Tribhuvan University
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The Interplay Between Measure Theory And Topology

Thesis

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Student's Declaration

I hereby declare that I am the only author of this work and that no sources other than that listed here have been used in this work.

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Supervisor's Recommendation

I hereby recommend that this thesis prepared under my supervision by **Mr. Jit Bahadur Rana** entitled **The Interplay Between Measure Theory And Topology** during the period prescribed by the rules and regulations of this institute in partial fulfillment of the requirements for the degree of M.A./M.Sc. in Mathematics be processed for the evaluation.

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ABSTRACT

A connection between measure theory and topology is established when a $\sigma-$field $\mathscr{F}$ is defined in terms of topological properties. More precisely, we defined $\mathscr{F}$ as the smallest $\sigma-$field containing all the open sets of a topological space Ω, then there are interesting interrelation between measure theory and topology. We study the interrelation between topological space, open sets and continuous functions in one hand and measure space, measurable set and measurable function on the other.

List of Notation

A^c	compliment of$A$
$A_n \uparrow A$	$A_n$increase to$A$
$A_n \downarrow A$	$A_n$decrease to$A$
$\Re$	set of real numbers
$\Re^n$	$n-$ dimensional Euclidean space
$\overline{\Re}$	set of extended real numbers
$\overline{\Re}^n$	corresponding $n-$dimensional Euclidean space
$f^{-1}(B)$	preimage of B under f
$\{f \leq g\}$	a set $\{\omega \in \Omega : f(\omega) \leq g(\omega)\}$
$\{f \in B\}$	a set $\{\omega \in \Omega : f(\omega) \in B\}$
I_A	indicator of A
f^+	$\max(f, 0)$
f^-	$\max(-f, 0)$
$\lim_{n\to\infty} \sup x_n$	limit superior of x_n
$\lim_{n\to\infty} \inf x_n$	limit inferior of x_n
$\vert\vert$	modulus
$\Vert\Vert$	norm
$\mathscr{F}$	collection of subsets
$\mathscr{S}$	collection of subsets
$\sigma(\mathscr{S})$	smallest $\sigma-$field containing $\mathscr{S}$
μ	measure
$(\Omega, \mathscr{F}, \mu)$	measure space
$\mathscr{B}(\Re)$	class of Borel sets of $\Re$
$(\Omega, \mathscr{F}_\mu, \mu)$	completion of $(\Omega, \mathscr{F}, \mu)$
F	distributive function
$\underline{\int_a^b} f$	Riemann integral
$\overline{\mathscr{B}}(\Re)$	class of Lebesgue measurable sets
$(\Omega, \mathscr{F})$	measurable space
$h : (\Omega_1, \mathscr{F}_1) \to (\Omega_2, \mathscr{F}_2)$	$h : \Omega_1 \to \Omega_2$ is measurable relative to $\mathscr{F}_1$ and $\mathscr{F}_2$
$\sup_n \inf_{k\geq n} f_k$	lower limit
$\inf_n \sup_{k\geq n} f_k$	upper limit
$\{x_n : n \in D\}$	net

$\int_\Omega h d\mu$	Lebesgue integral
$a.e.[\mu]$	a condition holds almost everywhere with respect to μ
λ^+	upper variation or positive part of λ
λ^-	lower variation or negative part of λ
$\lvert\lambda\rvert$	total variation of λ
τ	topology
G_δ	countable intersection of open sets
F_σ	countable union of closed sets
d	metric
$B_d(x,\epsilon)$	$\epsilon-$ball centered at x
D	directed set
$\prod_{i\in I}\Omega_i$	Cartesian product of Ω_i
$p_i(x)$	projection mapping
LSC	lower semicontinuous
USC	upper semicontinuous
$\underline{\underline{f}}$	lower envelope
$\overline{\overline{f}}$	upper envelope
$\mathscr{D}$	Dynkin system
L	vector space
$f \vee g$	$\max(f,g)$
$f \wedge g$	$\min(f,g)$
E	positive linear functional
H^+	$\{f \in H : f \geq o\}$
L'	collection of functions $\lim_n f_n$, f_n an increasing sequence of functions in L^+
L''	collection of nets $\lim_n f_n$, f_n form an increasing sequence of nets in L^+
$\sigma(H)$	smallest $\sigma-$field making every functions in H Borel measurable
$L^p(\Omega,\mathscr{F},\mu)$	collection of all complex valued Borel measurable functions
π_μ	projection mapping
$\mathscr{F}^n$	measurable rectangle
B_n	cylinder with base B^n
$\mu_n \xrightarrow{w} \mu$	weak convergence
$\partial(a,b]$	boundary of $(a,b]$

Contents

Chapter 1

Introduction

In this chapter, we state some definitions and theorems which will be used in later chapter of the thesis. we begin with the definition of increasing and decreasing sequences.

Definition 1.1. *Let $A_1, A_2, ...$ be subsets of a set Ω. If $A_1 \subset A_2 \subset ...$ and $\bigcup_{n=1}^{\infty} A_n = A$, we say that the A_n form an increasing sequence of sets with limit A, or that the A_n increase to A; we write $A_n \uparrow A$. If $A_1 \supset A_2 \supset ...$ and $\bigcap_{n=1}^{\infty} A_n = A$, we say that the A_n form a decreasing sequence of sets with limit A, or that the A_n decrease to A; we write $A_n \downarrow A$.*

The set of real numbers is denoted by $\Re$, and $\Re^n$ denotes n dimensional Euclidean Space.
In $\Re$ the interval $(a, b]$ is defined as

$$(a, b] = \{x \in \Re : a < x \leq b\}$$

and the interval (a, ∞) is defined as

$$(a, \infty) = \{x \in \Re : x > a\}$$

Other types of intervals are defined similarly.
If $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n) \in \Re^n$, then $a \leq b$ will mean $a_i \leq b_i$ for all i. The interval $(a, b]$ in $\Re^n$ will be defined as

$$(a, b] = \{x \in \Re^n : a_i < x_i \leq b_i, i = 1, 2, ..., n\}$$

and other types of intervals are defined similarly.
The set of extended real numbers is the two-point compactificatoin $\Re \cup \{\infty\} \cup \{-\infty\}$, denoted by $\overline{\Re}$. The set of n-tuples $(x_1, x_2, ..., x_n)$, with each $x_i \in \overline{\Re}$,

is denoted by $\overline{\Re}^n$. If f is a function from Ω to Ω', written as $f : \Omega \to \Omega'$ and $B \subset \Omega'$,then the pre-image of B under f is defined as

$$f^{-1}(B) = \{\omega \in \Omega : f(\omega) \in B\}$$

If $f : \Re \to \Re$ is a function,then f is increasing iff

$$x < y \Rightarrow f(x) \leq f(y),$$

and f is decreasing iff

$$x < y \Rightarrow f(x) \geq f(y).$$

If $f_n : \Omega \to \overline{\Re}, n = 1, 2, ...$, then the f_n are said to form an increasing sequence iff

$$f_n(\omega) \leq f_{n+1}(\omega) \text{ for all } n \text{ and } \omega$$

and f_n are said to form a decreasing sequence iff

$$f_n(\omega) \geq f_{n+1}(\omega) \text{ for all } n \text{ and } \omega$$

If $f_1, f_2, ...$ form an increasing sequence of functions with limit f, that is, $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ for all ω, then we write $f_n \uparrow f$. Similarly, we write $f_n \downarrow f$ for a decreasing sequence.
We abbreviate $\{f \leq g\}$ for a set $\{\omega \in \Omega : f(\omega) \leq g(\omega)\}$, similarly the pre-image $\{\omega \in \Omega : f(\omega) \in B\}$ is abbreviated as $\{f \in B\}$.
For a subset A of Ω, the indicator of A, denoted by I_A is the function defined by

$$I_A(\omega) = \begin{cases} 1 & \text{for } \omega \in A \\ 0 & \text{for } \omega \notin A \end{cases}$$

If $f : \Omega \to \overline{\Re}$ is a function, then the positive and negative parts of f are defined by

$$\begin{aligned} f^+ &= \max(f, 0) \text{ and} \\ f^- &= \max(-f, 0) \end{aligned}$$

We next define limit superior and limit inferior.

Definition 1.2. *Let $\{x_n\}$ be a sequence of real numbers. Suppose there is a real number b satisfying the following conditions:*

(a) For every $\epsilon > 0$ *there exists an integer* N *such that* $n > N$ *implies*

$$x_n < b + \epsilon.$$

(b) Given $\epsilon > 0$ *and given* $m > 0$*, there exists an integer* $n > m$ *such that*

$$x_n > b - \epsilon.$$

then b *is called the limit superior or upper limit of* $\{x_n\}$ *and write*

$$b = \lim_{n\to\infty} \sup x_n.$$

The limit inferior or lower limit of $\{x_n\}$ *is defined as follows:*

$$\lim_{n\to\infty} \inf x_n = -\lim_{n\to\infty} \sup y_n.$$

where $y_n = -x_n, n = 1, 2, .$
For the sequence of sets $A_1, A_2, ...$ *of subsets of* Ω *we define*

$$\lim_n \sup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

and

$$\lim_n \inf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

Thus, $\omega \in \limsup_n A_n$ *iff for every* n*,* $\omega \in A_k$ *for some* $k \geq n$ *and* $\omega \in \liminf_n A_n$ *iff for some* n*,* $\omega \in A_k$ *for all* $k \geq n$*.*

The following is the useful theorem for our thesis.

Theorem 1.1. *De Morgan's Law Let* $A_1, A_2, ...$ *be subsets of a set* ω*. Then*

$$\left(\bigcup_n A_n\right)^c = \bigcap_n A_n^c \quad \text{and} \quad \left(\bigcap_n A_n\right)^c = \bigcup_n A_n^c.$$

Following is also the useful definition.

Definition 1.3. *A norm on a vector space* L *is a function* $\|\|$ *from* L *to the nonnegative reals satisfying*

$$\|ax\| = |a| \, \|x\| \quad \text{for all } c \in C, x \in L.$$

$$\|x + y\| \leq \|x\| + \|y\| \quad \text{for all } x, y \in L.$$

$$\|x\| = 0 \;\; \text{implies} \;\; x = 0.$$

Chapter 2

Measure and Integration Theory

In order to understand the interplay between the measure theory and topology, we need to learn the measure theory in detail. Chapter 2 will be devoted to measure theory. In this chapter, we state definitions and give examples and state standard theorems with proofs of measure theory followed by integration theory. First we begin with the definition of field.

Definition 2.1. *Let $\mathscr{F}$ be a collection of subsets of a set Ω. Then $\mathscr{F}$ is called a field iff*

(a) $\Omega \in \mathscr{F}$

(b) If $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$

(c) If $A_1, A_2, ..., A_n \in \mathscr{F}$, then $\bigcup_{i=1}^n A_i \in \mathscr{F}$

It follows that $\mathscr{F}$ is closed under finite intersection because by De Morgan Law if $A_1, A_2, ..., A_n \in \mathscr{F}$, then

$$\bigcap_{i=1}^{n} A_i = (\bigcup_{i=1}^{n} A_i^c)^c \in \mathscr{F}.$$

If (c) is replaced by

$$\text{If } A_1, A_2, ... \in \mathscr{F}, \text{ then } \bigcup_{i=1}^{\infty} A_i \in \mathscr{F},$$

then $\mathscr{F}$ is called a σ-field.

We note that if $\mathscr{F}$ is a field, then a countable union of a set in $\mathscr{F}$ can be expressed as the limits of an increasing sequence of sets in $\mathscr{F}$, and conversely. To see this if $\bigcup_{i=1}^{\infty} A_i = A$, then by definition $\bigcup_{i=1}^{n} A_i \uparrow A$ and conversely if $A_n \uparrow A$, then $A = \bigcup_{n=1}^{\infty} A_n$ a countable union of sets in $\mathscr{F}$. This argument shows that a σ-field is a field that is closed under limits of increasing sequences.

We next give some examples of field and $\sigma-$field.

(a) The largest σ-field of subsets of a fixed set Ω is the collection of all subsets of Ω. The smallest σ-field consists of the two subsets ϕ and Ω.

(b) Let A be a non empty proper subset of Ω. Then $\mathscr{F} = \{\phi, \Omega, A, A^c\}$ is the smallest σ-field containing A.

(c) Let $\Omega = \Re$. Let $\mathscr{F}$ consist of all finite disjoint unions of right-semiclosed intervals(for convention we also count $(a, \infty), -\infty \leq a < \infty$ as right-semiclosed). Then $\mathscr{F}$ is a field but not a σ-field because if $A_n = \left(0, 1 - \frac{1}{n}\right] \in \mathscr{F}$ for $n = 1, 2...$ and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left(0, 1 - \frac{1}{n}\right] = (0, 1) \notin \mathscr{F}$.

Definition 2.2. *Let $\mathscr{S}$ be a class of sets. Then the smallest σ-field containing the sets of $\mathscr{S}$ is denoted by $\sigma(\mathscr{S})$ and called the minimal σ-field over $\mathscr{S}$.*

Definition 2.3. *A measure on a σ-field $\mathscr{F}$ is a non-negative, extended real-valued function μ on $\mathscr{F}$ such that whenever $A_1, A_2, ...$ form a finite or countably infinite collection of disjoint sets in $\mathscr{F}$, we have*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

If $\mu(\Omega) = 1$, μ is called a probability measure.

Definition 2.4. *A measure space is a triple $(\Omega, \mathscr{F}, \mu)$ where Ω is a set, $\mathscr{F}$ is a $\sigma-$field of subsets of Ω, and μ is a measure on $\mathscr{F}$. If μ is a probability measure, then $(\Omega, \mathscr{F}, \mu)$ is called a probability measure space.*

Some examples of measures are as follows:

(a) Let Ω be any set and let $\mathscr{F}$ consists of all subsets of Ω. Define $\mu(A)$ as the number of points in A. Then if A has n members, then $\mu(A) = n, n = 1, 2, ...$ and if A is infinite set, then $\mu(A) = \infty$. The set function μ is a measure on $\mathscr{F}$, called a counting measure.

(b) Let $\Omega = \{x_1, x_2, ...\}$ be a finite or countably infinite set and let $p_1, p_2, ...$be non-negative numbers. Let $\mathscr{F}$ be a collection of subsets of Ω and define

$$\mu(A) = \sum_{x_i \in A} p_i$$

Then the set function μ is a measure on $\mathscr{F}$. If $\sum p_i = 1$ then μ is a probability measure and if all $p_i = 1$, then μ is a counting measure.

Definition 2.5. *The class of Borel sets of $\Re$ denoted by $\mathscr{B}(\Re)$, is defined as the smallest σ-field of subsets of $\Re$ containing all intervals $(a, b], a, b \in \Re$.*

By definition, $\mathscr{B}(\Re)$ can be described as the intersection of all $\sigma-$fields containing the intervals $(a, b], a, b \in \Re$. Since

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right) \text{and} (a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right]$$

and by definition of $\sigma-$field, $\mathscr{B}(\Re)$ is the smallest $\sigma-$field containing all open intervals. Similarly, $\mathscr{B}(\Re)$ can be described as the smallest $\sigma-$field containing the class of all intervals of $\Re$. Moreover since an open set is a countable union of open intervals, $\mathscr{B}(\Re)$ is the smallest $\sigma-$field containing all open sets of $\Re$.

The following is the basic properties of the set functions:

Theorem 2.1. *Let μ be a finitely additive set function on the field $\mathscr{F}$.*

(a) $\mu(\phi) = 0$.

(b) $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ for all $A, B \in \mathscr{F}$.

(c) If $A, B \in \mathscr{F}$ and $B \subset A$, then $\mu(A) = \mu(B) + \mu(A - B)$ (hence $\mu(A - B) = \mu(A) - \mu(B)$ if $\mu(B)$ is finite, and $\mu(B) \leq \mu(A)$ if $\mu(A - B) \geq 0$).

(d) If μ is nonnegative,

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i) \text{ for all } A_1, ..., A_n \in \mathscr{F}.$$

If μ is a measure,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n) \text{ for all } A_1, A_2 ... \in \mathscr{F} \text{ such that } \bigcup_{n=1}^{\infty} A_n \in \mathscr{F}.$$

Proof. (a) Let $A \in \mathscr{F}$ such that $\mu(A)$ is finite. Then

$$\begin{aligned} \mu(A) &= \mu(A \cup \phi) \\ &= \mu(A) + \mu(\phi.) \end{aligned}$$

So $\mu(\phi) = 0$.

(b) We know

$$A = (A \cap B) \cup (A - B)$$

and

$$B = (A \cap B) \cup (B - A).$$

Thus by finite additivity of μ we have

$$\mu(A) = \mu(A \cap B) + \mu(A - B)$$

and

$$\mu(B) = \mu(A \cap B) + \mu(B - A).$$

By adding

$$\begin{aligned} \mu(A) + \mu(B) &= \mu(A \cap B) + \mu(A \cap B) + \mu(A - B) + \mu(B - A) \\ &= \mu(A \cap B) + \mu(A \cup B). \end{aligned}$$

(c) We have

$$A = B \cup (A - B)$$

So

$$\mu(A) = \mu(B) + \mu(A - B)$$

If $\mu(B)$ is finite, then

$$\mu(A) - \mu(B) = \mu(A - B)$$

Morever, if $\mu(A - B) \geq 0$, then

$$\mu(A) \geq \mu(B).$$

(d) We know

$$\bigcup_{i=1}^{n} A_i = A_1 \cup (A_1^c \cap A_2) \cup ... \cup (A_1^c \cap ... \cap A_{n-1}^c \cap A_n)$$

and the sets on the right are disjoint. Also, since μ is nonnegative, by (c)

$$\mu(A_1^c \cap ... \cap A_{n-1}^c \cap A_n) \leq \mu(A_n).$$

Thus,

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i).$$

Morever, since

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_1^c \cap ... \cap A_{n-1}^c \cap A_n),$$

same as the proof of first part of (d) we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

if μ is a measure.

□

Definition 2.6. *A set function μ defined on $\mathscr{F}$ is said to be finite iff $\mu(A)$ is finite, that is, not $\pm\infty$, for each $A \in \mathscr{F}$. A nonnegative, finitely additive set function μ on the field $\mathscr{F}$ is said to be σ-finite on $\mathscr{F}$ iff Ω can be written as $\bigcup_{n=1}^{\infty} A_n$ where the A_n belong to $\mathscr{F}$ and $\mu(A_n) < \infty$ for all n.*

Countably additive set functions have a basic continuity property, which as follows:

Theorem 2.2. *Let μ be a countably additive set function on the σ-field $\mathscr{F}$.*

(a) If $A_1, A_{2,}, ... \in \mathscr{F}$ and $A_n \uparrow A$, then $\mu(A_n) \to \mu(A)$ as $n \to \infty$.

(b) If $A_1, A_{2,}, ... \in \mathscr{F}$, $A_n \downarrow A$, and $\mu(A_1)$ is finite, then $\mu(A_n) \to \mu(A)$ as $n \to \infty$.

Proof. (a) If $\mu(A_n) = \infty$ for some n, then

$$\mu(A) = \mu(A_n) + \mu(A - A_n) = \infty.$$

Replacing A by A_k, we have $\mu(A_k) = \infty$ for all $k \geq n$. So in this case the statement is trivial.
Similar case arises if $\mu(A_n) = -\infty$ for some n.
Thus we assume that $\mu(A_n)$ is finite for all n. Since A_n form an increasing sequence, we can write

$$A = A_1 \cup (A_2 - A_1) \cup ... \cup (A_n - A_{n-1}) \cup ...$$

Thus

$$\begin{aligned} \mu(A) &= \mu(A_1) + \mu((A_2) - \mu(A_1) + ... + \mu((A_n) - \mu(A_{n-1}) + ... \\ &= \lim_{n\to\infty} \mu(A_n) \end{aligned}$$

that is, $\mu(A_n) \to \mu(A)$ as $n \to \infty$ as desired.

(b) If $A_n \downarrow A$, then $A_1 - A_n \uparrow A_1 - A$. So by (a), $\mu(A_1 - A_n) \uparrow (A_1 - A)$. Thus

$$\mu(A_1 - A) = \mu(A_1) - \mu(A) \ \text{ (as } \mu(A_1) \text{ is finite, so is } \mu(A))$$

$$\begin{aligned} \mu(A) &= \mu(A_1) - \mu(A_1 - A) \\ &= \mu(A_1) - \lim_{n\to\infty} \mu(A_1 - A_n) \\ &= \lim_{n\to\infty} \mu(A_n) \ \text{ (by Th 2.1(c))} \end{aligned}$$

Hence $\mu(A_n) \to \mu(A)$ as $n \to \infty$.

□

The following theorem ensures that finite additivity plus continuity implies countable additivity.

Theorem 2.3. *Let μ be a finitely additive set function on the field $\mathscr{F}$.*

(a) Assume that μ is continuous from below at each $A \in \mathscr{F}$, that is, $A_1, A_2, .. \in \mathscr{F}, A = \bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$ and $A_n \uparrow A$, then $\mu(A_n) \to \mu(A)$. It follows that μ is countably additive on $\mathscr{F}$.

(b) Assume that μ is continuous from above at the empty set, that is, if $A_1, A_2, ... \in \mathscr{F}$ and $A_n \downarrow \phi$, then $\mu(A_n) \to 0$. It follows that μ is countably additive on $\mathscr{F}$.

Proof. (a) Let $A_1, A_2, ...$ be disjoint sets in $\mathscr{F}$ such that $A = \bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$. Let $B_n = \bigcup_{i=1}^{n} A_i$. Then $B_n \uparrow A$. So by hypothisis $\mu(B_n) \to \mu(A)$. By finite addivity of μ

$$\mu(B_n) = \sum_{i=1}^{n} \mu(A_i).$$

So

$$\mu(A) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i)$$

as desired.

(b) Let $A_1, A_2, ...$ be disjoint sets in $\mathscr{F}$ such that $A = \bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$. Let $B_n = \bigcup_{i=1}^{n} A_i$. So by Th 2.1

$$\mu(A) = \mu(B_n) + \mu(A - B_n)$$

But $A - B_n \downarrow \phi$, so by hypothesis, $\mu(A - B_n) \to 0$. Thus $\mu(B_n) \to \mu(A)$. So

$$\begin{aligned}
\mu(A) &= \lim_{n\to\infty} \mu(B_n) \\
&= \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} A_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i). \quad \text{(by finite additivity of } \mu)
\end{aligned}$$

□

Definition 2.7. *An outer measure on Ω is a nonnegative, extended real-valued set function λ on the class of all subsets of Ω, satisfying*

1. $\lambda(\phi) = 0$.
2. $A \subset B$ *implies* $\lambda(A) \leq \lambda(B)$ *(monotonicity).*
3. $\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \lambda(A_n)$ *(countable subadditivity).*

Definition 2.8. *Let λ be an outer measure on the set Ω. We say that the set E is λ-measurable iff*

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c) \text{for all } A \subset \Omega.$$

In the definition if $\mathscr{M}$ is the class of all $\lambda-$measurable sets then λ is a measure on $\mathscr{M}$.

Definition 2.9. *A measure μ on a $\sigma-$field $\mathscr{F}$ is said to be complete iff whenever $A \in \mathscr{F}$ and $\mu(A) = 0$, we have $B \in \mathscr{F}$ for all $B \subset A$.*
The completion of a measure space $(\Omega, \mathscr{F}, \mu)$ is defined as follows:
Let $\mathscr{F}_\mu$ be the class of sets $A \cup N$, where A ranges over $\mathscr{F}$ and N, over all subsets of measure 0 in $\mathscr{F}$. Then $\mathscr{F}_\mu$ is a $\sigma-$field including $\mathscr{F}$. Now we extend μ to $\mathscr{F}_\mu$ by setting $\mu(A \cup N) = \mu(A)$. This definition is well defined because if $A_1 \cup N_1 = A_2 \cup N_2 \in \mathscr{F}_\mu$, then

$$\begin{aligned} \mu(A_1) &= \mu(A_1 \cap A_2) + \mu(A_1 - A_2) \\ &= \mu(A_1 \cap A_2) \;\;, (\textit{since } A_1 - A_2 \subset N_2 \textit{ so that } \mu(A_1 - A_2) = 0) \end{aligned}$$

Thus,

$$\mu_(A_1) \leq \mu(A_2)$$

Similarly

$$\begin{aligned} \mu(A_2) &= \mu(A_2 \cap A_1) + \mu(A_2 - A_1) \\ &= \mu(A_2 \cap A_1) \;\;, (\textit{since } A_2 - A_1 \subset N_1 \textit{ so that } \mu(A_2 - A_1) = 0) \end{aligned}$$

Thus,

$$\mu_(A_2) \leq \mu(A_1)$$

So $\mu(A_1) = \mu(A_2)$. So μ is well defined.
The measure space $(\Omega, \mathscr{F}_\mu, \mu)$ is called the completion of $(\Omega, \mathscr{F}, \mu), \mathscr{F}_\mu$ the completion of $\mathscr{F}$ relative to μ. We note that the completion is in fact complete. For let $M \subset A \cup N \in \mathscr{F}_\mu$ where $A \in \mathscr{F}, \mu(A) = 0, N \subset B \in \mathscr{F}, \mu(B) = 0$, then $M \subset A \cup B \in \mathscr{F}, \mu(A \cup B) = 0$, hence $M \in \mathscr{F}_\mu$.

Definition 2.10. *A class $\mathscr{C}$ of subsets of Ω is said to be monotone if $\mathscr{C}$ satisfies the following condition:*

$$\textit{if } A_n \in \mathscr{C} \textit{ and } A_n \uparrow A \textit{ or } A_n \downarrow A \;, \textit{then } A \in \mathscr{C}.$$

Theorem 2.4. *Monotone class theorem.[6] Let $\mathscr{F}_0$ be a field of subsets of Ω, and $\mathscr{C}$ a class of subsets of Ω that is monotone. If $\mathscr{C} \supset \mathscr{F}_0$, then $\mathscr{C} \supset \sigma(\mathscr{F}_0)$, the minimal $\sigma-$field over $\mathscr{F}_0$.*

The following is the useful fundamental extension theorem.

Theorem 2.5. *Caratheodory extension theorem.[6] Let μ be a measure on the field $\mathscr{F}_0$ of subsets of Ω, and assume that μ is $\sigma-$finite on $\mathscr{F}_0$, so that Ω can be decomposed as $\bigcup_{n=1}^{\infty} A_n$, where $A_n \in \mathscr{F}_0$ and $\mu(A_n) < \infty$ for all n. Then μ has a unique extension to a measure on the minimal $\sigma-$field $\mathscr{F}$ over $\mathscr{F}_0$.*

Now we state Jordan-Hahn decomposion theorem which gives the definition of signed measure.

Theorem 2.6. *Jordan-Hahn Decomposition Theorem.[6] Let λ be a countably additive extended real-valued set function on the $\sigma-$field $\mathscr{F}$. Define*

$$\lambda^+(A) = \sup\{\lambda(B) : B \in \mathscr{F}, B \subset A\}$$

$$\lambda^-(A) = -\inf\{\lambda(B) : B \in \mathscr{F}, B \subset A\}$$

Then λ^+ and λ^- are measure on $\mathscr{F}$ and $\lambda = \lambda^+ - \lambda^-$.

Definition 2.11. *In the Jordan-Hahn Decomposition Theorem we call λ^+ the upper variation or positive part of λ, λ^- the lower variation or negative part, and $|\lambda| = \lambda^+ + \lambda^-$ the total variation. The difference $\lambda = \lambda^+ - \lambda^-$ is sometimes called the signed measure.*

Theorem 2.7. *Borel-Cantelli Lemma.[6] If $A_1, A_2, ... \in \mathscr{F}$ and $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup_n A_n) = 0$.*

Before defining Lebesgue integral, we define Riemann integral.

Definition 2.12. *Let f be a bounded real valued function defined on the closed, bounded interval $[a, b]$. Let $P : a = x_0 < x_1 < ... < x_n = b$ is a partition of $[a, b]$. Then we define the lower and upper Darboux sums for f with respect to P, respectively, by*

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

where for $1 \leq i \leq n$,

$$m_i = \inf\{f(x) : x_{i-1} < x < x_i\} \text{ and}$$

$$M_i = \sup \{f(x) : x_{i-1} < x < x_i\}.$$

We define the lower and upper Riemann integrals of f over $[a,b]$, respectively, by

$$\underline{\int_a^b} f = \sup \{L(f,P) : P \text{ a partition of } [a,b]\}$$

$$\overline{\int_a^b} f = \inf \{U(f,P) : P \text{ a partition of } [a,b]\}.$$

By definition of lower and upper integrals, the upper integral is always at least as large as the lower integral. If two integrals are equal, we say that f is Reimann integrable over $[a,b]$ and this common value is called the Reimann integral of f over $[a,b]$. We denote it by $\int_a^b f$.

Next we define Lebesgue-Stieltjes measure on real line.

Definition 2.13. *A Lebesgue-Stieltjes measure on $\Re$ is a measure μ on $\mathscr{B}(\Re)$ such that $\mu(I) < \infty$ for each bounded interval I. A distribution function on $\Re$ is a map $F : \Re \to \Re$ that is increasing and right continuous($\lim_{x \to x_0^+} F(x) = F(x_0)$)*

The following two theorems are the standard results related to Lebesgue-Stieltjes measure.

Theorem 2.8. *[6] Let μ be a Lebesgue-Stieltjes measure on $\Re$. Let $F : \Re \to \Re$ be defined, up to an additive constant, by $F(b) - F(a) = \mu(a,b]$. Then F is a distribution function.*

Theorem 2.9. *[6] Let F be a distribution function on $\Re$, and let $\mu(a,b] = F(b) - F(a), a < b$. There is a unique extension of μ to a Lebesgue-Stieltjes measure on $\Re$.*

Here theorem 2.8 and theorem 2.9 show that the formula $\mu(a,b] = F(b) - F(a), a < b$ sets up a one-to-one correspondence between Lebesgue-Stieltjes measures and distribution functions, where two distribution functions that differ by a constant are identified.

Let F be a distribution function and μ the corresponding Lebesgue-Stieltjes measure. Then $\mu(a,b] = F(b) - F(a), a < b$. Morever the measure of any interval, right semi-closed or not, may be expressed in terms of F. For if $F(x^-) = \lim_{y \to x^-} F(y)$, then

(1) $\mu(a,b] = F(b) - F(a)$

(2) $\mu(a,b) = F(b^-) - F(a)$

(3) $\mu[a,b] = F(b) - F(a^-)$

(4) $\mu[a,b) = F(b^-) - F(a^-)$

(5) F is continuous at x iff $\mu\{x\} = 0$. Hence the magnitude of discontinuity of F at x coincides with the measure of $\{x\}$

To establish (2), we have

$$\begin{aligned}\mu(a,b) &= \lim_{n\to\infty} \mu(a, b - \frac{1}{n}] \\ &= \lim_{n\to\infty} [F(b - \frac{1}{n}) - F(a)] \\ &= F(b^-) - F(a).\end{aligned}$$

To establish (3), we have

$$\begin{aligned}\mu[a,b] &= \lim_{n\to\infty} \mu(a - \frac{1}{n}, b] \\ &= \lim_{n\to\infty} [F(b), F(a - \frac{1}{n})] \\ &= F(b) - F(a^-).\end{aligned}$$

Similarly (4) will follow because

$$\begin{aligned}\mu[a,b) &= \lim_{n\to\infty} \mu(a - \frac{1}{n}, b - \frac{1}{n}] \\ &= \lim_{n\to\infty} [F(b - \frac{1}{n}) - F(a - \frac{1}{n})] \\ &= F(b^-) - F(a^-).\end{aligned}$$

(5) also holds because if $a = b = x$, then by the proof of expression (3), we have $\mu\{x\} = \mu[x, x^-] = F(x) - F(x^-)$.

If F is continuous at a and b, then all four expressions (1-4) are equal.

Now let $f : \Re \to \Re, f > 0$, and f is Riemann integrable on any finite interval. Then if we fix $F(0)$ arbitrarily and define

$$F(x) - F(0) = \int_0^x f(t)dt, \;\; x > 0$$

$$F(0) - F(x) = \int_x^0 f(t)dt, \ x < 0,$$

then F is a continuous distribution function and so there exist a Lebesgue-Stieltjes measure, specifically,

$$\mu(a, b] = \int_a^b f(x)dx,$$

In particular we may take $f(x) = 1$ for all x, and $F(x) = x$; then

$$\mu(a, b] = \int_a^b 1dx, = b - a.$$

Here the set function μ is called the Lebesgue measure on $\mathscr{B}(\Re)$. The completion of $\mathscr{B}(\Re)$ relative to Lebesgue measure is called the class of Lebesgue measurable sets, written as $\overline{\mathscr{B}}(\Re)$. Thus, Lebesgue measurable set is the union of a Borel set and a subset of a Borel set of Lebesgue measure 0. The extension of Lebesgue measure to $\overline{\mathscr{B}}(\Re)$ is called Lebesgue measure also.

Now we define the measurable function.

Definition 2.14. *Let $h : \Omega_1 \to \Omega_2$ be a function. Then h is said to be measurable relative to the $\sigma-$fields $\mathscr{F}_j$ of subsets of $\Omega_j, j = 1, 2$, iff $h^{-1}(A) \in \mathscr{F}_1$ for each $A \in \mathscr{F}_2$.*

It is sufficient to show that $h^{-1}(A) \in \mathscr{F}_1$ for each $A \in \mathscr{C}$ where $\mathscr{C}$ is the class of subsets of Ω_2 such that the minimal $\sigma-$field over $\mathscr{C}$ is $\mathscr{F}_2$.
The notation $h : (\Omega_1, \mathscr{F}_1) \to (\Omega_2, \mathscr{F}_2)$ will mean that $h : \Omega_1 \to \Omega_2$, measurable relative to $\mathscr{F}_1$ and $\mathscr{F}_2$.
If $\mathscr{F}$ is a $\sigma-$field of subsets of Ω, $(\Omega, \mathscr{F})$ is sometimes called a measurable space, and the sets in $\mathscr{F}$ are sometimes called measurable sets.

Definition 2.15. *Let $(\Omega, \mathscr{F})$ be a measurable space and $h : \Omega \to \Re^n$ (or $\overline{\Re}^n$). Then h is said to be Borel measurable on $(\Omega, \mathscr{F})$ iff h is measurable relative to the $\sigma-$fields $\mathscr{F}$ and $\mathscr{B}$, the class of Borel sets. If Ω is a Borel subset of $\Re^k$ (or $\overline{\Re}^k$) and we use the term Borel measurable, we always assume that $\mathscr{F} = \mathscr{B}$.*

Some of the consequences are in order:

(a) A continuous map $h : \Re^k \to \Re^n$ is Borel measurable. To prove this let $\mathscr{C}$ be the class of open subsets of $\Re^n$, then by definition of continuous function $h^{-1}(A)$ is open, hence $h^1(A) \in \mathscr{B}(\Re^k)$, for each $A \in \mathscr{C}$. Thus h is Borel measurable.

(b) If A is a subset of $\Re$ that is not a Borel set and I_A is the indicator of A, then I_A is not Borel measurable. For $\{\omega : I_A(\omega) = 1\} = A \notin \mathscr{B}(\Re)$.

(c) To show that a function $h : \Omega \to \Re$ (or $\overline{\Re}$) is Borel measurable, it is sufficient to show that $\{\omega : h(\omega) > c\}$ belongs to $\mathscr{F}$ for each real c. For this let $\mathscr{C}$ be the class of sets $\{x : x > c\}$, $c \in \Re$. Then $\mathscr{C}$ consists all types of intervals. So $\sigma(\mathscr{C}) = \mathscr{B}(\Re)$. Thus if

$$\begin{aligned} h^{-1}\{x : x > c\} &= h^{-1}(c, \infty) \\ &= \{\omega : h(\omega) \in (c, \infty)\} \\ &= \{\omega : h(\omega) > c\} \end{aligned}$$

belongs to $\mathscr{F}$, then h is Borel measurable as desired. Similarly $\{\omega : h(\omega) > c\}$ can be replaced by $\{\omega : h(\omega) \geq c\}$, $\{\omega : h(\omega) < c\}$, or $\{\omega : h(\omega) \leq c\}$, or equally well $\{\omega : a \leq h(\omega) \leq b\}$ for all real a, and b and so on.

(d) If $(\Omega, \mathscr{F}, \mu)$ is a measure space, the terminology "h is measurable on $(\Omega, \mathscr{F}, \mu)$" will mean that h is Borel measurable on $(\Omega, \mathscr{F})$ and μ is a measure on $\mathscr{F}$.

Definition 2.16. *Let $(\Omega, \mathscr{F})$ be a measurable space. Let $h : \Omega \to \overline{\Re}$ be a function. Then h is said to be simple iff h is Borel measurable and takes only finitely many distinct values. Equivalently, h is simple iff it can be written as a finite sum $\sum_{i=1}^{r} x_i I_{A_i}$, where the A_i are disjoint sets in $\mathscr{F}$ and I_A is the indicator of A_i; x_i need not be distinct.*

We note that sums, differences, products, and quotients of simple functions are simple provided the operations are well defined.

Now we define Lebesgue integral.

Definition 2.17. *Let $(\Omega, \mathscr{F})$ be a measure space and μ be a measure on $\mathscr{F}$. Let $h : \Omega \to \overline{\Re}$ be Borel measurable. For a simple function $h = \sum_{i=1}^{r} x_i I_{A_i}$, where the A_i are disjoint sets in $\mathscr{F}$, we define the Lebesgue integral of h with respect to μ, written as $\int_\Omega h d\mu$ or $\int_\Omega h(\omega) d\mu(\omega)$, by*

$$\int_\Omega h d\mu = \sum_{i=1}^{r} x_i \mu(A_i)$$

provided $+\infty$ and $-\infty$ do not both appear in the sum. If they do, we say that the integral does not exist.
If h is nonnegative Borel measurable, we define

$$\int_\Omega h d\mu = sup \left\{ \int_\Omega s d\mu : s \ simple, \ 0 \leq s \leq h \right\}.$$

Finally if h is an arbitrary Borel measurable function, then we define

$$\int_\Omega h d\mu = \int_\Omega h^+ d\mu - \int_\Omega h^- d\mu$$

provided the right hand side is not of the form $+\infty - \infty$ where $h^+ = max(h, 0)$ and $h^- = max(-h, 0)$.

Here the function h is said to be $\mu-$integrable (or simply integrable if μ is understood) iff $\int_\Omega h d\mu$ is finite, that is, iff $\int_\Omega h^+ d\mu$ and $\int_\Omega h^- d\mu$ are both finite.
If $A \in \mathscr{F}$, then we define

$$\int_A h d\mu = \int_\Omega h I_A d\mu.$$

If h is a step function from $\Re \to \Re$ and μ Lebesgue measure, $\int_\Omega h d\mu$ is same as the Riemann integral. However, the integral of h with respect to Lebesgue measure exists for many functions that are not Riemann integrable. The following is an example of it.

Let the function f be defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{for } x \text{ is irrational} \\ 0 & \text{for } x \text{ is rational} \end{cases}$$

Then f is integrable with respect to Lebesgue measure. But f is not Riemann integrable. For let P be a partition on $[a, b]$. The by the density of the rationals and the irrationals, we have $L(P, f) = 0$ and $U(P, f) = 1$. Thus

$$\underline{\int_0^1} f = 0 < 1 = \overline{\int_0^1} f.$$

Hence f is not Riemann integrable.

The following are some theorems related on Borel measurable function and Lebesgue integration that are useful to establish the result on "the interplay between measure theory and topology".

Theorem 2.10. *If $h_1, h_2, ..$ are Borel measurable function from Ω to $\overline{\Re}$ and $h_n(\omega) \to h(\omega)$ for all $\omega \in \Omega$, then h is Borel measurable*

Proof. To prove h is Borel measurable, we show that the set $\{\omega : h(\omega) > c\}$ is Borel measurable, that is, $\{\omega : h(\omega) > c\} \in \mathscr{F}$ for each real c. We have

$$\begin{aligned}
\{\omega : h(\omega) > c\} &= \left\{\omega : \lim_{n\to\infty} h_n(\omega) > c\right\} \\
&= \left\{\omega : h_n(\omega) > c + \frac{1}{r} \text{ for some } r = 1, 2, ...\right\} \\
&= \bigcup_{r=1}^{\infty} \left\{\omega : h_n(\omega) > c + \frac{1}{r} \text{ for all but finitely many } n\right\} \\
&= \bigcup_{r=1}^{\infty} \liminf_n \left\{\omega : h_n(\omega) > c + \frac{1}{r}\right\} \quad \text{(by definition of lower limit)} \\
&= \bigcup_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{\omega : h_n(\omega) > c + \frac{1}{r}\right\} \quad \text{(by definition of lower limit)}
\end{aligned}$$

which belongs to $\mathscr{F}$ because $\mathscr{F}$ is a $\sigma-$field. Thus h is Borel measurable. □

Theorem 2.11. *(a) A nonnegative Borel measurable function h is the limit of an increasing sequence of nonnegative, finite valued, simple functions h_n.*

(b) An arbitrary Borel measurable function f is the limit of a sequence of finite valued simple functions f_n, with $|f_n| \leq |f|$ for all n.

Proof. (a) Let us define h_n by

$$h_n(\omega) = \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq h(\omega) < \frac{k}{2^n}, k = 1, 2, ..., n2^n \\ n & \text{if } h(\omega) \geq n. \end{cases}$$

Then h_n is increasing, finite-valued, nonnegative and simple. Next, if $h_n(\omega) < \infty$, then for some n,

$$0 \leq h(\omega) - h(\omega_n) \leq \frac{1}{2^n}.$$

So

$$\lim(h(\omega) - h_n(\omega)) = 0 \Rightarrow \lim_{n\to\infty} h_n(\omega) = h(\omega).$$

So the theorem is proved for in this case.
If $h(\omega) = \infty$, then $h_n(\omega) = n$ for every n. Hence as $n \to \infty$, we have $h_n(\omega) \to h(\omega)$.
This completes the proof of the (a).

(b) Let g_n and h_n be nonnegative, finite valued, simple functions with $g_n \uparrow f^+$ and $h_n \uparrow f^-$. Let us define f_n by $f_n = g_n - h_n$. Then f_n is also nonnegative, finite valued and simple functions and $f_n \uparrow f^+ - f^- = f$. Morever, $|f_n| \leq f$ for all n since f_n form an increasing sequence of functions with limit f.

□

Theorem 2.12. *[6] If h_1, and h_2 are Borel measurable functions from Ω to $\Re$, so are $h_1 + h_2, h_1 - h_2, h_1 h_2$, and h_1/h_2 assuming these are well defined.*

Theorem 2.13. *A composition of measurable function is measurable.*

Proof. Let $g : (\Omega_1, \mathscr{F}_1) \to (\Omega_2, \mathscr{F}_2)$ and $h : (\Omega_2, \mathscr{F}_2) \to (\Omega_3, \mathscr{F}_3)$. Then we have to show that hog is measurable. For, let $B \in \mathscr{F}$. Then

$$(hog)^{-1}(B) = g^{-1}(h^{-1}(B))$$

which belongs to $\mathscr{F}$ (because $h^{-1}(B)$ belongs to $\mathscr{F}$ and hence $g^{-1}(h^{-1}(B))$ belongs to $\mathscr{F}$ as g and h are measurable.) □

Theorem 2.14. *[6]*

(a) If $\int_\Omega h d\mu$ exists and $c \in \Re$, then $\int_\Omega c h d\mu$ exists and

$$\int_\Omega c h d\mu = c \int_\Omega h d\mu.$$

(b) If $g(\omega) \geq h(\omega)$ for all ω, then

$$\int_\Omega g d\mu \geq \int_\Omega h d\mu.$$

(c) If $\int_\Omega h d\mu$ exists, then

$$\left| \int_\Omega h d\mu \right| \leq \int_\Omega |h| \, d\mu.$$

(d) If $h \geq 0$ and $B \in \mathscr{F}$, then

$$\int_B h d\mu = \sup \left\{ \int_B s d\mu : 0 \leq s \leq h, s \textit{ simple} \right\}$$

(e) If $\int_\Omega h d\mu$ exists, so does $\int_A h d\mu$ for each $A \in \mathscr{F}$; if $\int_\Omega h d\mu$ is finite, then $\int_A h d\mu$ is also finite for each $A \in \mathscr{F}$.

Theorem 2.15. *Monotone Convergence Theorem. [2] Let $h_1, h_2, ...$ form an increasing sequence of nonnegative Borel measurable functions, and let $h(\omega) = \lim_{n\to\infty} h_n(\omega), \omega \in \Omega$, then $\int_\Omega h_n d\mu \to \int_\Omega h d\mu$.*

Theorem 2.16. *Additivity Theorem. [2] Let f and g be Borel measurable, and assume that $f + g$ is well defined. If $\int_\Omega f d\mu$ and $\int_\Omega g d\mu$ exists and $\int_\Omega f d\mu + \int_\Omega g d\mu$ is well defined, then*

$$\int_\Omega (f+g) d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu.$$

In particular, if f and g are integrable, so is $f + g$.

Corollary 2.1. *[2]*

(a) If $h_1, h_2, ...$ are nonnegative Borel measurable,

$$\int_\Omega \left(\sum_{n=1}^{\infty} h_n \right) d\mu = \sum_{n=1}^{\infty} \int_\Omega h_n d\mu.$$

(b) If h is Borel measurable, then h is integrable iff $|h|$ is integrable.

(c) If g and h are Borel measurable with $|g| \leq h$, h is integrable, then g is integrable.

Definition 2.18. *A condition is said to hold almost everywhere with respect to the measure μ (written a.e. $[\mu]$) iff there is a set $B \in \mathscr{F}$ of $\mu-$measure 0 such that the condition holds outside of B.*

In the integration theory, the functions that differ only on a set of measure 0 can be identified. This is established by the following theorem.

Theorem 2.17. *[6] Let f, g, and h be Borel measurable functions.*

(a) If $f = 0$ a.e. $[\mu]$, then $\int_\Omega f d\mu = 0$.

(b) If $g = h$ a.e. $[\mu]$ and $\int_\Omega g d\mu$ exists, then so does $\int_\Omega h d\mu$ and $\int_\Omega g d\mu = \int_\Omega h d\mu$.

Theorem 2.18. *[6] Let h be Borel measurable.*

(a) If h is integrable, then h is finite a.e. $[\mu]$.

(b) If $h \geq 0$ and $\int_\Omega h d\mu = 0$, then $h = 0$ a.e. $[\mu]$.

The following theorem asserts that under appropriate conditions, the limit of the integral of sequence of functions is the integral of the limit functions.

Theorem 2.19. *Extended monotone convergence theorem. [2] Let* $g_1, g_2, ..., g, h$ *be Borel measurable.*

(a) If $g_n \geq h$*, for all* n*, where* $\int_\Omega h d\mu > -\infty$*, and* $g_n \uparrow g$*, then*

$$\int_\Omega g_n d\mu \uparrow \int_\Omega g d\mu.$$

(b) If $g_n \geq h$*, for all* n*, where* $\int_\Omega h d\mu < \infty$*, and* $g_n \downarrow g$*, then*

$$\int_\Omega g_n d\mu \downarrow \int_\Omega g d\mu.$$

Definition 2.19. *If* $f_1, f_2, ...$ *are functions from* Ω *to* $\overline{\Re}$*, we define* $\liminf_{n\to\infty} f_n$ *and* $\limsup_{n\to\infty} f_n$ *by*

$$\left(\liminf_{n\to\infty} f_n\right)(\omega) = \sup_n \inf_{k\geq n} f_k(\omega).$$

$$\left(\limsup_{n\to\infty} f_n\right)(\omega) = \inf_n \sup_{k\geq n} f_k(\omega).$$

Theorem 2.20. *Fatou's Lemma. [2] Let* $f_1, f_2, ..., f$ *be Borel measurable.*

(a) If $f_n \geq f$ *for all* n*, where* $\int_\Omega f d\mu > -\infty$*, then*

$$\liminf_{n\to\infty} \int_\Omega f_n d\mu \geq \int_\Omega \left(\liminf_{n\to\infty} f_n\right) d\mu.$$

(b) If $f_n \leq f$ *for all* n*, where* $\int_\Omega f d\mu < \infty$*, then*

$$\limsup_{n\to\infty} \int_\Omega f_n d\mu \leq \int_\Omega \left(\limsup_{n\to\infty} f_n\right) d\mu.$$

Theorem 2.21. *Dominated Convergence Theorem. [2] If* $f_1, f_2, ...f, g$ *are Borel measurable functions,* $|f_n| \leq g$*for all* n *where* g *is* $\mu-$*integrable and* $f_n \to f$ *a.e.* $[\mu]$*, then* f *is* $\mu-$*integrable and* $\int_\Omega f_n d\mu \to \int_\Omega f d\mu$*.*

Chapter 3

Topology

In this chapter, we focus our attention on the basics about the topology which will be helpful to study the interplay between measure theory and topology. For, first we define topology.

Definition 3.1. *A collection τ of subsets of a set Ω is said to be a topology in Ω if τ has the following three properties:*

(a) $\phi, \Omega \in \tau$.

(b) If $V_i \in \tau$ for $i = 1, 2, ..., n$, then $\bigcap_{i=1}^{n} V_i \in \tau$.

(c) If V_α is an arbitrary collection of members of τ, then $\bigcup_\alpha V_\alpha \in \tau$.

If τ is a topology in Ω, Ω is called a topological space, and the members of τ are called the open sets of Ω. A subset A of a topological space Ω is said to be closed if the set $\Omega - A$ is open.

If Ω_1 and Ω_2 are topological spaces and f is a mapping of Ω_1 into Ω_2, then f is said to be continuous provided that $f^{-1}(V)$ is an open set in Ω_1 for every open set V in Ω_2.

Definition 3.2. *Let Ω_1, Ω_2 be topological spaces. A map $f : \Omega_1 \to \Omega_2$ is said to be an open map if for every open set U of Ω_1, the set $f(U)$ is open in Ω_2.*

Definition 3.3. *A neighborhood of a point $\omega \in \Omega$ is an open set containing ω; an over neighborhood of ω is an overset of a neighborhood of ω.*

Definition 3.4. *Let Ω be a set. A basis for a topology on Ω is a collection $\mathscr{K}$ of subsets of Ω (called basis element) such that*

(a) For each $\omega \in \Omega$, there is at least one basis element K containing ω.

(b) *If ω belongs to the intersection of two basis element K_1 and K_2, then there is a basis element K_3 containing ω such that $K_3 \subset K_1 \cap K_2$.*

Definition 3.5. *A metric space is a set Ω with a function d (called a metric) from $\Omega \times \Omega$ to the nonnegative reals, satisfying $d(x, y) \geq 0, d(x, y) = 0$ iff $x = y, d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$*

Definition 3.6. *Let $\epsilon > 0$ be given. Consider the set*

$$B_d(x, \epsilon) = \{y : d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. Then the set $B_d(x, \epsilon)$ is called the $\epsilon-$ball centered at x.

Definition 3.7. *Let Ω be a metric space with metric d. A subset A of Ω is said to be bounded if there is some number M such that $d(x, y) \leq M$ for every pair of points x, y of A.*

Definition 3.8. *A directed set is a set D on which there is defined a pre-ordering (a reflexive and transitive relation), denoted by $\leq$, with the property that whenever $a, b \in D$, there is a $c \in D$ with $a \leq c$ and $b \leq c$. A net in a topological space Ω is a function from a directed set D into Ω. A net is denoted by $\{x_n, n \in D\}$ or simply by $\{x_n\}$. The net $\{x_n\}$ is said to converge to the point x iff for every neighborhood U of x, there is an $n_0 \in D$ such that $x_n \in U$ for all $n \in D$ such that $n \geq n_0$.*

Definition 3.9. *Let Ω be a topological space. Then Ω is said to have a countable basis at ω if there is a countable collection $\mathscr{C}$ of neighborhoods of ω such that each neighborhood of ω contains at least one of the elements of $\mathscr{C}$. A space that has a countable basis at each of its points is said to satisfy first countability axiom or to be first countable.*

Definition 3.10. *If a topological space Ω has a countable basis for its topology, then Ω is said to satisfy the second countability axiom or to be second countable.*

Definition 3.11. *Let $\Omega_i, i \in I$ be an arbitrary collection of topological spaces. Then the Cartesian product of $\Omega_i, i \in I$, denoted by $\prod_{i \in I} \Omega_i = \Omega$(say), is defined as the collection of all families $(x_i, i \in I)$, that is, all functions on I such that $x_i \in \Omega_i$ for each i. The product topology (also called the topology of point wise convergence) on $\Omega = \prod_{i \in I} \Omega_i$ has as a base all sets of the form*

$$\{x \in \Omega : x_{i_k} \in U_{i_k}, k = 1, 2, ..., n\}$$

where U_{i_k} are open in Ω_{i_k} and n is an arbitrary positive integer.

Definition 3.12. *Let $p_i : \prod_{i\in I} \Omega_i \to \Omega_j$be the function assigning to each element of the product topological space its j^{th} coordinate,*

$$p_i(x) = x_j.$$

Then p_i is called the projection mapping associated with the index j.

Definition 3.13. *A collection $\mathscr{C}$ of subsets of a topological space Ω is said to cover Ω, or to be a covering of Ω, if the union of elements of $\mathscr{C}$ is equal to Ω. It is called an open covering of Ω if its elements are open subsets of Ω. A topological space Ω is said to be compact iff every open covering of Ω has a finite subcovering.*

Definition 3.14. *A topological space Ω is said to be Hausdorff if for each pair x, y of distinct points of Ω, there exist disjoint open sets containing x and y, respectively. A Hausdorff space is said to be locally compact iff each $x \in \Omega$ has a neighborhood whose closure is compact.*

Definition 3.15. *Let Ω be a topological space. Assume that one point sets are closed in Ω. The space Ω is said to be normal if for each pair A, B of disjoint closed sets of Ω, there exists disjoint open sets containing A and B, respectively.*

The following are useful theorems for our main part "The interplay between measure theory and topology".

Theorem 3.1. *Urysohn Lemma. [1] Let Ω be a Hausdorff space. Then Ω is normal iff for each pair of disjoint closed sets A and B, there is a continuous function $f : \Omega \to (0, 1)$ with $f = 0$ on A and $f = 1$ on B.*

Theorem 3.2. *Tietze Extension Theorem. [1] Let Ω be a Hausdorff space. Then Ω is normal iff for every closed set $A \subset \Omega$ and every continuous real-valued function f defined on A, f has an extension to a continuous real valued function F on Ω.Furthermore, if $|f| < c$ on A, then Fcan be taken less than c on Ω.*

Theorem 3.3. *[1] Let A be a closed subset of the normal space Ω. There is a continuous function $f : \omega \to (0, 1)$ such that $A = f^{-1}(0)$ iff A is a G_δ, that is, a countable intersection of open sets.*

Theorem 3.4. *Every metric space is normal.*

Proof. Let Ω be a metric space with metric d. Let A and B be disjoint closed subset of Ω. For each $a \in A$, choose $\epsilon_a > 0$ so that the ball $B(a, \epsilon_a)$ does not intersect B. Similarily, for each $b \in B$, choose $\epsilon_b > 0$ so that the ball $B(b, \epsilon_b)$ does not intersect A. Define

$$U = \bigcup_{a \in A} B\left(a, \frac{\epsilon_a}{2}\right), V = \bigcup_{b \in B} B\left(b, \frac{\epsilon_b}{2}\right).$$

Then U and V are open containing A and B, respectively. Now we asserts that $U \cap V = \phi$.

For, if possible let us suppose $U \cap V \neq \phi$. Then there exist a point z such that $z \in U \cap V$. Then

$$z \in B\left(a, \frac{\epsilon_a}{2}\right) \cap B\left(b, \frac{\epsilon_b}{2}\right)$$

for some $a \in A$ and some $b \in B$. So by triangle inequality

$$d(a, b) \leq d(a, z) + d(z, b) < \frac{\epsilon_a}{2} + \frac{\epsilon_b}{2}.$$

If $\epsilon_a \leq \epsilon_b$, then $d(a, b) < \epsilon_b$ so that $a \in B(b, \epsilon_b)$, a contradiction.
If $\epsilon_b \leq \epsilon_a$, then $d(a, b) < \epsilon_a$ so that $b \in B(a, \epsilon_a)$, again a contradiction.
Hence $U \cap V = \phi$.
So Ω is normal.

□

Theorem 3.5. *[1] Every compact Hausdorff space is normal.*

Definition 3.16. *Let Ω be topological space. The function $f : \Omega \to \overline{\Re}$ is said to be lower semicontinuous (LSC) on Ω iff $\{x \in \Omega : f(x) > a\}$ is open in Ω for each $a \in \overline{\Re}$. Similarly the function $f : \Omega \to \overline{\Re}$ is said to be upper semicontinuous (USC) on Ω iff $\{x \in \Omega : f(x) < a\}$ is open in Ω for each $a \in \overline{\Re}$. Thus f is LSC iff $-f$ is USC. The function $f : \Omega \to \overline{\Re}$ is continuous iff it is both LSC and USC.*

The following is the criterion for a function to be semicontinuious.

Theorem 3.6. *The function f is LSC on a topological space Ω iff, for each net $\{x_n\}$ converging to a point $x \in \Omega$, we have $\liminf_n f(x_n) \geq f(x)$. Similarly, f is USC iff $\limsup_n \leq f(x)$, whenever $x_n \to x$.(In first countable space 'net' may be replaced by 'sequence.')*

Proof. Let f be LSC on Ω. Let $x_n \to x$ and $b < f(x)$. Then by definition of LSC $x \in f^{-1}(b, \infty]$, an open subset of Ω. Hence $x_n \in f^{-1}(b, \infty]$ for all but finitely many n, that is, $f(x_n) > b$ for all but finitely many n. Thus, $\liminf_n f(x_n) \geq f(x)$.

Conversely, Let $x_n \to x$ implies $\liminf_n f(x_n) \geq f(x)$. Then we have to show that f is LSC on Ω. For this it is sufficient to show that $V = \{f(x) > a\}$ is open in Ω for any $a \in \overline{\Re}$. For, let $x_n \to x$, where $f(x) > a$. Then $\liminf_n f(x_n) > a$, hence $f(x_n) > a$ for all but finitely many n. Thus $x_n \in V$ for all but finitely many n. So V is open. □

Following are some properties of semicontinuous functions.

Theorem 3.7. *Let f be a LSC on the compact space Ω. Then f attains its infimum.(Hence if f is USC on the compact space Ω, f attains its supremum.)*

Proof. Let $b = \inf f$. Then there is a sequence of points $x_n \in \Omega$ such that $f(x_n) \to b$. But since Ω is compact, we have a subnet x_{n_k} converging to some $x \in \Omega$. Since f is LSC, $\liminf_k f(x_{n_k}) \geq f(x)$. (by preceeding theorem) But as $f(x_{n_k}) \to b$, we must have $f(x) \leq b$. Since b is infimum of such functions, we must have $b = f(x)$ as required. □

Theorem 3.8. *If f_i is LSC on Ω for each $i \in I$, then $\sup_i f_i$ is LSC; if I is finite, then $\min_i f_i$ is LSC.(Hence if f_i is USC for each i, then $\inf_i f_i$ is USC; and if I is finite, then $\max_i f_i$ is USC.)*

Theorem 3.9. *[6] Let $f : \Omega \to \overline{\Re}, \Omega$ any topological space, f arbitrary. Define*

$$\underline{f}(x) = \liminf_{y \to x} f(y), \;\; x \in \Omega;$$

that is,

$$\underline{f}(x) = \sup_V \inf_{y \in V} f(y),$$

where V ranges over all neighborhoods of x. Then $\underline{f}$ is LSC on Ω and $\underline{f} \leq f$; furthermore if g is LSC on Ω and $g \leq f$, then $g \leq \underline{f}$.
Similarly, if $\overline{f}(x) = \limsup_{y \to x} f(y) = \inf_V \sup_{y \in V} f(y)$, then $\overline{f}$, is USC and $\overline{f} \geq f$; in fact $\overline{f}$ is is the inf *of all USC functions that are less than or equal to f.*

Definition 3.17. *In the above theorem, $\underline{f}$, the* sup *of all LSC functions that are less than or equal to f is called the lower envelope of f and $\overline{f}$, the* inf *of all USC functions that are greater than or equal to f is called the upper envelope of f.*

Theorem 3.10. *[6] Let Ω be a metric space, f a LSC function on Ω. There is a sequence of continuous functions $f_n : \Omega \to \overline{\Re}$ such that $f_n \uparrow f$. (Thus, if f is USC, there is a sequence of continuous functions $f_n \downarrow f$.) If $|f| \leq M < \infty$, the f_n may chosen so that $|f_n| \leq M$ for all n.*

Chapter 4

The Interplay Between Measure Theory And Topology

4.1 Introduction

In this chapter, we will focus on the interplay between measure theory and topology. More precisely, we will establish the natural relation between measure theoritic and topological questions. We will investigate these natural relation in details. We begin with the situation in which there is connection between measure theory and topology.

We already defined $\sigma-$field in previous chapter. Let us define a $\sigma-$field $\mathscr{F}$ as the smallest $\sigma-$field containing all open sets of a topological space Ω. Doing this we can connect the measure theory and topology and further connections can be established. For example, if μ is a measure on $\mathscr{F}$ and $A \in \mathscr{F}$, then we wish to know whether A can be approximated by compact subsets by using measure μ. In other words, we wish to know the relation :

$$\mu(A) = \sup\{\mu(K) : K \text{ is a compact subset of } A\}$$

Another example, we will focus on the question that whether a function in the space $L^P(\Omega, \mathscr{F}, \mu)$ can be approximated by a continuous function or not. In Other words, whether the set of continuous functions are dense in the space $L^P(\Omega, \mathscr{F}, \mu)$.

The results in the first two sections are not topological but they serve as basic tools in the later development.

Definition 4.1. *Let $\mathscr{D}$ be a class of subsets of a set Ω. Then $\mathscr{D}$ is said to be a Dynkin system (D-system for short) iff the following conditions hold:*

(a) $\Omega \in \mathscr{D}$.

(b) If $A, B \in \mathscr{D}, B \subset A$, *then* $A - B \in \mathscr{D}$. *Thus* $\mathscr{D}$ *is closed under the proper differences.*

(c) If $A_1, A_2, ... \in \mathscr{D}$ *and* $A_n \uparrow A$, *then* $A \in \mathscr{D}$.

In definition, by (a) and (b), $\mathscr{D}$ is closed under complementation; hence by (c) $\mathscr{D}$ is a monotone class. Thus if $\mathscr{D}$ is closed under finite union (or closed under finite intersection), then $\mathscr{D}$ is a field, and hence a $\sigma-$field.

Theorem 4.1. *Dynkin System Theorem. Let* $\mathscr{S}$ *be a class of subsets of* Ω *and assume that* $\mathscr{S}$ *is closed under finite intersection. If* $\mathscr{D}$ *is Dynkin system and* $\mathscr{D} \supset \mathscr{S}$, *Then* $\mathscr{D}$ *includes the minimal* $\sigma-$*field* $\mathscr{F} = \sigma(\mathscr{S})$.

Proof. Let $\mathscr{F}_0$ be the smallest $D-$system including $\mathscr{S}$. Then we asserts that $\mathscr{D}_0 = \mathscr{F}$. Since $\mathscr{D}_0 \subset \mathscr{D}$, the result will follow.
Now, since $\mathscr{F}$ itself is a $D-$system containing $\mathscr{S}$, we have $\mathscr{D}_0 \subset \mathscr{F}$. so it sufficient to show that $\mathscr{F} \subset \mathscr{D}_0$. For this let us define

$$\mathscr{C} = \{A \in \mathscr{D}_0 : A \cap B \in \mathscr{D}_0 \text{ for all } B \in \mathscr{S}\}.$$

Then $\mathscr{S} \subset \mathscr{C}$ since $\mathscr{S}$ is closed under finite intersection. Also since $\mathscr{D}_0$ is a $D-$system, $\mathscr{C}$ is also a $D-$system. Thus $\mathscr{D}_0 \subset \mathscr{C}$. Hence $\mathscr{D}_0 = \mathscr{C}$.
Now let $\mathscr{C}' = \{C \in \mathscr{D}_0 : C \cap D \in \mathscr{D}_0 \text{ and } D \in \mathscr{D}_0\}$. Since $\mathscr{C} = \mathscr{D}_0$, same as above $\mathscr{S} \subset \mathscr{C}'$ and since $\mathscr{C}'$ is a $D-$system, we have $\mathscr{D}_0 \subset \mathscr{C}'$. Hence $\mathscr{D}_0 = \mathscr{C}'$. This follows that $\mathscr{D}_0$ is closed under finite intersection. Hence $\mathscr{D}_0$ is a field; so that $\mathscr{F} \subset \mathscr{D}_0$ as desired. □

Corollary 4.1. *Let* $\mathscr{S}$ *be a class of subsets of* Ω *and let* μ_1 *and* μ_2 *be finite measure on* $\sigma(\mathscr{S})$. *Assume* $\Omega \in \mathscr{S}$ *and* $\mathscr{S}$ *is closed under finite intersection. If* $\mu_1 = \mu_2$ *on* $\mathscr{S}$, *then* $\mu_1 = \mu_2$ *on* $\sigma(\mathscr{S})$.

Proof. Let $\mathscr{D}$ be a class of sets $A \in \sigma(\mathscr{S})$ such that $\mu_1(A) = \mu_2(A)$. Then $\mathscr{S} \subset \mathscr{D}$. Now we asserts that $\mathscr{D}$ is a $D-$system. For $\Omega \in \mathscr{D}$ because $\Omega \in \mathscr{S}$. Let $A, B \in \mathscr{D}$ with $B \subset A$. Then $A - B \in \sigma(\mathscr{S})$ and

$$\mu_1(A - B) = \mu_1(A) - \mu_1(B) = \mu_2(A) - \mu_2(B) = \mu_2(A - B)$$

Thus $A - B \in \mathscr{D}$.
further, let $A_1, A_2, ... \in \mathscr{D}$ and $A_n \uparrow A$. Then $\mu_1(A_n) \to \mu_1(A)$ and $\mu_2(A_n) \to \mu_2(A)$ as $n \to \infty$. But for all n, $A_n \in \mathscr{D}$. So $\mu_1(A_n) = \mu_2(A_n)$ for all n; hence

$\mu_1(A) = \mu_2(A)$. This implies that $A \in \mathscr{D}$.
Thus $\mathscr{D}$ is a $D-$system.
Hence $\mathscr{D}$ is a $D-$system including $\mathscr{S}$. So by above theorem, $\sigma(\mathscr{S}) \subset \mathscr{D}$. But $\mathscr{D} \subset \sigma(\mathscr{S})$. Thus $\mathscr{D} = \sigma(\mathscr{S})$.
This proves the corollary. □

Corollary 4.2. *Let $\mathscr{S}$ be a class of subsets of Ω; assume that $\Omega \in \mathscr{S}$ and $\mathscr{S}$ is closed under finite intersection. Let H be a vector space of real-valued functions on Ω such that $I_A \in H$ for each $A \in \mathscr{S}$. Suppose that whenever $f_1, f_2, ...$ are nonnegative functions in H, $|f_n| \leq M < \infty$ for all n, and $f_n \uparrow f$, the limit function f belongs to H. Then $I_A \in H$ for all $A \in \sigma(\mathscr{S})$.*

Proof. Let us define $\mathscr{D}$ by

$$\mathscr{D} = \{A \subset \Omega : I_A \in H\}.$$

Then by hypothesis $\mathscr{S} \subset \mathscr{D}$ and hence $\Omega \in \mathscr{D}$. First we show that $\mathscr{D}$ is a $D-$system. For this let $A, B \in \mathscr{D}$. Then $I_{A-B} = I_A - I_B \in H$ (because H is a vector space). So, $A - B \in \mathscr{D}$. Also if A_n be a sequence of sets in $\mathscr{D}$ such that $A_n \uparrow A$, then $I_{A_n} \uparrow I_A$. Hence by hypothesis $I_A \in H$. So $A \in \mathscr{D}$. Thus $\mathscr{D}$ is a $D-$system including $\mathscr{S}$. So by above Theorem, $\mathscr{D} \supset \sigma(\mathscr{S})$ which proves the corollary. □

4.2 The Daniel Integral

We know that one of the properties of integration is linearity; if f and g are $\mu-$integrable and a, b are real or complex, then

$$\int_\Omega (af + bg)d\mu = a \int_\Omega f d\mu + b \int_\Omega g d\mu.$$

Thus, the integral can be regarded as a linear functional on the vector space of integrable functions. We use this idea to define integration theory. We begin with a linear functional E on a vector space and extend E to a larger space. We finally show that there is a measure μ such that E is infact the integral with respect to μ.

Some notations are in order as follows:

Throughout this section, a vector space of real-valued functions on a set Ω will be denoted by L. We also assume that L is closed under the lattice operations, that is, if $f, g \in L$, then $f \vee g = \max(f, g), f \wedge g = \min(f, g) \in L$. We can find several examples of such spaces. For example, if L is the class of continuous real-valued functions, then L is closed under lattice operations.

E will denote the positive linear functional on L. This implies that E is monotone, that is, $f \leq g$ implies that $E(f) \leq E(g)$.

If H is any class of functions from Ω to $\Re$ (or $\overline{\Re}$), then H^+ denotes $\{f \in H : f \geq 0\}$. The collection of functions $f : \Omega \to \overline{\Re}$ of the form $\lim_n f_n$ where f_n form an increasing sequence of functions in L^+, is denoted by L'. If f_n form an increasing net in L^+, then the resulting class is denoted by L''. If H is defined as above, then $\sigma(H)$ is defined as the smallest $\sigma-$field making every function in H Borel measurable.

Throughout this section the following hypothesis will be assumed.

Hypothesis A: If $f_n \downarrow 0$ in L, then $E(f_n) \downarrow 0$. Equivalently, if $f_n \in L$ and $f_n \uparrow f$ in L, then $E(f_n) \uparrow E(f)$.

Hypothesis B: if the function f_n form a net in L decreasing to 0, then $E(f_n)$ decrease to 0. Equivalently, if the nets $f_n \in F$ and increase to a net $f \in L$, then $E(f_n) \uparrow E(f)$.

The main purpose for this section is to establish Daniel representation theorem which is important for our next section "The interplay between measure theory and topology." To establish Daniel representation theorem we need following some Lemmas.

Lemma 4.1. *Let $\{f_m\}$ and $\{f'_n\}$ be sequences in L increasing to f and f' ,respectively, with $f \leq f'$(f and f' need not belong to L). Then*

$$\lim_m E(f_m) \leq \lim_n E(f'_n).$$

Hence E may be extended to L' by defining $E(\lim_n f_n) = \lim_n E(f_n)$.(Under hypothesis B and with 'sequence' replaced by 'net' in the above statement, E may be extended to L'' in the same fashion.)

Proof. As $n \to \infty$, we have $f_m \wedge f'_n \uparrow f_m \wedge f'$. But both $\{f_m\}$ and $\{f'_n\}$ are increasing sequences in L increasing to f and f', respectively, with $f \leq f'$. So $f_m \wedge f'_n \uparrow f_m \wedge f' = f_m$. Hence

$$\lim_n E(f'_n) \geq \lim_n E(f_m \wedge f'_n) \geq E(f_m \wedge f') = E(f_m).$$

This inequality holds for all m. So as $m \to \infty$, we have

$$\lim_m E(f_m) \leq \lim_n E(f'_n)$$

as desired. □

Lemma 4.2. *The extension of E to L' has the following properties:*

(a) $0 \leq E(f) \leq \infty$ *for all* $f \in L'$.

(b) If $f, g \in L', f \leq g$, *then* $E(f) \leq E(g)$.

(c) If $f \in L'$ *and* $0 \leq c < \infty$, *then* $cf \in L'$ *and* $E(cf) = cE(f)$.

(d) If $f, g \in L'$, *then* $f + g, f \vee g, f \wedge g \in L'$ *and*

$$E(f + g) = E(f \vee g) + E(f \wedge g) = E(f) + E(g).$$

(e) If $\{f_n\}$ *is a sequence in* L' *increasing to* f, *then* $f \in L'$ *and* $E(f_n)$ *increase to* $E(f)$.

(Under the hypothesis B, the extension of E *to* L'' *has exactly the same properties; 'sequence' is replaced by 'net' in (e).)*

Proof. (a) Let $f \in L'$. Then there is a sequence of functions $\{f_n\}$ in L^+ such that $\lim_n f_n = f$. But by lemma4.1, we have

$$0 \leq \lim_n E(f_n) = E(\lim_n f_n) = E(f)$$

Thus $o \leq E(f) \leq \infty$ for all $f \in L'$.

(b) Here $f, g \in L'$ with $f \leq g$. So there are sequence of functions $\{f_n\}$ and $\{g_n\}$ in L^+ such that $f_n \uparrow f$ and $g_n \uparrow g$. So by lemma4.1

$$\lim_n E(f_n) \leq \lim_n E(g_n).$$

It follows that $E(f) \leq E(g)$.

(c) Here $f \in L'$ and $0 \leq c < \infty$. So $cf \geq 0$. Thus by the definition of L', there is a sequence of functions $f_n \in L^+$ such that $\lim_n cf_n = cf$. But it follows that there is a sequence of functions $\{cf_n\}$ in L^+ such that $cf_n \uparrow cf$. So $cf \in L'$.
Also by lemma4.1

$$\begin{aligned} E(\lim_n cf_n) &= \lim_n E(cf_n) \\ &= \lim_n cE(f_n) \\ &= c \lim_n E(f_n). \end{aligned}$$

Hence $E(cf) = cE(f)$ as required.

(d) Here $f, g \in L'$, so there are sequence of functions $\{f_n\}$,$\{g_n\}$ in L^+ such that $f_n \uparrow f$ and $g_n \uparrow g$. Then

$$\begin{aligned} E(f_n \vee g_n) + E(f_n \wedge g_n) &= E((f_n \vee g_n) + (f_n \wedge g_n)) \text{ (Since } E \text{ is linear)} \\ &= E(f_n + g_n) \\ &= E(f_n) + E(g_n) \end{aligned}$$

Now as $n \to \infty$, then

$$E(f \vee g) + E(f \wedge g) = E(f) + E(g) = E(f + g)$$

as desired.

(e) Let $\{f_{nm}\}$ be a sequence in L^+ such that $f_{nm} \uparrow f_n$ as $m \to \infty$. Let us define g_m by

$$g_m = f_{1m} \vee f_{2m} \vee \ldots \vee f_{mm}.$$

Then $\{g_m\}$ form an increasing sequence in L^+, and satisfy

$$f_{nm} \leq g_m \leq f_m \text{ for } n \leq m. \tag{4.1}$$

So by (b)

$$E(f_{nm}) \leq E(g_m) \leq E(f_m) \text{ for } n \leq m. \tag{4.2}$$

Now, let $m \to \infty$ in 4.1, we get

$$\lim_m f_{nm} \leq \lim_m g_m \leq \lim_m f_m$$

This implies that

$$f_n \leq \lim_m g_m \leq f$$

Now as $n \to \infty$, then

$$\lim_n f_n \leq \lim_n \left(\lim_m g_m\right) \leq f.$$

So

$$f \leq \lim_m g_m \leq f.$$

Thus $\lim_m g_m = f$, that is, $g_m \uparrow f$. Hence by hypothesis B, $E(g_m) \uparrow E(f)$.
Next let $m \to \infty$ in 4.2, then

$$\lim_m E(f_{nm}) \leq \lim_m E(g_m) \leq \lim_m E(f_m).$$

So

$$E(f_n) \leq E(f) \leq \lim_m E(f_m).$$

Now as $n \to \infty$

$$\lim_n E(f_n) \leq E(f) \leq \lim_n E(f_n).$$

Thus

$$\lim_n E(f_n) = E(f)$$

Hence $E(f_n) \uparrow E(f)$.

□

Lemma 4.3. *Let $\mathscr{G} = \{G \subset \Omega : I_G \in L'\}$ and define $\mu(G) = E(I_G), G \in \mathscr{G}$. Then $\mathscr{G}$ satisfies the following conditions:*

(a) $\phi, \Omega \in \mathscr{G}, \mu(\phi) = 0, \mu(\Omega) = 1, 0 \leq \mu(A) \leq 1$ *for all* $A \in \mathscr{G}$

(b) If $G_1, G_2 \in \mathscr{G}$*, then* $G_1 \cup G_2, G_1 \cap G_2 \in \mathscr{G}$*, and*

$$\mu(G_1 \cup G_2) + \mu(G_1 \cap G_2) = \mu(G_1) + \mu(G_2).$$

(c) If $G_1, G_2 \in \mathscr{G}$ *and* $G_1 \subset G_2$*, then* $\mu(G_1) \leq \mu(G_2)$.

(d) If $G_n \in \mathscr{G}, n = 1, 2, ...$ *and* $G_n \uparrow G$*, then* $G \in \mathscr{G}$ *and* $\mu(G_n) \uparrow \mu(G)$.

Thus $\mu^*(A) = \inf\{\mu(G) : G \in \mathscr{G}, G \supset A\}$ *is a probability measure on the* $\sigma-$*field*

$$\mathscr{H} = \{H \in \Omega : \mu^*(H) + \mu^*(H^c) = 1\}$$

and $\mu^* = \mu$ *on* $\mathscr{G}$*.(Under hypothesis B, we take* $\mathscr{G} = \{G \in \Omega : I_G \in L^+\}$ *and replace sequences by nets in (d). The class* $\mathscr{G}$ *then has exactly the same properties.)*

Proof. (a) We have $I_\phi = 0, I_\Omega = 1$. So I_ϕ and I_Ω are constant functions and hence belong to L^+(and so L'.) Thus $\phi, \Omega \in \mathscr{G}$. Also

$$\mu(\phi) = E(I_\phi) = E(0) = 0.$$
$$\mu(\Omega) = E(I_\Omega) = E(1) = 1.$$

If $A \in \mathscr{G}$, then $A \subset \Omega$ implies that $I_\phi \leq I_A \leq I_\Omega$. So by 4.2(b), $E(I_\phi) \leq E(I_A) \leq E(I_\Omega)$. Hence, $0 = \mu(\phi) \leq \mu(A) \leq \mu(\Omega) = 1$ for all $A \in \mathscr{G}$.

(b) Here $G_1, G \in \mathscr{G}$, so $I_{G_1}, I_{G_2} \in L'$. So by lemma4.2(d) $I_{G_1} \vee I_{G_2}, I_{g_1} \wedge I_{G_2} \in L'$. But we have

$$I_{G_1 \cup G_2} = I_{G_1} \vee I_{G_2} \text{ and } I_{G_1 \cap G_2} = I_{G_1} \wedge I_{G_2}.$$

Thus $I_{G_1 \cup G_2}, I_{G_1 \cap G_2} \in L'$. Hence $G_1 \cup G_2, G_1 \cap G_2 \in \mathscr{G}$. Moreover,

$$\begin{aligned} \mu(G_1 \cup G_2) + \mu(G_1 \cap G_2) &= E(I_{G_1 \cup G_2}) + E(I_{G_1 \cap G_2}) \\ &= E(I_{G_1} \vee I_{G_2}) + E(I_{G_1} \wedge I_{G_2}) \\ &= E(I_{G_1}) + E(I_{G_2}) \\ &= \mu(G_1) + \mu(G_2). \end{aligned}$$

(c) If G_1, G_2 and $G_1 \subset G_2$, then $I_{G_1}, I_{G_2} \in L'$ and $I_{G_1} \leq I_{G_2}$. So by the monotonicity of $E, E(I_{G_1}) \leq E(I_{G_2})$. Hence $\mu(G_1) \leq \mu(G_2)$.

(d) If $G_n \in G, n = 1, 2, ...\ G_n \uparrow G$, then $I_{G_n} \in L'$ for $n = 1, 2, ...$ and $\{I_{G_n}\}$ is an increasing sequence in L' such that $I_{G_n} \uparrow I_G$. So by lemma4.2(e), $I_G \in L'$ and $\mu(G_n) \uparrow \mu(G)$.

$\square$

Lemma 4.4. *If $f \in L'$ and $a \in \Re$, then $\{\omega : f(\omega) > a\} \in \mathscr{G}$. Hence*

$$f : (\Omega, \sigma(\mathscr{G})) \to (\overline{\Re}, \mathscr{B}(\overline{\Re})).$$

(The same result holds for $f \in L^+$ under hypothesis B)

Proof. Here $f \in L'$, so there are sequence of functions $\{f_n\}$ in L^+ such that $f_n \uparrow f$.Thus the sequence $(f_n - a)^+ = (f_n - a) \vee 0 \in L^+ \subset L'$. Hence $(f_n - a)^+ \uparrow (f - a)^+$. Thus by definition of L', we have $(f - a)^+ \in L'$ and by lemma4.2(e), $k(f - a)^+ \in L'$ for $k \geq 0$. But as $k \to \infty$, we have $1 \wedge k(f - a)^+ \uparrow I_{\{f > a\}}$. So by lemma4.2(e), we have $I_{\{f>a\}} \in L'$. Thus by definition of $\mathscr{G}$, we have $\{f > a\} \in L'$ as desired. $\square$

Lemma 4.5. *The* $\sigma-$*fields* $\sigma(L), \sigma(L')$, *and* $\sigma(\mathscr{G})$ *are identical. (Under the hypothesis B, we only have* $\sigma(L'') = \sigma(\mathscr{G})$ *and* $\sigma(L) \subset \sigma(L'')$.*)*

Proof. Under our hypothesis $\sigma(L')$ is the smallest $\sigma-$field of subsets of Ω making every function in L' Borel measurable. But by lemma4.4, we have$\sigma(\mathscr{G})$makes every function in L' Borel measurable Thus $\sigma(L') \subset \sigma(\mathscr{G})$. To prove converse inclusion, let $G \in \mathscr{G}$. Thus $I_G \in L'$. Hence $G = \{I_G = 1\} \in \sigma(L')$. Therefore, $\sigma(\mathscr{G}) \subset \sigma(L')$. Thus $\sigma(\mathscr{G}) = \sigma(L')$.
Now we assert that $\sigma(L') = \sigma(L)$. For let $f \in L$. Then $f = f^+ - f^-$ where $f^+, f^- \in L^+ \subset L'$.Since f^+ and f^- are $\sigma(L')-$measurable, f is also $\sigma(L')-$measurable. Thus $\sigma(L')$ makes every function in L Borel measurable. So $\sigma(L) \subset \sigma(L')$. Next let $f \in L'$. Then there is a sequence $\{f_n\}$ in $L^+ \subset L$ such that f is a limit of the sequence $\{f_n\}$. But since $\{f_n\}$ are $\sigma(L)-$measurable, we have f is also $\sigma(L)-$measurable. Thus $\sigma(L') \subset \sigma(L)$. Hence $\sigma(L') = \sigma(L)$.
Thus we have $\sigma(\mathscr{G}) = \sigma(L') = \sigma(L)$. □

Lemma 4.6. *For any* $A \subset \Omega, \mu^*(A) = \inf\{E(f) : f \in L', f \geq I_A\}$. *The result is the same under hypothesis B, with* L' *is replaced by* L''.

Proof. Let $A \in \Omega$. Then by the definition of μ^*

$$\begin{aligned}
\mu^*(A) &= \inf\{\mu(G) : G \in \mathscr{G}, G \supset A\} \\
&= \inf\{E(I_G) : G \in \mathscr{G}, G \supset A\} \\
&= \inf\{E(f) : f = I_G \in L', f \geq I_A\} \\
&\geq \inf\{E(F) : f \in L', f \geq I_A\}.
\end{aligned}$$

Next let $f \in L', f \geq I_A$. Then for $0 \leq a \leq 1$, we have $A \subset \{f > a\}$. But by lemma4.4, $\{f > a\} \in \mathscr{G}$. Thus $\mu^*(A) \leq \mu^*\{f > a\} = \mu\{f > a\} = E(I_{\{f>a\}})$. But since $f \geq 0$, we have $f \geq aI_{\{f>a\}}$. Hence $aE(I_{\{f>a\}}) \leq E(f)$, that is $E(I_{\{f>a\}}) \leq \frac{E(f)}{a}$.
Now as $a \to 1$

$$E(I_{\{f>a\}}) \leq E(f)$$

that is,

$$\mu^*(A) \leq E(I_{\{f>a\}}) \leq E(f)$$

Now taking inf over all $f \in L'$, we get

$$\mu^*(A) \leq \inf\{E(f) : f \in L', f \geq I_A\}.$$

This completes the proof of the lemma. □

Lemma 4.7. *If* $\mathscr{H} = \{H \subset \Omega : \mu^*(H) + \mu^*(H^c) = 1\}$, *then* $\mathscr{G} \subset \mathscr{H}$, *hence* $\sigma(\mathscr{G}) \subset \mathscr{H}$. *(The result is the same under the hypothesis B)*

Proof. If $G \in \mathscr{G}$, then $I_G \in L'$. So by definition of L', there is a sequence $\{f_n\}$ of functions in L^+ such that $f_n \uparrow I_G$. Thus

$$\mu^*(G) = \mu(G) = E(I_G) = \lim E(f_n) \text{ by lemma4.2(e)}$$

But since $G \in \mathscr{G}, G^c \in \mathscr{G}$. So by lemma4.6

$$\mu^*(G^c) \leq \inf \{E(f) : f \in L', f \geq I_{G^c}\}.$$

But since $f_n \leq I_G$ and $I_G + I_{G^c} = 1$, we have $1 - f_n \geq I_{G^c}$. Also since $1 - f_n \geq 0$, we have $1 - f_n \in L^+ \subset L'$.
Hence

$$\begin{aligned}
\mu^*(G^c) \leq \inf E(1 - f_n) &= \inf_n (E(1) - E(f_n)) \\
&= E(1) - inf_n E(f_n) \\
&= 1 - \lim_n E(f_n) \\
&= 1 - E(I_G) \\
&= 1 - \mu^*(G)
\end{aligned}$$

Thus $\mu^*(G^c) + \mu^*(G) \leq 1$. But since $\mu^*(G^c) + \mu^*(G)$ is always at least 1, we have

$$\mu^*(G^c) + \mu^*(G) = 1.$$

Hence $G \in \mathscr{H}$. Thus $\mathscr{G} \subset \mathscr{H}$ and hence $\sigma(\mathscr{G}) \in \mathscr{H}$. □

Now we prove the Daniel Representation theorem.

Theorem 4.2. *Let* L *be a vector space of real-valued functions on the set* Ω*; assume that* L *contains the constant functions and is closed under lattice operations. Let* E *be a Daniel Integral on* L*, that is, a positive linear functional on* L *such that* $E(f_n) \downarrow 0$ *for each sequence of functions* $f_n \in L$ *with* $f_n \downarrow 0$*; assume that* $E(1) = 1$.
Then there is a unique probability measure P *on* $\sigma(L)(= \sigma(L') = \sigma(\mathscr{G}))$ *such that each* $f \in L$ *is* $P-$*integrable and* $E(f) = \int_\Omega f dP$.

Proof. Let P be the restriction of μ^* to $\sigma(L)$. Then P is a probability measure on $\sigma(L)(= \sigma(L') = \sigma(\mathscr{G}))$. If $G \in \mathscr{G}$, then

$$E(I_G) = \mu(G) = \mu^*(G) = P(G) = \int_\omega I_G dP \tag{4.3}$$

Now first let $f \in L'$. let us define h_n by

$$h_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} I_{\left\{\frac{k-1}{2^n} < f \leq \frac{k}{2^n}\right\}} + nI_{\{f>n\}}$$

Then h_n form a sequence of nonnegative simple functions increasing to f. Also since $I_{\{a<f<b\}} = I_{\{f>a\}} - I_{\{f>b\}}$ for $a < b$, and $\{f > a\}, \{f > b\} \in \mathscr{G}$, we have $I_{\left\{\frac{k-1}{2^n} < f \leq \frac{k}{2^n}\right\}}, I_{\{f>n\}} \in \mathscr{G}$. Hence

$$\begin{aligned}
E(h_n) &= E\left(\sum_{k=1}^{n2^n} \frac{k-1}{2^n} I_{\left\{\frac{k-1}{2^n} < f \leq \frac{k}{2^n}\right\}} + nI_{\{f>n\}}\right) \\
&= \sum_{k=1}^{n2^n} \frac{k-1}{2^n} E(I_{\left\{\frac{k-1}{2^n} < f \leq \frac{k}{2^n}\right\}}) + nE(I_{\{f>n\}}) \\
&= \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \int_\Omega I_{\left\{\frac{k-1}{2^n} < f \leq \frac{k}{2^n}\right\}} dP + n \int_\Omega I_{\{f>n\}} dP \\
&= \int_\Omega \left(\sum_{k=1}^{n2^n} \frac{k-1}{2^n} I_{\left\{\frac{k-1}{2^n} < f \leq \frac{k}{2^n}\right\}} + nI_{\{f>n\}} \right) dP \\
&= \int_\Omega h_n dP.
\end{aligned}$$

Thus by monotone convergence theorem

$$E(f) = \int_\Omega f dP.$$

Now let $f \in L$. Then $f = f^+ - f^-$, so that $f^+, f^- \in L^+ \subset L'$. Then

$$\begin{aligned}
E(f) &= E(f^+) - E(f^-) \\
&= \int_\Omega f^+ dP - \int_\Omega f^- dP \\
&= \int_\Omega (f^+ - f^-) dP \\
&= \int_\Omega f dP.
\end{aligned}$$

Thus for all $f \in L$, we have $E(f) = \int_\Omega f dP$.
Now since $f^+, f^- \in L$, we have the integrals are finite. So each $f \in L$ is $P-$integrable.

This establish the existence of desire probability measure P.
Now we show that P is unique. For, let P' be another such measure. Then

$$\int_\Omega f dP = \int_\Omega f dP' \text{ for all } f \in L$$

But by monotone convergence theorem

$$\int_\Omega f dP = \int_\Omega f dP' \text{ for all } f \in L'$$

Setting $f = I_G$ for $G \in \mathscr{G}$, then we have

$$\int_\Omega I_G dP = \int_\Omega I_G dP'$$

So $P(G) = P'(G)$; hence $P = P'$ on $\mathscr{G}$. But by lemma 4.3(b), $\mathscr{G}$ is closed under finite intersection. So by corollary 1, we have $P = P'$ on $\sigma() = \sigma(L) = \sigma(L')$. Thus P is unique. □

Following is also the Daniel Representation Theorem under the hypothesis B.

Theorem 4.3. *Let L be a vector space of real valued functions on the set Ω; assume that L contains the constant functions and is closed under Lattice operations. Let E be a positive linear functional on L such that $E(f_n) \downarrow 0$ for each net of functions $f_n \in L$ with $f_n \downarrow 0$; assume that $E(1) = 1$.*
Then there is a unique probability measure P on $\sigma(L'')(= \sigma(\mathscr{G}))$ such that:

(a) Each $f \in L$ is $P-$integrable and $E(f) = \int_\Omega f dP$.

(b) If $\{G_n\}$ is a net of sets in $\mathscr{G}$ and $G_n \uparrow G$, then $G \in \mathscr{G}$ and $P_n(G) \uparrow P(G)$.

Proof. Let P be the restriction of μ^* to $\sigma(L'')$. Then same as the proof of the Daniel Representation Theorem P satisfies (a) with L' is replaced by L'' and sequence is replaced by net. Then P satisfies (b) as well by lemma 4.3(d).
Now it remains to show that P is unique(Actually we can not use the method of proving uniqueness part of Daniel representation Theorem for this theorem because the monotone convergence theorem fails in general for nets.) To prove uniqueness of P, let P' be another probability measure satisfying (a) and (b).
Let $f \in L''$. Then there is a net of functions $f_\alpha \in L^+$such that $f_\alpha \uparrow f$. Let us define $h_{n\alpha}$ by

$$h_{n\alpha} = \frac{1}{2^n} \sum_{j=1}^{n2^n} I_{\{f_\alpha > j2^{-1}\}}, \; n = 1, 2, ... \tag{4.4}$$

Then if $\frac{k-1}{2^n} < f_\alpha(\omega) \leq \frac{k}{2^n}, k = 1, 2, ..., n2^n$, then $h_{n\alpha} = \frac{k-1}{2^n}$, and if $f_\alpha(\omega) > n$, then $h_{n\alpha}(\omega) = n$. Then $h_{n\alpha}, n = 1, 2, ...$ are the sequence of nonnegative simple functions increasing to f_α.
Similarly, if

$$h_n = \frac{1}{2^n} \sum_{j=1}^{n2^n} I_{\{f > j2^{-1}\}}, \ n = 1, 2, ... \tag{4.5}$$

then h_n are nonnegative simple functions increasing to f.
Now

$$\begin{aligned}
\int_\Omega h_n dP' &= \int_\Omega \frac{1}{2^n} \sum_{j=1}^{n2^n} I_{\{f>j2^{-1}\}} dP' \text{ (using 4.5)} \\
&= \frac{1}{2^n} \sum_{j=1}^{n2^n} \int_\Omega I_{\{f>j2^{-1}\}} dP' \\
&= \frac{1}{2^n} \sum_{j=1}^{n2^n} P'(\{f > j2^{-1}\}) \\
&= \frac{1}{2^n} \sum_{j=1}^{n2^n} \lim_\alpha P'(\{f_\alpha > j2^{-1}\}) \text{ (Using (b))} \\
&= \lim_\alpha \frac{1}{2^n} \sum_{j=1}^{n2^n} P'(\{f_\alpha > j2^{-1}\}) \text{ (Since the sum on } j \text{ is finite)}
\end{aligned}$$

So

$$\int_\Omega h_n dP' = \lim_\alpha \int_\Omega h_{n\alpha} dP' \text{ Using 4.4.} \tag{4.6}$$

But

$$\begin{aligned}
\int_\Omega f dP' &= \lim_n \int_\Omega h_n dP' \text{ (By the monotone convergence theorem)} \\
&= \lim_n \lim_\alpha \int_\Omega h_{n\alpha} dP' \text{ (Using 4.6)} \\
&= \lim_\alpha \lim_n \int_\Omega h_{n\alpha} dP' \text{ (since } h_{n\alpha} \text{ is monotone in each variable)} \\
&= \lim_\alpha \int_\Omega f_\alpha dP' \text{ (By the monotone convergence theorem)} \\
&= \lim_\alpha E(f_\alpha) \text{ (By hypothesis since } P' \text{ satisfies (a))} \\
&= \lim_\alpha \int_\Omega f_\alpha dP \text{ (Using (a))} \\
&= \int_\Omega f dP \text{ (Same as above if we replace } P' \text{ by } P).
\end{aligned}$$

So

$$\int_\Omega f dP' = \int_\Omega f dP.$$

Again setting $f = I_G, G \in \mathscr{G}$, then same as proof in Daniel Representation Theorem, we have $P = P'$ in $\sigma(\mathscr{G}) = \sigma(L'')$ proving the uniqueness of P. □

The following approximation theorem will be useful to establish some results in the next section.

Theorem 4.4. *Assume the hypothesis of the Daniel Representation Theorem, and in addition assume that L is closed under limits of uniformly convergent sequences. Let*

$$\mathscr{G}' = \left\{ G \subset \Omega : G = \{f > 0\} \text{ for some } f \in L^+ \right\}$$

Then

(a) $\mathscr{G}' = \mathscr{G}$.

(b) If $A \in \sigma(L)$, *then* $P(A) = \inf \{P(G) : G \in \mathscr{G}', G \supset A\}$.

(c) If $G \in \mathscr{G}$, *then* $P(G) = \sup \{E(f) : f \in L^+, f \leq I_G\}$.

Proof. (a) By Lemma 4.4, if $f \in L'$ and $a \in \Re$, then $\{f > a\} \in \mathscr{G}$. Then $\mathscr{G}' \subset \mathscr{G}$. For converse, suppose that $G \in \mathscr{G}$ and let $f_n \in L^+$ with $f_n \uparrow I_G (\in L')$. Let us define f by

$$f = \sum_{n=1}^{\infty} 2^{-n} f_n.$$

Since $0 \leq f_n \leq 1$, we have $0 \leq 2^{-n} f_n < 1$. Thus the series is uniformly convergent. Hence by hypothesis $f \in L^+$. But

$$\begin{aligned} \{f > 0\} &= \bigcup_{n=1}^{\infty} \{f_n > 0\} \\ &= \{I_G = 1\} \quad (\text{since } f_n \uparrow I_G) \\ &= G \end{aligned}$$

So by definition of $\mathscr{G}'$, we have $G \in \mathscr{G}'$. Hence $\mathscr{G} \subset \mathscr{G}'$.
Therefore, $\mathscr{G} = \mathscr{G}'$.

(b) Let $G \in \mathscr{G}', G \supset A$. Then $P(G) \geq P(A)$. So $P(A)$ is lower bound for $P(G), G \in \mathscr{G}' = \mathscr{G}$ Also let $A \in \sigma(L)$ with $G \in \mathscr{G}, G \supset A$. Then

$$\begin{aligned} P(A) &= \mu^*(A) \ (\text{since } P = \mu^* \text{ on } \sigma(L)) \\ &= \inf \{\mu(G) : G \in \mathscr{G}, G \supset A\} \ \ (\text{By Lemma 4.3}) \\ &= \inf \{\mu(G) : G \in \mathscr{G}', G \supset A\} \end{aligned}$$

as desired.

(c) Let $f \in L^+, f \leq I_G$. Then

$$E(f) \leq E(I_G) = P(G) \tag{4.7}$$

Next, let $G \in \mathscr{G}$ with $f_n \in L^+, f_n \uparrow I_G$. Then

$$P(G) = E(I_G) = \lim_n E(f_n) = \sup_n E(f_n)$$

Hence

$$P(G) \leq \sup \left\{E(f) : f \in L^+, f \leq I_G\right\}.$$

But by 4.7, $P(G)$ is upper bound for $E(f), f \in L^+, f \leq I_G$. Hence for $G \in \mathscr{G}$, we have

$$P(G) = \sup \left\{E(f) : f \in L^+, f \leq I_G\right\}.$$

□

4.3 Measures on topological spaces

First we define the Baire $\sigma-$field which is as follows:

Definition 4.2. *Let Ω be a topological space. The Baire $\sigma-$field, which we denote by $\mathscr{A}(\Omega)$ or simply by $\mathscr{A}$ is defined as the smallest $\sigma-$field of subsets of Ω for which the functions in $C(\Omega)$ are Borel measurable. In other words $\mathscr{A}$ is the minimal $\sigma-$field containing all sets $f^{-1}(B)$ where B ranges over $\mathscr{B}(\Re)$ and f ranges over the class $C(\Omega)$ of continuous maps from Ω to $\Re$. Every sets in $\mathscr{A}(\Omega)$ are called Baire sets.*

Some consequences are:

(a) The class of Baire sets $\mathscr{A}$ is the minimal $\sigma-$field making all bounded continuous functions Borel measurable. For let $\mathscr{F}$ be a $\sigma-$field that makes all bounded continuous functions measurable. If $f \in C(\Omega)$, then $f^+ \wedge n$ is a bounded continuous function and $f^+ \wedge n \uparrow f^+$ as $n \to \infty$. Thus f^+ is measurable. Similarly f^- is also is measurable. Hence $f = f^+ - f^-$ is $\mathscr{F}-$measurable. Thus $\mathscr{A} \subset \mathscr{F}$. But we have $\mathscr{F} \subset \mathscr{A}$. Hence $\mathscr{F} = \mathscr{A}$ as required.

(b) If V is an open subset of Ω and $f \in C(\Omega)$, then by definition of continuous function $f^{-1}(V)$ is open in Ω. Hence $f^{-1}(V) \in \mathscr{B}(\Omega)$, the class of Borel sets. Since any set containing the sets $f^{-1}(V)$ for all open set V must contain the sets $f^{-1}(B)$ for all Borel set B, we have the sets $f^{-1}(V)$ generate $\mathscr{A}(\Omega)$. Thus we must have $\mathscr{A}(\Omega) \subset \mathscr{B}(\Omega)$.

(c) An F_σ set in Ω is a countable union of closed sets and a G_δ set in Ω is a countable intersection of closed sets.

We now establish the precise result on the interplay between measure theory and topology.

Theorem 4.5. *Let Ω be a normal topological space. Then $\mathscr{A}(\Omega)$ is the minimal $\sigma-$field containing the open F_σ sets (or equally well, the minimal $\sigma-$field containing the closed G_δ sets.)*

Proof. Let $\mathscr{H}$ be the minimal $\sigma-$field over the F_σ sets in Ω. Then we have to show that $\mathscr{A}(\Omega) = \mathscr{H}$. For this let $f \in C(\Omega)$. Then

$$\{f > a\} = \bigcup_{n=1}^{\infty} \left\{ f \geq a + \frac{1}{n} \right\}.$$

Since $f \in C(\Omega)$, we have $\{f > a\}$ is open in Ω. Similarly, $\left\{f \geq a + \frac{1}{n}\right\}$ is closed in Ω for all $n, n = 1, 2, ...$ So $\{f > a\}$ is an open F_σ set. But the sets

$\{f > a\}, f \in C(\Omega), a \in \Re$ are Borel measurable sets; and hence generate the class of Baire sets $\mathscr{A}(\Omega)$. Thus $\mathscr{A}(\Omega) \subset \mathscr{H}$.
For converse, let $H = \bigcup_{n=1}^{\infty} F_n, F_n$ closed, be an open F_σ set. Then H^c is closed and disjoint from F_n for each n. Since Ω is the normal topological space, we have by Urysohn's lemma, there are continuous functions $f_n : \Omega \to [0, 1]$ such that

$$f_n = 0 \text{ on } H^c \text{ and}$$

$$f_n = 1 \text{ on } F_n.$$

Now let us define f by

$$f = \sum_{n=1}^{\infty} 2^{-n} f_n$$

Then $f \in C(\Omega)$ with $0 \leq f \leq 1$, and

$$\{f > 0\} = \bigcup_{n=1}^{\infty} \{f_n > 0\} = H$$

Thus, since $f \in C(\Omega), \{f > 0\} \in \mathscr{A}$. Hence $H \in \mathscr{A}$. Therefore, $\mathscr{H} \subset \mathscr{A}$.
This completes the proof of the theorem. □

Corollary 4.3. *If Ω is a normal topological space, then open F_σ sets are precisely the sets $\{f > 0\}$ where $f \in C_b(\Omega)$, the class of bounded real valued function of Ω, $f \geq 0$.*

Proof. If $f \in C_b(\Omega)$ and $f \geq 0$, then by the proof of the theorem 4.5, we have

$$\{f > 0\} = \bigcup_{n=1}^{\infty} \left\{f_n \geq \frac{1}{n}\right\}$$

is an open F_σ set.
Conversely, if $H = \bigcup_{n=1}^{\infty} F_n, F_n$ closed, be an open F_σ set. Then again by the proof of the theorem 4.5, there is a bounded continuous function f with $o \leq f \leq 1$ such that $H = \{f > 0\}$.
This completes the proof of the corollary. □

The following corallary ensures that in a metric space the class of Borel sets and the class of Baire sets are same.

Corollary 4.4. *If Ω is a metric space, then $\mathscr{A}(\Omega) = \mathscr{B}(\Omega)$.*

Proof. We know that every metric space is normal and $\mathscr{A}(\Omega) \subset \mathscr{B}(\Omega)$. So it suffices to show that $\mathscr{B}(\Omega) \subset \mathscr{A}(\Omega)$.
For this, let F be a closed subset of Ω. Since Ω is a metric space, we have

$$F = \bigcup_{n=1}^{\infty} \left\{ \omega : dist(\omega, F) < \frac{1}{n} \right\}.$$

Thus F is a closed G_δ set. So by theorem 4.5,$\mathscr{A}(\Omega)$ contains F. Since F was arbitrary, $\mathscr{A}$ contains all closed sets of Ω. So $\mathscr{B}(\Omega) \subset \mathscr{A}(\Omega)$. Therefore, $\mathscr{B}(\Omega) = \mathscr{A}(\Omega)$. □

Corollary 4.4 proved that in a metric space, the Baire and Borel sets coincide.

Lemma 4.8. *Let A be an open F_σ set in the normal space Ω. Then I_A is the limit of an increasing sequence of continuous functions.*

Proof. Since A is an open F_σ set, by corollary 4.3

$$A = \{f > 0\} = \bigcup_{n=1}^{\infty} \left\{ f \geq \frac{1}{n} \right\}.$$

Since the set $A = \{f > 0\}$ is open, $A^c = \{f = 0\}$ is the closed set and disjoint from $\left\{ f \geq \frac{1}{n} \right\}$ for each n. So by Urysohn's lemma, there are functions $f_n \in C(\Omega)$ with $0 \leq f_n \leq 1$, such that

$$f_n = 0 \text{ on } \{f = 0\},$$

$$f_n = 1 \text{ on } \left\{ f \geq \frac{1}{n} \right\}.$$

Define g_n by

$$g_n = \max\{f_1, f_2, ..., f_n\}$$

Then g_n is an increasing sequence of continuous functions such that $g_n \uparrow I_{\{f>0\}}$ where $A = \{f > 0\}$ as required. □

Now by using Daniel Representation Theorem we prove the following approximation theorem.

Theorem 4.6. *Let P be any probability measure on $\mathscr{A}(\Omega)$, where Ω is a normal topological space.If $A \in \mathscr{A}$, then*

(a) $P(A) = \inf\{P(V) : V \supset A, V \text{ an open } F_\sigma \text{ set}\}$.

(b) $P(A) = \sup\{P(C) : C \subset A, C \text{ a closed } G_\delta \text{ set}\}$.

Proof. Let $L = C_b(\Omega)$ and let us define $E(f) = \int_\Omega f dP, f \in L$. Here we note that $\sigma(L) = \mathscr{A}$, so that each $f \in L$ is $\mathscr{A}-$measurable. Furthermore, since f is bounded, the integral $\int_\Omega f dP$ is finite. So E is well defined. Also E is a positive linear functional on L. hence by dominated convergence theorem, E is a Daniel integral. Thus this satisfies all the conditions for theorem 4.4. Thus by theorem 4.4(b),

$$P(A) = \inf\{P(G) : G \in \mathscr{G}', G \supset A\}$$

where

$$\mathscr{G}' = \left\{G \subset \Omega : G = \{f > 0\} \text{ for some } f \in L^+\right\}$$

But by corollary 4.3, $\mathscr{G}'$ is the class of open F_σ sets. Thus

$$P(A) = \inf\{P(V) : V \supset A, V \text{ an open } F_\sigma\text{set}\}$$

This proves (a).
Next, we know that, the complement of open set is closed and that of closed set is open. Thus the complement of open F_σ set is a closed G_δ set. We also know that for any set A and B, if $A \supset B$, then $A^c \subset B^c$. Thus by (a), we have

$$P(A) = \sup\{P(C) : C \subset A, C \text{ a closed } G_\delta\text{set}\}.$$

∎

Corollary 4.5. *If Ω is a metric space and P is a probability measure on $\mathscr{B}(\Omega)$, then for each $A \in \mathscr{B}(\Omega)$,*

(a) $P(A) = \inf\{P(V) : V \supset A, V \text{ open}\}$.

(b) $P(A) = \sup\{P(C) : C \subset A, C \text{ closed}\}$.

Proof. If G is a closed subset of Ω, then

$$G = \bigcap_{n=1}^{\infty} \left\{\omega : dist(\omega, G) < \frac{1}{n}\right\}.$$

So G is a G_δ set.
If F is an open subset of Ω, then F^c is closed and hence is a G_δ set. Since complement of a G_δ set is an F_σ, we have F is also an F_σ.
Thus in a metric space, every closed set is a G_δ and every open set is an F_σ.
Also by corollary 4.4, in a metric space we have $\mathscr{A}(\Omega) = \mathscr{B}(\Omega)$. Thus by theorem 4.6, we have for each $A \in \mathscr{B}(\Omega)$,

$$P(A) = \inf\{P(V) : V \supset A, V \text{ open}\}.$$

$$P(A) = \sup\{P(C) : C \subset A, C \text{ closed}\}.$$

□

By corollary 4.5, we conclude that in metric space every Borel set can be approximate by an open set (as well as a closed set).

Definition 4.3. *A metric space Ω with metric d, is said to be complete iff each Cauchy sequence converges to a point in the space Ω.*

Definition 4.4. *A subset A of a topological space Ω is said to be dense in Ω if $\overline{A} = \Omega$, where $\overline{A}$ is the closure of A in Ω. A topological space Ω is said to be separable if there is a countable dense subset of Ω.*

If Ω is a complete separable metric space, then the following theorem tells us that each Borel set can be approximate by a compact subset of Ω.

Theorem 4.7. *Let Ω be a complete separable metric space. If P is a probability measure on $\mathscr{B}(\Omega)$, then for each $A \in \mathscr{B}(\Omega)$,*

$$P(A) = \sup\{P(K) : K \ \textit{compact subset of } A\}.$$

Proof. To prove the theorem we show that if $\epsilon > 0$, there is a compact set K_ϵ such that $P(K_\epsilon) \geq 1 - \epsilon$. This implies the theorem. For if C is closed subset of Ω, then $C \cap K_\epsilon$ is compact and

$$\begin{aligned} P(C) &= P(C \cap K_\epsilon) + P(C - (C \cap K_\epsilon)) \\ &= P(C \cap K_\epsilon) + P(C \cap K_\epsilon). \end{aligned}$$

So

$$\begin{aligned} P(C) - P(C \cap K_\epsilon) &= P(C - K_\epsilon) \\ &\leq P(\Omega - K_\epsilon) \\ &= P(\Omega) - P(K_\epsilon) \\ &= 1 - P(K_\epsilon) \\ &\leq \epsilon. \end{aligned}$$

So $P(C) - P(C \cap K_\epsilon) \leq \epsilon$ for any $\epsilon > 0$.
This implies that C can be approximate by K_ϵ. But by theorem 4.5, A can be approximate by C for each Borel set $A \in \mathscr{B}(\Omega)$. Hence A can be approximate by compact subset K_ϵ of Ω.
Since Ω is separable, there is a countable dense set $\{\omega_1, \omega_2, ...\}$. Let $B(\omega_n, r)$ (respectively, $\overline{B}(\omega_n, r)$) be open (respectively, closed) ball with center at ω_n and radius r. Then since Ω is separable, for every $r > 0$,

$$\Omega = \bigcup_{n=1}^{\infty} \overline{B}(\omega_n, r)$$

so that $\bigcup_{k=1}^{m} \overline{B}(\omega_k, r) \uparrow \Omega$ as $m \to \infty$ (n fixed).
Thus, for given $\epsilon > 0$ and a positive integer n, there is a positive integer $m(n)$ such that

$$P(\bigcup_{k=1}^{m} \overline{B}(\omega_k, \frac{1}{n})) \geq 1 - \epsilon 2^{-n} \text{ for all } m \geq m(n).$$

Let $K_\epsilon = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{m(n)} \overline{B}(\omega_k, \frac{1}{n})$. Then K_ϵ is closed(being countable intersection of closed sets) and

$$\begin{aligned}
P(K_\epsilon^c) &\leq \sum_{n=1}^{\infty} P\left(\bigcup_{k=1}^{m(n)} \overline{B}(\omega_k, \frac{1}{n})\right)^c \\
&\leq \sum_{n=1}^{\infty} \epsilon 2^{-n} \\
&= \epsilon \quad \left(\text{since } \sum_{n=1}^{\infty} 2^{-n} = 1\right)
\end{aligned}$$

Therefore

$$P(K_\epsilon) + P(K_\epsilon^c) = P(\omega)$$

$$P(K_\epsilon) = 1 - P(K_\epsilon^c) \geq 1 - \epsilon.$$

Now it remains to show that K_ϵ is compact. For this,let $\{x_1, x_2, ...\}$ be a sequence in K_ϵ. Then we show that there is a subsequence of $\{x_p\}$ in K_ϵ that is Cauchy.
For we have $x_p \in K_\epsilon = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{m(n)} \overline{B}(\omega_k, \frac{1}{n})$ for all p. Hence $x_p \in \bigcup_{k=1}^{m(1)} \overline{B}(\omega_k, 1)$

for all p. Thus we conclude that for some integer $k_1, x_p \in \overline{B}(\omega_{k_1}, 1)$ for infinitely many p, say, for $p \in T_1$ an infinite set of positive integers.
Also $x_p \in \bigcup_{k=1}^{m(2)} \overline{B}\left(\omega_{k_1}, \frac{1}{2}\right)$ for all p, in particular for all $p \in T_1$. hence for some k_2

$$x_p \in \overline{B}(\omega_k, 1) \cap \overline{B}\left(\omega_{k_2}, \frac{1}{2}\right)$$

for infinitely many $p \in T_1$, say, for $p \in T_2 \subset T_1$.
Continue inductively to obtain integers $k_1, k_2, ...$ and infinite sets $T_1, T_2, ...$ such that

$$x_p \in \bigcap_{j=1}^{i} \overline{B}\left(\omega_{k_j}, \frac{1}{j}\right) \text{ for all } p \in T_i.$$

Pick $p_i \in T_i, i = 1, 2, ...$ with $p_1 < p_2 < ...$ Then for $j < i$, we have

$$x_{p_i}, x_{p_j} \in \overline{B}\left(\omega_{k_j}, \frac{1}{j}\right).$$

So $d(x_{p_i}, x_{p_j}) \leq \frac{2}{j} \to 0$ as $j \to \infty$.
Thus $\{x_{p_i}\}$ is a Cauchy sequence, hence converges to a point in K_ϵ since K_ϵ is closed and Ω is complete. Thus $\{x_p\}$ has a subsequence converging to a point of K_ϵ. So K_ϵ is compact. □

The following is the representation theorem of positive linear functional in a topological context.

Theorem 4.8. *Let Ω be a compact Hausdorff space, and let E be a positive linear functional on $C(\Omega)$, with $E(1) = 1$.There is a unique probability measure P on $\mathscr{A}(\Omega)$ such that*

$$E(f) = \int_\Omega f dP \text{ for all } f \in C(\Omega).$$

Proof. Let $L = C(\Omega)$. Then first we show that for $f_n \in L$ with $f_n \downarrow 0$, then $f_n \to 0$ uniformly.. For this, let $\delta > 0$ be given. Then we have $\Omega = \bigcup_{n=1}^{\infty} \{f_n < \delta\}$.
But since Ω is compact, there is a natural number N such that

$$\Omega = \bigcup_{n=1}^{N} \{f_n < \delta\}.$$

But since Ω is compact , there is a natural number N such that

$$\begin{aligned}\Omega &= \bigcup_{n=1}^{N}\{f_n<\delta\}\\ &= \{f_N<\delta\}.\end{aligned}$$

for some integer N. Thus $n \geq N$ implies that $0 \leq f_n(\omega) \leq f_N(\omega) < \delta$ for all ω. Since $\delta > 0$ was arbitrary, we conclude that $f_n \to 0$ uniformly.
Thus if for $f \in L, f_n \downarrow 0$ and $\delta > 0$ is given, then $0 \leq f_n < \delta$. So by monotonicity of E, we have $0 \leq E(f_n) \leq E(\delta) = \delta$. Therefore, $E(f_n) \downarrow 0$. So E is a Daniel integral. Thus, by Daniel Representation Theorem, there is a unique probability measure P on $\mathscr{A}(\Omega) = \sigma(L)$ such that $E(f) = \int_\Omega f dP$ for all $f \in L = C(\Omega)$. $\square$

If we use the Daniel Theory with hypothesis B, then the following somewhat different result is obtained.

Theorem 4.9. *Let Ω be a compact Hausdorff space, and let E be a positive linear functional on $C(\Omega)$, with $E(1) = 1$. There is a unique probability measure P on $\mathscr{B}(\Omega)$ such that*

(a) $E(f) = \int_\Omega f dP$ for all $f \in C(\Omega)$, and

(b) for all $A \in \mathscr{B}(\Omega)$,

$$P(A) = \inf\{P(V) : V \supset A, V \ \ \textit{open}\}.$$

Or equivalently,

$$P(A) = \sup\{P(K) : K \subset A, K \ \ \textit{compact}\}.$$

(Compact may be replaced by closed since Ω is compact Hausdorff.)

Proof. Let $L = C(\Omega)$. Let $\{f_n, n \in D\}$ is a net in L and assume that $f_n \downarrow 0$. Then for any $\delta > 0$, we have $\Omega = \bigcup_{n\in D}\{f_n < \delta\}$. But since ω is compact, we have

$$\Omega = \bigcup_{j\in F}\{f_j<\delta\}$$

for some finite set $F \subset D$.
If $N \in D$ and $N \geq j$ for all $j \in F$, then by the monotonicity of the net, we have $\Omega = \{f_N < \delta\}$. Thus $n \geq N$ implies that $0 \leq f_n \leq f_N < \delta$. Since $\delta > 0$ was arbitrary, we have $f_n \to 0$ uniformly. So if $\delta > 0$ is given, then

$0 \leq f_n < \delta$ so that $0 \leq E(f_n) < E(\delta) = \delta$. Therefore, $E(f_n \downarrow 0)$.
Therefore, these all meets the criteria for the theorem 4.3. Hence there is a probability measure P on $\sigma(L'') = \sigma(\mathscr{G})$ such that $E(f) = \int_\Omega f dP$ for all $f \in L$.
Now we prove $\sigma(\mathscr{G}) = \mathscr{B}(\Omega)$, which proves part (a) of our theorem. For let $f \in L''$. Then there is a net of continuous functions f_n such that $f_n \uparrow f$. Hence for each real a,

$$\{f > a\} = \bigcup_n \{f_n > a\}$$

is an open set.
Thus if $G \in \mathscr{G}$, then $I_G \in L''$; so that $G = \{I_G > 0\}$ is open. Thus $\sigma(\mathscr{G}) \subset \mathscr{B}(\Omega)$. Conversely, if G is open and $\omega \in \mathscr{G}$, then by Urysohn Lemma, there is a continuous function $f_\omega : \Omega \to [0,1]$ such that $f_\omega(\omega) = 1$ and $f_\omega = 0$ on G^c. Thus $I_G = \sup_\omega f_\omega$, so that if for each finite set $F \subset G$ we define $g_F = \max \{f_\omega : \omega \in F\}$, and the sets by inclusion, Then we obtain the monotone net of nonnegative continuous functions increasing to I_G. Therefore, $I_G \in L''$, so that $G \in \mathscr{G}$. Hence we have $\mathscr{B}(\Omega) \subset \sigma(\mathscr{G})$. This proves that $\sigma(\mathscr{G}) = \mathscr{B}(\Omega)$.
Thus we have established the existence of a probability measure P on $\mathscr{B}(\Omega)$ satisfying (a).
For part (b), we know $P = \mu^*$ on $\sigma(\mathscr{G})$ and by above discussion, $\mathscr{G}$ is the class of open sets. So by lemma 4.3, we have for each $A \in \mathscr{G}$,

$$P(A) = \inf \{P(V) : V \supset A, V \text{ open}\}.$$

If we replace A by the complement of A, then we get

$$P(A) = \sup \{P(C) : C \subset A, C \text{ closed}\}$$

and in a Hausdorff space every compact subset is closed, so we have

$$P(A) = \sup \{P(K) : K \subset A, K \text{ compact}\}.$$

Now it remains to show that P is unique. For let P' is another probability measure satisfying (a) and (b) of the theorem. Now we show that P' satisfies all the requirement for the theorem 4.3(b). Then uniqueness part of the theorem 4.3(b) follows that $P = P'$. Thus, let $\{G_n\}$ be a net of open sets with $G_n \uparrow G$. Since $G = \bigcup_n G_n$, we have G is open. Then by part (b) of the theorem, there is a compact set $K \subset G$ such that $P'(G) \leq P'(K) + \delta$. Thus $G_n \cup K^c \uparrow G \cup K^c = \Omega$. Hence by compactness and the monotonicity of $\{G_n\}, G_m \cup K^c = \Omega$ for some m, so that $K \subset G_m$. Consequently

$$\begin{aligned} P'(G) &\leq P'(K) + \delta \\ &\leq P'(G_m) + \delta \\ &\leq \lim_n P'(G_n) + \delta. \end{aligned}$$

This implies that $P'(G_n) \uparrow P'(G)$.
Thus by using uniqueness part of the theorem 4.3(b) we conclude that $P = P'$. □

Definition 4.5. *If μ is a measure on $\mathscr{B}(\Omega)$, where Ω is normal topological space, μ is said to be regular iff for each $A \in \mathscr{B}(\Omega)$*

$$\mu(A) = \inf\{\mu(V) : V \supset A, V \ \ \textit{open}\}$$

and

$$\mu(A) = \sup\{\mu(C) : C \subset A, C \ \ \textit{closed}\}.$$

If μ is finite, then either of one condition implies other. Moreover, the probability measure P is regular in theorem 4.9.
If $\mu = \mu^+ - \mu^-$ is a finite signed measure on $\mathscr{B}(\Omega), \Omega$ normal, we say that μ is regular iff μ^+ and μ^- are regular(equivalently, iff the total variation $|\mu|$ is regular).

The following theorem connects the theorem 4.8 and theorem 4.9.

Theorem 4.10. *If P is a probability measure on $\mathscr{A}(\Omega), \Omega$ compact Hausdorff, then P has a unique extension to a regular probability measure on $\mathscr{B}(\Omega)$.*

Proof. Define E by

$$E(f) = \int_\Omega f dP, f \in L = C(\Omega).$$

Then E is a positive linear functional on L. Thus if $\{f_n\}$ is a net in L decreasing to 0, then $E(f_n) \downarrow 0$. But since $f \in L = C(\Omega)$, by theorem 4.9, there is a unique regular probability measure P' on $\mathscr{B}(\Omega)$ such that $E(f) = \int_\Omega f dP'$. So we have

$$\int_\Omega f dP = \int_\Omega f dP' \text{ for all } f \in L.$$

But by definition of $\mathscr{A}(\Omega)$, each $f \in L$ is measurable:$(\Omega, \mathscr{A}(\Omega)) \to (\Re, \mathscr{B}(\Re))$. Hence $\int_\Omega f dP'$ is determined by the values of P' on Baire sets. Thus the condition that $\int_\Omega f dP = \int_\Omega f dP'$ for all $f \in L$ is equivalent to $P = P'$ on $\mathscr{A}(\Omega)$, by the uniqueness part of the theorem 4.8. □

Definition 4.6. *Let $p > 0$. Then we define the space $L^p = L^p(\Omega, \mathscr{F}, \mu)$ as the collection of all complex-valued Borel measurable function f such that $\int_\Omega |f|^p d\mu < \infty$.*

We define $\|f\|_p$

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu \right)^{\frac{1}{p}}, \ f \in L^p.$$

Following is the result on the approximation of Borel measurable functions by continuous functions.

Theorem 4.11. *[6] Consider the measure space $(\Omega, \mathscr{F}, \mu)$, where Ω is a normal topological space, $\mathscr{F} = \mathscr{B}(\Omega)$, and μ is a regular measure on $\mathscr{F}$. If $0 < p < \infty, \epsilon > 0$, and $f \in L^p(\Omega, \mathscr{F}, \mu)$, there is a continuous complex valued function $g \in L^p(\Omega, \mathscr{F}, \mu)$ such that $\|f - g\|_p < \epsilon$; furthermore, g can be chosen so that $\sup|g| \le \sup|f|$. Thus the continuous functions are dense in L^p.*

Theorem 4.12. *Let μ be a regular finite measure on $\mathscr{B}(\Omega), \Omega$ normal. If f is a complex valued Borel measurable function on Ω and $\delta > 0$, there is a continuous complex valued function g on Ω such that*

$$\mu \{\omega : f(\omega) \neq g(\omega)\} < \delta.$$

Furthermore it is possible to choose g so that $\sup|g| \le \sup|f|$.

Proof. Let us define h_n by

$$h_n(\omega) = \frac{k-1}{2^n} \ \text{ if } \ \frac{k-1}{2^n} \le f(\omega) \le \frac{k}{2^n}, k = 1, 2, ..., n2^n$$

and

$$h_n(\omega) = n \ \text{ if } \ f(\omega) \ge n.$$

Then h_n are nonnegative simple functions increasing to f. Let $f_n = h_n - h_{n-1}, n = 1, 2, ...$ with $h_0 = 0$. Then

$$f = \sum_{n=1}^{\infty} f_n.$$

We note that f_n has only two possible values 0 and 2^{-n}. If $A_n = \{f_n \neq 0\}$, let C_n be a closed subset of A_n and V_n an open Overset of A_n such that $\mu(V_n -$

$C_n) < \delta 2^{-n}$. Since Ω is normal, by Urysohn lemma, there is a continuous function $g_n : \Omega \to [0,1]$ such that

$$g_n = 1 \text{ on } C_n$$

$$g_n = 0 \text{ off } V_n.$$

If $g = \sum_{n=1}^{\infty} 2^{-n} g_n$, then by Weierstrass M-test, g is continuous map of Ω onto $[0,1]$. We claim that if $\omega \notin \bigcup_{n=1}^{\infty}(V_n - C_n)$, a set of measure less than δ, then $f(\omega) = g(\omega)$. To see this we observe that for each $n, \omega \in C_n$ or $\omega \notin V_n$. If $\omega \in C_n \subset A_n$ then $2^{-n} g_n(\omega) = 2^{-n} = f_n(\omega)$ and if $\omega \notin V_n$, then $2^{-n} g_n(\omega) = 0 = f_n(\omega)$ since $\omega \notin A_n$.
This proves the existence of g when $0 \le f < 1$.
The extension to a complex valued bounded f is immediate.
If f is unbounded , we write

$$f = f I_{\{|f|<n\}} + f I_{\{|f|\ge n\}} = f_1 + f_2$$

where f_1 is bounded and $\mu\{f_2 = 0\} = \mu\{|f| \ge n\}$, which can be made less than $\frac{\delta}{2}$ for sufficiently large n. Thus if g is continuous and and $\mu\{f_1 \ne g\} < \frac{\delta}{2}$ then $\mu\{f \ne g\} < \delta$.
Finally, if $|f| \ge M < \infty$, and g approximates f as above, we define

$$g_1(\omega) = g(\omega) \text{ if } |g(\omega)| \le M \text{ and}$$

$$g_1(\omega) = \frac{M g(\omega)}{|g(\omega)|} \text{ if } |g(\omega)| > M$$

Then g_1 is continuous, $|g_1| \le M$, and $f(\omega) = g(\omega)$ implies that $|g(\omega)| \le M$. Hence $g_1(\omega) = g(\omega) = f(\omega)$.
Therefore, $\mu\{f \ne g_1\} \le \mu\{f \ne g\} < \delta$.
This completes the proof of the theorem. □

Corollary 4.6. *Assume the hypothesis of the theorem 4.11.*

(a) *There is a sequence of continuous complex valued functions f_n on Ω converging to f a.e.$[\mu]$, with $|f_n| \le \sup|f|$ for all n.*

(b) *Given $\epsilon > 0$, there is a closed set $C \subset \Omega$ and a continuous complex valued function g on Ω such that $\mu(C) \ge \mu(\Omega) - \epsilon$ and $f = g$ on C, hence the restriction of f to C is continuous. If μ has the additional property that $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ compact}\}$ for each $A \in \mathscr{B}(\Omega)$, then C may be taken as compact.*

Proof. (a) By theorem 4.12, there is a continuous function f_n such that $|f_n| \leq M = \sup|f|$ and $\mu\{f_n = f\} < 2^n$. If $A_n = \{f_n \neq f\}$ and $A = \limsup_n A_n$, then by Borel-Cantelli lemma

$$\mu(A) = \mu(\limsup_n A_n) = 0$$

But if $\omega \in A$, then $f_n(\omega) = f(\omega)$ for sufficiently large m. This means that $\lim_n f_n = f$ a.e.$[\mu]$ as desired.

(b) For given $\epsilon > 0$, by theorem 4.11, there is a continuous function g such that

$$\mu\{f \neq g\} < \frac{\epsilon}{2}.$$

But by the regularity of μ, there is a closed set $C \subset \{f = g\}$ with $\mu(C) \geq \mu\{f = g\} - \frac{\epsilon}{2}$ (Here $\{f = g\}$ is a Borel measurable set, so it can be approximate by a closed subset of Ω).
Thus

$$\mu(C) \geq \mu\{f = g\} - \frac{\epsilon}{2}$$

But

$$\mu\{f = g\} + \mu\{f \neq g\} = \mu(\Omega).$$

So,

$$\begin{aligned} \mu\{f = g\} &= \mu(\Omega) - \mu\{f \neq g\} \\ &= \mu(\Omega) - \frac{\epsilon}{2} \end{aligned}$$

Thus

$$\mu(C) \geq \mu(\Omega) - \epsilon \text{ and } f = g \text{ on C}$$

as desired.
If $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ compact}\}$, for each $A \in \mathscr{B}(\Omega)$, then since every closed set is Borel set, the closed set discussed as above can be approximate by K. This means that for given $\epsilon_1 > 0$, there is a compact set K such that

$$\begin{aligned} \mu(K) &\geq \mu(C) - \epsilon_1 \\ &\geq \mu(\Omega) - \epsilon - \epsilon_1 \end{aligned}$$

So

$$\mu(K) \geq \mu(\omega) - \epsilon' \text{ where } \epsilon' = \epsilon + \epsilon_1.$$

and $f = g$ on K as desired.

□

Corollary (b) is also called Lusins theorem. This corollary shows that the Borel measurable function can be approximate by a continuous function.

4.4 Measures on Uncountably Infinite Product Spaces

In this section we consider probability measure on countably infinite product spaces and extend them to uncountable products under certain topological assumptions about the individual factor spaces. For this first we define some basics.

Definition 4.7. *Let $\mathscr{F}_j$ be a $\sigma-$field of subsets of $\Omega_j, j = 1, 2, ..., n$, and let $\Omega = \Omega_1 \times \Omega_2 \times ... \times \Omega_n$. A measurable rectangle in Ω is a set $A = A_1 \times A_2 \times ... \times A_n$, where $A_j \in \mathscr{F}_j$ for each $j = 1, 2, ..., n$. The smallest $\sigma-$field containing the measurable rectangles is called the product $\sigma-$field, written $\mathscr{F}_1 \times \mathscr{F}_2 \times ... \times \mathscr{F}_n$. If all $\mathscr{F}_j$ coincide with a fixed $\sigma-$field $\mathscr{F}$, the product $\sigma-$field is denoted by $\mathscr{F}^n$. Note that the notation $\mathscr{F}_1 \times \mathscr{F}_2 \times ... \times \mathscr{F}_n$ is not the Cartesian product of the $\mathscr{F}_j$. The product $\sigma-$field is the minimal σfield over the measurable rectangles. Note also that the collection of finite disjoint unions of measurable rectangles forms a field.*

Definition 4.8. *For each $j = 1, 2, ...$, let $(\Omega_j, \mathscr{F}_j)$ be a measurable space. Let $\Omega = \prod_{j=1}^{\infty} \Omega_j$, the set of all sequences $(\omega_1, \omega_2, ...)$ such that $\omega_j \in \Omega_j, j = 1, 2, ...$ If $B^n \subset \prod_{j=1}^{\infty} \Omega_j$, we define*

$$B_n = \{\omega \in \Omega : (\omega_1, ..., \omega_n) \in B^n\}.$$

The set B_n is called the cylinder with base B^n. The cylinder is said to be measurable if $B^n \in \prod_{j=1}^{n} \mathscr{F}_j$. If $B^n = A_1 \times ... \times A_n$, where $A_i \subset \Omega_i$ for each i, B_n is called a rectangle, a measurable rectangle if $A_i \in \mathscr{F}_i$ for each i.

Remarks: A cylinder with an $n-$dimensional base may always be regarded as having a higher dimensional base. For example

$$B = \left\{\omega \in \Omega : (\omega_1, \omega_2, \omega_3) \in B^3\right\},$$

then

$$\begin{aligned} B &= \left\{\omega \in \Omega : (\omega_1, \omega_2, \omega_3) \in B^3, \omega_4 \in \Omega_4\right\} \\ &= \left\{\omega \in \Omega : (\omega_1, \omega_2, \omega_3, \omega_4) \in B^3 \times \Omega_4\right\}. \end{aligned}$$

It follows that the measurable cylinders form a field. It is also true that finite disjoint union of measurable rectangles form a field.

Definition 4.9. *The minimal $\sigma-$field over the measurable cylinders is called the product of σfields $\mathscr{F}_j$, written $\prod_{j=1}^{\infty} \mathscr{F}_j$; $\prod_{j=1}^{\infty} \mathscr{F}_j$ is also called the minimal $\sigma-$field over the measurable rectangles. If all $\mathscr{F}_j$ coincide with a fixed $\sigma-$field $\mathscr{F}$, then $\prod_{j=1}^{\infty} \mathscr{F}_j$ is denoted by $\mathscr{F}^{\infty}$, and if all Ω_j coincide with a fixed set $S, \prod_{j=1}^{\infty} \Omega_j$ is denoted by S^{∞}.*

Now we defined the product of uncountably many $\sigma-$fields

Definition 4.10. *For t in the arbitrary index set T, let $(\Omega_t, \mathscr{F}_t)$ be a measurable space. Let $\prod_{t\in T} \Omega_t$ be the set of all functions $\omega = (\omega(t), t \in T)$ on T such that $\omega(t) \in \Omega_t$ for each $t \in T$. If $t_1, ...t_n \in T$ and $B^n \subset \prod_{i=1}^{n} \Omega_{t_i}$, we define the set $B^n(t_1, ..., t_n)$ as*

$$\left\{\omega \in \prod_{t\in T} \Omega_t : (\omega(t_1), ..., \omega(t_n) \in B^n)\right\}.$$

We call $B^n(t_1, ..., t_n)$ the cylinder with base B^n at $(t_1, ..., t_n)$; the cylinder is said to be measurable iff $B^n \in \prod_{i=1}^{n} \mathscr{F}_{t_i}$. If $B^n = B_1 \times ... \times B_n$, then the cylinder is called a rectangle, a measurable rectangle iff $B_i \in \mathscr{F}_{t_i}, i = 1, ..., n$. If all $\Omega_t = \Omega$, then we write $\prod_{t\in T} \Omega_t = \Omega^T$.

Example:
Let $T = [0, 1], \Omega_t = \Re$ for all $t \in T, B^2 = \{(u, v) : u > 3, 1 < v < 2\}$. Then

$$B^2\left(\frac{1}{2}, \frac{3}{4}\right) = \left\{x \in \Re^T : x\left(\frac{1}{2}\right) > 3, 1 < x\left(\frac{3}{4}\right) < 2\right\}.$$

Note:The measurable cylinders form a field, as do the finite disjoint unions of measurable rectangles.

Definition 4.11. *The minimal $\sigma-$field over the measurable cylinders is denoted by $\prod_{t\in T} \mathscr{F}_t$, and called the product of the $\sigma-$fields $\mathscr{F}_t$. If $\Omega_t = S$ and $\mathscr{F}_t = \mathscr{S}$ for all $t, \prod_{t\in T} \mathscr{F}_t$ is denoted by $\mathscr{S}^T$. $\prod_{t\in T} \mathscr{F}_t$ is also the minimal $\sigma-$field over the measurable rectangles.*

Definition 4.12. *If $v = \{t_1, ..., t_n\}, t_1 < ... < t_n$, the space $(\prod_{i=1}^n \Omega_{t_i}, \prod_{i=1}^n \mathscr{F}_{t_i})$ is denoted by $(\Omega_v, \mathscr{F}_v)$. If $u = \{t_{i1}, ..., t_{ik}\}$ is a non empty subset of v and $y = (y(t_1), ..., y(t_n)) \in \Omega_v$, the $k-$tuple $(y(t_1), ..., y(t_k))$ is denoted by y_u. Similarly, if $\omega = (\omega(t), t \in T)$ belongs to $\prod_{t\in T} \Omega_t$, the notation ω_v will be used for $(\omega(t_1), ..., \omega(t_n))$.*

Definition 4.13. *If P_v is a probability measure on $\mathscr{F}_v$, the projection of P_v on $\mathscr{F}_u$ is the probability measure $\pi_u(P_v)$ on $\mathscr{F}_u$ defined by*

$$[\pi_u(P_v)](B) = P_v \{y \in \Omega_v : y_u \in B\}, \ \ B \in \mathscr{F}_u.$$

Similarly, if Q is a probability measure on $\prod_{t\in T} \mathscr{F}_t$, the projection of Q on $\mathscr{F}_v$ is defined by

$$[\pi_v(Q)](B) = Q \left\{ \omega \in \prod_{t\in T} \Omega_t : \omega_v \in B \right\} = Q(B(v)) \ \ B \in \mathscr{F}_v.$$

To prove main result of this section we first prove preliminary result.

Theorem 4.13. *For each $n = 1, 2, ...$, suppose that $\mathscr{F}_n$ is the class of Borel sets of a separable metric space Ω_n. Let $\Omega = \prod_n \Omega_n$, with the product topology and let $\mathscr{F} = \mathscr{B}(\Omega)$. Then $\mathscr{F}$ is the product $\sigma-$field $\prod_n \mathscr{F}_n$.*

Proof. We note that the separability and second countability are equivalent in metric spaces. So the sets $\{\omega \in \Omega : \omega_1 \in A_1, ..., \omega_n \in A_n\}, n = 1, 2, ...$, where A_i ranges over the countable base for Ω_i, form a countable base for Ω. Since the sets $\{\omega \in \Omega : \omega_1 \in A_1, ..., \omega_n \in A_n\}, n = 1, 2, ...$, defined as above are measurable rectangles and since every open subset of Ω is the countable union of such sets, every open subset of Ω belongs to $\prod_n \mathscr{F}_n$. Hence $\mathscr{F} \subset \prod_n \mathscr{F}_n$. On the other hand, for a fixed positive integer i, let

$$\mathscr{C} = \{B \in \mathscr{B}(\Omega_i) : \{\omega \in \Omega : \omega_i \in B\} \in \mathscr{F}\}.$$

Then $\mathscr{C}$ is a $\sigma-$field containing the open sets of Ω_i, hence $\mathscr{C} = \mathscr{B}(\Omega_i)$. Thus every measurable rectangle with one-dimensional base belong to $\mathscr{F}$. Since an arbitrary measurable rectangle is a finite intersection of such sets, it follows that $\prod_n \mathscr{F}_n \subset \mathscr{F}$. □

We now establish the main result of this section.

Theorem 4.14. *Kolmogorov Extension Theorem. For each t in the arbitrary index set T, let Ω_t be a complete, separable metric space, and $\mathscr{F}_t$ the class of Borel sets of Ω_t. Assume that for each finite nonempty subset v of T, we are given a probability measure P_v on $\mathscr{F}_v$. Assume that P_v are consistent, that is, $\pi_u(P_v) = P_u$ for each nonempty $u \subset v$.*
Then there is a unique probability measure P of $\mathscr{F} = \prod_{t\in T} \mathscr{F}_t$ such that $\pi_v(P) = P_v$ for all v.

Proof. We define the set function P on measurable cylinders by

$$P(B^n(v)) = P_v(B^n), \ \ B^n \in \mathscr{F}_v.$$

Now we prove this set function P satisfies our requirements.
First we show that P is well defined because a given measurable cylinder can be represented in several ways. For this it is sufficient to consider dual representation of the same measurable cylinder in the form $B^n(v) = B^k(u)$ where $k < n$ and $u \subset v$. Then

$$\begin{aligned} P_u(B^k) &= [\pi_u(P_v)](B^k) \ \text{ by the consistency hypothesis} \\ &= P_v\left\{y \in \Omega_v : y_u \in B^k\right\}. \ \text{ by the definition of projection} \end{aligned}$$

But the assumption $B^n(v) = B^k(u)$ implies that if $y \in \Omega_v$, then $y \in B^n$ iff $y_u \in B^k$, hence $P_u(B^k) = P_v(B^n)$. Thus P is well defined on the measurable cylinders. Since the class $\mathscr{F}_0$ of measurable cylinders form a field and $\sigma(\mathscr{F}_0) = \mathscr{F}$, P is well defined on $\mathscr{F} = \prod_{t\in T} \mathscr{F}_t$.
Next we asserts that P is countably additive. For let $A_1, ..., A_m$ be disjoint sets in $\mathscr{F}_0$. Then we may write $A_i = B_i^n(v), i = 1, ..., m$, where $v = \{t_1, ..., t_n\}$ is fixed and the $B_i^n, i = 1, ..., m$ are disjoint sets in $\mathscr{F}_v$. Thus

$$\begin{aligned} P\left(\bigcup_{i=1}^m A_i\right) &= P\left(\bigcup_{i=1}^m B_i^n(v)\right) \\ &= P_v\left(\bigcup_{i=1}^m B_i^n\right) \quad \text{(by definition of } P\text{)} \\ &= \sum_{i=1}^m P_v(B_i^n) \quad \text{(since } P_v \text{ is a measure)} \\ &= \sum_{i=1}^m P(A_i). \quad \text{(by definition of } P\text{)} \end{aligned}$$

Therefore P is finitely additive on $\mathscr{F}_0$. To show that P is countably additive on $\mathscr{F}_0$, it is sufficient to show that P is continuous from above at ϕ.
Let $A_k, k = 1, 2, ...$ be a sequence of measurable cylinders decreasing to ϕ. For contrary let us suppose $P(A_k)$ does not approach to 0. Then for some $\epsilon > 0, P(A_k) \geq \epsilon > 0$ for all k. Suppose $A_k = B^{n_k}(v_k)$. By taking on extra factors, we may assume that the numbers n_k and the sets v_k increase with k. But by theorem 4.13, each Ω_{v_k} is a complete, separable metric space and $\mathscr{F}_{v_k} = \mathscr{B}(\Omega_{v_k})$. So by theorem 4.7 there exist a compact set $C^{n_k} \subset B^{n_k}$ such that $P_{v_k}(B^{n_k} - C^{n_k}) < \frac{\epsilon}{2^{k+1}}$. Define $A'_k = C^{n_k}(v_k) \subset A_k$. Then

$$P(A_k - A'_k) = P_{v_k}(B^{n_k} - C^{n_k}) < \frac{\epsilon}{2^{k+1}}.$$

In this way we can approximate the given cylinders by cylinders with compact bases. Now take

$$D_k = A'_1 \cap ... \cap A'_k \subset A_1 \cap ... \cap A_k = A_k.$$

Then

$$\begin{aligned} P(A_k - D_k) &= P\left(A_k \cap \bigcup_{i=1}^{k} A_i'^{c}\right) \quad \text{(De Morgans law)} \\ &\leq \sum_{i=1}^{k} P(A_k \cap A_i'^{c}) \\ &\leq \sum_{i=1}^{k} P(A_i - A'_i) \\ &\leq \sum_{i=1}^{k} \frac{\epsilon}{2^{i+1}} \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Also, since $D_k \subset A'_k$, we have $P(A_k - D_k) = P(A_k) - P(D_k)$. Consequently, $P(A_k) - P(D_k) = P(A_k - D_k) < \frac{\epsilon}{2}$. So $P(D_k) > P(A_k) - \frac{\epsilon}{2}$. Thus, in particular D_k is nonempty.
Now pick $x^k \in D_k, k = 1, ...$, say, $A'_i = C^{n1}(t_{11}, t_{21}..., t_{n1}) = C^{n1}(v_1)$. Consider the sequence

$$(x^1_{t_1}, ..., t^1_{n_1}), (x^2_{t_1}, ..., t^2_{n_1}), (x^3_{t_1}, ..., t^3_{n_1}), ...$$

that is,

$$x^1_{v_1}, x^2_{v_1}, x^3_{v_1}, ...,$$

Since $x^n_{v_1}$ belongs to C^{n_1}, a compact subset of Ω_{v_1}, we have a convergent subsequence $x^{r_{1n}}_{v_1}$ approaching some $x_{v_1} \in C^{n_1}$. If $A'_2 = C^{n_1}(v_2)$ (so that $D_k \subset A'_2$), consider the sequence $x^{r_{i1}}_{v_2}, x^{r_{i2}}_{v_2}, x^{r_{i3}}_{v_2}, ... \in C_{n_2}$ and extract a convergent subsequence $x^{r_{2n}}_{v_2} \to x_{v_2} \in C^{n_2}$.
We note that $(x^{r_{2n}}_{v_2})_{v_1} = x^{r_{2n}}_{v_1}$. Also as $n \to \infty$, the left side approaches $(x_{v_1})_{v_1}$, and since $\{r_{2n}\}$ is a subsequence of $\{r_{1n}\}$, the right side approaches x_{v_1}. Hence $(x_{v_2})_{v_1} = x_{v_1}$.
Continue in this fashion, at step i we have a subsequence

$$x^{r_{in}}_{v_i} \to x_{v_i} \in C^{n_i} \ \text{ and } \ (x_{v_i})_{v_j} = x_{v_j} \ \text{ for } j < i$$

Since $(x_{v_i})_{v_j} = x_{v_j}, j < i$, we can choose $\omega \in \prod_{t \in T} \Omega_t$ such that $\omega_{v_j} = x_{v_j}$ for all $j = 1, 2, ...$ Hence

$$\omega \in \bigcap_{j=1}^{\infty} A'_j \subset \bigcap_{j=1}^{\infty} A_j = \phi,$$

a contradiction.
Thus, $P(A_k) \to 0$. So by theorem 1.3(b) P is countably additive. Also by Caratheodory extension theorem P may be extends to $\mathscr{F}$. Hence by construction of P

$$\pi_v(P) = P_v \text{ for all } v.$$

Now it remains to show that P is unique. For let P and Q be two probability measure on $\mathscr{F}$ such that $\pi_v(P) = \pi_v(Q)$ for all finite $v \subset T$. Then for any $B^n \in \mathscr{F}_v$,

$$\begin{aligned} P(B^n(v)) &= [\pi_v(P)](B^n) \\ &= [\pi_v(Q)](B^n) \\ &= Q(B^n(v)). \end{aligned}$$

Thus P and Q are equal on the measurable cylinders. Hence again by the uniqueness part of Caratheodory extension theorem $P = Q$ on $\mathscr{F}$.
This completes the proof of the theorem. □

4.5 Weak Convergence

By theorem 4.8 and theorem 4.9(also called representation theorem), a continuous linear functional on $C(\Omega)$, where Ω is a compact Housdorff space can be identified with a regular finite measure on $\mathscr{B}(\Omega)$. Thus, if $\{\mu_n\}$ is a sequence of such measures, weak convergence of the sequence to the measure μ means that $\int_\Omega f d\mu_n \to \int_\Omega f d\mu$ for all $f \in C(\Omega)$.

In this section, we establish such type of convergence in different context. For this, we start with the following theorem which gives us somewhat different definition of weak convergence.

Theorem 4.15. *Let $\mu, \mu_1, \mu_2, ...$ be finite measures on the Borel sets of metric space Ω. The following conditions are equivalent:*

(a) $\int_\Omega f d\mu_n \to \int_\Omega f d\mu$ for all bounded continuous $f : \Omega \to \Re$.

(b) $\liminf_{n\to\infty} \int_\Omega f d\mu_n \geq \int_\Omega f d\mu$ *for all bounded lower semi continuous* $f : \Omega \to \Re$.

(b') $\limsup_{n\to\infty} \int_\Omega f d\mu_n \leq \int_\Omega f d\mu$ *for all bounded upper semi continuous* $f : \Omega \to \Re$.

(c) $\int_\Omega f d\mu_n \to \int_\Omega f d\mu$ *for all bounded* $f : (\Omega, \mathscr{B}(\Omega)) \to (\Re, \mathscr{B}(\Re))$ *such that* f *is continuous a.e.* $[\mu]$

(d) $\liminf_{n\to\infty} \mu_n(A) \geq \mu(A)$ *for every open set* $A \subset \Omega$*, and* $\mu_n(\Omega) \to \mu(\Omega)$.

(d') $\limsup_{n\to\infty} \mu_n(A) \leq \mu(A)$ *for every closed set* $A \subset \Omega$*, and* $\mu_n(\Omega) \to \mu(\Omega)$.

(e) $\mu_n(A) \to \mu(A)$ *for every* $A \in \mathscr{B}(\Omega)$ *such that* $\mu(\partial A) = 0(\partial A$ *denotes the boundary of* A*).*

Proof. (a) $\Rightarrow$ (b)
Let $f : \Omega \to \Re$ be a lower semi continuous (LSC). If g is bounded continuous with $g \leq f$, then

$$\begin{aligned} \lim_{n\to\infty}\inf \int_\Omega f d\mu_n &\geq \lim_{n\to\infty}\inf \int_\Omega g d\mu_n \\ &= \int_\Omega g d\mu. \quad \text{(by hypothesis since } g \text{ is bounded continuous)} \end{aligned}$$

But since f is LSC, by theorem 3.10 f is a limit of a sequence of continuous functions. Also since f is bounded, $|f| \leq M < \infty$. So again by theorem 3.10 all functions in the sequence may be chosen so that all less than or equal to M in absolute value.Thus taking sup over g in the above equations we get

$$\lim_{n\to\infty}\inf \int_\Omega f d\mu_n \geq \sup \int_\Omega g d\mu = \int_\Omega f d\mu.$$

as desired.
(b)$\Leftrightarrow$ (b')
We know that a function f is LSC iff $-f$ is USC. Thus (b) holds iff (c) holds.
(b) $\Rightarrow$ (c)

Let $\underline{f}$ be the lower envelope of f(the sup of all LSC functions g such that $g \leq f$) and $\overline{f}$ the upper envelope of f(the inf of all USC functions such that $g \geq f$). Then by the definition of $\underline{f}$ and $\overline{f}$, we have

$$\underline{f} = \lim_{y\to x}\inf f(y) \quad \text{and} \quad \overline{f} = \limsup_{y\to x} f(y)$$

But since f is continuous at x we have

$$\underline{f}(x) = f(x) = \overline{f}(x).$$

Also by theorem 3.8, $\underline{f}$ and $\overline{f}$ are LSC and USC, respectively. Thus if f is bounded and continuous a.e. $[\mu]$,

$$\begin{aligned}
\int_\Omega f d\mu = \int_\Omega \underline{f} d\mu &\leq \liminf_{n\to\infty} \int_\Omega \underline{f} d\mu_n \quad \text{(by hypothesis)} \\
&\leq \liminf_{n\to\infty} \int_\Omega f d\mu_n \quad (\text{since } \underline{f} \leq f) \\
&\leq \limsup_{n\to\infty} \int_\Omega f d\mu_n \\
&\leq \limsup_{n\to\infty} \int_\Omega \overline{f} d\mu_n \\
&\leq \int_\Omega \overline{f} d\mu \\
&= \int_\Omega f d\mu
\end{aligned}$$

This proves (c).
(c) $\Rightarrow$ (d)
Suppose (c) holds. Then (a) also holds, consequently (b) also holds. Now let A be open subset of Ω. Then I_A is LSC, so by (b) $\limsup_{n\to\infty} \int_\Omega I_A d\mu_n \geq \int_\Omega I_A d\mu$, that is, $\limsup_{n\to\infty} \mu_n(A) \geq \mu(A)$. Since $A \subset \Omega$ is arbitrary, we have shown that $\limsup_{n\to\infty} \mu_n(A) \geq \mu(A)$ for all open subset A of Ω.
Next, we know that $I_\Omega \equiv 1$, so that I_Ω is bounded and measurable:$(\Omega, \mathscr{B}(\Omega)) \to (\Re, \mathscr{B}(\Re))$. Thus by (c)

$$\int_\Omega I_\Omega d\mu_n \to \int_\Omega I_\Omega d\mu$$

$$\Rightarrow \mu_n(\Omega) \to \mu(\Omega).$$

(d) $\Leftrightarrow$ (d')
We know that a set $A \subset \Omega$ is open iff A^c(where A^c denote complement of A) is closed in Ω.Thus (d) holds iff (d') holds.
(d) $\Rightarrow$ (e)
Let $A \in \mathscr{B}(\Re)$. Let A^0 be the interior of A and $\overline{A}$ be the closure of A. Then by definition, A^0 is open and $\overline{A}$ is closed. So

$$\begin{aligned}
\limsup_{n\to\infty} \mu_n(A) &\leq \limsup_{n\to\infty} \mu_n(\overline{A}) \quad (\text{since } A \subset \overline{A}) \\
&\leq \mu(\overline{A}) \quad \text{(by (d'))} \\
&= \mu(A). \quad (\text{since } \mu(\partial A) = 0)
\end{aligned}$$

Also

$$\begin{aligned}
\liminf_{n\to\infty} \mu_n(A) &\geq \liminf_{n\to\infty} \mu_n(A^0) \quad (\text{since } A^0 \subset A) \\
&\geq \mu(A^0) \quad (\text{by (e)}) \\
&= \mu(A). \quad (\text{since } \mu(\partial A) = 0)
\end{aligned}$$

So

$$\mu(A) \leq \liminf_{n\to\infty} \mu_n(A) \leq \limsup_{n\to\infty} \mu_n(A) \leq \mu(A).$$

Hence

$$\mu_n(A) \to \mu(A).$$

(e) $\Rightarrow$ (a)
Let f be a bounded continuous functions on Ω. Suppose that $|f| < M$ for some $0 < M < \infty$. Let us define a set A by

$$A = \left\{ c \in \Re : \mu(f^{-1}\{c\}) \neq 0 \right\}.$$

Then A is countable since the sets $f^{-1}\{c\}$ are disjoint and μ is finite. Let us construct a partition of $[-M, M]$, say, $-M = t_0 < t_1 < ... < t_j = M$, with $t_i \notin A, i = 0, 1, ..., j$($M$ may be increased if necessary). Such a partition is exist since A is countable. Let

$$B_i = \{x : t_i \leq f(x) < t_{i+1}\} \quad i = 1, ..., j.$$

Since $f^{-1}(t_i, t_{i+1})$ is open, we have $\partial f^{-1}[t_i, t_{i+1}) \subset f^{-1}\{t_i, t_{i+1}\} = 0$(since $t_i, t_{i+1} \notin A$). Thus from (e), we have

$$\sum_{i=1}^{j-1} t_i \mu_n(B_i) \to \sum_{i=1}^{j-1} t_i \mu(B_i).$$

Now

$$\left| \int_\Omega f d\mu_n - \int_\Omega f d\mu \right| \leq \left| \int_\Omega f d\mu_n - \sum_{i=1}^{j-1} t_i \mu_n(B_i) \right| + \left| \sum_{i=1}^{j-1} t_i \mu_n(B_i) - \sum_{i=1}^{j-1} t_i \mu(B_i) \right| + \left| \sum_{i=1}^{j-1} t_i \mu(B_i) - \int_\Omega \right.$$

The first term on the right can be written as

$$\left| \sum_{i=0}^{j-1} \int_{B_i} (f(x) - t_i) d\mu_n(x) \right|$$

which is bounded by $\max_i(t_{i+1} - t_i)\mu_n(\Omega)$, which can be made arbitrarily small by choice of the partition since $\mu_n(\Omega) \to \mu(\Omega) < \infty$.
The second term on the right approaches 0 as $n \to \infty$.
The third term on the right is bounded by $\max_i(t_{i+1} - t_i)\mu(\Omega)$ which can be made arbitrarily small by the choice of the partition since $\mu_n(\Omega) \to \mu(\Omega) < \infty$.
Thus for given $\epsilon > 0$, we have

$$\left| \int_\Omega f d\mu_n - \int_\Omega f d\mu \right| < \epsilon$$

$$\Rightarrow \int_\Omega f d\mu_n \to \int_\Omega f d\mu$$

proving (a).
This completes the poof of the theorem. □

Definition 4.14. *The convergence described in theorem 4.15 is sometimes called weak or vague convergence of measures and we write* $\mu_n \xrightarrow{w} \mu$.

Definition 4.15. *A continuity point of a distribution function* F *on* $\Re$ *is a point* $x \in \Re$ *such that* F *is continuous at* x, *or* $\pm\infty$*(thus by convention,* ∞ *and* $-\infty$ *are continuity points.)*

If the measures μ_n and μ are defined on $\mathscr{B}(\Re)$, then there exist corresponding distribution functions F_n and F on $\Re$. The following theorem relate the convergence of measure to convergence of distribution functions.

Theorem 4.16. *Let* $\mu, \mu_1, \mu_2, ...$ *be finite measures on* $\mathscr{B}(\Re)$ *with corresponding distribution functions* $F, F_1, F_2, ...$ *Then the following are equivalent:*

(a) $\mu_n \xrightarrow{w} \mu$.

(b) $F_n(a, b] \to F(a, b]$ *for all continuity points* a, b *of* F, *where* $F(a, b] = F(b) - F(a), F(\infty) = \lim_{x\to\infty} F(x), F(-\infty) = \lim_{x\to-\infty} F(x)$.

If all distributions are 0 at $-\infty$, *conditions (b) equivalent to the statement* $F_n(x) \to F(x)$ *at all points* $x \in \Re$ *at which* F *is continuous, and* $F_n(\infty) \to F(\infty)$.

Proof. (a)⇒(b)
Suppose (a) holds. Let $a, b \in \Re$ be continuity points of F. Then $(a, b]$ is a Borel set with $\mu(\partial(a, b]) = 0$, where $\partial(a, b]$ denotes the boundary of $(a, b]$. So by theorem 4.15 (g), $\mu_n(a, b] \to \mu(a, b]$, that is, $F_n(a, b] \to F(a, b]$.

If $a = -\infty$, then same as above $(-\infty, b]$ is a Borel set whose boundary has μ measure 0. So by theorem 4.15 (e),$\mu_n(-\infty, b] \to \mu(-\infty, b]$, that is, $F_n(-\infty, b] \to F(-\infty, b]$.
If $b = \infty$ then (a, ∞) is also a Borel set whose boundary has $\mu-$measure 0, and so $F_n(a, \infty) \to F(a, \infty)$ same as above.
(b) $\Rightarrow$ (a)
Suppose (b) holds. Let A be an open subset of $\Re$. Then we can express A as the disjoint union of open intervals, $I_1, I_2...$ Then

$$\begin{aligned}
\liminf_{n\to\infty} \mu_n(A_k) &= \liminf_{n\to\infty} \mu_n\left(\bigcup_{n=1}^{\infty} I_k\right) \\
&= \liminf_{n\to\infty} \sum_{n=1}^{\infty} \mu_n(I_k) \quad \text{(countable additivity)} \\
&\geq \sum_{k=1}^{\infty} \liminf_{n\to\infty} \mu_n(I_k). \quad \text{(by Fatou's}\mathit{Lemma})
\end{aligned}$$

Let $\epsilon > 0$ be given. For each k, let I'_k be a right semi closed subinterval of I_k such that the end points of I'_k are continuity points of F, and $\mu(I'_k) \geq \mu(I_k) - \epsilon 2^{-k}$. Here I'_k can be choose since F has only countably many discontinuities. So

$$\liminf_{n\to\infty} \mu_n(I_k) \geq \liminf_{n\to\infty} \mu_n(I'_k).$$

Thus

$$\begin{aligned}
\liminf_{n\to\infty} \mu_n(A) &\geq \sum_k \mu(I'_k) \\
&\geq \sum_k \mu(I_k) - \epsilon \\
&= \mu(A) - \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we have

$$\liminf_{n\to\infty} \mu_n(A) \geq \mu(A)$$

for every open set $A \subset \Omega$. Thus by theorem 4.15(d), we have $\mu_n \xrightarrow{w} \mu$. □

Definition 4.16. *Condition (b) of theorem 4.16 is called weak convergence of the sequence $\{F_n\}$ to F, and written $F_n \xrightarrow{w} F$.*

Chapter 5

Conclusion

The class of Borel measurable functions plays a fundamental role in integration theory. It has some basic properties in common with another most important class, the class of continuous functions. More precisely, there are interrelation between the concepts topological space, open set, and continuous function, on the one hand, and measurable space, measurable set, and measurable function, on the other. After discussing "The interplay between measure theory and topology " we arrive at the following conclusions.

- In a metric space, Borel sets and Baire sets are same.
- In complete separable metric space Ω and for a finite measure μ, a Borel set can be approximated by an open set (or by a closed set and hence by a compact set) in Ω.
- In a compact Hausdorff space, a positive linear functional in $C(\Omega)$ can be represented in terms of Lebesgue integration.
- The continuous functions are dense in L^p.
- In a normal space, a complex valued Borel measurable function can be approximated by a complex valued continuous functions.
- Probability measure on countably infinite product spaces can be extended to uncountable products under some topological assumptions.
- In the class of Borel sets in real numbers, the weak convergence of finite measures and weak convergence of corresponding distribution functions are equivalent.

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