+1} d_{k+2} ... d_{k+r} d_{k+1} d_{k+2} ... d_{k+r} ....$$

a block of decimal part is repeated infinitely,then

$$\alpha = m.d_1 d_2 ... d_k \overline{d_{k+1} d_{k+2} ... d_{k+r}},$$

where $m = [\alpha]$ *is the integral part*

Definition 2.1.3. *A number α is said to be algebraic if it is a root of a polynomial $f(x) = a_n x^n + ... + a_1 x + a_0, f(x) \neq 0$ with rational coefficients. To prove that a given number α is algebraic, we need to find a non-zero polynomial $f(x) \in \mathbb{Q}[x]$such that $f(\alpha) = 0$*

Definition 2.1.4. *The finite continued fraction expansion for a real number α,*

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

is denoted by

$$\alpha = [a_0; a_1, a_2, a_3, ..., a_n].$$

where, the a_is are integer parts, clearly $a_i \in \mathbb{Z}$ for all i and $a_i > 0$ for all $i \geq 1$. If the terms of continued fraction goes infinite in number then it is called simple infinite continued fraction of real number. It is denoted by

$$\alpha = [a_0; a_1, a_2, a_3, ...].$$

Definition 2.1.5. *The number*

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, a_3, ..., a_n] = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$$

is called the n_{th} convergent of α, where $p_{-2} = 0, p_{-1} = 1, q_{-2} = 1, q_{-1} = 0$ gives $p_0 = a_0, q_0 = 1$. The integer a_n in the continued fraction expansion is called n_{th}partial quotient and $\alpha_n = [a_n, a_{n+1}, ...]$ is called the n_{th} total quotient.

Definition 2.1.6. *Two irrational numbers α and β are called $GL\,(2, \mathbb{Z})$ equivalent if there exists integers a, b, c, d with $ad - bc = \pm 1$ such that $\beta = \dfrac{a\alpha + b}{c\alpha + d}$. This relation is an equivalence relation.*

The irrational number $\phi = \frac{1+\sqrt{5}}{2} = [1; 1, 1, ...] = [\bar{1}]$ which is purely periodic(only one integer repeated from begining and forever) and $\alpha = \frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, ...] = [0; \bar{1}]$ are equivalent.

Definition 2.1.7. *An irrational number whose continued fraction expansion has from stage onwards exclusively $1'$s as its partial entries is called Noble number. The irrational number $\frac{\sqrt{5}+1}{2}$ and all its equivalent irrational numbers are Noble numbers.*

2.2 Results Related with Convergents and Partial Quotients

In this section we discuss about the basic results of continued fraction connected with convergents and partial quotient. We based on[17][23] for these results. Let $\alpha = [a_0; a_1, ..., a_n]$ and $\alpha = [a_0; a_1, ...]$ are two finite and infinite continued fractions of real number α according as α is rational or irrational respectively.

Lemma 2.2.1. *For*

$$n \geq 2, p_n = a_n p_{n-1} + p_{n-2}, q_n = a_n q_{n-1} + q_{n-2}$$

Here for an irrational $\alpha = [a_0; a_1, a_2, ...]$. $\frac{p_{n-2}}{q_{n-2}}$, $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$ are the three consecutive convergents of the continued fraction expansion of α. Since $p_0 = a_0, q_0 = 1$ then $C_0 = a_0$. Similarly $C_1 = \frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1}$ For n=2 we have

$$\begin{aligned}
\frac{p_2}{q_2} &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} \\
&= \frac{a_2\left(a_1 a_0 + 1\right) + a_0}{a_2 a_1 + 1} \\
&= \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0}
\end{aligned} \tag{2.1}$$

Now suppose that the theorem is true for $n = 3, 4, ..., n-1$ we need to show that it is true for n . Since it is true for n-1 then

$$\frac{p_{n-1}}{q_{n-1}} = \frac{a_{n-1}p_{n-2} + p_{n-3}}{a_{n-1}q_{n-2} + q_{n-3}}$$

Now

$$\begin{aligned} \frac{p_n}{q_n} = [a_0, a_1, ..., a_{n-2}, a_{n-1} + \frac{1}{a_n} &= \frac{\left(a_{n-1} + \frac{1}{a_n}\right) p_{n-2} + p_{n-3}}{\left(a_{n-1} + \frac{1}{a_n}\right) q_{n-2} + q_{n-3}} \\ &= \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} \end{aligned} \tag{2.2}$$

Lemma 2.2.2. *For each k with $1 \leq k \leq n$. Let*

$$r_k = [a_k, a_{k+1}, ..., a_n]$$

then

$$[a_0; a_1, ...a_k, ..., a_n] = [a_0, a_1, ..., a_{k-1}, [a_k, ..., a_n]] = \frac{r_k p_{k-1} + p_{k-2}}{r_k q_{k-1} + q_{k-2}}$$

Lemma 2.2.3. *1.*

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n \text{ , } \mathit{for} n \geq -1$$

2.

$$r_{n-1} - r_n = \frac{(-1)^n}{q_{n-1}q_n}$$

3.

$$p_{n-2}q_n - p_n q_{n-2} = (-1)^{n-1} a_n, \mathit{for} n \geq 0$$

4.

$$r_{n-2} - r_n = \frac{(-1)^{n-1} a_n}{q_n q_{n-2}}$$

The fraction $\frac{p_i}{q_i}$ is reduced,that is

$$(p_i, q_i) = 1$$

Lemma 2.2.4. *For $n \geq 1$, we have $\dfrac{q_n}{q_{n-1}} = [a_n, a_{n-1}, ..., a_1]$*

Theorem 2.2.1. *Suppose $a_0, a_1, a_2, ...$ are real numbers numbers with $a_1, a_2, ..$ positive. Then*

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < ...\frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

In words These convergent's for $0, 2, 4, ..$ form a monotonically increasing sequence with α as a limit. Similarly, for the odd convergents 1,3,5.... form a monotonically decreasing sequence tending to limit α.

Since we have

$$\frac{p_{n-2}}{q_{n-2}} - \frac{p_n}{q_n} = \frac{(-1)^{n-1}a_n}{q_{n-2}q_n}.$$

For $n \geq 2$, n even, we get

$$\frac{p_{n-2}}{q_{n-2}} - \frac{p_n}{q_n} < 0.$$

Hence even convergents are increasing. But for $n \geq 3$, n odd, we get

$$\frac{p_{n-2}}{q_{n-2}} - \frac{p_n}{q_n} > 0.$$

Hence the odd convergents are decreasing. Now it remains to show that

$$\frac{p_n}{q_n} < \frac{p_m}{q_m}$$

if n is even and m is odd. Suppose say, $n < m$ then

$$\frac{p_n}{q_n} < \frac{p_{m-1}}{q_{m-1}}.$$

Now to prove the lemma it is sufficient to prove

$$\frac{p_{m-1}}{q_{m-1}} < \frac{p_m}{q_m}.$$

From the lemma for m odd we have

$$q_m p_{m-1} - p_m q_{m-1} = (-1)^m < 0.$$

Hence combining these results we have

$$\frac{p_{m-1}}{q_{m-1}} < \frac{p_m}{q_m}.$$

Similarly the result is true for $n > m$.

2.3 Properties of Continued Fraction of Rational and Irrational

Theorem 2.3.1. *A real number is rational if and only if it's continued fraction is finite.*

Let n be a rational number then $n = \frac{p}{q}$ for some integers p and q. Suppose that p and q are in lowest terms . To prove the theorem , we use the Euclid's algorithm. By applying the algorithm we have

$$
\begin{aligned}
p &= a_1 q + r_1, \quad 0 \leq r_1 < q \\
q &= a_2 r_1 + r_2, \quad 0 \leq r_2 < r_1 \\
r_1 &= a_3 r_2 + r_3, \quad 0 \leq r_3 < r_2 \\
&\ldots \\
&\ldots \\
&\ldots \\
r_{n-3} &= a_{n-1} r_{n-2} + r_{n-1}, \quad 0 \leq r_{n-1} < r_{n-2} \\
r_{n-2} &= a_n r_{n-1}
\end{aligned}
$$

The sequence $r_1, r_2, ..., r_{n-1}$ forms a strictly decreasing sequence of non negative integers that must converge to zero in finite number of steps so there are at most n a_i's.

The next step involves the rearranging the algorithm in the following manner.

$$\frac{p}{q} = a_1 + \frac{1}{\frac{q}{r_1}}$$
$$\frac{q}{r_1} = a_2 + \frac{1}{\frac{r_1}{r_2}}$$
$$\frac{r_1}{r_2} = a_3 + \frac{1}{\frac{r_2}{r_3}}$$
$$\dots$$
$$\dots$$
$$\dots$$
$$\frac{r_{n-2}}{r_{n-1}} = a_n + \frac{1}{\frac{r_{n-1}}{r_n}}$$
$$\frac{r_{n-1}}{r_n} = a_{n+1}$$

Now, substituting each equation into previous, we find

$$n = \frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + ... + a_n + \cfrac{1}{a_{n+1}}}}}$$

Conversely, we prove by induction that if a simple continued fraction has n terms, it is rational. Let α represent the value of the continued fraction, we first check the base case $n = 1$. Then $\alpha = a_1$, but then $\alpha = a_1$ is clearly a rational, since a_1 is an integer. We now prove the inductive case. Assume the theorem is true for all i, $i \leq n$.

we show that the theorem also true for $n + 1$. Let α be a continued fraction that is represented by $n+1$ terms. We wish to show that α is rational. So, we have

$$\alpha = \frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + ... + a_n + \cfrac{1}{a_{n+1}}}}}.$$

Here we can rewrite the expression

$$B = a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + ... + a_n + \cfrac{1}{a_{n+1}}}}$$

But B is a continued fraction with n terms and by our induction hypothesis, it can be written as a rational $\frac{p}{q}$. This implies that $\alpha = a_1 + \frac{1}{\frac{p}{q}}$. By applying same simple algebra, we arrive at the following equality. $\alpha = \frac{a_1 p + q}{p}$. Since a_1, as well as p and q are integer, α must be a rational. Thus the theorem is true for n+1 and by induction, it must hold for all integers[14].

Lemma 2.3.1. *1. Suppose that r is an integer. Then there are precisely two expansions of r into a simple continued fraction as above, namely $r = [r]$ and $r = [r-1, 1]$*

2. For rational but not integral, r has precisely two simple continued fraction expansions: one is of the form $[a_0; a_1, a_2, ..., a_{n-1}, a_n]$ with the last partial quotient $a_n \geq 2$ and the other is $[a_0; a_1, a_2, ..., a_{n-1}, a_n - 1, 1]$

Theorem 2.3.2. *The denominators q_n of the convergents are an increasing sequence of positive integers $n > 0$. Finally, with α_n defined by*

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n}$$

,

$$a_n = [\alpha_n]$$

we have $[a_0; a_1, a_2, ...] = [a_0; a_1, a_2, ..., \alpha_n]$ and $\alpha_n = [a_n, a_{n+1}, a_{n+2}, ...]$

Theorem 2.3.3. *The continued fraction expansion of real number is infinite if and only if it is irrational. This theorem can be stated alternatively as follows. Continued fraction* $[a_0; a_1, ...]$ *represents an irrational number.*

Conversely, given an irrational number α *there is unique representation of* α *as an infinite simple continued fraction* $\alpha = [a_0; a_1, ...]$

The continued fraction expansion of quadratic irrational is periodic. In particular $\sqrt{2} = [1, \bar{2}]$ and conversely if the continued fraction expansion of a real number is periodic then we can write $\alpha = a + \frac{1}{\alpha}$ and leads to the equation of the form $a\alpha^2 + b\alpha + c = 0$. In general in 1770, Lagrange established the result.

Theorem 2.3.4. *If* α *is a quadratic irrational number, then its continued fraction* $\alpha = [a_0; a_1, ..., a_n]$ *is ultimately periodic.*

The irrationals numbers $\sqrt{2}$ and $\sqrt{2} + 1$ are equivalent and the continued fraction of $\sqrt{2} = [1, \bar{2}$ and $\sqrt{2} + 1 = [\bar{2}]$. In general in 1878, Serret proved the theorem.

Theorem 2.3.5. *Let* α *and* β *be two irrational numbers with continued fractions* $\alpha = [a_0; a_1, ..., a_n, ...]$ *and* $\beta = [b_0; b_1, ..., b_m, ...]$ *respectively. Then the two following properties are equivalent.*

1. *There exists a matrix of order* 2×2 *with with rational integer coefficients* a, b, c, d *and determinant* ± 1 *such that* $\beta = \dfrac{a\alpha + b}{c\alpha + d}$
2. *There exists* $n_0 \geq 0$ *and* $m_0 \geq 0$ *such that* $a_{n_0+k} = b_{m_0+k}$ *for all* $k \geq 0$.

Chapter 3

Continued fraction and Pell's Fermat Equation

3.1 Introduction

Diophantine equation is the equation of the form $x^2 - Dy^2 = N$, where D is a given positive square free integer, N is non zero integer and integer solutions are sought, is called Pell's Fermat equation. In Cartesian coordinates, the equation has the form of a hyperbola, solutions occur whenever the curve passes through a point whose x and y coordinates are both integers, such as the trivial solution with x = 1 and y = 0. Joseph Louis Lagrange proved that, as long as D is not a perfect square, Pell's Fermat equation has infinitely many distinct integer solutions. These solutions may be used to accurately approximate the square root of D by rational numbers of the form $\frac{x}{y}$.

This equation was first studied extensively in India, starting with Brahmagupta, who developed the Chakravala method to solve the equation and other quadratic indeterminate equations in his Brahma Sphuta Siddhanta in 628, about a thousand years before Pell's time. His Brahma Sphuta Siddhanta was translated into Arabic in 773 and was subsequently translated into Latin in 1126. Bhaskara II in the 12th century and Narayana Pandit in the 14th century both found general solutions to the equation and other quadratic indeterminate equations. Solutions to specific examples of the equation, such as the Pell's numbers arising from the equation with D = 2, had been known for much longer, since the time of Pythagoras in Greece and to a similar date in India. The name of Pell's Fermat equation arose from Leonhard Euler's mistakenly attributing Lord Brouncker's solution of the equation to John Pell[14].

3.2 Some Useful Definitions

Definition 3.2.1. *A real number $x \in R$ is a quadratic irrational , if there exist $a, b, c \in Z$ and $a \neq 0$ such that $ax^2 + bx + c = 0$ and $D = b^2 - 4ac > 0$ and D is not a perfect square . Consequently, the solutions of this equation are quadratic irrational numbers, that can be expressed as:* $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

From this definition quadratic irrationals have to satisfy two conditions: they must be the solution of a quadratic equation and be irrational.

Definition 3.2.2. *A quadratic irrational Q is said to be reduced if $Q > 1$ is the root of a quadratic equation with integer coefficients whose conjugate root $\bar{Q}$ lies between -1 and 0.*

Definition 3.2.3. *A continued fraction which is periodic from the first partial quotient is called purely periodic. We denote as:*

$$[a_0; a_1, a_2, a_3, ..., a_{k-1}, a_k, a_0, a_1, a_2, a_3, ..., a_{k-1}, a_k, ...] = \overline{[a_0; a_1, a_2, a_3, ..., a_{k-1}, a_k]}$$

If the period starts with the second partial quotient, the continued fraction is called simply periodic which is represented by:

$$[a_0; a_1, a_2, a_3, ..., a_{k-1}, a_k, a_1, a_2, a_3, ..., a_{k-1}, a_k, ...] = [a_0; \overline{a_1, a_2, a_3, ..., a_{k-1}, a_k}]$$

Then, a purely periodic is also simplify periodic but the reverse is not true.

3.3 Continued Fraction of $\sqrt{D}$ and Pell's Fermat equation

Quadratic irrational are the irrational root α of a quadratic equation $ax^2 + bx + c = 0$,where a, b, c are integers. The second root of the equation will be denoted α_0 and called the (algebraic) conjugate of α. In order to state the theorem describing continued fractions of quadratic irrationals, we need to recall that a continued fraction $[a_0; ..., a_n, ...]$ is called even-tually periodic if

$$[a_0; ..., a_n, ...] = [a_0; ..., a_k, a_{k+1}, ..., a_l]$$

starts with a preperiod $[a_0; ..., a_k]$ and then a period $a_{k+1}, ..., a_l]$ is repeated an infinite number of times.

Theorem 3.3.1. *(Lagrange) Let α is a real number. The continued fraction of α is eventually periodic if and only if $\alpha$is a quadratic irrational.*

Theorem 3.3.2. *(Galois) Let α be a quadratic irrational and α_0 its conjugate. The continued fraction of α is purely periodic if and only if $\alpha > 1$ and α_0 is in $(-1, 0)$*

Example Let $\alpha = \dfrac{1+\sqrt{5}}{2}$ i.e., the so-called Golden ratio, then it is the root of $x^2 - x - 1 = 0$ and $\alpha_0 = \dfrac{1-\sqrt{5}}{2}$ is in $(-1, 0)$. The continued fraction of α is indeed purely periodic since $\alpha = \dfrac{1+\sqrt{5}}{2} = \bar{1}$. if we restrict our consideration to square roots of natural numbers, we will make use of the following lemma.

Lemma 3.3.1. *Let α be a quadratic irrational and α_0 its conjugate. If α has a purely periodic continued fraction $\overline{[a_0; a_1, ..., a_n]}$, then $\frac{-1}{\alpha_0} = \overline{[a_n, ..., a_1, a_0]}$*

Theorem 3.3.3. *Quadratic irrationals are the real numbers that can be exactly represented by periodic continued fractions.*

Theorem 3.3.4. *There is a one-to-one correspondence between a real number and a con- tinued fraction, which is either finite or infinite.*

Theorem 3.3.5. *The continued fraction which represents a quadratic irrational Q is purely periodic if and only if Q is a reduced surd.*

Lemma 3.3.2. *For any positive integer D that is not a perfect square, the continued fraction of $\sqrt{D}$ is simply periodic and more precisely it has the form*

$$\sqrt{D} = [a_0; \overline{a_1, a_2, ...a_n, 2a_0}]$$

Let a_0 = Lower integral part of $\sqrt{D}$. Since D is a positive integer, $\sqrt{D} + a_0 > 1$. Because D is not a perfect square, we have $0 < \sqrt{D} - a_0 < 1$ and conjugate of $a_0 + \sqrt{D}$ lies between -1 and 0, which means $-1 < -\sqrt{D} + a_0 < 0$ then it is the reduced irrationals and using above theorem it must be purely periodic.

$$\sqrt{D} + a_0 = \overline{[2a_0; a_1, a_2, ..., a_n = 4a_0]}$$

which is equivalent to

$$\sqrt{D} + a_0 = [2a_0; \overline{a_1, a_2, ..., a_n, 2a_0}]$$

Consequently, we have

$$\sqrt{D} = [a_0; \overline{a_1, a_2, ...a_n, 2a_0}]$$

.

Theorem 3.3.6. *There is a non trivial solution (x, y) in positive integers to the equation $x^2 - Dy^2 = \pm 1$. Hence there are infinitely many solutions in positive integers and there is a smallest one, the fundamental solution (x_1, y_1). For any integer n and every choice of the sign $\pm$, a solution (x, y) in rational integers is given by $\left(x_1 + y_1\sqrt{D}\right)^n = x + \sqrt{D}y$.*

If the fundamental solution $x_1^2 - Dy_1^2 = \pm 1$ produces the positive sign, then the equation $x^2 - Dy^2 = -1$ has no solution. If the fundamental solution $x_1^2 - Dy_1^2 = \pm 1$ produces the negative sign then the fundamental solution of the equation $x^2 - Dy^2 = 1$ is (x_2, y_2) with

$$\left(x_2 + y_2\sqrt{D}\right) = \left(x_1 + \sqrt{D}y_1\right)^2$$

Hence $x_2 = x_1^2 + Dy_1^2$ and $y_2 = 2x_1y_1$. The solutions of $x^2 - Dy^2 = 1$ are the (x_n, y_n) with n is even. The solutions of $x^2 - Dy^2 = -1$ are obtained with n is odd.

If (x, y) is a solution, then the equation $x^2 - Dy^2 = \pm 1$ written as

$$\frac{x}{y} - \sqrt{D} = \pm \frac{1}{y\left(x + y\sqrt{D}\right)}$$

shows that $\frac{x}{y}$ is a good rational approximation to $\sqrt{D}$. All the problem now is to find the fundamental solution. Let D be a positive integer which is not a square .

Then the continued fraction of the number $\sqrt{D}$ is periodic. If k is the smallest period of length (that means that the length of any period is a positive integer multiple of k) this continued fraction can be written $\sqrt{D} = [a_0; \overline{a_1, a_2, ..., a_k}]$with $a_k = 2a_0$ and a_0=lower integral part of $\sqrt{D}$. Further $(a_0; a_1, ..., a_{k-1})$ is a palindrome $a_j = a_{k-j}$ for $1 \leq j < k - 1$.

The rational number given by the continued fraction $[a_0; a_1, ..., a_{k-1}]$ is a good rational approximation to $\sqrt{D}$ If k is even, the fundamental solution of the equation $x^2 - Dy^2 = 1$ is given by $[a_0; a_1, ..., a_{k-1}] = \frac{x_1}{y_1}$. In this case the equation $x^2 - Dy^2 = -1$ has no solution.

If k is odd, the fundamental solution (x_1, y_1) of the equation $x^2 - Dy^2 = -1$ is given by the fraction and $[a_0; a_1, ..., a_{k-1}] = \frac{x_1}{y_1}$ and the fundamental solution $\frac{x_2}{y_2}$of the equation $x^2 - Dy^2 = 1$ by the fraction $[a_0; a_1, ..., a_{k-1}, a_k, a_1, ..., a_{k-1}] = \frac{x_2}{y_2}$.

In both cases where k is either even or odd, we obtain the sequence $(x_n, y_n)_{n \geq 1}$ of all solutions by repeating n-1 times $a_1, a_2, ..., a_k$ followed by $a_1, a_2, ..., a_{k-1}$

Here, we try to find the fundamental solutions and general solutions of some important pell's Fermat equations with some examples.

Example 3.3.1. *1. Find the fundamental solution of the pell's equation $x^2 - 2y^2 = \pm 1$*

Here $D = 2$ and $\sqrt{D} = \sqrt{2} = [1; \bar{2}]$. Hence the continued fraction expansion is periodic with period length 1 which is odd . Hence the fundamental solution of

$x^2-2y^2=-1$ is $x_1=1$ and $y_1=1$ and the fundamental solution of $x^2-2y^2=1$ is given by $[1,2]=1+\frac{1}{2}=\frac{3}{2}$

2. *Solve the pell's equation $x^2-3y^2=1$. Here D=3 and $\sqrt{D}=\sqrt{3}=[1;\overline{1,2}]$. The continued fraction expansion with period of length 2 which is even so the the equation $x^2-3y^2=1$ has fundamental solution is $[1,1]=1+1=\frac{2}{1}=\frac{x_1}{y_1}$. In this case the equation $x^2-3y^2=-1$ has no solution.*

3. *Find the fundamental solution of Brahmagupta's equation $x^2-92y^2=\pm1$. The continued fraction expansion of*
$$\sqrt{92}=[9;\overline{1,1,2,4,2,1,1,18}].$$
Hence the continued fraction expansion is periodic with the length of period 8 which is even so the equation $x^2-92y^2=1$ has fundamental solution
$$\frac{x_1}{y_1}=[9;1,1,2,4,2,1,1]=\frac{1151}{120}.$$
Since the length is even so the equation $x^2-92y^2=-1$ has no solution

4. *Solve the Narayana's Equation $x^2-92y^2=1$. Here*
$$\sqrt{103}=[10;\overline{6,1,2,11,9,1,1,2,1,6,20}],$$
with period of length 12 which is even and the fundamental solution of the equation $x^2-92y^2=1$ is given by
$$\frac{x_1}{y_1}=[10;6,1,2,11,9,1,1,2,1,6]=\frac{227528}{22419}$$

5. *Solution of Bhaskara Equation $x^2-61y^2=\pm1$, here*
$$\sqrt{61}=[7;\overline{1,4,3,1,2,2,1,3,4,1,14}],$$
with period of length 11 which is odd and the fundamental solution of the equation $x^2-61y^2=-1$ is given by
$$\frac{x_1}{y_1}=[7;1,4,3,1,2,2,1,3,4,1]=\frac{29718}{3805}$$

and the fundamental solution to the equation $x^2 - 61y^2 = 1$ *is*

$$\frac{x_1}{y_1} = [7; 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1] = \frac{1766319049}{226153980}$$ [24][14]

.

3.3.1 Ramanujan, Real World Problem and its Solution Using Continued Fraction

The famous mathematician Ramanujan possessed a real world problem and its solution by Pell's fermat equation. The problem is Imagine there are a bunch of houses on a street, the house numbers of which are $1, 2, 3....$ Now, your friend lives in a house where the sum of the house numbers to the left of his house and to the right of his house is the same (his house is not included in either sum).

If there are fewer than 10 houses on the street, how many houses are there and what is the house number of your friend? By simple arithmetic the house number of your friend is 6 and there are 8 houses on the street. History reports that Ramanujan solved this same problem with the boundaries between 50 and 500 houses on the street in a matter of seconds. There must be a connection to continued fraction in his solution, so observe how laboriously the common math mind would attempt this problem. Letting m = the number of houses on the street and n = the particular house number, notice the connecting equation:

$$1 + 2 + ... + (n - 1) = (n + 1) + (n + 2) + ... + m$$

$$\frac{n(n-1)}{2} = \frac{m(m+1)}{2} - \frac{n(n+1)}{2}$$

The left-hand side reflects the number of house before the desired one multiplied by the desired house number divided by 2 to get the sum of the house numbers up to but not including the desired house number. The right-hand side reflects the sum of all the house numbers combined minus the desired house number and the overlap to avoid double counting.

$$\frac{m(m+1)}{2} = \frac{n(n-1)+n(n+1)}{2}$$

$$\frac{m(m+1)}{2} = n^2$$

Multiplying both sides of the equation by 8 then

$$8\frac{m(m+1)}{2} = 8n^2$$

The above equation can be written as $(2m+1)^2 - 2\,(2n)^2 = 1$ To put this equation into simpler terms of x and y, write $2m+1 = x$ and $2n = y$ resulting in: $x^2 - 2y^2 = 1$ By Lagranges method using continued fractions, in order to solve an equation in the format of $x^2 - Dy^2 = 1$, express $\sqrt{D}$ as a periodic infinite continued fraction. Our D = 2.

$\sqrt{2} = [1; 2, 2, 2, 2, ...]$ Since the period is 1 so all convergents are solutions. Since an infinite continued fraction is an expression that represents the sum of its integer part and the reciprocal of another number, written as a sum of its integer and another reciprocal, and so on, the convergents of the $\sqrt{2}$ continued fraction would represent the number of houses on the street, the numerator and the house number, the denominator. The first 10 convergents of $\sqrt{2}$ are:

$$1 = \frac{1}{1}, 1 + \frac{1}{2} = \frac{3}{2}, [1; 2, 2] = \frac{7}{5}, [1; 2, 2, 2] = \frac{17}{12}, [1; 2, 2, 2, 2] = \frac{41}{29}$$
$$[1; 2, 2, 2, 2, 2] = \frac{99}{70}, [1; 2, 2, 2, 2, 2, 2] = \frac{239}{169}, [1; 2, 2, 2, 2, 2, 2, 2] = \frac{577}{408},$$
$$[1; 2, 2, 2, 2, 2, 2, 2, 2] = \frac{1393}{985}, [1; 2, 2, 2, 2, 2, 2, 2, 2, 2] = \frac{3363}{2378}$$

Every other convergent has an even denominator. To show the solution pairs that solve the problem posed to Ramanujan, notice for the convergents $\frac{x}{y}$with even denominators there are $m = \frac{x-1}{2}$ houses and friend lives in the house number$n = \frac{y}{2}$.

Convergent- 2:$m = \dfrac{3-1}{2} = 1$ and$n = \dfrac{2}{2} = 1$ so the solution $(m,n) = (1,1)$.The convergent-4 the solution is $(8,6)$. Similarly from the successive convergents 6,8 and 10 respectively we can obtain the solutions $(49,35)$,$(288,204)$ and $(1681,1189)$[11]. Thus one could find the number of houses and the house number of the friend for a street that was infinitely long with the stated stipulations.

3.3.2 Pell's Fermat Equation with Different values of D

Here,we considered some specific Pell's Fermat equations and their integer solutions. Further, we focus to find the fundamental and general solution of these Pell's Fermat equations with the continued fraction expansion of $\sqrt{D}$, for some specific values of D namely $D = k^2+1, k^2-1, k^2+2, k^2-2, k^2+k$ and k^2-k , where k is any positive integer.

Theorem 3.3.7. *1. Let $k \geq 1$ be any integer, and let $D = k^2+1$. The continued fraction expansion of $\sqrt{D}$ is $\sqrt{D} = [1;\bar{2}]$ if $k = 1$ and $[k;\bar{2k}]$. if $k > 1$ and $(x_1, y_1) = (2k^2+1, 2k)$ is the fundamental solution.*

2. *Set (x_n, y_n), where $\dfrac{x_n}{y_n} = [k; 2k, ..., 2k]$, $2k$ in $2n-1$ number of times for $n \geq 2$. Then (x_n, y_n) is a solution of $x^2 - (k^2+1)y^2 = 1$.*

3. *The consecutive solutions (x_n, y_n) and (x_{n+1}, y_{n+1}) satisfy*

$$x_{n+1} = (2k^2+1)x_n + (2k^3+2k)y_n$$

$$y_{n+1} = 2kx_n + (2k^2+1)y_n$$

for $n \geq 1$.

1. Let $D = k^2+1$. If $k = 1$, then it is easily seen that $\sqrt{2} = [1;\bar{2}]$. Let $k > 1$, then

we easily get

$$\begin{aligned}\sqrt{k^2+1} &= k+\sqrt{k^2+1}-k \\ &= k+\cfrac{1}{\frac{1}{\sqrt{k^2+1}-k}} \\ &= k+\cfrac{1}{\sqrt{k^2+1}+k} \\ &= k+\cfrac{1}{2k+\sqrt{k^2+1}-k}\end{aligned} \tag{3.1}$$

So $\sqrt{D} = [k;\bar{2k}]$. Again let $\sqrt{D} = [a_0;\overline{a_1, a_2, ..., a_n}]$ denote the continued fraction expansion of period length l. Set $p_{-2} = 0, p_{-1} = 1, q_{-2} = 1, q_{-1} = 0$ and $p_n = a_n p_{n-1} + p_{n-2}, q_n = a_n q_{n-1} + q_{n-2}$ for non negative integer n. Then it is given that $C_n = \frac{p_n}{q_n}$ is the nth convergent of $\sqrt{D}$, and the fundamental solution of $x^2 - Dy^2 = 1$ is $(x_1, y_1) = (p_{l-1}, q_{l-1})$ if l is even and (p_{2l-1}, q_{2l-1}) if l is odd. Moreover, if l is odd, then the fundamental solution of $x^2 - Dy^2 = -1$ is $(x_1, y_1) = (p_{l-1}, q_{l-1})$. we see as above that $\sqrt{D} = [k;\bar{2k}]$. $p_0 = k, p_1 = 2k^2 + 1, q_0 = 1$ and $q_1 = 2k$.

Therefore

$$\begin{aligned}(x_1, y_1) &= (p_{2l-1}, q_{2l-1}) \\ &= (p_1, q_1) \\ &= (2k^2+1, 2k)\end{aligned} \tag{3.2}$$

is the fundamental solution.

2. Indeed $(2k^2 + 1)^2 - (k^2 + 1)(2k)^2 = 1$.
   Now we assume that $(x_n, y_n)$is a solution of $x^2 - (k^2 + 1)y^2 = 1$. Then$x_n^2 -$

$(k^2+1)y_n^2 = 1$ then we have from

$$
\begin{aligned}
\frac{x_{n+1}}{y_{n+1}} &= k+\cfrac{1}{2k+\cfrac{1}{2k+\cfrac{1}{2k+\cfrac{1}{2k+\cfrac{1}{2k+\cfrac{1}{\ddots+2k+\frac{1}{2k}}}}}}} \\
&= k+\cfrac{1}{2k+\cfrac{1}{k+k+\cfrac{1}{2k+\cfrac{1}{2k+\cfrac{1}{2k+\cfrac{1}{\ddots+2k+\frac{1}{2k}}}}}}} \\
&= k+\cfrac{1}{2k+\cfrac{1}{k+\cfrac{x_n}{y_n}}} \\
&= \frac{(2k^2+1)x_n+(2k^3+2k)y_n}{2kx_n+(2k^2+1)y_n}
\end{aligned}
\tag{3.3}
$$

is also a solution of the equation. For $n+1$th solution, $2k$'s are repeated in $2(n+1)-1 = 2n+1$ number of times but in n th solution it repeates for $2n-1$ number of times so in transition from $n+1$th solution in to nth solution it loose two terms of $2k$.

Since

$$x_{n+1}^2-(k^2+1)y_{n+1}^2 = [(2k^2+1)x_n+(2k^3+2k)y_n]^2-(k^2+1)[2kx_n+(2k^2+1)y_n]^2$$

$$= (2k^2+1)^2x_n^2+2(2k^2+1)(2k^3+2k)y_nx_n$$

$$+[(2k^3+2k)y_n]^2-(k^2+1)(4k^2x_n^2+4k(2k^2+1)y_nx_n+(2k^2+1)^2y_n^2$$

$$= (2k^2+1)^2x_n^2 - 4k^2(k^2+1) + y_nx_n(2(2k^2+1)(2k^3+2k) - 4k(k^2+1)(2k^2+1))$$

$$+y_n^2((2k^3+2k)^2 - (k^2+1)(2k^2+1)^2)$$
$$= x_n^2 - (k^2+1)y_n^2$$
$$= 1$$

3. The assertion is clear by 2 since $x_{n+1} = (2k^2+1)x_n + (2k^3+2k)y_n$ and $y_{n+1} = 2kx_n + (2k^2+1)y_n$

Example 3.3.2. *Let* $k = 4$*, then* $D = k^2 + 1 = 17$*. Find the fundamental solution and the other solutions.*

Continued fraction expansion of $\sqrt{17} = [4, \bar{8}]$. Further, the fundamental solution of $x^2 - 17y^2 = 1$ is $(x_1, y_1) = (33, 8)$ and since

$$\frac{2177}{528} = [4; 8, 8, 8]$$
$$\frac{143649}{34840} = [4; 8, 8, 8, 8, 8]$$
$$\frac{9478657}{2298912} = [4; 8, 8, 8, 8, 8, 8, 8]$$
$$\frac{625447713}{151693352} = [4; 8, 8, 8, 8, 8, 8, 8, 8, 8]$$
$$\frac{41270070401}{10009462320} = [4; 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8]$$

the other solutions are

$$(x_2, y_2) = (2177, 528), (x_3, y_3) = (143649, 34840)$$

$$(x_4, y_4) = (9478657, 2298912), (x_5, y_5) = (625447713, 151693352)$$
$$(x_6, y_6) = (41270070401, 10009462320).$$

The fundamental solution of $x^2 - Dy^2 = -1$ for $D = k^2 + 1$ also exist since the continued fraction expansion of $\sqrt{D} = \sqrt{17}$ is periodic with odd period length 1. The fundamental solution is $\dfrac{p_{l-1}}{q_{l-1}} = \dfrac{p_0}{q_0} = \dfrac{k}{1}$. Which shows that $x = 4$ and $y = 1$ is the fundamental solution of the equation $x^2 - 17y^2 = -1$. The other solutions are obtained from $\dfrac{p_3}{q_3} = 4 + \dfrac{1}{8 + \dfrac{1}{8}} = \dfrac{268}{65}$,$\dfrac{p_5}{q_5}$ and so on.

Theorem 3.3.8. *Let $k \geq 2$ be any integer, and let $D = k^2 - 1$*

1. *The continued fraction expansion of $\sqrt{D}$ is $\sqrt{D} = [k-1; \overline{1, 2k-2}]$*
2. *The fundamental solution, $(x_1, y_1) = k - 1 + \frac{1}{1} = (k, 1)$.*
3. *Set (x_n, y_n) where $\frac{x_n}{y_n} = [k-1; 1, 2k-2, ..., 1, 2k-2, 1, 2k-1]$ $n-2$ times, for $n \geq 2$.Then (x_n, y_n) is a solution of $x^2 - (k^2-1)y^2 = 1$*
4. *The consecutive solutions (x_n, y_n) and (x_{n+1}, y_{n+1})satisfy $x_{n+1} = kx_n + (k^2 - 1)y_n$ $y_{n+1} = x_n + ky_n, n \geq 1$*

Proof: Here

$$\begin{aligned}
\sqrt{k^2-1} &= k - 1 + \sqrt{k^2+1} - (k-1) \\
&= k - 1 + \frac{1}{\frac{1}{\sqrt{k^2-1}-(k-1)}} \\
&= k - 1 + \frac{1}{\frac{\sqrt{k^2-1}+k-1}{2k-2}} \\
&= k - 1 + \frac{1}{1 + \frac{1}{\sqrt{k^2-1}+k-1}} \\
&= k - 1 + \frac{1}{1 + \frac{1}{2k-2+\sqrt{k^2-1}-(k-1)}}
\end{aligned} \tag{3.4}$$

$\sqrt{D} = [k-1; \overline{1, 2k-2}]$. Since the period $l = 2$ which is even so the fundamental solution,(p_{l-1}, q_{l-1})= $(x_1, y_1) = k - 1 + \frac{1}{1} = (k, 1)$

$$\begin{aligned}
\frac{x_{n+1}}{y_{n+1}} &= k-1+\cfrac{1}{1+\cfrac{1}{2k-2+\cfrac{1}{1+\cfrac{1}{2k-2+\cfrac{1}{1+\cfrac{1}{\ddots+1+\frac{1}{2k-1}}}}}}} \\
&= k-1+\cfrac{1}{1+\cfrac{1}{k-1+k-1+\cfrac{1}{1+\cfrac{1}{2k-2+\cfrac{1}{1+\cfrac{1}{\ddots+1+\frac{1}{2k-1}}}}}}} \\
&= k-1+\cfrac{1}{1+\cfrac{1}{k-1+\cfrac{x_n}{y_n}}}
\end{aligned}$$

$$= \frac{(k^2-1)y_n + kx_n}{ky_n + x_n} \tag{3.5}$$

In n+1 solutions the period repeat for $n-1$ times and in nth solution it repeat for $n-2$ times so in the transition from $n+1$ to n we have loose one value of $2k-2$.

Example 3.3.3. *Taking $k = 2$ then $D = k^2 - 1 = 3$, find the fundamental solution and other solutions.*

$$
\begin{aligned}
\sqrt{3} &= 1+\sqrt{3}-1 \\
&= 1+\frac{1}{\frac{\sqrt{3}+1}{2}} \\
&= 1+\frac{1}{1+\frac{1}{2+\sqrt{3}-1}} \\
&= [1,\overline{1,2}] \qquad (3.6)
\end{aligned}
$$

The period of the continued fraction expansion is 2 which is even so the fundamental solution is

$$\frac{p_{l-1}}{q_{l-1}} = \frac{p_1}{q_1} = k-1+\frac{1}{1} = \frac{k}{1} = \frac{2}{1}.$$

The other solutions are

$$(x_2, y_2 = (2^2-1)y_1 + 2x_1, 2y_1 + x_1) = (3\times 1 + 2\times 2, 2\times 1 + 2) = (7,4)$$

and so on[4].

Chapter 4

Different Approach of Rational Approximation

This chapter covered the different approach of rational approximation. In this context, the rational approximation by using the Dirichlet's box principle, continued fraction expansion of real number and solution of Pell's Fermat equation are discussed.

4.1 Computational Approach of Rational Approximation using Continued Fraction

We have defined some basic terms of continued fraction in chapter two. To further our exploration, we want to consider some infinite continued fractions and offer evidence that continued fractions provide, in some sense, the best approximation for any given real number. It is true that any real number can be expressed as a continued fraction.

If we start with an infinite continued fraction, a convergent is the sequence of values we get if we truncate the infinite continued fraction to compute corresponding finite continued fractions, i.e. if our infinite continued fraction is $[a_0; a_1, a_2, ..]$, then in general, the nth convergent or approximant is the value obtained by evaluating the finite continued fraction, $[a_0; a_1, a_2, ..., a_n]$. These values will limit to the real number represented by the infinite continued fraction. Evaluating the convergent to n decimal places adds a level of exactness. Consider the example $\sqrt{5} \approx 2.236067977$.

The corresponding continued fraction is $\sqrt{5} = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{\ddots}}}} = [2; 4, 4, 4, ...]$.

Within the first four terms the convergents are
$2 = 2, \frac{9}{4} = 2.25, \frac{38}{17} = 2.2352941176, \frac{161}{72} = 2.2361111$.
we are rapidly approaching $\sqrt{5} \approx 2.236067977$. As is seen by this example, using the convergents is one of the fastest ways to approximate irrational numbers. Consequently, there are well-known constants that mathematicians have explored through continued fractions, for example π, e, and ϕ. Each of those values and any other irrational number, have infinitely many rational approximations. We can think of the corresponding finite continued fractions as the best rational estimates for the irrational number in the sense that expressing the irrational number as a continued fraction finds a much better approximation for the number with the same number of terms than the 11 to 13 digits a calculator.

The continued fractions provide more accurate estimation depending upon which convergent the value is computed. As you move from left to right in the continued fraction the estimates get better. In the above example of $\sqrt{5}, \frac{161}{72}$ is a better approximation than $\frac{38}{17}$, which is a better approximation than $\frac{9}{4}$.

It is interesting to note that the convergents of a rational number alternate between being larger and smaller values than the exact value of the irrational number. The convergent will never equal the exact value of the irrational number but they do limit to the exact value. Think of convergents as the behavior the finite continued fractions produce as terms are added. We can see for the numbers π,e andϕ see in[11].

4.2 Basic Results of Approximation using Dirichlet's Box Principle

In this section we discuss the basic method of rational approximation by using Dirichlet's box principle.In this context we state some basic results of rational approximation.

Theorem 4.2.1. *Given an irrational* α *there exist an integer p such that*

$$|\alpha - p| < \frac{1}{2} \tag{4.1}$$

Let α is given irrational number and p is the nearest integer of α then

$$\frac{-1}{2} < \alpha - p < \frac{1}{2}$$

Theorem 4.2.2. *Let* α *be any irrational number and* $'q'$ *be any positive integer. Then there is a rational number with denominator* q *say* $\frac{p}{q}$ *such that*

$$\frac{-1}{2q} < \alpha - \frac{p}{q} < \frac{1}{2q} \tag{4.2}$$

Theorem 4.2.3. *Given any irrational number* α *and any positive integer* $'k'$*, there is a rational number* $\frac{p}{q}$ *whose denominator* $'q'$ *does not exceed* $'k$*,such that*

$$\frac{-1}{kq} < \lambda - \frac{p}{q} < \frac{1}{qk}, q \leq k \tag{4.3}$$

Our objective is to try for better approximation of irrational numberα.
From the approximation of α by $\frac{p}{q}$ to within $\frac{1}{2q}$ for any 'q' in the previous theorem. Now the approximation is within $\frac{1}{q^2}$

Theorem 4.2.4. *Given any irrational number* α*, there are infinitely many rational numbers* $\frac{p}{q}$ *in lowest terms such that*

$$\frac{-1}{q^2} < \alpha - \frac{p}{q} < \frac{1}{q^2} \tag{4.4}$$

First we observe that any rational number $\dfrac{p}{q}$ satisfying the inequality of theorem

$$\frac{-1}{kq} < \alpha - \frac{p}{q} < \frac{1}{kq}$$

where $q \leq k$ satisfies the inequality of this theorem

$$\frac{-1}{q^2} < \alpha - \frac{p}{q} < \frac{1}{q^2}$$

since

$$\frac{1}{k} \leq \frac{1}{q}$$

implies

$$\frac{1}{kq} < \frac{1}{q^2}$$

Hence any number which lies in between

$$\frac{-1}{kq}, \frac{1}{kq}$$

must certainly lies between the range between $\dfrac{-1}{q^2}, \dfrac{1}{q^2}$ It is possible to prove the following stronger version of theorem [4.2.4]

Theorem 4.2.5. *Given any irrational number α, there are infinitely many rational numbers $\frac{p}{q}$ in lowest terms such that*

$$\frac{-1}{q(q+1)} < \alpha - \frac{p}{q} < \frac{1}{q(q+1)}$$

4.2.1 Limitations on Approximations

We proved in theorem [4.2.2] that, corresponding to any irrational number α, there are infinitely many rational numbers $\dfrac{p}{q}$ such that

$$\frac{-1}{2q} < \alpha - \frac{p}{q} < \frac{1}{2q}$$

Then, in the theorem [4.2.4] we established that there are infinitely many $\dfrac{p}{q}$ such that

$$\frac{-1}{q^2} < \alpha - \frac{p}{q} < \frac{1}{q^2}$$

Is it possible to prove that there are infinitely many $\frac{p}{q}$ such that

$$\frac{-1}{2q^2} < \alpha - \frac{p}{q} < \frac{1}{2q^2}$$

The answer is Yes, although we shall not prove it here,In fact, there is a famous theorem which states that there are infinitely many $\dfrac{p}{q}$ corresponding to any irrational number α such that

$$\frac{-1}{\sqrt{5}q^2} < \alpha - \frac{p}{q} < \frac{1}{\sqrt{5}q^2}$$

and furthermore that $\sqrt{5}$ is the constant which yields the best possible approximation of this kind. This means that if $\sqrt{5}$ is replaced by any larger constant, the statement becomes false.

To give some idea as to how it is possible to prove that there is a limit on the size of the constant, we establish the following result. There are not infinitely many rational numbers $\dfrac{p}{q}$ such that

$$\frac{-1}{5q^2} < \sqrt{2} - \frac{p}{q} < \frac{1}{5q^2} \tag{4.5}$$

In fact we prove that this inequality is impossible for any integer 'q' greater than 10.

Proof:

We assume that the inequality is impossible for any integer 'q' greater than 10. We assume that the inequality holds for some integers p and q, with $q > 10$. The inequality $\frac{-1}{5q^2} < \sqrt{2} - \frac{p}{q}$ implies, for $q > 10$,that

$$\frac{p}{q} < \frac{1}{5q^2} + \sqrt{2} < \sqrt{2} + \frac{1}{500} < 2 \tag{4.6}$$

On the other hand, the inequality

$$\sqrt{2}-\frac{p}{q}<\frac{1}{5q^2}$$

implies, for $q>10$

$$\frac{p}{q}>\frac{-1}{5q^2}+\sqrt{2}>\sqrt{2}-\frac{1}{500}>1 \tag{4.7}$$

Now if we add $\frac{p}{q}$ to the members of the inequalities of this theorem, we get

$$\frac{p}{q}-\frac{1}{5q^2}<\sqrt{2}<\frac{p}{q}+\frac{1}{5q^2} \tag{4.8}$$

From (4.7) we see that

$$\frac{p}{q}-\frac{1}{5q^2}>1-\frac{1}{5q^2}>1-\frac{1}{500}>0 \tag{4.9}$$

$$\left(\frac{p}{q}-\frac{1}{5q^2}\right)^2<2<\left(\frac{p}{q}+\frac{1}{5q^2}\right) \tag{4.10}$$

$$\frac{p^2}{q^2}-\frac{2p}{5q^3}+\frac{1}{25q^4}<2<\frac{p^2}{q^2}+\frac{2p}{5q^3}+\frac{1}{25q^4}$$

Multiplying by q^2,we get

$$p^2-\frac{2p}{5q}+\frac{1}{25q^2}<2q^2<p^2+\frac{2p}{5q}+\frac{1}{25q^2} \tag{4.11}$$

Now by equation (4.6) we see that

$$p^2+\frac{2p}{5q}+\frac{1}{25q^2}<p^2+\frac{4}{5}+\frac{1}{25q^2}<p^2+\frac{4}{5}+\frac{1}{2500}<p^2+1 \tag{4.12}$$

On the other hand, by equation(4.6), we can write

$$p^2-\frac{2p}{5q}+\frac{1}{25q^2}>p^2-\frac{2p}{5n}>p^2-\frac{4}{5}>p^2-1 \tag{4.13}$$

Applying equations (4.11) and (4.12) to (4.13) , we obtain

$$p^2-1<p^2-\frac{2p}{5q}+\frac{1}{25q^2}<2q^2<p^2+\frac{2p}{5q}+\frac{1}{25q^2}<p^2+1, \tag{4.14}$$

Then $p^2-1<2q^2<p^2+1$. But $2q^2$ is an integer, so if it lies between the integers p^2-1 and p^2+1, it must equal to p^2. Hence we conclude that$2q^2=p^2, 2=\frac{p^2}{q^2}, \sqrt{2}=\frac{p}{q}$. This is a contradiction, since $\sqrt{2}$ is irrational, while p and q were assumed to be integers[8].

4.3 Application of solution of Pell's Fermat Equation for Rational Approximation of $\sqrt{D}$

In this section we state some results of Pell's Fermat equation and their application for rational approximation of quadratic irrationals. Given a positive integer D which is not a square, there exists $(x, y) \in Z^2$ with x and y both are positive such that $x^2 - Dy^2 = 1$. The first step of the proof is to show that there exists a non zero integer k such that the Diophantine equation $x^2 - Dy^2 = k$ has infinitely many solutions in Z^2. The main idea behind the proof. which will be made explicit the two lemmas and one corollary below is to relate the integer solution of such a Diophantine equation with rational approximation $\frac{x}{y}$ of $\sqrt{D}$.

We deduce that there are infinitely many (x, y) in Z^2 with $y > 0$ and hence $x > 0$ satisfying $|\sqrt{D} - \frac{x}{y}| < \frac{1}{y^2}$. For such a (x, y), we have $0 < x < y\sqrt{D} + 1 < y\left(\sqrt{D} + 1\right)$. Hence $0 < |x^2 - Dy^2| = |x - y\sqrt{D}||x + y\sqrt{D}| < 2\sqrt{D} + 1$.Since there are only finitely integers $k \neq 0$ in the range $-\left(2\sqrt{D} + 1\right) < k < 2\sqrt{D} + 1$. At least one of them is of the form $x^2 - Dy^2$ for infinitely many (x, y)

Lemma 4.3.1. *Let D be a positive integer which is not a square. Let x and y be positive rational integers . The following conditions are equivalent.*

1.
$$x^2 - Dy^2 = 1$$

2.
$$0 < \frac{x}{y} - \sqrt{D} < \frac{1}{2y^2\sqrt{D}}$$

3.
$$0 < \frac{x}{y} - \sqrt{D} < \frac{1}{y^2\sqrt{D} + 1}$$

Let $x^2 - Dy^2 = 1$ is true, we need to show that(2) is true.

We claim that

$0 < \frac{x}{y} - \sqrt{D}$. Since $x^2 - Dy^2 = 1$,then $\left(x + \sqrt{D}y\right)\left(x - y\sqrt{D}\right) = 1$.

Again, since $\left(x + \sqrt{D}y\right) > 0$

$\Rightarrow \left(x - y\sqrt{D}\right) > 0$

$\Rightarrow 0 < \frac{x}{y} - \sqrt{D}$.

Here,

$$\left(x - y\sqrt{D}\right) = \frac{1}{\left(x + \sqrt{D}y\right)}$$

$\Rightarrow$

$$\begin{aligned} \frac{x}{y} - \sqrt{D} &= \frac{1}{\left(x + \sqrt{D}y\right)y} \\ &= \frac{1}{xy + y^2\sqrt{D}} < \frac{1}{2y^2\sqrt{D}} \end{aligned} \tag{4.15}$$

Claim:

$$\begin{gathered} 2y^2\sqrt{D} < xy + y^2\sqrt{D} \\ \Rightarrow y^2\sqrt{D} < xy \\ \Rightarrow y\sqrt{D} < x \\ \Rightarrow \sqrt{D} < \tfrac{x}{y} \end{gathered}$$

which is true.

Now let (2)is true. We need to show that (3) is also true.

Since we have, $0 < \dfrac{x}{y} - \sqrt{D} < \dfrac{1}{2y^2\sqrt{D}}$. $\Rightarrow 0 < \dfrac{x}{y} - \sqrt{D} < \dfrac{1}{y^2\sqrt{D} + 1}$.

Claim:

$$y^2\sqrt{D} + 1 < 2y^2\sqrt{D}.$$

$\Rightarrow 1 < y^2\sqrt{D}$. Which is true

Now (3) implies (1)

Since

$$\begin{aligned} 0 &< \frac{x}{y} - \sqrt{D} \\ \Rightarrow 0 &< \frac{x^2}{y^2} - D \\ \Rightarrow 1 &\leq x^2 - Dy^2 \end{aligned} \tag{4.16}$$

Now

$$\begin{aligned} 1 \leq (x - \sqrt{D}y)(x + \sqrt{D}y) &< \frac{y(x + \sqrt{D}y)}{y^2\sqrt{D} + 1} \\ &< 2 \end{aligned} \tag{4.17}$$

Claim:

$$\begin{aligned} xy + \sqrt{D}y^2 &< 2y^2\sqrt{D} + 2 \\ \Rightarrow xy &< y^2\sqrt{D} + 2 \\ \Rightarrow x &< y \quad \sqrt{D} + \frac{2}{y} \\ \Rightarrow \frac{x}{y} - \sqrt{D} &< \frac{2}{y^2} \end{aligned} \tag{4.18}$$

is true since $\dfrac{x}{y} - \sqrt{D} < \dfrac{1}{y^2\sqrt{D} + 1} < \dfrac{2}{y^2}$
Therefore $1 \leq x^2 - Dy^2 < 2$ and hence $x^2 - Dy^2 = 1$.

Lemma 4.3.2. *Let D be a positive integer which is not a square, x and y be positive rational integers then the following conditions are equivalent.*

1. $x^2 - Dy^2 = -1$.

2. $0 < -\frac{x}{y} + \sqrt{D} < \frac{1}{2y^2\sqrt{D}-1}$.

3. $0 < -\frac{x}{y} + \sqrt{D} < \frac{1}{y^2\sqrt{D}}$.

Corollary 4.3.1. *Let D be a positive integer which is not a square. Let x and y be positive rational integers. The following conditions are equivalent.*

1. $x^2 - Dy^2 = \pm 1$.

2. $|\sqrt{D} - \frac{x}{y}| < \frac{1}{2y^2\sqrt{D}-1}$.

3. $|\sqrt{D} - \frac{x}{y}| < \frac{1}{y^2\sqrt{D}+1}$.

It is instructive to compare with Liouville's inequality

Lemma 4.3.3. *Let D be a positive integer which is not a square. Let x and y be positive rational integers. Then $|\sqrt{D} - \dfrac{x}{y}| > \dfrac{1}{2y^2\sqrt{D}+1}$ [14].*

Chapter 5

Development of Rational Approximation

In this chapter we discuss about application of continued fraction in rational approximation. In this context we will discss about the Hurwitz's theorem. Further it also covered the advances of Hurwitz's result for irrationals.

5.1 Basic Theorems of Rational Approximation

Definition 5.1.1. *Two irrational numbers α and β are called $GL\,(2, \mathbb{Z})$ equivalent if there exists integers a, b, c, d with $ad - bc = \pm 1$ such that $\beta = \dfrac{a\alpha + b}{c\alpha + d}$. This relation is an equivalence relation.*

The irrational number $\phi = \dfrac{1 + \sqrt{5}}{2} = [1; 1, 1, ...] = [\bar{1}]$ which is purely periodic and $\alpha = \dfrac{\sqrt{5} - 1}{2} = [0; 1, 1, 1, ...] = [0; \bar{1}]$ are equivalent.

Definition 5.1.2. *An irrational number whose continued fraction expansion has from stage onwards exclusively $1'$s as its partial entries is called Noble number. The irrational number $\dfrac{\sqrt{5} + 1}{2}$ and all its equivalent irrational numbers are Noble numbers.*

Definition 5.1.3. *An irrational number α is badly approximable if there is a constant $c = c(\alpha) \geq 0$ such that*

$$|\alpha - \frac{p}{q}| > \frac{c}{q^2},$$

for every rational $\frac{p}{q}$, such a constant c must satisfy $0 < c \leq \frac{1}{\sqrt{5}}$[17].

A badly approximable numbers are precisely the numbers whose partial quotients are bounded. A rational number $\alpha = \frac{a}{b}$ may be obviously and perfectly approximated by

$$\frac{p_i}{q_i} = \frac{ia}{ib}$$

for every positive integer i. we have

$$\frac{p}{q} \neq \alpha = \frac{a}{b},$$

$$|\frac{a}{b} - \frac{p}{q}| = |\frac{aq - bp}{bq}| \geq \frac{1}{bq}$$

because $|aq - bp|$ is a positive integer and is thus not lower than 1. Thus the accuracy of the approximation is bad relative to irrational numbers. In summary, a rational number is perfectly approximated by itself, but is badly approximated by any other rational number. The Golden number and silver numbers are also badly approximable irrationals numbers but golden number are more bad then the silver number. If the distance between given irrational number α and a rational number $\frac{p}{q}$ is less than $\frac{1}{2q^2}$ then the rational number $\frac{p}{q}$ is one of the convergents of the simple continued fraction expansion of the irrational α In the continued fraction expansion of irrational real numbers, one of the two successive convergents is the best rational approximate. Vahlen in 1895 proved the result.

Theorem 5.1.1. *Let $\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}$ be consecutive convergents to α. Then at least one of them satisfies*

$$|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$$

Since the numbers

$$\alpha - \frac{p_{n-1}}{q_{n-1}}, \alpha - \frac{p_n}{q_n}$$

are of opposite sign,hence we have

$$|\alpha - \frac{p_{n-1}}{q_{n-1}}| + |\alpha - \frac{p_n}{q_n}| = |\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n}| = \frac{1}{q_n q_{n-1}} \leq \frac{1}{2q_n^2} + \frac{1}{2q_{n-1}^2}$$

Since $ab < \dfrac{a^2+b^2}{2}$ and $a \neq b$. Hence, either

$$|\alpha - \frac{p_n}{q_n}| < \frac{1}{2q_n^2}$$

or

$$|\alpha - \frac{p_{n-1}}{q_{n-1}}| < \frac{1}{2q_{n-1}^2}.$$

This completes the proof of the theorem.

In the continued fraction expansion of irrational number at least one of the three successive convergents is the more refined best rational approximation then from the Vahelen rational approximation. E.Borel in 1903 established the result.

Theorem 5.1.2. *Let* $\dfrac{p_{n-2}}{q_{n-2}}, \dfrac{p_{n-1}}{q_{n-1}}, \dfrac{p_n}{q_n}$ *be three consecutive convergents to* α*. Then at least one of them satisfies*

$$|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}.$$

Proof: Let $\alpha = [a_0; a_1, ...]$ and $\alpha_i = [a_i, a_{i+1}, ...]$ and $\beta_i = \dfrac{q_{i-2}}{q_{i-1}}$ for $i = n-1, n, n+1$,we have

$$\alpha = [a_0; a_1, ..., a_n, \alpha_{n+1}]$$

so

$$\alpha q_n - p_n = \frac{(\alpha_{n+1} p_n + p_{n-1})\, q_n}{\alpha_{n+1} q_n + q_{n-1}} - p_n = \frac{(-1)^n}{\alpha_{n+1} q_n + q_{n-1}}.$$

Thus

$$|\alpha - \frac{p_n}{q_n}| = \frac{1}{q_n(\alpha_{n+1} q_n + q_{n-1})} = \frac{1}{q_n^2(\alpha_{n+1} + \beta_{n+1})}$$

Now to prove the theorem, it suffices to show that there cannot be three integers $i = n-1, n, n+1 with$[17].

$$\alpha_i + \beta_i \leq \sqrt{5} \tag{5.1}$$

Supposee that inequality(5.1) were true for $i = n, n-1$. Now

$$\alpha_{n-1} = a_{n-1} + \frac{1}{\alpha_n}$$

and

$$\frac{1}{\beta_n} = \frac{q_{n-1}}{q_{n-2}} = a_{n-1} + \frac{q_{n-3}}{q_{n-2}} = a_{n-1} + \beta_{n-1}$$

. Therefore

$$\frac{1}{\alpha_n} + \frac{1}{\beta_n} = \alpha_{n-1} + \beta_{n-1} \leq \sqrt{5}.$$

Therefore

$$1 = \alpha_n \frac{1}{\alpha_n} \leq (\sqrt{5} - \beta_n)(\sqrt{5} - \frac{1}{\beta_n}),$$

which is equivalent to

$$\beta_n^2 - \sqrt{5}\beta_n + 1 \leq 0,$$

it follows that $\beta_n \geq \frac{\sqrt{5}-1}{2}$, since β_n is rational so $\beta_n > \frac{\sqrt{5}-1}{2}$. If the inequality also were true for $i = n, n+1$, then $\beta_{n+1} > \frac{\sqrt{5}-1}{2}$ and therefore

$$1 \leq a_n = \frac{q_n}{q_{n-1}} - \frac{q_{n-2}}{q_{n-1}} = \frac{1}{\beta_{n+1}} - \beta_n < \frac{2}{\sqrt{5}-1} - \frac{\sqrt{5}-1}{2} = 1,$$

a contradiction. This implies that for at least one $i, \alpha_i + \beta_i > \sqrt{5}$. This completes the proof.

Theorem 5.1.3 (Legendre)**.** *Suppose p,q are relatively prime integers with $q > 0$ and if the rational $\frac{p}{q}$ satisfies*

$$|\alpha - \frac{p}{q}| \leq \frac{1}{2q^2},$$

then $\frac{p}{q}$ is a convergent of α.

If we assume that $\alpha \neq \frac{p}{q}$, else the theorem is trivially true. Then we may write

$$\alpha - \frac{p}{q} = \frac{\epsilon\theta}{q^2},$$

where $0 < \theta < \frac{1}{2}$ and $\epsilon = \pm 1$. Since there is an expansion of

$$\frac{p}{q} = [b_0, b_1, ..., b_{n-1}],$$

where n is so chosen that $(-1)^{n-1} = \epsilon$. Now define w by

$$\alpha = \frac{wp_{n-1} + p_{n-2}}{wq_{n-1} + q_{n-2}}$$

so that $\alpha = [b_0, b_1, ..., b_{n-1}, w]$. Note that the definition of w is equivalent to

$$(\alpha q_{n-1} - p_{n-1})w = p_{n-2} - \alpha q_{n-2}.$$

We may assume that $\alpha q_{n-1} - p_{n-1} \neq 0$, else

$$\alpha = \frac{p_{n-1}}{q_{n-1}} = \frac{p}{q}$$

then

$$\frac{\epsilon\theta}{q^2} = \alpha - \frac{p}{q} = \frac{\alpha q_{n-1} - p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{(wq_{n-1} + q_{n-2})q_{n-1}}[17]$$

and therefore

$$\theta = \frac{q_{n-1}}{wq_{n-1} + q_{n-2}}.$$

Solving this for w, we obtain

$$w = \frac{q_{n-1} - \theta q_{n-2}}{\theta q_{n-1}} = \frac{1}{\theta} - \frac{q_{n-2}}{q_{n-1}},$$

it follows that $w > 2 - 1 = 1$. Now expand w into finite or infinite continued fraction $w = [b_n, b_{n+1}, ...]$. Since $w > 1$, each of these integers $b_j, j = n, n+1...$ is positive and therefore

$$\alpha = [b_0, b_1, ..., b_{n-1}, [b_n, b_{n+1}, ...]] = [b_0, b_1, ..., b_{n-1}, b_n, b_{n+1}, ...].$$

By lemma (2.2.2) passing to the limit if necessary. This is a simple continued fraction for α, and

$$\frac{p}{q} = \frac{p_{n-1}}{q_{n-1}} = [b_0, b_1, ..., b_{n-1}]$$

is a convergent to α, so the theorem is proved.

5.2 Hurwitz Theorems for Irrationals

Theorem 5.2.1 (Hurwitz, 1891)**.** *The nth convergent of* $\frac{1}{\alpha}$ *is the reciprocal of the* $(n-1)$*st convergent of* α *if* α *is any real number bigger than 1.*

Lemma 5.2.1. *if* α *is real,* $\alpha > 1$*, and* $\alpha + \frac{1}{\alpha} < \sqrt{5}$*, then* $\alpha < \frac{\sqrt{5}+1}{2}$ *and* $\frac{1}{\alpha} > \frac{\sqrt{5}-1}{2}$

Adolf Hurwitz was born Born: 26 March 1859 in Hildesheim, Lower Saxony, Germany into a Jewish family. His father, Salomon Hurwitz, was in the manufacturing business but was not particularly well off. Salomon had three sons, Max, Julius and Adolf, but their daughter Jenny died at the age of one. Sadly, Adolf's mother Elise Wertheimer died when he was only three years old. Salomon Hurwitz offered his sons a good education, encouraging them to engage in music, gymnastics, Jewish traditions and smoking, "as he could scarcely imagine a proper gentleman without a cigar or even better a pipe. The three brothers all had a particular talent for mathematics. He Died, 18 November 1919 in Zrich, Switzerland

Theorem 5.2.2 (Hurwitz)**.** *Given any irrational number*α*,there exist infinitely many rational numbers* $\frac{p}{q}$ *such that*

$$|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2} \tag{5.2}$$

To prove the theorem it is sufficient to prove that for any three consecutive convergents of simple continued fraction expansion of α at least one of them satisfies the inequality of the theorem. Let $\beta_n = \frac{q_n}{q_{n-1}}$ we first claim that

$$\beta_n + \frac{1}{\beta_n} < \sqrt{5} \tag{5.3}$$

If (5.2) is false for both convergents $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$. Suppose that (5.2) is false for these two values of $\frac{p}{q}$. We have

$$|\alpha - \frac{p_{n-1}}{q_{n-1}}| + |\alpha - \frac{p_n}{q_n}| \geq \frac{1}{\sqrt{5}q_{n-1}^2} + \frac{1}{\sqrt{5}q_n^2}$$

But α lies between $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$ and hence we have

$$|\alpha - \frac{p_{n-1}}{q_{n-1}}| + |\alpha - \frac{p_n}{q_n}| = |\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n}| = \frac{1}{q_{n-1}q_n}$$

Combining these results we get

$$\frac{1}{q_{n-1}q_n} \geq \frac{1}{\sqrt{5}q_{n-1}^2} + \frac{1}{\sqrt{5}q_n^2}$$

and it reduced in

$$\frac{q_n}{q_{n-1}} + \frac{q_{n-1}}{q_n} \leq \sqrt{5}.$$

Since the left hand side is rational we actually have a strict inequality,and we claim (5.3). Now if (5.2) is false for $\frac{p}{q} = \frac{p_j}{q_j}, j = n-1, n, n+1$ we then have (5.3) for both $j = n$ and $j = n+1$ then by lemma we see that $\frac{1}{\beta_n} > \frac{\sqrt{5}-1}{2}$ and $\beta_{n+1} < \frac{\sqrt{5}+1}{2}$. Since we have $\beta_{n+1} = a_{n+1} + \frac{1}{\beta_n}$.

This gives that

$$\frac{\sqrt{5}+1}{2} > \beta_{n+1} = a_{n+1} + \frac{1}{\beta_n} > a_{n+1} + \frac{\sqrt{5}-1}{2} \geq 1 + \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2}.$$

Which is a contradiction. This completes the proof of the theorem[25][17][3][23].

5.2.1 Error Analysis for Best Approximation

If $E < \frac{1}{2q^2}$ be the error and $M = \frac{1}{\sqrt{5}q^2}$ is the Hurwitz's bound. then the rational approximation with $\frac{E}{M} < 1.118$ is a best approximation of the given irrationals.

This table suggest that π admits much better rational approximations than $\sqrt{2}$. In fact no rational approximates to $\sqrt{2}$ ever gets an $\frac{E}{M}$ ratio as small as 0.13, let alone 0.007, and $\sqrt{2}$ is really harder to approximate with rationals than π. In this precise sense $\sqrt{2}$ is a more irrational number than π[3]

5.3 Advances of Hurwitz Theorem

Theorem 5.3.1. *Suppose α has a continued fraction expansion of type*

$$\alpha = [a_0; a_1, ..., a_N, 1, 1, 1, ...],$$

then

$$\lim_{n\to\infty} q_{n^2}|\alpha - \frac{p_n}{q_n}| = \frac{1}{\sqrt{5}}$$

We have

$$|\alpha - \frac{p_n}{q_n}| = \frac{1}{q_n^2\,(\alpha_{n+1} + \beta_{n+1})}$$

Here if n is sufficiently large,

$$\alpha_{n+1} = [1, 1, 1, ...] = \frac{\sqrt{5}+1}{2}$$

and $\dfrac{1}{\beta_{n+1}} = \dfrac{q_n}{q_{n-1}} = [a_n, a_{n-1}, ..., a_N, ..., a_1, a_0] = [1, 1, 1..., 1, a_N, ..., a_1, a_0]$. Since $[1, 1, 1..., 1, 1]$with $n - N - 1$ terms and $[1, 1, ..., 1]$ with n-N terms are consecutive convergents to $\dfrac{1}{\beta_{n+1}}$, the number $\dfrac{1}{\beta_{n+1}}$ lies in between these convergents, and therefore $\dfrac{1}{\beta_{n+1}}$ approaches to $[1, 1, ..., 1] = \frac{\sqrt{5}+1}{2}$ as n tends to infinity. Hence β_{n+1} tends to $\frac{\sqrt{5}-1}{2}$and $\alpha_{n+1} + \beta_{n+1}$ approach to $\sqrt{5}$. This completes the proof of theorem.

Theorem 5.3.2 (Hurwitz Theorem)**.** *For the rational $\alpha = \dfrac{\sqrt{5}+1}{2} = [1; 1, 1, ...]$. The inequality*

$$|\alpha - \frac{p_n}{q_n}| < \frac{1}{Cq_n^2} \tag{5.4}$$

is satisfied for at most finitely many reduced $\dfrac{p_n}{q_n}$ if and only if $C > \sqrt{5}$

Let $\alpha = \dfrac{\sqrt{5}-1}{2} = [0; 1, 1, 1, ...]$ and fix $C > \sqrt{5}$ and suppose that

$$|\alpha - \frac{p}{q}| < \frac{1}{Cq^2}$$

has infinitely many solutions. Here q may be as large as we wish. In other words

$$\alpha = \frac{p}{q} + \frac{\delta}{q^2}$$

has infinitely many solutions. $\frac{p}{q}$ is rational and δ is in R,$|\delta| < \frac{1}{C}$. Rewrite this as

$$\frac{\delta}{q} - \frac{q\sqrt{5}}{2} = -\frac{q}{2} - p.$$

After simplification we have

$$\frac{\delta^2}{q^2} - \delta\sqrt{5} = p^2 + pq - q^2$$

For sufficiently big q the left hand side is absolute value less than 1 and hence $p^2 + pq - q^2 = 0$ has integer solution. which gives rise $(2p+q)^2 = 5q^2$ leads in to $\sqrt{5} = 1 + 2\frac{p}{q}$ is rational. Again this is a contradiction.

Theorem 5.3.3 (Hurwitz Theorem, 1891)**.** *Suppose $\alpha = [a_0; a_1, a_2, ...]$with $a_n \geq 2$ for infinitely many q. Then there are infinitely many distinct rationals $\dfrac{p}{q}$ with*

$$|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{8}q^2}$$

Corollary 5.3.1. *The inequality*

$$|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{8}q^2}$$

has infinitely many rational solutions $\dfrac{p}{q}$ whenever α is not equivalent to $\frac{\sqrt{5}+1}{2}$

Let $\beta = \sqrt{2} - 1 = [0; 2, 2, 2......]$ and fix $A > \sqrt{8}$ and suppose that

$$|\beta - \frac{p}{q}| < \frac{1}{Aq^2}$$

Here q may be as large as we wish. In other words

$$\beta = \frac{p}{q} + \frac{\delta}{q^2}$$

has infinitely many solutions. $\dfrac{p}{q}$ is rational and δ is in R, $|\delta| < \frac{1}{A}$

Rewrite this as

$$\frac{\delta}{q} = \beta q - p = \sqrt{2}q - q - p$$

Then $\frac{\delta}{q} - \sqrt{2}q = -p - q$ After simplification we have

$$\frac{\delta^2}{q^2} - \delta 2\sqrt{2} = p^2 + 2pq - q^2$$

For sufficiently big q the left hand side is absolute value less than 1 and hence $p^2+2pq-q^2 = 0$ has integer solution. which gives rise $(p+q)^2 = 2q^2$ leads in to $\sqrt{2} = 1 + \frac{p}{q}$ is rational. Again this is a contradiction[12].

Let $\beta = \frac{\sqrt{13}-3}{2} = [0; 3, 3, 3, ...]$ and fix $A > \sqrt{13}$ and suppose that

$$|\beta - \frac{p}{q}| < \frac{1}{Aq^2}$$

Here q may be as large as we wish. In other words

$$\beta = \frac{p}{q} + \frac{\delta}{q^2}$$

has infinitely many solutions. $\dfrac{p}{q}$ is rational and δ is in R ,$|\delta| < \frac{1}{A}$.
Rewrite this as

$$\frac{\delta}{q} - \frac{q\sqrt{5}}{2} = -\frac{q}{2} - p$$

. After simplification we have

$$\frac{\delta^2}{q^2} - \delta\sqrt{13} = p^2 + 3pq - q^2$$

For sufficiently big q the left hand side is absolute value less than 1 and hence p^2+3pq-

$q^2 = 0$ has integer solution. which gives rise $(p+\frac{3q}{2})^2 = \frac{13q^2}{4}$ leads in to $\sqrt{13} = 3+2\frac{p}{q}$ is rational. Again this is a contradiction[12]

In general if we take $\beta = \dfrac{\sqrt{n^2+4}-n}{2}$ the best possible constant is $K = \sqrt{n^2+4}$. For the constant $A > K$ the equation

$$|\beta - \frac{p}{q}| < \frac{1}{Aq^2}$$

has only finitely many solutions.

Definition 5.3.1.

$$\nu(\alpha) = \lim_{q\to\infty} q|q\alpha - p|$$

Remark:$\nu(\alpha) = 0$ whenever α is rational. It is clear that $\nu(\alpha) > 0$ if and only if α is badly approximable. It is clear that from Hurwitz's theorem $\nu(\alpha) \leq \dfrac{1}{\sqrt{5}}$ for every real number. There are numbers α with $\nu(\alpha) = \dfrac{1}{\sqrt{5}}$ for $\alpha = \dfrac{\sqrt{5}+1}{2}$ and $\nu(\alpha) \leq \dfrac{1}{\sqrt{8}}$ whenever α is not equivalent to $\alpha = \dfrac{\sqrt{5}+1}{2}$.

Theorem 5.3.4 (Markoff,1879, Hurwitz, 1906)**.** *There exists numbers $\mu_1 = \dfrac{1}{\sqrt{5}} > \mu_2 = \dfrac{1}{\sqrt{8}} > \mu_3 > \mu_4..$ with limit $\frac{1}{3}$ such that for every μ_i there are finitely many equivalence classes of numbers such that $\nu(\alpha) = \mu_i$, precisely if α lies in such a class. Furtheremore, if $\mu > \frac{1}{3}$ and $\mu \neq \mu_i, i = 1, 2, 3, ...$, there is no α with $\nu(\alpha) = \mu$*

Theorem 5.3.5. *Let $k \geq 1$ and α be a real irrational number and not equivalent to the element in $F(k-1)$. Then there exists infinitely many rational numbers $\dfrac{p}{q}$ such that*

$$|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{k^2+4}q^2}$$

The constant $\dfrac{1}{\sqrt{k^2+4}}$ is the best possible.

For k=1,this theorem gives Hurwit'z theorem. For k=2 the constant is $\frac{1}{\sqrt{8}}$,k=3 the constant is $\dfrac{1}{\sqrt{13}}$. Similarly for k=4, the largest constant is $\dfrac{1}{\sqrt{20}}$ and so on.
Facts for $\theta = \dfrac{\sqrt{k^2+4}-k}{2} = [0; k, k, k.......]$ the largest constant is $\dfrac{1}{\sqrt{k^2+4}}$ for k=1 the irrational number is equivalent to $\frac{\sqrt{5}-1}{2}$ the best possible constant is $\frac{1}{\sqrt{5}}$. Now for k=2, the rational numbers are equivalent with $\sqrt{2}-1 = [0; 2, 2, 2, ...]$ or not equivalent with $\dfrac{\sqrt{5}-1}{2}$ the best possible constant is $\dfrac{1}{\sqrt{8}}$

Similarly for k=3, the irrationals which are equivalent with$\dfrac{\sqrt{13}-3}{2} = [0; 3, 3, 3, ...]$ or not equivalent with $\sqrt{2}-1$ the best possible constant is $\dfrac{1}{\sqrt{13}}$ Again for k=4, the irrationals $\sqrt{5}-2 = [0; 4, 4, 4.....]$ that are not equivalent with $\dfrac{\sqrt{13}-3}{2}$ the best possible constant is $\dfrac{1}{\sqrt{20}}$[7]

Theorem 5.3.6. *For any $n \geq 0$,*

$$|\alpha - \frac{h_n}{k_n}| < \frac{1}{k_n k_{n+1}}$$

and

$$|\alpha k_n - h_n| < \frac{1}{k+1}$$

Theorem 5.3.7 (Lagrange)**.** *The sequence*

$$|q_n \alpha - p_n|$$

for $n \geq 0$ is strictly decreasing and for $n \geq 1$ and $1 \leq q \leq q_n$ and if $(p, q) \neq (p_n, q_n)$ and $(p, q) \neq (p_{n-1}, q_{n-1})$, then

$$|\alpha q - p| > |\alpha q_{n-1} - p_{n-1}|.$$

Remark: It follows from the theorem that if $1 \leq q \leq q_n, (p, q) \neq (p_n, q_n)$,then

$$|\alpha q - p| > |\alpha q_n - p_n|$$

which is sometimes is called the law of best approximation.

$$|q_n\alpha - p_n| = \frac{1}{\alpha_{n+1}q_n + q_{n-1}} < \frac{1}{q_n + q_{n-1}}$$

and

$$|\alpha q_{n-1} - p_{n-1}| = \frac{1}{\alpha_n q_{n-1} + q_{n-2}} > \frac{1}{(a_n + 1)q_{n-1} + q_{n-2}} = \frac{1}{q_n + q_{n-1}}$$

which proves the first part.

Now define a and b by the equations

$$ap_n + bp_{n-1} = p, aq_n + bq_{n-1} = q.$$

The matrix determined by these two equations has determinant ± 1 hence a, b are integers if b=0 then $(p,q) = a(p_n, q_n)$, but this is impossible since $0 \leq q \leq q_n$ and $(p,q) \neq (p_n, q_n)$. if a=0, then $(p,q) = b(p_{n-1}, q_{n-1})$. But since $(p,q) \neq (p_{n-1}, q_{n-1})$, we have $b \geq 2$,and therefore

$$|\alpha q - p| \geq 2(|\alpha q_{n-1} - p_{n-1}|) > \alpha q_{n-1} - p_{n-1}.$$

If both $a \neq 0$ and $b \neq 0$ then since $1 \leq q \leq q_n$, a and b are of opposite sign. Hence $a(\alpha q_n - p_n)$ and $b(\alpha q_{n-1} - p_{n-1})$ are of the same sign, and therefore

$$|\alpha q - p| = |a(\alpha q_n - p_n)| + |b(\alpha q_{n-1} - p_{n-1})|.$$

Then

$$|\alpha q - p| > |\alpha q_{n-1} - p_{n-1}|.$$

Since $ab \neq 0$ and the result is established.

Theorem 5.3.8. *If $\frac{p}{q}$ is a rational number with positive denominator such that*

$$|\alpha - \frac{p}{q}| < |\alpha - \frac{p_n}{q_n}|$$

for some$n \geq 1$*, then* $q > q_n$*. In fact if*

$$|\alpha q - p| < |\alpha q_n - p_n|$$

for some $n \geq 0$*, then* $q \geq q_{n+1}$*[17].*

Chapter 6

Two Directions of Hurwitz's Theorem

In this chapter we discuss about the two directions of Hurwitz's theorem. Hurwitz's theorem gives the idea of general approximation of irrationals by rationals. If we want to discuss this result then, specifically there are two different directions. The first approach is by increasing the size of denominator so that the distance between the given irrational and rational approximations goes to arbitrarily small. This approach was discussed in chapter five. On the other hand by increasing the size of numerator so that the rational approximations are the linear combinations of numerator and denominator of convergents of the continued fraction with some conditions.

6.1 Finite Diophantine Approximation

Now in case only one rational approximation, and not infinitely many, is asked for, the bound $\frac{1}{\sqrt{5}}$ and $\frac{1}{\sqrt{8}}$ of Hurwitz's theorem can be improved. A.V Prasad in 1948 proved that to every irrational α there is at least one rational approximation with the bound $\frac{3+\sqrt{5}}{2}$.L.C Eggan gave the extension of the theorem of Prasad. He proved that if α is not equivalent with golden ratio and only one rational approximation is required then the bound can be improved with $\frac{\frac{3}{2}+\sqrt{2}}{2}$.

Theorem 6.1.1 (A.V Prasad 1948)**.** *For any irrational α there are at least m pairs of relatively prime integers p, q with $q > 0$ satisfying*

$$|\alpha - \frac{p}{q}| \leq \frac{1}{c_m q^2} \tag{6.1}$$

where $c_m = \frac{\sqrt{5}+1}{2} + \frac{h_{2m-1}}{k_{2m-1}}$ *where* $\frac{h_j}{k_j}$ *is the jth convergent to* $\frac{\sqrt{5}-1}{2}$*. Further if* $\alpha = \frac{\sqrt{5}-1}{2}$*, then there are exactly m solutions.*

A.V Prasad showed that the constant in (6.2) cannot be improved. In case $\alpha = \frac{\sqrt{5}-1}{2}$ In fact it is not hard to see that many irrational numbers, all equivalent to $\frac{\sqrt{5}-1}{2}$ also require this constant.

What about those irrationals not equivalent to $\frac{\sqrt{5}-1}{2}$? Can the constant be improved for these irrationals? The affirmative answer is a consequence of the following theorem.

Theorem 6.1.2 (Extension theorem of Prasad, L.C Eggan)**.** *Let $\alpha_2 = \sqrt{2} - 1$ and $\frac{p_j}{q_j}$ denote the jth convergent to α_2. Then for any irrationals α which are not equivalent to $\frac{\sqrt{5}-1}{2}$ and any positive integer m, there are at least 'm' solutions in relatively prime integers p, q with $q > 0$ to the inequality*

$$|\alpha - \frac{p}{q}| \leq \frac{1}{k_m q^2} \tag{6.2}$$

Where

$$k_m = \sqrt{2} + 1 + \frac{p_{2m-1}}{q_{2m-1}}.$$

Moreover if $\alpha = \alpha_2$ there are exactly 'm' solutions. Both this result and Prasad's theorem are special cases of the next theorem.

Theorem 6.1.3. *Let 'n' be a positive integer and let* $\alpha_n = [0; n, n, n....] = \frac{\sqrt{n^2+4}-n}{2}$*. For positive integer m, let*

$$c_m = \alpha_n + n + \frac{p_{2m-1}}{q_{2m-1}}$$

where $\frac{p_j}{q_j}$ is the jth convergent to α_n.

Then if $\alpha = [a_0; a_1, a_2, ...]$ is irrational and if $a_j \geq n$ for finitely many values of j, there are at least m solutions in relatively prime integers p, q with $q > 0$, to the inequality

$$|\alpha - \frac{p}{q}| \leq \frac{1}{c_m q^2} \tag{6.3}$$

Moreover the constant c_m cannot be improved since $\alpha = \alpha_n$ there are exactly 'm' solutions and equality attained.

Corollary 6.1.1. *If α is not equivalent to $\frac{\sqrt{5}-1}{2}$, then there is at least one pair of relatively prime integers p, q with $q > 0$ satisfying*

$$|\alpha - \frac{p}{q}| \leq \frac{1}{\left(\sqrt{2} + \frac{3}{2}\right) q^2} \tag{6.4}$$

Moreover $\alpha = \sqrt{2} - 1$ there is precisely one pair.

Corollary 6.1.2 (Perron)**.** *For positive integer n,if $\alpha = [0; n, n, n....$ Then there are infinitely many pairs of relatively prime integers p, q and $q > 0$.satisfying*

$$|\alpha - \frac{p}{q}| < \frac{1}{\left(\sqrt{n^2 + 4}\right) q^2,} \tag{6.5}$$

Moreover the constant can not be improved for this α

Corollary 6.1.3. *For any positive integer n, if α is as in theorem (6.1.3) but not equivalent to α_n, then there are infinitely many pairs of relatively prime integers p, q with $q > 0$ satisfying*

$$|\alpha - \frac{p}{q}| < \frac{1}{\left(\frac{\sqrt{n^2+4}+n}{2} + \frac{1}{n}\right) q^2} \tag{6.6}$$

[13]

6.2 One Approximation in Some Other Setting

As we noted in the preceding section, A.V Prasad considered the question of one approximation corresponding to the Hurwitz's theorem. There are other theorem stating the existence of infinitely many approximations about which the question of one approximation may be asked. In this section we consider three such theorems. Let

$$|\alpha - \frac{p}{q}| < \frac{k}{q^2}.$$

Scott[26] showed that if we restrict the fractions $\frac{p}{q}$ to be any one of the three classes (i) p,q both odd (ii)p odd and q even or (iii) p even and q odd, then there are infinitely many such $\frac{p}{q}$ satisfying the inequality. Other proofs of this result have been given by Robinson[20], Oppenheim, kuipers and Meulenbeld. Robinson also showed that if any pair of these classes were used, there would be infinitely many satisfying $k = \frac{1}{2}$. Here there are two results about which we can ask if we can improve the constants 1 and $\frac{1}{2}$ if we wish only one solution to the inequality. The answer is contained in the following theorem.

Theorem 6.2.1. *Let α be an irrational number and consider the three types of fractions $\frac{p}{q}$, where (i) both p and q are odd(ii) p odd and q even and (iii) p even and q odd,*

1. *If any one of three types of chosen, then there is at least one fraction $\frac{p}{q}$ of the chosen type satisfying*
$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}.$$
Moreover the constant 1 cannot be diminished.

2. *If any two of the three types are chosen, then there is at least one fraction of chosen type satisfying*
$$|\alpha - \frac{p}{q}| < \frac{1}{2q^2}.$$
Moreover the constant 2 can not be increased.

The first statement in each of (1) and (2) follows trivially from the Scott[26] and Robinson[20]. The last statement follow the following theorems

Theorem 6.2.2. *For any $\sigma > 0$ and for any choice of one of the three types, there is an irrational number α so that no fraction of the chosen types satisfy*

$$|\alpha - \frac{p}{q}| \leq \frac{1-\sigma}{q^2}.$$

Theorem 6.2.3. *For any $\sigma > 0$ and for any choice of two of three types, there is an irrational number α so that no fraction of the chosen types satisfies*

$$|\alpha - \frac{p}{q}| \leq \frac{1}{(2+\sigma)q^2}.$$

In 1945, Serge[6] proved the following theorem.

Theorem 6.2.4. *Every irrational α has infinitely many rational approximations $\frac{p}{q}$ such that*

$$\frac{-\gamma}{(\sqrt{1+4\gamma})q^2} < \alpha - \frac{p}{q} < \frac{1}{(\sqrt{1+4\gamma})q^2},$$

where $\gamma \geq 0$ is arbitrary.

Note that any even convergent satisfies the result for $\gamma = 0$. While $\gamma = 1$ the result is the classical theorem of Hurwitz. Since that time a number of people, including olds, Negoescu, Robinson, Levegue and Tornheim, we have considered this problem of symmetric approximation. Robinson showed further that for any $\sigma > 0$ and any irrational α, the inequality

$$\frac{-1}{(\sqrt{5}-\sigma)q^2} < \frac{p}{q} - \alpha < \frac{1}{(\sqrt{5}+1)q^2},$$

has infinitely many solutions. Thus on side of Hurwitz's theorem can be strengthened without essentially weakening the other are there inequalities of above form when only one solution is desired? In particular, is it possible to strengthened one side of Prasad's theorem without esentially weakening the other? The answer to these questions in the negative is contained in the following theorem.

Theorem 6.2.5. *Any statement of the form " For any irrational α, there is at least one pair of integers p, q so that*

$$\frac{-c}{q^2} \leq \frac{p}{q} - \alpha < \frac{d}{q^2}".$$

For some d, $0 \leq d < \frac{3-\sqrt{5}}{2}$ is false unless $c > \frac{1}{\sqrt{5}}$

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6.3 Worley's Results of Diophantine Approximation

The classical Legendre's theorem in Diophantine approximations states that if a real number α and a rational number $\frac{p}{q}$ we will always assume that $q \geq 1$ and satisfy the inequality

$$|\alpha - \frac{p}{q}| < \frac{1}{2q^2,} \tag{6.7}$$

then $\frac{p}{q}$ is a convergent of the continued fraction expansion of $\alpha = [a_0; a_1, ...]$. This result has been extended by Fatou in[18], who showed that if

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}.$$

Then $\frac{p}{q} = \frac{p_m}{q_m}$ or $\frac{p_{m+1} \pm p_m}{q_{m+1} \pm q_m}$, where $\frac{p_m}{q_m}$ denote the mth convergent ofα

In 1981, Worley [21] generalized these results to the inequality

$$|\alpha - \frac{p}{q}| < \frac{k}{q^2},$$

where k is an arbitrary positive real number, Worley's result was slightly improved by Dujella in 2004[**?**]

Theorem 6.3.1 (Worley [21], Dujella[2])**.** *Letα be a real number and let p and q be coprime nonzero integers, satisfying*

$$|\alpha - \frac{p}{q}| < \frac{k}{q^2,} \tag{6.8}$$

where k is a positive real number. Then $(p, q) = (rp_{m+1} \pm sp_m, rq_{m+1} \pm sq_m)$ for some $m \geq -1$ and non negative integers r and s such that $rs < 2k$

Theorem 6.3.2 (Worley [21], Theorem 2)**.** *If α is an irrational number, $k \geq \frac{1}{2}$ and $\frac{p}{q}$ is a rational approximation to α (in reduced form) for which the inequality (6.8) holds, then either $\frac{p}{q}$ is a convergent$\frac{p_m}{q_m}$ to α or $\frac{p}{q}$ has one of the following forms.*

1. $\frac{p}{q} = \frac{rp_{m+1} + sp_m}{rq_{m+1} + sq_m}$, $r > s$ *and* $rs < 2k$, *or* $r \leq s$ *and* $rs < k + \frac{r^2}{a_{m+2}}$

2. $\dfrac{a}{b} = \dfrac{sp_{m+1} - tp_m}{sq_{m+1} - tq_m}$, $t > s\, and\, ts < 2k$, $or\, t \leq s\, and\, ts(1 - \dfrac{t}{2s}) < k$

where r, s, t are positive integers.

Since the fraction $\dfrac{p}{q}$ is in reduced form, it is clear that in the statements of Theorems (6.2.1) and (6.2.2) we may assume that gcd $(r, s) = 1$ and gcd $(s, t) = 1$. Worley [[21], Corollary, p.206] also gave the explicit version of his result for k = 2: $|\alpha - \frac{p}{q}| < \frac{2}{q^2}$ implies

$$\frac{p}{q} = \frac{p_m}{q_m}, \frac{p_{m+1} \pm p_m}{q_{m+1} \pm q_m},$$

$$\frac{2p_{m+1} \pm p_m}{2q_{m+1} \pm q_m}, \frac{3p_{m+1} + p_m}{3q_{m+1} + q_m},$$

$$\frac{p_{m+1} \pm 2p_m}{q_{m+1} \pm 2q_m}, \frac{p_{m+1} - 3p_m}{q_{m+1} - 3q_m},$$

6.3.1 Explicit versions of Worley's theorem

We start by few details from the proof of Theorem (6.2.1), which will be useful in our future arguments. In particular, we will explain how the integer m appearing in the statement of Theorem (6.2.2) can be found. We assume that $\alpha < \dfrac{p}{q}$, since the other case is completely analogous. Let m be the largest odd integer satisfying $\alpha < \dfrac{p}{q} < \dfrac{p_m}{q_m}$. If $\dfrac{p}{q} > \dfrac{p_1}{q_1}$, we take $m = -1$, following the convention that $p_1 = 1, q_1 = 0$. Since $|p_{m+1}q_m - p_m q_{m+1}| = 1$, the numbers r and s defined by $p = rp_{m+1} + sp_m, q = rq_{m+1} + sq_m$ are integers, and since $\frac{p_{m+1}}{q_{m+1}} < \frac{p}{q} < \frac{p_m}{q_m}$. we have that $r \geq 0$ and $s > 0$.

From the maximality of m, we find that

$$\frac{sa_{m+2} - r}{qq_{m+2}} = |\frac{p_{m+2}}{q_{m+2} - \frac{a}{b}}| < |\alpha - \frac{p}{q}| < \frac{k}{q^2} \tag{6.9}$$

From (6.9) we immediately have

$$a_{m+2} > \frac{r}{s} \tag{6.10}$$

and we can derive the inequality

$$r^2 - sr_{am+2} + ka_{m+2} > 0 \tag{6.11}$$

Let us define a positive integer t by $t = sa_{m+2} - r$. Then we have $p = rp_{m+1} + sp_m = sp_{m+2} - tp_{m+1}$
$q = r_{qm+1} + sq_m = sq_{m+2} - tq_{m+1}$, and s and t satisfy analogs of (6.11) and (6.12)

$$a_{m+2} > \frac{t}{s} \tag{6.12}$$

$$t^2 - sta_{m+2} + ka_{m+2} > 0 \tag{6.13}$$

If $r > t$, i.e. $rs > st$, then we will represent p and q in terms of s and t Worley 1981 gave the explicit version of his result for k = 2. He showed, if a real number α and a rational number $\frac{p}{q}$ satisfy the inequality

$$|\alpha - \frac{p}{q}| < \frac{2}{q^2},$$

then

$$\frac{p}{q} = \frac{rp_{m+1} + sp_m}{rq_{m+1} + sq_m},$$

where

$$(r, s) \in R_2 = \{(0, 1), (1, 1), (1, 2), (2, 1), (3, 1)\},$$

or

$$\frac{p}{q = }\frac{p}{q} = \frac{sp_{m+2} - tp_{m+1}}{sq_{m+2} - tq_{m+1}},$$

where

$$(s, t) \in T_2 = \{(1, 1), (1, 2), (1, 3), (2, 1)\}$$

(for an integer $m \geq -1$).

In 2008, Andrej Dujella, Bernadin Ibrahimpasic[1] extended the Worley's work and gave the explicit and sharp versions of Theorems (6.2.1) and (6.2.2) for $k =$

$3, 4, 5, ..., 12$. Where, the list the pairs (r, s) which appear in the expression of solutions of (6.8) in the form

$$(p, q) = (rp_{m+1} \pm sp_m, rq_{m+1} \pm sq_m).$$

In 2013 bernadin Ibrahimpasic[5] extend the worley's work(and also the work of Dujella and Ibrahimpasic)and gave explicit and sharp version of theorem (6.2.1) and (6.2.2) for k=13. In their work the list of the pairs (r,s) which appear in the expression of the solution of inequality (6.8) in the form

$$(p, q) = (rp_{m+1} \pm sp_m, rq_{m+1} \pm sq_m),$$

and showed by explicit examples that all pairs from the list are indeed necessary. They proved some pattern in pairs(r, s) and (s, t) which appear in representation

$$(p, q) = (rp_{m+1} + sp_m, rq_{m+1} + sq_m)$$

and

$$(p, q) = (sp_{m+2} - tp_{m+1}, sq_{m+2} - tq_{m+1})$$

of solutions of the inequality (6.8)

6.4 Explicit Version of Worley's work in Diophantine Approximation

Theorem 6.4.1. *Let $k \geq 3$ be a integer. There exist a real number α and rational numbers $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ such that*

$$|\alpha - \frac{p_1}{q_1}| < \frac{k}{q_1^2}$$

and

$$|\alpha - \frac{p_2}{q_2}| < \frac{k}{q_2^2},$$

where
$$\frac{p_1}{q_1} = (rp_{m+1} + 2p_m, rq_{m+1} + 2q_m)$$
and
$$\frac{p_2}{q_2} = (2p_{m+2} - tp_{m+1}, 2q_{m+2} - tq_{m+1}),$$
for some $m \geq -1$ *and integers r and t such that* $1 \leq r, t \leq k - 1.$

Dujella and Ibrahimpasic [1] gave the following result.

Proposition

Let
$$k \in \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$
If a real number α and a rational number $\frac{p}{q}$ satisfy the inequality (6.8),
$$\frac{p}{q} = \frac{rp_{m+1} + sp_m}{rq_{m+1} + sq_m},$$
where
$$(r, s) \in R_k = R_{k-1} \cup \acute{R_k},$$
or
$$\frac{p}{q} = \frac{sp_{m+2} - tp_{m+1}}{sq_{m+2} - tq_{m+1}},$$
where
$$(s, t) \in T_k = T_{k-1} \cup \acute{T_k}$$
(for an integer $m > -1$), where the sets $\acute{R_k}$ and $\acute{T_k}$ are given in the following table. More- over, if any of the elements in sets R_k or T_k is omitted, the statement will no longer valid.

k	$\acute{R_k}$
4	(1, 4), (3, 2), (6, 1), (7, 1)
5	(1, 5), (2, 3), (8, 1), (9, 1)
6	(1, 6), (5, 2), (10, 1), (11, 1)
7	(1, 7), (2, 5), (4, 3), (12, 1) (13, 1)
8	(1, 8), (3, 4), (7, 2), (14, 1), (15, 1)
9	(1, 9), (5, 3), (16, 1), (17, 1)
10	(1, 10), (9, 2), (18, 1), (19, 1)
11	(1, 11), (2, 7), (3, 5), (20, 1), (21, 1)
12	(1, 12), (5, 4), (7, 3) (11, 2), (22, 1), (23, 1)

k	$\acute{T_k}$
4	(4, 1), (2, 3), (1, 6), (1, 7)
5	(5, 1), (3, 2), (1, 8), (1, 9)
6	(6, 1), (2, 5), (1, 10), (1, 11)
7	(7, 1), (5, 2), (3, 4), (1, 12), (1, 13)
8	(8, 1), (4, 3), (2, 7), (1, 14), (1, 15)
9	(9, 1), (3, 5), (1, 16), (1, 17)
10	(10, 1), (2, 9), (1, 18), (1, 19)
11	(11, 1), (7, 2), (5, 3), (1, 20), (1, 21)
12	(12, 1), (4, 5), (3, 7) (2, 11), (1, 22), (1, 23)

By Theorem (6.2.1), we have to consider only pairs of non negative integers (r, s) and (s, t) satisfying $rs < 2k, st < 2k,$ gcd(r, s) = 1 and gcd(s, t) = 1. Furthermore, as in the case k = 3, it follows directly from the inequalities (6.11) and (6.13) for r = 1, resp. t = 1, that the pairs $(r, s) = (1, s)$ and $(s, t) = (s, 1)$ with $s > k + 1$ can be omitted. Similarly, for r = 2 or 3, resp. t = 2 or 3, we can exclude the pairs (r, s) = (2, s) and (s, t) = (s, 2) with $s \geq \frac{k}{2} + 2$, and the pairs (r, s) = (3, s) and (s, t) = (s, 3) with $s \geq \frac{k}{3} + 3$

Proposition

If a real number α and a rational number $\frac{p}{q}$ satisfy the inequality (6.8)

$$|\alpha - \frac{p}{q}| < \frac{13}{q^2}$$

then

$$\frac{p}{q} = \frac{rp_{m+1} + sp_m}{rq_{m+1} + sq_m},$$

where

$$(r, s) \in R_{13}$$

= $R_{12} \bigcup \{(1, 13), (3, 7), (4, 5), (24, 1), (25, 1)\}$or

$$\frac{a}{b} = \frac{sp_{m+2} - tp_{m+1}}{sq_{m+2} - tq_{m+1}},$$

where

$$(s, t) \in T_{13} = T_{12} \bigcup \{(13, 1), (7, 3), (5, 4)(1, 24)(1, 25)\}$$

(for an integer $m \geq -1$). From the proof of the Theorem (6.2.1) in [1] (see also [5]) where m is the largest integer satisfying $\alpha < \frac{a}{b} \leq \frac{p_m}{q_m}$

By Theorem (6.2.1), we have to consider only pairs of non negative integers (r, s) and (s, t) satisfying $rs < 2k$, $st < 2k$, gcd(r, s) = 1 and gcd(s, t) = 1. The inequalities (6.11) and (6.13) for r = 1, resp. t = 1, imply that the pairs (r, s) = (1, s) and $(s, t) = (s, 1)$ with $s > k + 1 = 14$can be excluded. Similarly, for $r = 2or3$, respectively $t = 2$ or 3, we can exclude the pairs $(r, s) = (2, s)$ and $(s, t) = (s, 2)$ with $s \geq \frac{13}{2} + 2$, and the pairs $(r, s) = (3, s)$ and $(s, t) = (s, 3)$ with $s \geq \frac{13}{3} + 3$. In particular, the pairs $(r, s) = (2, 9), (2, 11), (3, 8)$, and the pairs $(s, t) = (9, 2), (11, 2), (8, 3)$ can be excluded.

Now we show that the pairs $(r, s) = (8, 3)$ and $(s, t) = (3, 8)$ can be replaced with other pairs with smaller products rs, respectively, st. For $(r, s) = (8, 3)$ and $k = 13$, from (6.11) and (6.13) we obtain

$$\frac{8}{3} < a_{m+2} < \frac{64}{11}$$

, and therefore we have three possibilities: $a_{m+2} = 3, 4or5$. If $a_{m+2} = 3$, then from (6.13) we obtain $t = 3 \times 3 - 8 = 1$, and we can replace $(r, s) = (8, 3)$ by $(s, t) = (3, 1)$.

If $a_{m+2} = 4$, we can replace it by $(s,t) = (3,4)$ and if $a_{m+2} = 5$, we can replace it by $(s,t) = (3,7)$.

The proof for pairs $(s,t) = (3,8)$ is completely analogous. We use the inequalities (6.12) and (6.13), instead of (6.10) and (6.11). We obtain

$$\frac{8}{3} < a_{m+2} < \frac{64}{11},$$

and therefore we have, again, three possibilities: $a_{m+2} = 3, 4$ or 5. If $a_{m+2} = 3$, we can replace $(s,t) = (3,8)$ by $(r,s) = (1,3)$, if $a_{m+2} = 4$, we can replace it by $(r,s) = (4,3)$ and if $a_{m+2} = 5$, we can replace it by $(r,s) = (7,3)$.

Our next aim is to show that if we exclude any of the pairs $(r,s) or (s,t)$ appearing in second Proposition, the statement of the proposition will no longer be valid. More precisely, if we exclude a pair $(r', s') \in R_{13}$, then there exist a real number α and a rational number $\frac{p}{q}$ satisfying (6.8), but such that $\frac{p}{q}$ cannot be represented in the form

$$\frac{p}{q} = \frac{rp_{m+1} + sp_m}{rq_{m+1} + sq_m}$$

nor

$$\frac{p}{q} = \frac{sp_{m+2} - tp_{m+1}}{sq_{m+2} - tq_{m+1}},$$

where $m > -1$. $(r,s) \in R_{13} - (\{(r', s')\})$, $(s,t) \in T_{13}$ (and similarly for an excluded pair $(s', t') \in T_{13})$. In the next table, we give explicit examples for each pair. There are many such examples of different form, but we give some numbers α of the form $\sqrt{d}$, where d is a non-square positive integer.

k = 13						
$\sqrt{\alpha}$	a	b	m	r	s	t
$\sqrt{5328}$	11533	158	1	1	13	12
$\sqrt{168}$	1063	82	1	3	7	4
$\sqrt{56}$	943	126	1	4	5	6
$\sqrt{626}$	30049	1201	0	24	1	26
$\sqrt{677}$	33851	1301	0	25	1	27
$\sqrt{5328}$	127957	1753	1	12	13	1
$\sqrt{168}$	1387	107	1	4	7	3
$\sqrt{56}$	1377	184	1	6	5	4
$\sqrt{626}$	32551	1301	0	26	1	24
$\sqrt{677}$	36557	1405	0	27	1	25

Let us consider $\alpha = \sqrt{56} = [7; 2, \bar{1}4.]$ The some convergents of $\sqrt{56}$ are

$$\frac{7}{1}, \frac{15}{2}, \frac{217}{29}, \frac{449}{60}, \frac{6503}{869}, \dots$$

Its rational approximation $\frac{943}{126}$ (the third row of the table) satisfies

$$|\sqrt{56} - \frac{943}{126}| \approx 0 : 0008123 < \frac{13}{126^2}.$$

We have that the only representation of the fraction $\frac{943}{126}$ in the form

$$\frac{rp_{m+1} + sp_m}{rq_{m+1} + sq_m}, (r, s) \in R_{13} \text{or} \frac{sp_{m+2} - tp_{m+1}}{sq_{m+2} - tq_{m+1}}, (s, t) \in T13$$

is

$$\frac{943}{126} = \frac{4 \times 217 + 5 \times 15}{4 \times 29 + 5} = \frac{4 \times \sqrt{2} + 5 \times \sqrt{1}}{4 \times q_2 + 5 \times q_1},$$

which implies that the pair $(4, 5)$ cannot be excluded from the set R_{13}.

Chapter 7

Conclusion with open problems

In this Dissertation, we have presented about the rational approximation of an irrational numbers with special focus on Hurwitz result of general rational approximation and its advances specifically in two different directions. We have studied about the advances of Hurwitz theorem by restriction of certain set of irrationals so that the approximation can be better than that of Hurwitz approximation. In this context we have studied [7] the paper of Hailiza Kamarul Haili,and Norhayati Rosli in 2005. similarly we have studied about the refinement of best approximations of irrationals in [10] and studied about the finite Diophantine approximation originated by A.V Prasad 1958 in [13]. On the other hand we have discussed about the rational approximation of irrationals by increasing the size of numerator. In this context we have studied the paper of Worley [21], Dujella[2], Bernadin Ibrahimpasic[1] in 2008 for k=2,3,...,12 and In 2013 bernadin Ibrahimpasic[5] for k=13. We have studied about the Pell's Fermat equation $x^2 - Dy^2 = \pm 1$ with the help of continued fraction for different types of D, rational approximation by using Dirichlet's box principle and by using solution of Pell's Fermat equation. Now, i am interested to proceed my research in anyone of the following directions.

1. **Conjecture-1(McMullen)** In any real quadratic field, there is a infinite periodic continued fractions formed only of the integers 1 and 2.

2. **Some work towards conjecture-1** For example,$Q[\sqrt{10}]$ contains the infinite sequence of continued fractions.
   $[\overline{1,1,2,1,1,2}]$ Rule$[2,2,0]$
   Counting the Number of 1's before each 2. For the last 2 taking both the number of 1's before and after . $[\overline{1,1,2,1,2,1,1,1,2,2,1,1}]$ Rule$[2,1,3,0,2]$
   $[\overline{1,1,2,1,2,1,1,2,1,1,1,2,2,1,1,2,1,1}]$ Rule$[2,1,2,3,0,2,2]$. General period length is $6n$ and pattern rule is
   $[2, 1,\ (n-2)$ times $2, 3, 0,\ (n-1)$ times $2]$ for $n \geq 2$.

3. (conjecture-2) There exists an integer m such that any real quadratic field $\sqrt{d}$ contains an infinite number of periodic continued fractions uniformly bounded by m.

4. (Zaremba conjecture-3) There is a constant m such that for any integer $q \geq 1$, there is a first integer p to q such that we have $\frac{p}{q} = [a_0, a_1, a_2, ...]$ where a_i are integers between 1 and m.

5. Is there a constant m such that in any real quadratic field,there is a periodic continued fraction of the form $[a_0, a_1, a_2, ..., a_2, a_1]$ uniformly bounded by m?[16]

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