REGULARITY OF 2D SURFACE QUASI-GEOSTROPHIC (SQG) EQUATIONS

A THESIS SUBMITTED TO THE

CENTRAL DEPARTMENT OF MATHEMATICS

INSTITUTE OF SCIENCE AND TECHNOLOGY

TRIBHUVAN UNIVERSITY

NEPAL

FOR THE AWARD OF

DOCTOR OF PHILOSOPHY

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BY

PAWAN SHRESTHA

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DEAN'S OFFICE
Kirtipur, Kathmandu, Nepal

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Asst. Dean

DECLARATION

Thesis entitled "**Regularity of 2D Surface Quasi-Geostrophic (SQG) Equations**" which is being submitted to the Central Department of Mathematics, Institute of Science and Technology (IOST), Tribhuvan University, Nepal for the award of the degree of Doctor of Philosophy (Ph.D.), is a research work carried out by me under the supervision of Assoc. Prof. Dr. Durga Jang KC, Central Department of Mathematics, Tribhuvan University, and Assoc. Prof. Dr. Ramjee Sharma, Department of Mathematics, University of North Georgia. This research is original and has not been submitted earlier in part or full in this or any other form to any university or institute, here or elsewhere, for the award of any degree.

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Pawan Shrestha

February 2023

RECOMMENDATION

This is to recommend that **Pawan Shrestha** has carried out research entitled "**Regularity of 2D Surface Quasi-Geostrophic (SQG) Equations**" for the award of Doctor of Philosophy (Ph.D.) in **Mathematics** under our supervision. To our knowledge, this work has not been submitted for any other degree.

He has fulfilled all the requirements laid down by the Institute of Science and Technology (IOST), Tribhuvan University, Kirtipur for the submission of the thesis for the award of Ph.D. degree.

....

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LETTER OF APPROVAL

On the recommendation of Assoc. **Prof. Dr. Durga Jang KC** and **Assoc. Prof. Dr. Ramjee Sharma**, this Ph.D. thesis submitted by **Pawan Shrestha,** entitled **"Regularity of 2D Surface Quasi-Geostrophic (SQG) Equations"** is forwarded by Central Department Research Committee (CDRC) to the Dean, IOST, T.U..

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Pawan Shrestha

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ABSTRACT

In this research, we delve into three distinct topics within the realm of non linear fluid dynamics, namely the generalized Korteweg-de Vries (KdV)-type equation, the regularity of solutions in the 2D Surface Quasi-Geostrophic (SQG) equation, and the behavior of water waves under indefinite boundary constraints.

Firstly, we undertake an analytical and numerical examination of the following generalized KdV-type equation.

$$u_t + au_x + 2buu_x + cu_{xxx} - du_{xx} = 0, u(x, 0) = u_0(x) \tag{1}$$

where a, b, c, d are real parameters. Our study involves allowing the coefficients a, b, c, and d to approach zero in the limiting sense, while contrasting the outcomes with the scenario in which each coefficient is precisely zero. By analyzing this nonlinear partial differential equation in one dimension, we trace the impact of the nonlinear term on the solution. Furthermore, we extend our findings to a two-dimensional equation with structures comparable to those in the 2D SQG equation.

Secondly, we focus on the regularity of solutions in the following 2D SQG equation

$$\begin{aligned} \partial_t\theta + u.\nabla\theta + \kappa\,(-\Delta)^{\alpha}\theta &= 0 \\ \nabla.u &= 0 \\ \theta(x, 0) &= \theta_0(x) \end{aligned} \tag{2}$$

where $\kappa \geq 0$ and $\alpha > 0$ are parameters, conducting a thorough analysis that addresses a notable gap in analytical and numerical research. The SQG equation exhibits numerous characteristics similar to the 3D Euler equation and the Navier-Stokes equation, with the regularity of the latter being recognized as one of the Clay Institute of Mathematics' millennium problems. To bridge this gap, we concentrate on various aspects of the SQG equation, exploring both inviscid and dissipative instances. In the dissipative case, we categorize the instances as subcritical, critical, and supercritical. Analytical solutions have recently been derived for the subcritical and critical scenarios, while the question of regularity in the supercritical case remains unresolved. Our research focuses on numerical calculations of the inviscid and supercritical SQG equations, with particular attention to the proximity of level curves, the L^2 norm, and the expansion of the $|\nabla^{\perp}\theta|$

quantity. We meticulously examine the nature of the solution, particularly in the region where $\alpha = \frac{1}{2}$.

Finally, we turn our attention to the study of the following water waves

$$\partial_t \vec{u} + (\vec{u}.\nabla)\vec{u} + \nabla P = -g\vec{e_y}, \text{ div } \vec{u} = 0 \tag{3}$$

where u is the velocity, P is the pressure, and g is the acceleration due to gravity, which are typically modeled using Euler equations with unit density. We address an outstanding open problem concerning the existence of closed orbits for water waves under indefinite boundary constraints. Our investigation begins with a discussion of advancements in water wave structure under finite bottom conditions. We then shift our focus to the behavior of water waves at the kinematic barrier of infinite depth. By employing the Crandall-Rabinowitz theorem to construct water wave profiles for scenarios with zero and constant vorticity, we present our findings as a contribution towards addressing this problem.

LIST OF ACRONYMS AND ABBREVIATIONS

DFT	: Discrete Fourier Transform
IDFT	: Inverse Discrete Fourier Transform
FFT	: Fast Fourier Transform
SQG	: Surface Quasi Geostrophic
KdV	: Korteweg de Vries
DE	: Differential Equation
ODE	: Ordinary Differential Equation
PDE	: Partial Differential Equation
BMO	: Bounded Mean Oscillation
2D	: Two Dimensional
3D	: Three Dimensional
NS	: Navier Stokes
MWR	: Method of Weighted Residuals
FEM	: Finite Element Method
FDS	: Finite Difference Schemes
RK	: Runge Kutta

LIST OF SYMBOLS

η	: Water profile height
ρ	: Fluid density
κ	: Viscosity
α	: Fractional power
∇^2	: Laplacian
Δ	: Operator
$(-\Delta)^{\alpha}$	: Fractional Laplacian of order α
θ	: Potential temperature
u	: Velocity
c	: Wave velocity
g	: Acceleration due to gravity
Ω	: Domain
$\frac{D}{Dt}$	: Material Derivative
ψ	: Stream function
$G[\eta][\psi]$	: Dirichlet Neumann Operator
ϕ	: Velocity potential
$\nabla.u$	: Divergence
$\hat{u}$	: Fourier transform of u
∂_t	: Partial derivative
R_1, R_2	: Riesz Transform
$W^{k,p}(\Omega)$	: Sobolev Space
H^S	: Homogenous Sobolev Space
$\dot{H}^S$	: Inhomogenous Sobolev Space
$L^p(\Omega)$	: Lebesgue Space
$D^\alpha f$	: Multi-index derivative
$\omega = \nabla \times u$	: Vorticity
L^1_{loc}	: Locally integrable function
$B^{s,p,\theta}(\mathbb{R}^N)$	: Besov Space
$C_{k,\gamma}(\bar{U})$	: Holder Space
$\forall$	: For all
L	: Differential Operator
B	: Differential Operator
$\langle ., . \rangle$	: Inner Product

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Chapter 1

Introduction

1.1 Introduction

A mathematical model is a description of natural or artificial systems using mathematical concepts and languages. The process of developing a mathematical model is known as mathematical modeling. These models are used in the natural sciences and engineering disciplines as well as in the social sciences. Mathematical models can take many forms, including dynamical systems, statistical models, and differential equations. In some cases, the quality of a scientific field depends on how well the mathematical models developed on the theoretical background. In physical sciences, a traditional mathematical model contains most of the following elements: governing equations, supplementary sub-models, assumptions, and constraints with initial and boundary conditions.

Fluid dynamics is the study of the flow of liquids and gases, usually into and around solid surfaces. It can be further subdivided into aerodynamics and hydrodynamics. Aerodynamics studies the air flows around airplanes and automobiles, whereas hydrodynamics deals with the flow of water in different situations, such as in pipes, around ships, underground, etc. Fluid flow is a part of fluid mechanics which deals with fluid dynamics. It involves the motion of a fluid subjected to unbalanced forces, and the motion continues as long as unbalanced forces are applied. There are different types of fluids, such as Newtonian, Non-Newtonian, compressible, incompressible, etc. Scientists use both experiments and mathematical models and calculations to understand fluid dynamics. The fluid flow is affected by the viscosity, density, and velocity of the fluid and changes in the fluid temperature. Daniel Bernoulli was the first person to study fluid flow mathematically. To study these phenomena, there are different models in the form of differential equations. Among these, our study is focused on the three types of the equations namely, Korteweg de Vries (KdV) equation, Surface Quasi-Geostrophic (SQG) equation and the water wave equation.

The first equation we have studied is the KdV equation. It was first introduced by Boussinesq and later rediscovered by D. Korteweg and G. de Vries in 1895 to model the water waves on shallow water surfaces. This KdV equation is an example of an exactly solvable nonlinear partial differential equation. We study the generalized KdV type equation by letting different coefficients go to zero and comparing it with the case when the coefficients are exactly zero. By choosing the particular values of constants, the equation is reduced to the transport, burger and KdV equations. The numerical study is carried out by using the pseudospectral method. Our study shows that there is no difference between the behavior of the solutions in the limiting case and when corresponding coefficients are exactly zero.

The second equation we have considered in our study is the 2D SQG equation. J.G. Charney derived the general 3D quasi geostrophic equations in 1940s. The 3D quasi-geostrophic system is a widely used model in oceanography and meteorology to describe large-scale oceanic and atmospheric circulation. These equations have been very successful in describing the major features of large-scale motions in the atmosphere and oceans in the midlatitudes (Pedlosky, 1987). The dynamics of these 3D geostrophic equations with uniform potential vorticity reduces to the Surface Quasi- Geostrophic (SQG) equation. This SQG equation models the evolution of buoyancy or the potential temperature on the 2D horizontal boundaries. The inviscid SQG equation is useful in modeling the atmospheric phenomenon such as frontogenesis, the formation of strong fronts between the masses of hot and cold air. Also, the SQG equation (2.13) with $\kappa = 0$ is an important example of an active scalar and an important testbed for turbulence theories due to some of its distinctive features (Blumen, 1978; Held, Pierrehumbert, Garner, & Swanson, 1995). The SQG equation with $\alpha = \frac{1}{2}$ and $\kappa > 0$ arises in geophysical studies of strongly rotating fluids (A. J. Majda & Tabak, 1996; Pedlosky, 1987). This equation shows similarity with 3D Euler equations and hence with Navier-Stokes equation in many aspects. We also note that the regularity of the solution of the Navier-Stokes equation is one of the millennium problems, as enlisted by the Clay Institute of Mathematics.

A detailed survey of Surface Quasi-Geostrophic Equations shows that there exists a gap in the numerical as well as the analytical study. It is basically classified into inviscid and dissipative cases. Further, the dissipative case is divided into subcritical, critical, and supercritical cases. The analytical solution for the subcritical and critical cases were obtained by the previous researchers. But the regularity issue for the supercritical case is still open. The numerical computations for inviscid SQG and supercritical SQG equations are presented in this work.

Finally, we have studied the water wave equation. Stokes water waves are nonlinear and periodic surface waves on an inviscid fluid layer of constant mean depth, which were introduced in the mid nineteenth century. These water waves, which propagate on the water surface of the sea or river, are progressive periodic two dimensional waves. The issue of whether there is a closed orbit in the water waves in an infinite boundary condition is an outstanding open problem. We first discuss the various developments in the structure of water waves in the context of finite bottom conditions, and then concentrate on the behavior of water for the kinematic boundary for the infinite depth. We have created water wave profile for the zero and constant vorticity conditions for the Newtonian fluid is created through the application of the Crandall Rabinowitz theorem.

We work on the KdV type equations. We use numerical techniques to compare the results when the coefficients tend to zero in the sense of limit with the case when the coefficients are exactly zero. Secondly, we work on the SQG equation, with the help of which we are able to identify the research gaps regarding the regularity of the solution of the SQG equation. The open problem is the regularity issue of inviscid and supercritical SQG equations. This is important because it shares similarity with 3D Euler equations, and hence with NS equations, which is a millennium dollar prize problem as enlisted by Clay Institute of Mathematics. If we explore the regularity issues of the solution of SQG equation, then it will be great contribution to the mathematical community to get ideas and hints about the solution of NS equations. Lastly, we work on the wave profile for the water waves. In this regard, we extend the finite depth boundary condition to the infinite depth boundary condition and guarantee the traveling wave solution for the Newtonian fluid with zero and constant vorticity using the Crandall Rabinowitz theorem.

Our thesis is divided into six chapters. The first chapter discusses the preliminaries. The second one deals with introduction. The issue concerned with KdV type equations and its solutions are discussed in the third chapter. Following this, in fourth chapter, we discuss mainly about the regularity of solutions of inviscid and the supercritical SQG equations numerically. In the fifth chapter, we discuss about the water wave profile with the extension of finite depth boundary condition to infinite depth. The sixth chapter includes the summary and conclusion of this research work and recommendations for further work.

1.2 Rationale

Our result will make a positive contribution to the scientific community. In our work, we study the regularity of the solution of the inviscid SQG and supercritical SQG equations numerically. With our findings and predictions, other interested researchers can begin their study to find the solution of 3D Euler equations and hence NS equations which is an outstanding open problem as well as a millennium dollar prize problem as enlisted by Clay Institute of Mathematics. Another contribution is that we have created traveling wave solution extending the finite depth to infinite depth boundary, which will give a hint for the extension of finite to the infinite boundary condition in other problems of fluid dynamics as well as from Newtonian to Non-Newtonian fluids.

1.3 Objectives

We pursue our research work to fulfill the following objectives:

1. To study the regularity of the solutions of inviscid SQG and supercritical SQG equations numerically.
2. To study the nature of the solution of generalized KdV type equations when the coefficients tend to zero in the limiting sense and compare the results with the case when the coefficients are exactly zero.
3. To create a traveling wave solution for water waves with constant and zero vorticity extending the result from a finite depth to an infinite depth boundary condition.

Chapter 2

Preliminaries

2.1 Basic Terminology

Let Ω be a domain in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$. We call a vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ a multi-index of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, where each component α_i is a non negative integer. The set $D(\Omega)$ represents the set of $C^\infty(\Omega)$ functions with compact support in Ω.

Locally Integrable Function:
A locally integrable function is a function which is integrable on every compact subset of its domain of definition.

Lebesgue Spaces:
For $1 \leq p < \infty$, the set of p-integrable measurable functions is denoted by $L^p(\Omega)$ and is defined by

$$L^p(\Omega) = \left\{ u : \int_\Omega |u(x)|^p dx < \infty \right\}$$

together with norm

$$||u||_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

When $p = \infty$, we set

$$||u||_\infty = \text{ess sup}|u|.$$

Weak Derivative:
Given a multi-index α, the derivative $D^\alpha_\omega f(x)$ is given by

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f.$$

We say that a given function $f \in L^1_{\text{loc}}$ has a weak derivative $D^f_\omega f$, provided there exists a function $g \in L^1_{\text{loc}}$ such that

$$\int_\Omega g(x)\phi(x)dx = (-1)^{|\alpha|}\int_\Omega f(x)\phi^{(\alpha)}(x)dx$$

for all test functions $\phi \in D(\Omega)$.

If such a g exists, we define $D^\alpha_\omega f = g$ and g is called the α^{th} weak partial derivative of f. The subscript ω is discarded from D^α_ω and simply written as D^α. Note that if weak derivative exists then it is unique upto a set of measure zero.

Lipschitz and Hölder Continuous functions:

Assume $U \subset \mathbb{R}^n$ is open and $0 < \gamma \leq 1$. A function $u : U \to \mathbb{R}$ is said to be Lipschitz continuous (Evans, 2010) if

$$|u(x) - u(y)| \leq C|x - y|; x, y \in U \tag{2.1}$$

for some constant C. Further, function $u : U \to \mathbb{R}$ is said to be Hölder continuous with exponent γ (Evans, 2010) if

$$|u(x) - u(y)| \leq C|x - y|^\gamma;\ x, y \in U \tag{2.2}$$

for some $0 < \gamma \leq 1$ and a constant C.

Hölder Space:

The Hölder space (Evans, 2010) is denoted by $C^{k,\gamma}(\bar{U})$ which consists of all functions $u \in C^k(\bar{U})$ together with norm

$$||u||_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha|\leq k} ||D^\alpha u||_{C(\bar{U})} + \sum_{|\alpha|=k}[D^\alpha u]_{C^{0,\gamma}(\bar{U})} < \infty \tag{2.3}$$

Here $C^{k,\gamma}(\bar{U})$ is the set of functions which are k-times continuously differentiable and k^{th} partial derivatives are bounded and Hölder continuous with exponent γ.

$C^\infty_c(\bar{U})$ is the space of infinitely differentiable functions $\phi : U \to \mathbb{R}$ with the compact support U and the function ϕ is called the test function.

Sobolev Space:

Sobolev Space (Evans, 2010) is defined as the set

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), 0 \leq |\alpha| \leq m\}$$

where m is a non-negative integer and $1 \leq p \leq \infty$ together with the norm

$$||u||^{m,p}_W(\Omega) = \left\{\sum_{0\leq|\alpha|\leq m} ||D^\alpha u||^p_{L^p}(\Omega)\right\}^{\frac{1}{p}}.$$

Homogeneous and Inhomogeneous Sobolev Space:

The homogeneous space $\dot{H}^s(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ together with norm

$$||f||_{\dot{H}^s} = |||\xi|^{2s}\hat{f}(\xi)||_{L^2(\mathbb{R}^n)}.$$

Similarly, the homogeneous space $H^s(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ together with norm

$$||f||_{H^s} = ||(1+|\xi|^2|)^{\frac{s}{2}}\hat{f}(\xi)||_{L^2(\mathbb{R}^n)}.$$

Here is s is non-negative number.

Besov Space:

Given a function $u : \mathbb{R}^N \to \mathbb{R}$, for every $h \in \mathbb{R}, i = 1, \cdots, N$, and $x \in \mathbb{R}^n$, we define

$$\Delta_i^h u(x) = u(x + he_i) - u(x_i) = u(x_i', x_i + h) - u(x_i', x_i) \tag{2.4}$$

where e_i is the i^{th} vector of the canonical basis in $\mathbb{R}^N$. If $N = 1$, we write $\Delta^h u = \Delta_1^h u$. Let $1 \leq p, \theta \leq \infty$ and $0 < s < 1$. A function $u \in L^1_{loc}(\mathbb{R}^N)$ belongs to Besov space $B^{s,p,\theta}(\mathbb{R}^N)$ (Leoni, 2017) if

$$||u||_{B^{s,p,\theta}(\mathbb{R}^N)} = ||u||_{L^p(\mathbb{R}^N)} + |u|_{B^{s,p,\theta}(\mathbb{R}^N)} < \infty.$$

Weak Solution:

Consider a boundary value problem as

$$Lu = f \ \text{ in } \ U\,; u = 0 \ \text{ on } \ \partial U \tag{2.5}$$

where U is open, bounded subset of $\mathbb{R}^n$ and $u : \bar{U} \to \mathbb{R}$ is unknown with $u = u(x)$. Also note that $f : U \to \mathbb{R}$ is given and L is a second order partial differential operator having the either form

$$Lu = -\sum_{i,j=1}^{n}(a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{2.6}$$

or

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{2.7}$$

for given coefficient functions $a^{ij}, b^i, c(i, j = 1 \cdots, n)$. Note that the PDE $Lu = f$ is in divergence form if L is given by (2.6) and is non-divergence form if L is given by (2.7).

The function $u \in H_0^1(U)$ is a weak solution (Evans, 2010) of the boundary value problem (2.5) if

$$B[u, v] = \langle f, v\rangle \tag{2.8}$$

for all $v \in H_0^1(U)$, where $\langle , \rangle$ denotes the inner product in $L^2(U)$. Here $B[,]$ is a bilinear form associated with (2.6) and is given by

$$B[u,v] = \int_U \left\{ \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + c(x)uv \right\} dx \tag{2.9}$$

In simple word, weak or generalized solution is a function for which the derivatives may or may not exist but which satisfy the differential equation in some precisely defined sense. The most important weak solution is based on the notion of distributions. The differential equation is rewritten such that there is no derivatives of the solution in the equation which is known as the weak formulation and the corresponding solutions are called weak solutions.

Leray Hopf weak solution:

θ is a Leray Hopf weak solution to initial value problem for the SQG equation on $[0, T)$ if

1. $\theta \in L^\infty([0,T), L^2(\mathbb{R}^2)) \cap L^2([0,T), H^\alpha(\mathbb{R}^2))$ then $\int_{\mathbb{R}^2} \theta^2(x,t_2)dx + \int_{t_1}^{t_2} \int_{\mathbb{R}^2} (\Lambda^\alpha \theta(x,\tau))^2 dx\ d\tau \leq \int_{\mathbb{R}^2} \theta^2(x,t_1)dx$ for $0 \leq t_1 < t_2 < \ T.$
2. θ and $u = R(\theta)$ satisfy the SQG equation in distribution sense where R is singular integral operator.

Fourier Transform:

The Fourier Transform of function $f(x)$ is denoted by F and is given by

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} dx.$$

Similarly, $f(x)$ is called the inverse Fourier Transform of F and is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{2\pi ikx} dk.$$

Here the functions f and F are taken as continuous in the given range.

2.2 Background

Fluid mechanics is the field of physics that deals with the physical mechanics of fluids such as plasma, gases, and liquids and the forces acting on them. It is broadly categorized into fluid statics and fluid dynamics. The first deals with the fluid at the state of rest and the second deals with the impacts of forces on the fluids in motion. It is a prolific research field that is generally mathematically complex. There are many problems that are wholly or partly unsolved and effectively addressed by numerical techniques with the use of computers. There is a numerical technique generally called computational fluid dynamics. There are many factors affecting the natural phenomena behave differently. Their activities can be modeled with the help of differential equations, as the parameters show similarity with those factors.

2.2.1 Mathematical Modeling

A mathematical model uses mathematical concepts and languages to describe natural or artificial systems. Mathematical modeling is the process of creating a mathematical model. These models are employed not only in the social sciences but also in the scientific sciences and engineering fields. Different types of mathematical models exist, such as dynamical systems, statistical models, and differential equations. The majority of the following components can be found in a standard mathematical model in the physical sciences: regulating equations, supporting submodels, presumptions, and constraints, including initial and boundary conditions. The following are the steps involved in mathematical modeling:

1. Assumptions and hypothesis
2. Mathematical formulation
3. Obtain the solution
4. Display the prediction of model
5. Check the models predictions with known facts
6. Make necessary change in assumptions if necessary

2.2.2 Navier-Stokes Equation

A mathematical description of fluid motion characterized by density, velocity, pressure together with conservation of mass, Newton's second law of motion and material property. The Navier–Stokes equations describe the dynamics of the incompressible fluid.

The 3D incompressible Navier- Stokes equation is given by

$$\partial_t u + (u.\nabla)u = -\frac{\nabla P}{\rho} + \nu\nabla^2 u, \ \nabla.u = 0 \tag{2.10}$$

where ν = viscosity, ρ = density, u = flow velocity, ∇ = divergence, ∇^2 = Laplacian, P = pressure, t = time.

Here the term $(u.\nabla)u$ is the divergence on a velocity. In a simpler term, it means how the divergence affects the velocity. The term $-\frac{\nabla P}{\rho}$ is thought to refer to how the particles move as pressure changes and their tendency to move away from areas of higher pressure. The term $\nu\nabla^2 u$ contains two key parts: viscosity (ν) and Laplacian(∇^2). It can be thought of as the difference between what a particle does and what its neighbors do. When the motion of a particle in a high viscous substance is considered, then it will induce particles to move, while in a less viscous fluid such as water or milk induce, it will have a less effect on its neighbors.

Whether the classical solutions of three dimensional incompressible Navier-Stokes equations can develop a finite time singularity or globally regular for all time from smooth initial data with finite energy is an outstanding open problem in fluid dynamics. There is no analytical proof as well as no numerical result for the finite time singularity for the NS equations. It is also a millennium problem as enlisted by Clay Institute of Mathematics and offered prize money of a million US dollars.

2.2.3 Euler Equation

The Euler equation is the set of quasilinear partial differential equations. This equation corresponds to NS equation with zero viscosity and zero thermal conductivity. These equations are equally applicable to incompressible as well as compressible fluids and basically reflect the conservation of mass, momentum, and kinetic energy. The first comprehensive mathematical model of a fluid was proposed by Euler in the 1750s. The 3D Euler equation is given by

$$\partial_t u + u.\nabla u = -\frac{\nabla P}{\rho}, \quad \nabla.u = 0 \tag{2.11}$$

where u, the velocity field; P, the fluid pressure and ρ is the density.

2.2.4 3D Quasi-Geostrophic Equations

J. G. Charney derived the general 3D quasi geostrophic equations in 1940s. It is given by

$$\begin{aligned} \partial_t \Delta\psi + \overline{\nabla}^{\perp}\psi \nabla\Delta\psi &= 0 \\ \partial_t(\partial_\nu\psi) + \overline{\nabla}^{\perp}\psi\nabla\Delta\partial_\nu\psi &= \Delta\psi \\ \psi(0,z,x) &= \psi_0(z,x) \end{aligned} \tag{2.12}$$

We note that $\overline{\nabla}^{\perp}\psi$ is the velocity field which transport the potential vorticity $\Delta\psi$ and Neumann derivative $\partial_\nu\psi$. The above system includes two coupled equations: Transport equation on the vorticity, and equation satisfied by $\partial_\nu\psi$. The 3D quasi-geostrophic system is a widely used model in oceanography and meteorology to describe large-scale oceanic and atmospheric circulation. These equations have been very successful in describing the major features of large-scale motions in the atmosphere and oceans in the midlatitudes (Pedlosky, 1987). The dynamics of these 3D geostrophic equations with uniform potential vorticity reduces to the SQG equation.

2.2.5 Surface Quasi-Geostrophic Equation

The two dimensional (2D) Surface Quasi Geostrophic (SQG) equation is given by

$$\begin{aligned} \partial_t\theta + u.\nabla\theta + \kappa\,(-\Delta)^{\alpha}\theta &= 0 \\ \nabla.u &= 0 \\ \theta(x,0) &= \theta_0(x) \end{aligned} \tag{2.13}$$

where $\kappa \geq 0$ and $\alpha > 0$ are parameters, $\theta = \theta(x_1, x_2, t)$ is a scalar representing the potential temperature and $u = (u_1, u_2)$ is the velocity field determined from θ by the stream function ψ with the auxiliary relations

$$(u_1, u_2) = (-\partial_{x_2}\psi, \partial_{x_1}\psi),\ (-\Delta)^{1/2}\psi = \theta.$$

2.2.6 KdV Equation

The KdV equation is given by

$$u_t + u_x + 2uu_x + \delta u_{xxx} = 0 \tag{2.14}$$

which is a nonlinear partial differential equation of third order. Here $u = u(x,t)$ is a scalar function of x and t where $x \in \mathbb{R}$ and $t \geq 0$ and δ is a positive parameter. u represents the amplitude of wave and term δu_{xxx} produces the dispersive broadening.

2.2.7 Water Wave Equation

The equation of motion of the wave is given by

$$\partial_t u + u.\nabla u = -\nabla P + F, \quad \nabla.u = 0 \tag{2.15}$$

where u is the velocity, P is the pressure, and $F = (0, -g)$. This is the Euler equation with $\rho = 1$. In Cartesian form, the equation (2.15) is given by

$$u_t + uu_x + vu_y = -P_x \tag{2.16}$$

$$v_t + uv_x + vv_y = -P_y - g \tag{2.17}$$

where $P(t,x,y)$ denotes the pressure and g is the gravitational constant of acceleration.

The boundary conditions for the water wave problem are

$$P = P_{\text{atm}} \text{ on } y = \eta(t,x) \tag{2.18}$$

P_{atm} being the constant atmospheric pressure. The kinematic boundary conditions

$$v = \eta_x + u\eta_y \text{ on } y = \eta(t,x) \tag{2.19}$$

and

$$v = 0 \text{ on } y = -d \tag{2.20}$$

In the presence of waves $y = \eta(t,x)$ will be the free surface and $(u(t,x,y), v(t,x,y))$ will be the velocity field.

2.3 Numerical Methods

There are many natural phenomena which can be modeled with the help of differential equations which may be partial or ordinary. Mathematicians usually try to obtain the analytical solution for the given equations. But many of the differential equations involve the terms which are nonlinear in nature and cannot be solved analytically. So an alternative approach will be used to get the approximate solution for the equations instead of actual solution. There are many approaches to obtain the approximate solution for the given instance. The numerical methods are used to approximate the actual solution of the given equation. With the help of these methods, approximate solutions are obtained. The various numerical methods are discussed.

2.3.1 Discrete Fourier Transform

Integral transforms are useful tools for solving problems involving partial differential equations especially when their solutions on the corresponding domains of definition are difficult to deal with. Hence there is algorithm scheme for solving PDEs on a defined domain by means of integral transform (Klamkin & Newman, 1961). Euler introduced integral transforms to solve the second-order differential equation (DE) problems (Deakin, 1985). Many integral transforms are suitable for solving the PDEs but most common transforms are Laplace and Fourier. The Fourier Transform has been proved to be extremely useful when applied to signal and image processing and for analyzing quantum mechanics phenomena. The details of it are found in (Franklin, 1933).

Jean Baptiste Joseph Fourier's was the first who introduced the Fourier Transform. It is the special case of continuous Fourier Transform, sometimes called the finite Fourier Transform. It is widely used in signal processing and related fields to analyze the frequencies contained in a sampled signal.

Consider a function $u(x)$ and $x \in [a,b]$. The interval is divided into n equal parts by the points $x_0, x_1, x_2, \cdots, x_n$. The Discrete Fourier Transform (DFT) of sequence $\{u(x_j)\}_{j=0}^{n-1}$ is given by the formula

$$\widehat{u(k)} = \sum_{j=0}^{n-1} u(x_j)\; e^{-\frac{2\pi i x_j k}{n}}; k = 0, 1, \cdots, n-1 \text{ where } i^2 = -1.$$

Similarly inverse Discrete Fourier transform (IDFT) of the sequence in the Fourier space is given by

$$u(x) = \frac{1}{n} \sum_{j=0}^{n-1} \widehat{u(x_j)} e^{\frac{2\pi i x_j k}{n}}; j = 0, 1, 2, \cdots, n-1.$$

The DFT is a linear transformation, so it can be considered as the transformation of a vector x to another vector X of the same dimension by the relation

$$X = \widehat{A_{mn}} x,$$

where

$$\widehat{A}_{mn} = \exp\left(-2\pi i \frac{(m-1)(n-1)}{N}\right).$$

2.3.2 Fast Fourier Transform

In DFT, N^2 complex multiplications are required for a set of N elements. But in 1942, it was discovered that DFT of length N could be rewritten as the sum of two Fourier Transforms of length $\frac{N}{2}$. This concept can be recursively applied to the data set until it is reduced to transforms of only two points. In 1965, Cooley and Tukey (Brigham, 1988) applied this concepts to filter noisy signals at Bell Labs. This method is known as Fast Fourier Transform which reduces the number of complex multiplications from N^2 to the order of $N \log_2 N$.

The DFT sequence $(\hat{x_k})$ of N complex valued numbers give another sequence of data $x(n)$ of length N according to formula

$$\widehat{x(k)} = \sum_{n=0}^{N-1} x(n) W_N^k; 0 \leq k \leq N-1$$

where $W_N = e^{-\frac{2\pi i}{N}}$. Similarly, inverse Discrete Fourier Transform is given by the formula

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} \widehat{x(k)} W_N^{-nk}; 0 \leq n \leq N-1.$$

For each value of k, direct computation of $x(\hat{k})$ involves N complex multiplications ($4N$ real multiplication) and $N-1$ complex additions ($4N-2$ real additions). Consequently to compute all N values of the DFT requires N^2 complex multiplications and $N^2 - N$ complex additions. For fastening the process of computation, we make the following two basic properties:
Symmetry property: $W_N^{k+\frac{N}{2}} = -W_N^k$
Periodicity property: $W_N^{k+N} = W_N^k$

In this process, the DFT of size N is divided into two intervaled DFT's of size $\frac{N}{2}$. One of those formed from the even numbered points of the original N whereas another from the odd numbered points. We can write

$$
\begin{aligned}
X_k &= \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}kn} \\
&= \sum_{n=0}^{\frac{N}{2}-1} x_{2n} e^{-\frac{2\pi i}{N}2nk} + \sum_{n=0}^{\frac{N}{2}-1} x_{2n+1} e^{-\frac{2\pi i}{N}(2n+1)k} \\
&= \sum_{n=0}^{\frac{N}{2}-1} x_{2n} e^{-\frac{2\pi i}{\frac{N}{2}}nk} + e^{-\frac{2\pi i}{N}k} \sum_{n=0}^{\frac{N}{2}-1} x_{2n+1} e^{-\frac{2\pi i}{\frac{N}{2}}nk} \\
&= X_k^e + W^k X_k^o
\end{aligned}
$$

Here X_k^e and X_k^o denote the DFTs of even and odd numbered indexed inputs respectively. The complex constant $W = e^{\frac{2\pi i}{N}}$ stands for twiddle factor. We note that X_k^e and X_k^o are periodic in k with length $\frac{N}{2}$.

2.3.3 Spectral Methods

Spectral methods are a class of techniques used in applied mathematics and scientific community to solve differential equations numerically which was first developed by Steven Orszag in 1969 in series of papers (Orszag, 1979) and also in (Boyd, 2001; Canuto, Hussaini, Quarteroni, & Zang, 2007). In this method, solution of the differential equation is expressed as the sum of a certain basis functions and the coefficients are chosen in order to satisfy the differential equation. This method uses the global smooth function to solve the ODEs, PDEs and eigenvalues problem. For the time dependent PDEs, the solution is written as sum of the basis function with time dependent coefficients and after substituting in PDEs yields a system of ODEs which can be solved by appropriate numerical methods.

2.3.4 Pseudospectral Method

Pseudospectral method is the special case of spectral method (Dutykh, 2016; Ohlsson, Schlatter, Mavriplis, & Henningson, 2011). This method is applied specially to time dependent PDEs and we find numerical solution for the given PDEs with certain boundary condition so that the residual is minimum. The test functions which are orthogonal to residuals and special points are considered with $\chi_n(x) = \delta(x - x_n), n = 0, 1, \cdots, N-1$ which are called collocation points. After the application of DFT, given PDEs are converted into a system of ODEs. We now explain procedure for solving PDEs with a

Pseudospectral or collocation method. Here $\chi_n(x) = \delta(x - x_n)$ where (x_n) constitute the collocation points. The smallness condition for the residuals is

$$\begin{aligned} & \langle \chi_n, R \rangle = 0 \; \forall n \in \{0, \cdots, N\} \\ \Longleftrightarrow \quad & \langle \delta(x - x_n), R \rangle = 0 \\ \Longleftrightarrow \quad & R(x_n) = 0 \\ \Longleftrightarrow \quad & Lu(x_n) = s(x_n) \\ \Longleftrightarrow \quad & \sum_{k=0}^{N} L\phi_k(x_n)\tilde{u_k} = s(x_n). \end{aligned}$$

The boundary condition is

$$\sum_{k=0}^{N}\sum_{p=0}^{M} \tilde{u_k} b_{pk} g_p(y) = 0.$$

$M + 1$ rows are dropped in the linear system and then solves the system

$$\sum_{k=0}^{N} L\phi_k(x_n)\tilde{u_k} = s(x_n), 0 \leq n \leq N - M - 1,$$

$$\sum_{k=0}^{N} b_{pk}\tilde{u_k} = 0, 0 \leq p \leq M.$$

2.3.5 Approximate Solution of PDEs using Numerical Methods

Consider a PDEs with boundary condition

$$Lu(x) = s(x), \text{ in } \Omega, Bu = 0 \text{ on } \partial\Omega \tag{2.21}$$

where L and B are linear differential operators. A function $\bar{u}$ is considered which satisfies the equation $Bu = 0$ so that the residuals $R = L\bar{u} - s$ becomes small. The residuals is made small in the frame work of method of weighted residuals(MWR). $\bar{u}$ is found in a finite dimensional subspace P_N of some Hilbert space W. The solution $\bar{u}$ is expanded in terms of trial functions as

$$\bar{u} = \sum_{n=0}^{N} \tilde{u_n}\phi_n(x)$$

where ϕ_n are the trial functions. Test functions $(\chi_1, \chi_2, \cdots, \chi_n)$ are used to define the smallness of residuals R by means of the Hilbert scalar product which is given by

$$\forall n \in \{0, \cdots, N\}, \langle \chi_n, R \rangle = 0$$

The trial functions vary according to numerical methods. For instance if the method is finite difference then the trial functions are overlapping local polynomials of low order whereas in case of finite element method they are smooth functions i.e. polynomials of fixed degree which are nonzero only on subdomains of Ω. Similarly, in case of spectral method the trial functions are the global smooth functions, in particular Fourier series.

In all the spectral methods, the trial functions are complete family of smooth global functions. For the Galerkin method, test and trial functions coincide, $\chi_n = \phi_n$ and each ϕ_n satisfy the boundary condition. Also, in case of collocation or pseudospectral method, the test functions are the delta functions at special points, called the collocation points which are given by $\chi_n = \delta(x - x_n)$.

Now we discuss how to solve the PDEs by Galerkin method. Since $\chi_n = \phi_n$, the smallness condition for the residuals , $\forall n \in \{0, \cdots, N\}$,

$$\begin{aligned}
& \langle \phi_n R \rangle = 0 \\
\Longleftrightarrow \quad & \langle \phi_n, L\bar{u} - s \rangle = 0 \\
\Longleftrightarrow \quad & \langle \phi_n, L \sum_{k=0}^{N} \widetilde{u_k} \phi_k \rangle - \langle \phi_n, s \rangle = 0 \\
\Longleftrightarrow \quad & \sum_{k=0}^{N} \widetilde{u_k} \langle \phi_n, L\phi_k \rangle - \langle \phi_n, s \rangle = 0 \\
\Longleftrightarrow \quad & \sum_{k=0}^{N} L_{nk} \widetilde{u_k} = \langle \phi_n, s \rangle
\end{aligned}$$

where L_{nk} denotes the matrix $L_{nk} = \langle \phi_n, L\phi_k \rangle$. Here solving the last steps means solving linear system which leads to the $(N+1)$ coefficients $\widetilde{u_k}$ of $\bar{u}$.

Chapter 3

KdV Type Equations

3.1 KdV Type Equations

The generalized Korteweg de Vries (KdV) is given by

$$u_t + au_x + 2buu_x + cu_{xxx} - du_{xx} = 0, u(x, 0) = u_0(x) \tag{3.1}$$

where a, b, c, d are real parameters. By choosing a particular value for the parameters, the equation (3.1) is reduced to transport, burger, and KdV equations, respectively. This KdV equation is a mathematical model of waves on the shallow water surface and is a notable example of exactly solvable nonlinear PDEs. This equation arises in the study of nonlinear dispersive waves, which was first derived by Bossinesq and then improved by Korteweg and de Vries in 1895. This equation is used to model waves in a shallow canal (Darrigol, 2005; Korteweg & De Vries, 1895). The KdV equation is given by

$$u_t + u_x + 2uu_x + \delta u_{xxx} = 0 \tag{3.2}$$

which is a nonlinear partial differential equation of third order. Here $u = u(x, t)$ denotes a scalar function of x and t, $x \in \mathbb{R}$ and $t \geq 0$ and δ is a positive parameter. Physically u represents the amplitude of the wave. The possibility of shock waves entering the solution is due to the nonlinear term. The dispersive broadening is produced by the term δu_{xxx} that can exactly compensate the narrowing caused by nonlinear term under proper condition.

The KdV equation has numerous applications in physical sciences and engineering fields, such as plasma physics, ion acoustic solitons (Das & Sarma, 1998), geophysical fluid dynamics, long wave in shallow seas and deep oceans (Osborne, 1995; Ostrovsky

& Stepanyants, 1989), and modeling waves in cold plasma. This equation has been studied by using various methods, such as tanh method in (Malfliet, 1992), sine-cosine method in (Yan & Zhang, 2001), and the homogeneous balance method (Lei & Ying-hai, 2002) with appropriate initial condition (Kruskal, Gardner, Green, & Miura, 1967). Similarly, other methods such as FEM (Aksan & Özdeş, 2006), FDS (Bahadır, 2005), and the spectral method (He-ping & Ben-yu, 1987) were introduced. The existence and uniqueness of solutions of the KdV equation was shown by (Kruskal et al., 1967). The author in (Rashid, 2007) introduced the artificial viscosity to reduce the round off error of pseudospectral method.

The KdV equation was studied in 2012 by Kolabaje and Oyewande analytically and numerically by using finite difference method and Adomian decomposition method. The approximate solution was obtained considering two possible scenarios, hyperbolic tangent and sinusoidal initial condition. With these considerations, they observed that the valid analytical solutions are restricted to the time values close to the initial time (Kolebaje & Oyewande, 2012).

The authors in (Alexander & Morris, 1979) used Galerkin techniques including cubic spline weight and interior functions with quintic polynomial boundary functions. The collocation method with septic splines was studied by (Soliman, 2004a) to obtain the solution of the KdV equation. The numerical solution of the KdV equation was obtained by using variational method by the authors in (Inc, 2007; Soliman, 2006). The modified Bernstein polynomials were used for the solitons type solution of it (Zabusky, 1967) and using the method of similarity reduction for PDEs were used to develop the schemes for solving the KdV equation by the authors in (Soliman, 2000, 2004b; Soliman & Ali, 2006) .

3.2 Exact Solution of KdV Equation

The special KdV equation is given by

$$u_t + 6uu_x + u_{xxx} = 0 \tag{3.3}$$

The solution of (3.3) is supposed to be of the form $u(x,t) = f(x - ct) = f(\eta)$. The ordinary differential equation is obtained after the substitution of this value in (3.3), which is given by

$$-c\frac{df}{d\eta}+6f\frac{df}{d\eta}+\frac{d^3f}{d\eta^3}=0 \tag{3.4}$$

On integration, we have,

$$-cf+3f^2+\frac{d^2f}{d\eta^2}=c_1 \tag{3.5}$$

where c_1 is the constant of integration. After multiplication with $\frac{df}{d\eta}$ and then integrating, we get

$$-\frac{c}{2}f^2+f^3+\frac{1}{2}\left(\frac{df}{d\eta}\right)^2=c_1f+c_2 \tag{3.6}$$

For $x\to\pm\infty$, we should have $z\to 0$, $\frac{df}{d\eta}\to 0$, $\frac{d^2f}{d\eta^2}\to 0$. For these requirements, we must have $c_1=c_2=0$. Then the above equation reduces to

$$\left(\frac{df}{d\eta}\right)^2=f^2(c-2f) \tag{3.7}$$

By separation of variables, we have

$$\int_0^f\frac{dy}{f\sqrt{c-2f}}=\int_0^\eta dz \tag{3.8}$$

Without loss of generality, the lower limit of integration can be chosen 0 as the starting point can be transformed linearly. With the substitution $y=\frac{1}{2}c\ sech^2\ w$, we have,

$$c-2y=c\tanh^2 w \tag{3.9}$$

and

$$\frac{dy}{dw}=-c\frac{\sinh w}{\cosh^3 w} \tag{3.10}$$

This implies that f changes to

$$w=sech^{-1}\ n\sqrt{\frac{2f}{c}} \tag{3.11}$$

After using the previous results, we have,

$$y=-\frac{2}{\sqrt{c}}\int_0^w\frac{1}{sech^2\ w\tanh w}\frac{\sinh w}{\cosh^3 w}dw=-\frac{2}{\sqrt{c}}w.$$

Further, simplifying we get, $f(\eta)=\frac{c}{2}sech^2\ \frac{\sqrt{c}}{2}y$.
Finally with our assumption, we get

$$u(x,t)=\frac{c}{2}sech^2\ \left[\frac{\sqrt{c}}{2}(x-ct)\right] \tag{3.12}$$

The same exact solution can be obtained by using Backlund transform (Brauer, 2000).

3.3 Spectral Method for the Generalized KdV Equation

The generalized KdV equation is given by

$$u_t + au_x + 2buu_x + cu_{xxx} - du_{xx} = 0, u(x,0) = u_0(x) \tag{3.13}$$

in the (x,t) space where $u = u(x,t)$. Consider $x \in [0, 2\pi], t > 0$. The interval is divided into N equal parts where N is power of 2. The discrete transform of $u(x,t)$ is given by

$$u(\hat{x,} t) = \sum_{j=0}^{N-1} u(x_j, t)e^{-\frac{2\pi i x_j k}{n}}; k = 0, 1, \cdots, N-1.$$

Taking the discrete Fourier transform on both sides of (3.13), we get

$$\hat{u}_t(k,t) + ika\hat{u}(k,t) + ikb\hat{u^2}(k,t) + (ik)^3 c\hat{u}(k,t) - (ik)^2 d\hat{u}(k,t) = 0, \hat{u}(k,0) = \hat{u_0}(k) \tag{3.14}$$

Solving (3.13) in (x,t) space is equivalent to solving (3.14) in interval $[0, 2\pi]$. To solve (3.14), we proceed as follows: Given initial function $u_0(x)$, we first find the discrete values at the N points and get a sequence $\{u_0(x_j)\}_{j=0}^{N-1}$. Then, we find the discrete Fourier transform $\{\widehat{u_0}(k)\}_{j=0}^{N-1}$ and $\left\{\widehat{u_0^2}(k)\right\}_{j=0}^{N-1}$, which are given by the formulas

$$\widehat{u_0}(k,t) = \sum_{j=0}^{N-1} u(x_j, t)e^{-\frac{2\pi i x_j k}{n}}; k = 0, 1, \cdots, N-1$$

and

$$\widehat{u_0^2}(k,t) = \sum_{j=0}^{N-1} u^2(x_j, t)e^{-\frac{2\pi i x_j k}{n}}; k = 0, 1, \cdots, N-1.$$

The Fast Fourier Transform is used to fasten the process. After finding the values of $\{\widehat{u_0}(k)\}_{j=0}^{N-1}$ and $\left\{\widehat{u_0^2}(k)\right\}_{j=0}^{N-1}$ from FFT, (3.14) can be solved numerically by using Euler method. For Euler method, $\widehat{u_t}(k,t)$ in equation (3.14) can be written as

$$\frac{\hat{u}(k,h) - \hat{u}(k,0)}{h} \approx -ika\hat{u}(k,0) - ikb\hat{u^2}(k,0) + ik^3 c\hat{u}(k,0) - k^2 d\hat{u}(k,0)$$

$$\hat{u}(k,h) \approx \hat{u}(k,0) - h[ika\hat{u}(k,0) + ikb\hat{u^2}(k,0) - ik^3 c\hat{u}(k,0) + k^2 d\hat{u}(k,0)]$$

$$\hat{u}(k,h) \approx \hat{u}(k) - h[ika\hat{u}(k) + ikb\hat{u^2}(k) - ik^3 c\hat{u}(k) + k^2 d\hat{u}(k)].$$

Additionally, we can also use the implicit Euler method. In this case, the formula from $t = jh$ to $t = (j+1)h$ is given by

$$\frac{\hat{u}(k,(j+1)h) - \hat{u}(k,jh)}{h} \approx -ika\hat{u}(k,jh) - ikb\hat{u^2}(k,jh) + ik^3 c\hat{u}(k,jh) - k^2 d\hat{u}(k,jh)$$

which is due to implicit Euler formula. Further simplification implies

$$\hat{u}(k,(j+1)h) \approx \frac{\hat{u}(k) - ikh\left[a\hat{u}(k,jh) + b\hat{u^2}(k,jh)\right]}{1 - h\left[ik^3 c - k^2 d\right]}$$

3.4 Numerical Results

The generalized KdV equation is given by

$$u_t + au_x + 2buu_x + cu_{xxx} - du_{xx} = 0, u(x,0) = f(x) \tag{3.15}$$

where a, b, c, d are parameters. Our main focus is in the behavior of the solution of the equation (3.15) as the different parameters a, b, c, d tend to zero in the limiting sense and compare the situation with the cases when the values of the parameters are exactly zero.

3.4.1 Transport Equation

With $b = c = d = 0$, the equation (3.15) becomes

$$u_t + au_x = 0, u(x,0) = f(x) \tag{3.16}$$

which is the transport equation. We have created different set of values of a and observe the scenario of the solution as the value of a tends to zero with the case when the value of a is exactly zero. The graphs of solution for the different values of a are presented in the following Figures[1, 2, 3, 4, 5, 6, 7, 8].

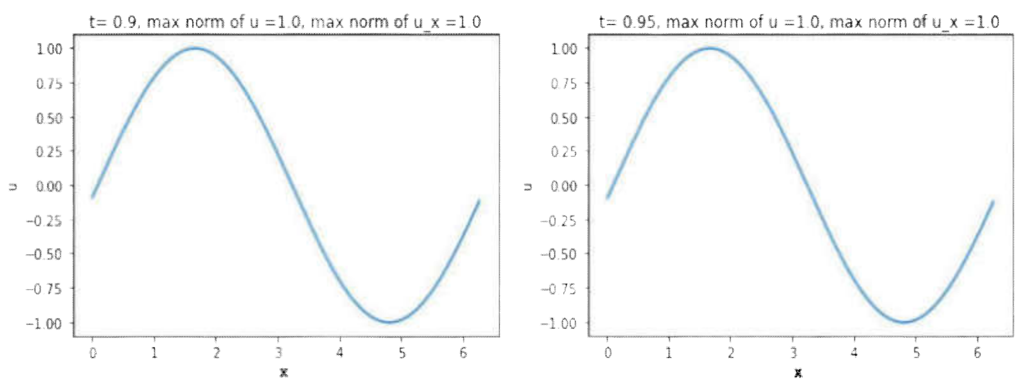

Figure 1: Plot of transport equation with a = 0.1

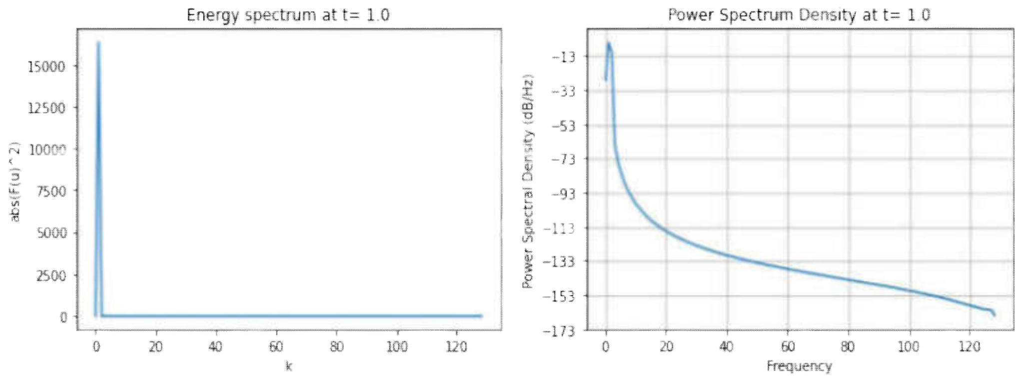

Figure 2: **Plot of energy and power spectrum**

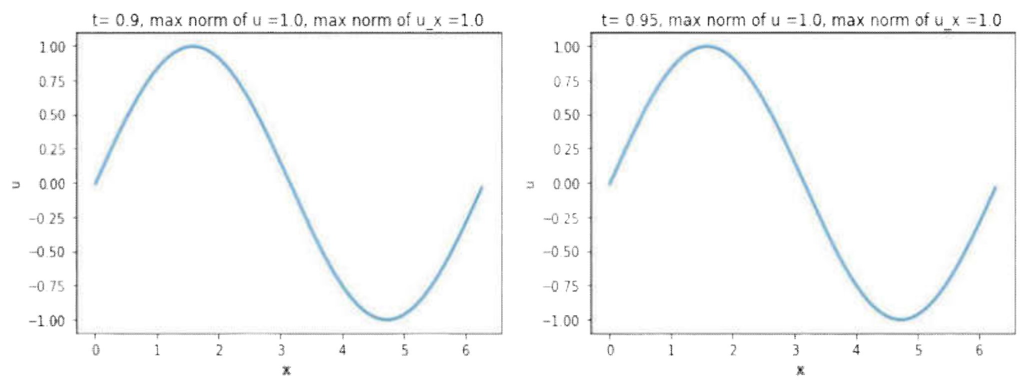

Figure 3: **Plot of transport equation with a = 0.01**

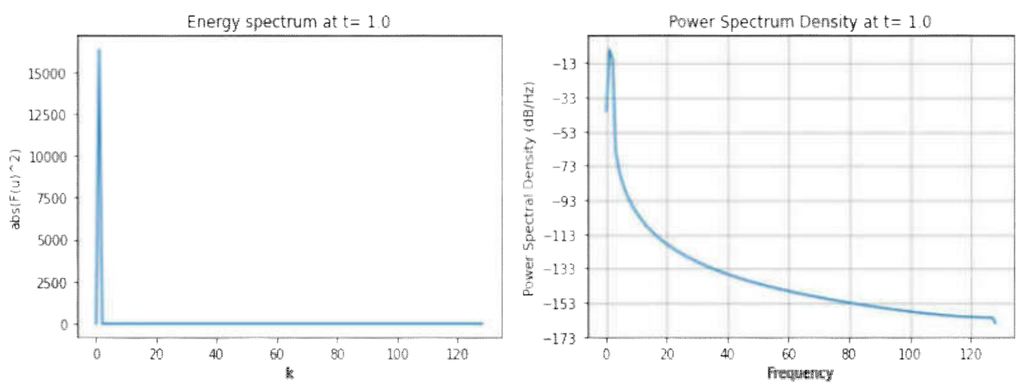

Figure 4: **Plot of energy and power spectrum**

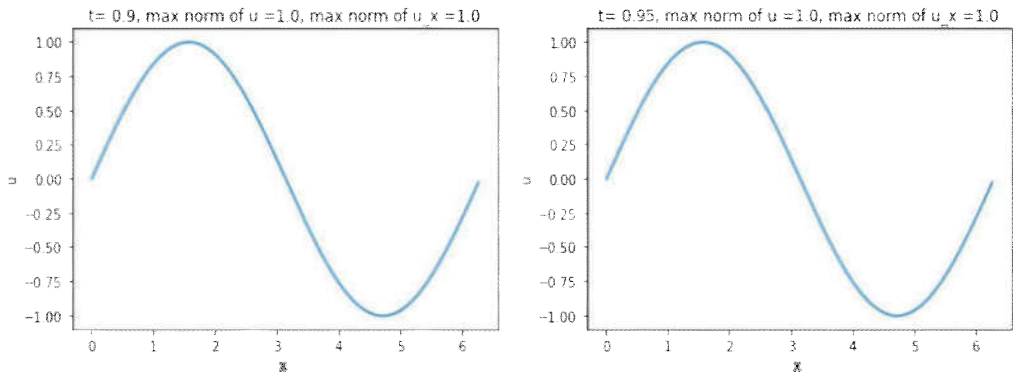

Figure 5: **Plot of transport equation with a= 0.001**

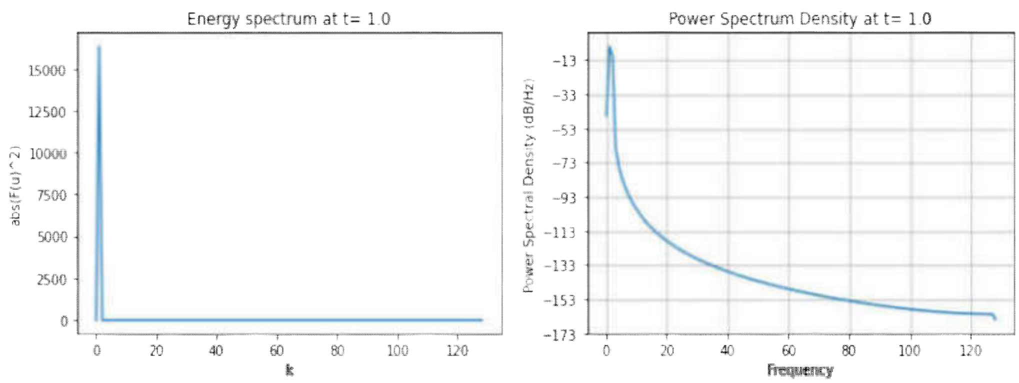

Figure 6: **Plot of energy and power spectrum**

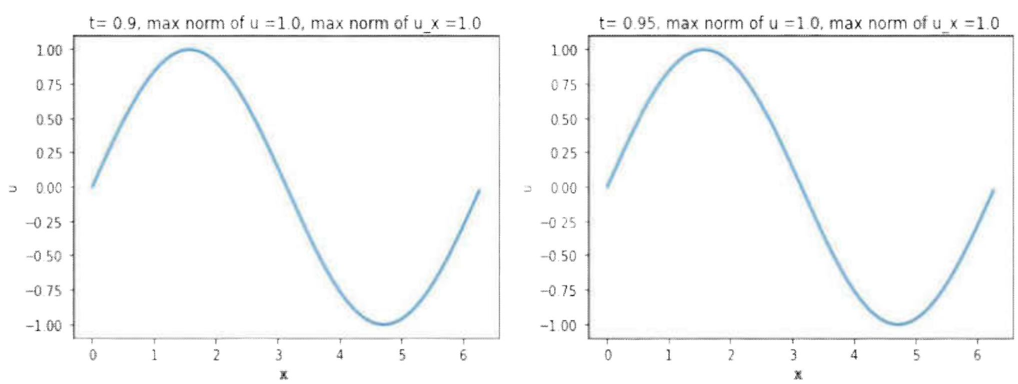

Figure 7: **Plot of transport equation with a= 0**

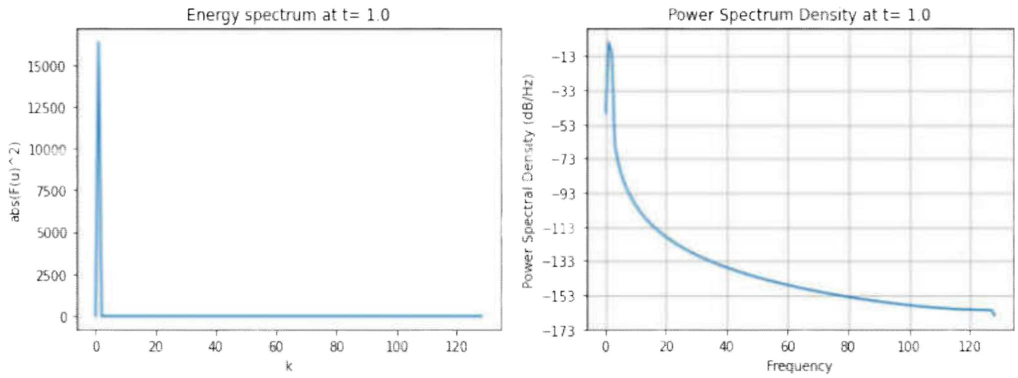

Figure 8: Plot of energy and power spectrum

The results of the numerical calculations show that the solution has a similar character when the coefficients are exactly zero and when the coefficients are zero in the limiting sense.

3.4.2 Burger Equation and Viscous Burger Equation

With $a = 0, b = 1, c = 0$, the equation (3.15) takes the form

$$u_t + 2uu_x - du_{xx} = 0, u(x, 0) = f(x) \tag{3.17}$$

This equation is called the viscous Burger equation and if $d = 0$ then equation (3.17) is called the inviscid Burger equation. The different data sets for the different values of d are obtained. Also, the nonlinearity in the solution is observed as the value of d tends to zero. Some of the graphs of the solutions are shown Figures [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

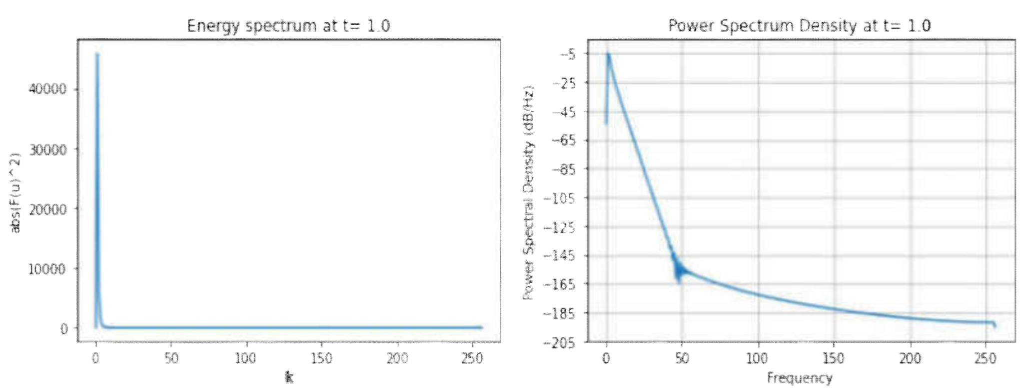

Figure 9: Plot of energy and power spectrum b = 0.5, d = 0.07

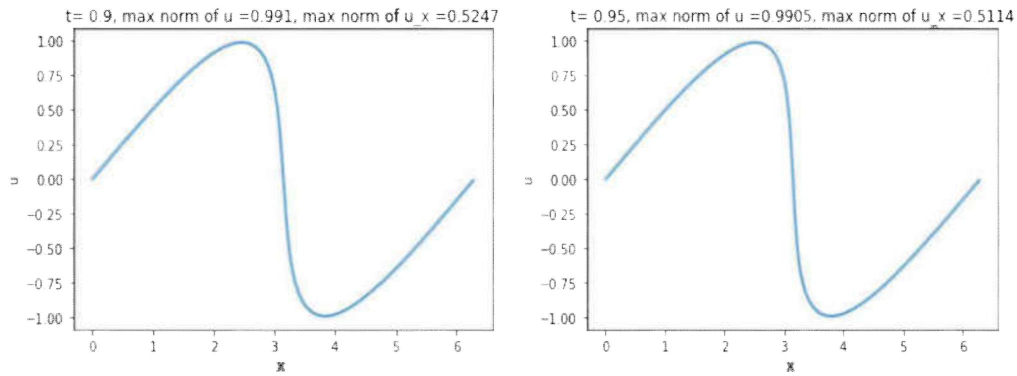

Figure 10: Plot of burger equation b = 0.5, d = 0.01

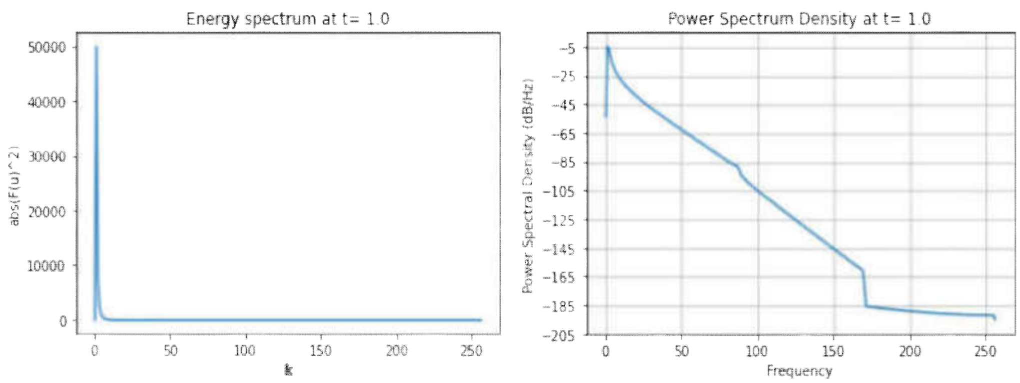

Figure 11: Plot of energy and power spectrum b = 0.5, d = 0.01

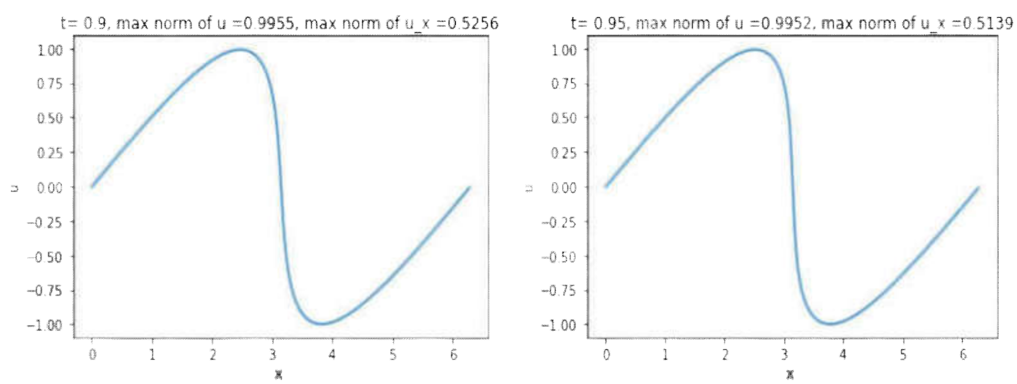

Figure 12: Plot of burger equation b = 0.5, d = 0.005

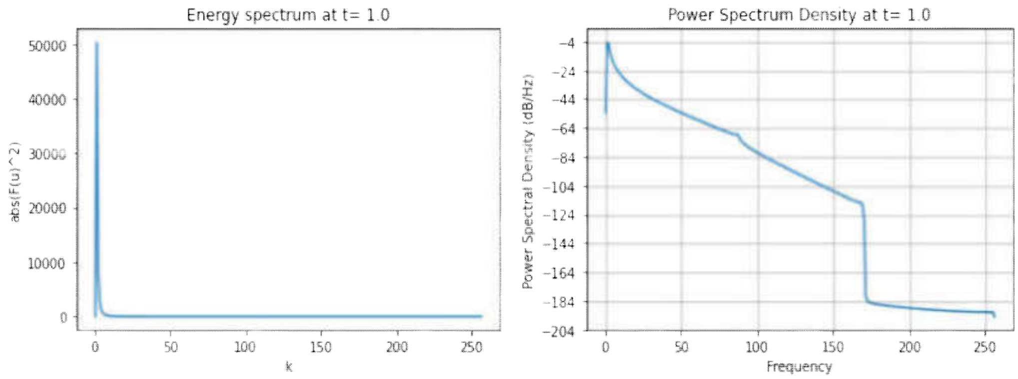

Figure 13: Plot of energy and power spectrum b = 0.5, d = 0.005

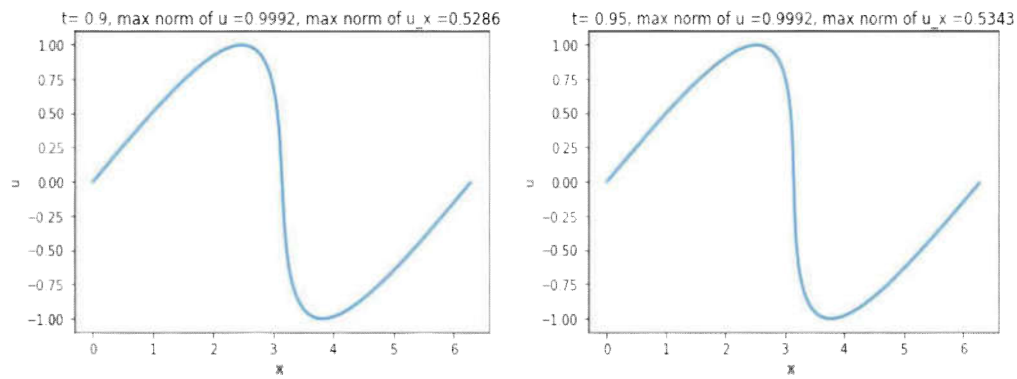

Figure 14: Plot of burger equation b = 0.5, d = 0.0009

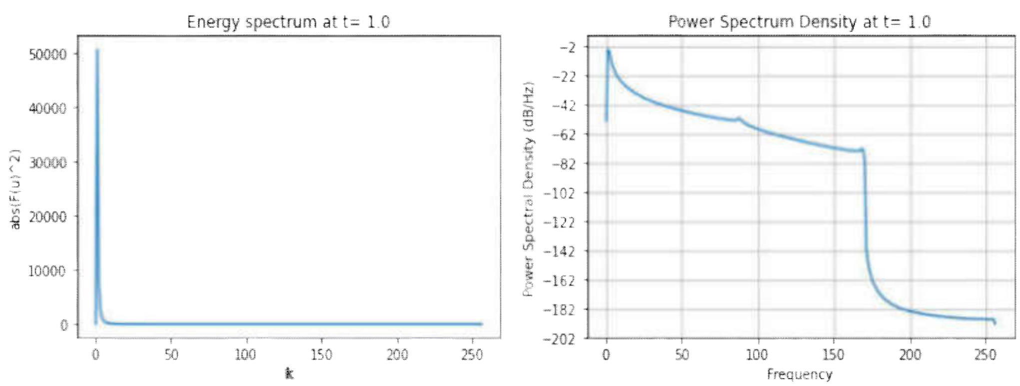

Figure 15: Plot of energy and power spectrum b = 0.5, d = 0.0009

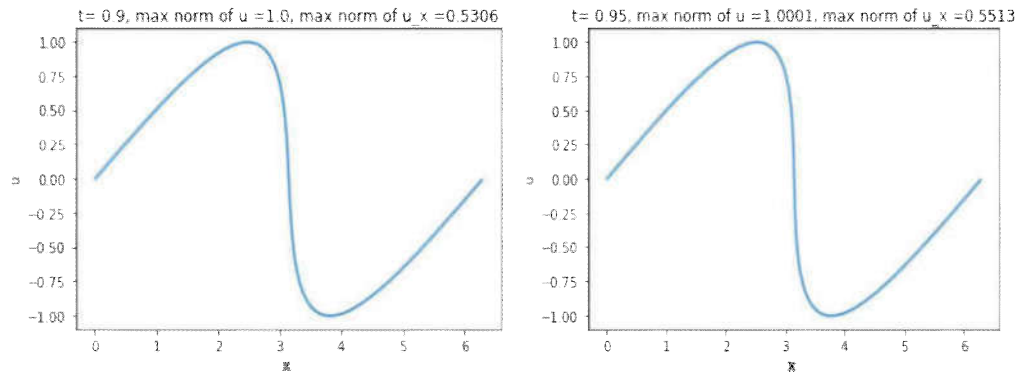

Figure 16: **Plot of burger equation b = 0.5, d = 0.0001**

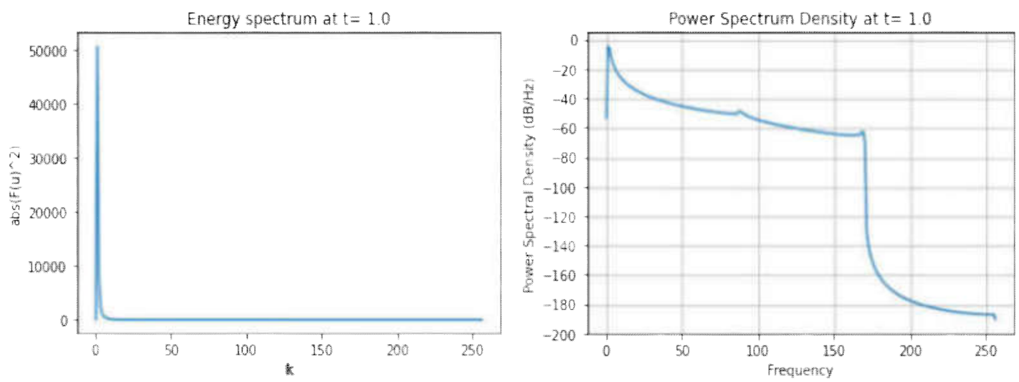

Figure 17: **Plot of energy and power spectrum b = 0.5, d = 0.0001**

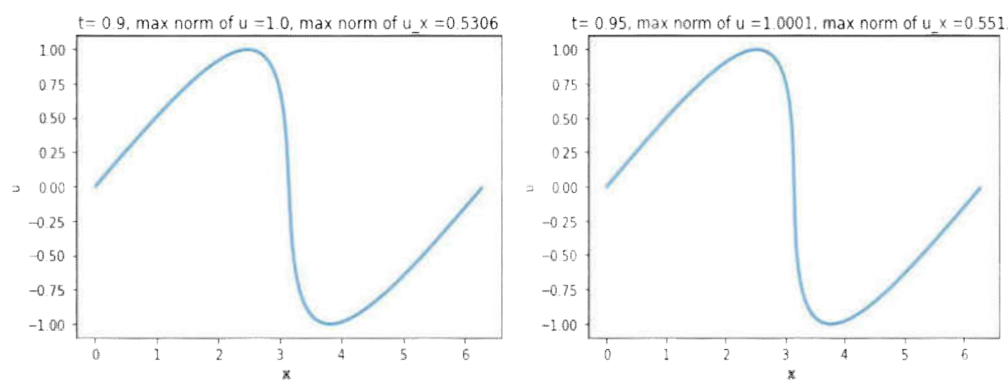

Figure 18: **Plot of burger equation b = 0.5, d = 0**

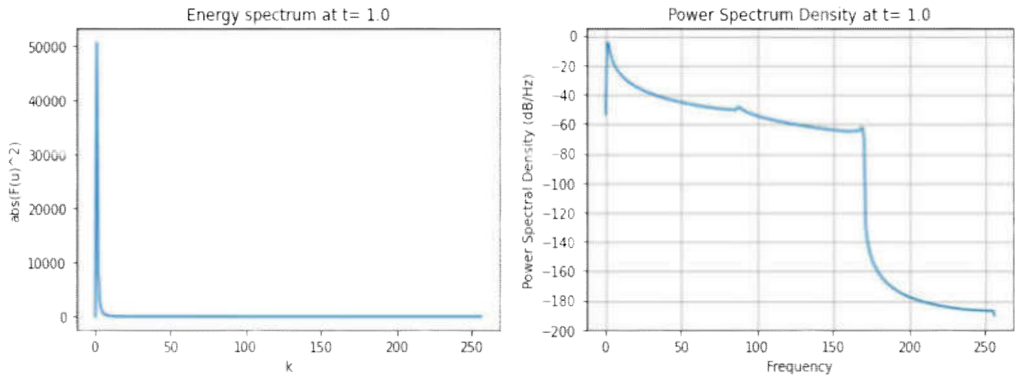

Figure 19: Plot of energy and power spectrum b = 0.5, d = 0

Based on the data obtained from numerical computations, it has been observed that there is no substantial difference in the solution when the coefficients are exactly zero compared to when the coefficients approach zero in a limiting sense.

3.4.3 The Linearized KdV Equation

With $b = d = 0$, the equation (3.15) takes the form

$$u_t + au_x + cu_{xxx} = 0, u(x, 0) = f(x) \tag{3.18}$$

which is known as the linearized Korteweg de Vries(KdV) equation. The different data sets are observed as the value of d varies and also, the nature of the solution is monitored when the value of d tends to zero in the limiting sense. Some of the graphs of the solutions are presented in Figures [20,21, 22, 23, 24, 25, 26, 27].

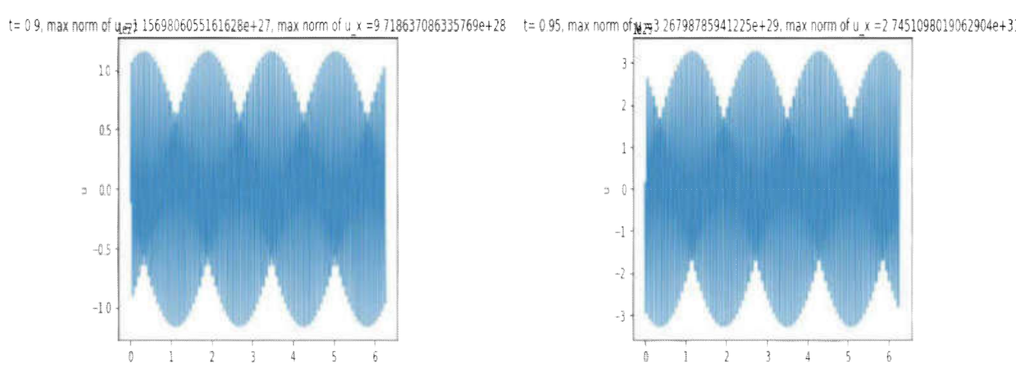

Figure 20: Plot of kdv equation with a = 1, c = 0.005

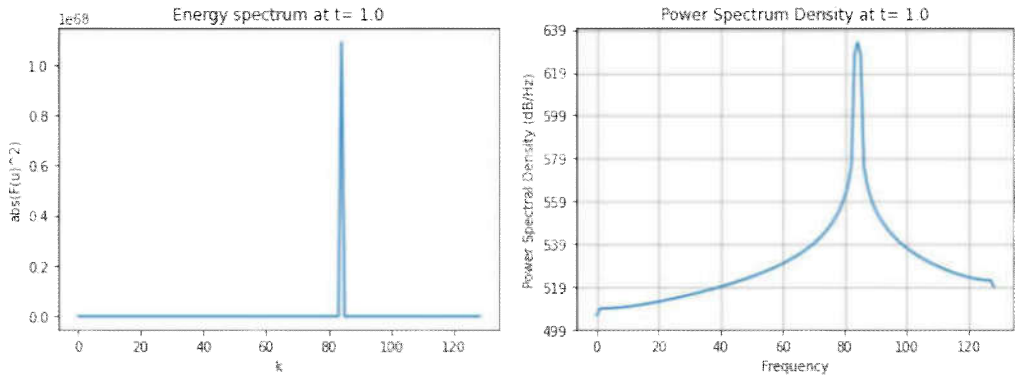

Figure 21: **Plot of energy and power spectrum, a = 1, c = 0.005**

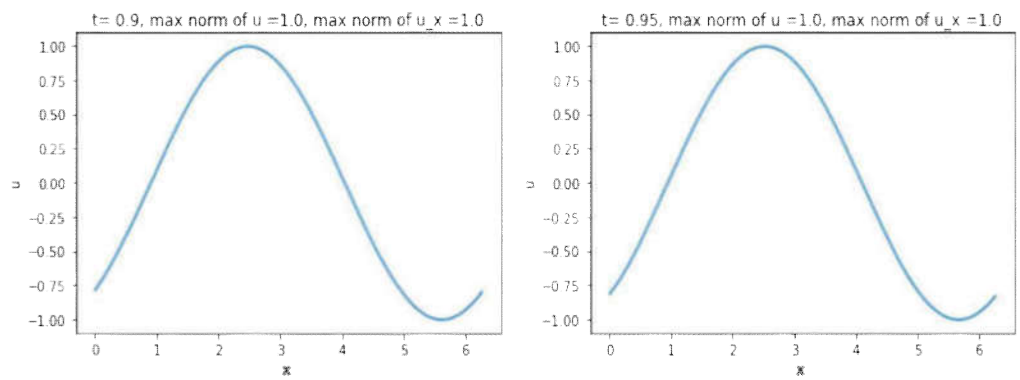

Figure 22: **Plot of kdv equation with a = 1, c = 0.003**

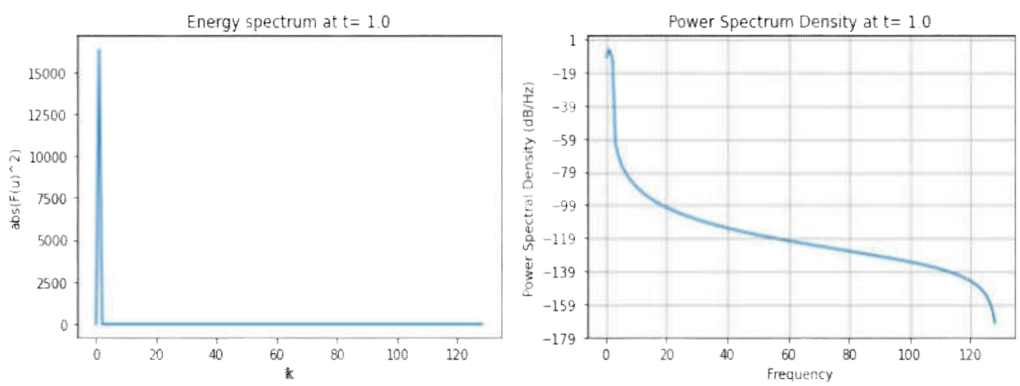

Figure 23: **Plot of energy and power spectrum, a = 1, c = 0.003**

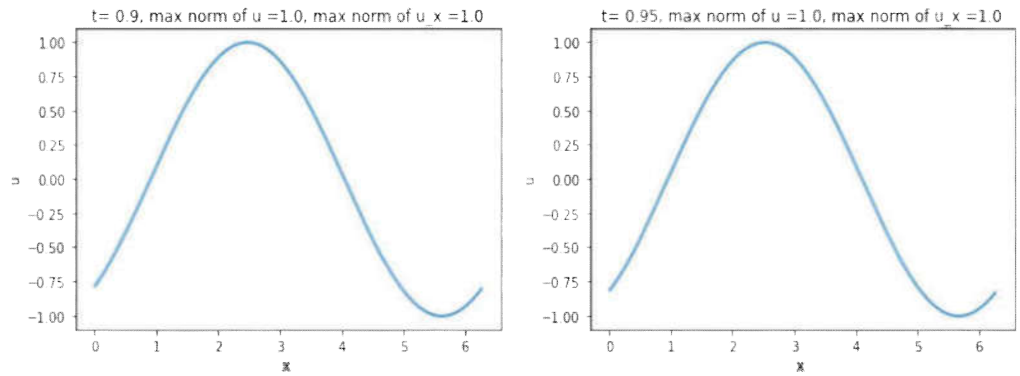

Figure 24: Plot of kdv equation with a = 1, c = 0.001

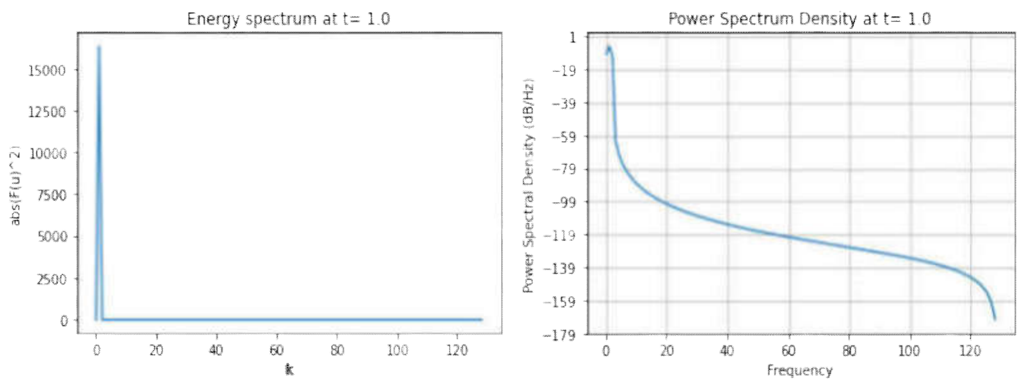

Figure 25: Plot of energy and power spectrum, a=1, c=0.001

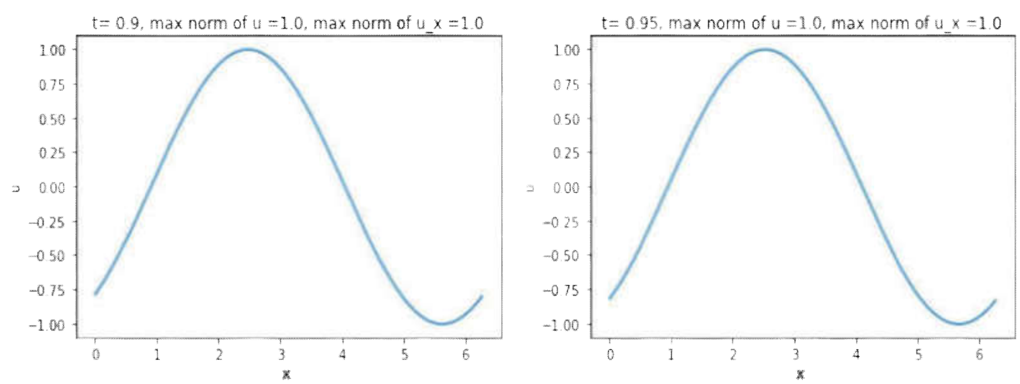

Figure 26: Plot of kdv equation with a = 1, c = 0

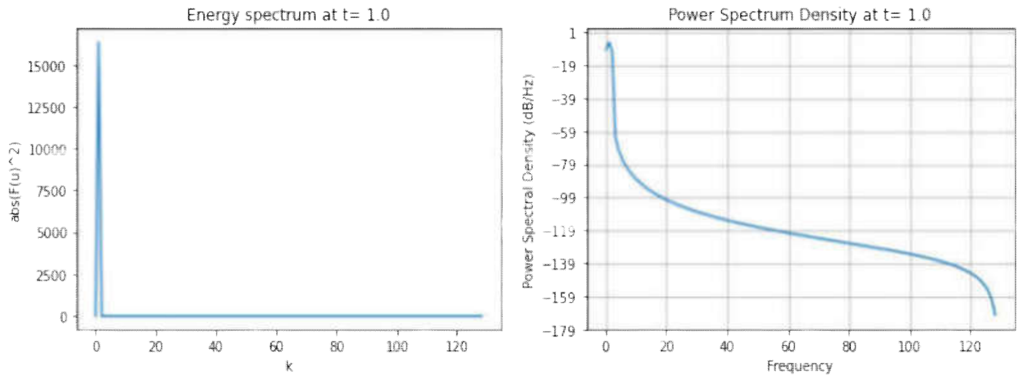

Figure 27: Plot of energy and power spectrum, a = 1, c = 0

From the data obtained from the numerical computations, it is observed that the two different cases one when the coefficients are exactly zero and other the coefficients are zero in limiting sense show no drastic difference in the nature of solution.

Chapter 4

2D Surface-Quasi Geostrophic Equation

4.1 SQG Equation

Atmospheric and oceanographic flows take place over horizontal lengths which are very large as compared to vertical lengths and hence they can be described using the shallow water equations. The quasi-geostrophic equations are approximations of the shallow water equations when the Rossby number is small. In particular, if the Rossby number is zero, then we have geostrophic flow. The quasi-geostrophic equations were first formulated by Jule Charney (see in (A. Majda & Wang, 2006)).

The 3D quasi geostrophic equations are given by

$$\begin{aligned} \partial_t \Delta\psi + \overline{\nabla}^{\perp}\psi \nabla\Delta\psi &= 0 \\ \partial_t(\partial_\nu \psi) + \overline{\nabla}^{\perp}\psi\nabla\Delta\partial_\nu\psi &= \Delta\psi \\ \psi(0,z,x) &= \psi_0(z,x) \end{aligned} \tag{4.1}$$

where ψ is the stream function. These equations (4.1) have been very successful in describing the major features of large-scale motions in the atmosphere and oceans in the midlatitudes (Pedlosky, 1987). The dynamics of these 3D geostrophic equations with uniform potential vorticity reduce to the SQG equation.

The two dimensional (2D) Surface Quasi-Geostrophic (SQG) equation is given by

$$\begin{aligned} \partial_t\theta + u.\nabla\theta + \kappa\,(-\Delta)^{\alpha}\theta &= 0 \\ \nabla.u &= 0 \\ \theta(x,0) &= \theta_0(x) \end{aligned} \tag{4.2}$$

where $\kappa \geq 0$ and $\alpha > 0$ are parameters, $\theta = \theta(x_1, x_2, t)$ is a scalar representing the potential temperature and $u = (u_1, u_2)$ is the velocity field determined from θ by the stream function ψ with the auxiliary relations

$$(u_1, u_2) = (-\partial_{x_2}\psi, \partial_{x_1}\psi), \, (-\Delta)^{1/2}\psi = \theta.$$

Assuming $\Lambda = (-\Delta)^{1/2}$ and $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$, the above relation can be written as

$$u = \nabla^{\perp}\Lambda^{-1}\theta = (-R_2\theta, R_1\theta),$$

where R_1 and R_2 are the usual Riesz transforms. The spatial domains concerned here is the periodic box T^2 or $\mathbb{R}^2$.

The SQG equation models the evolution of buoyancy or the potential temperature on the 2D horizontal boundaries. The inviscid SQG equation is useful in modeling the atmospheric phenomenon such as frontogenesis, the formation of strong fronts between the masses of hot and cold air. Also, the SQG equation with $\kappa = 0$ is an important example of an active scalar and an important testbed for turbulence theories due to some of its distinctive features (Blumen, 1978; Held et al., 1995). The SQG equation with $\alpha = \frac{1}{2}$ and $\kappa > 0$ arises in geophysical studies of strongly rotating fluids (A. J. Majda & Tabak, 1996; Pedlosky, 1987).

Depending upon k and α, the equation can be divided into the following categories:

1. When $k = 0$, the Equation (4.2) is called the inviscid SQG equation.
2. When $k > 0$, the Equation (4.2) is called the dissipative SQG equation.
   (a) When $\alpha > \frac{1}{2}$, the Equation (4.2) is called the subcritical SQG equation.
   (b) When $\alpha = \frac{1}{2}$, the Equation (4.2) is called the critical SQG equation.
   (c) When $\alpha < \frac{1}{2}$, the Equation (4.2) is called the supercritical SQG equation.

The subcritical and critical cases have been solved but the supercritical case is still an open problem regarding the regularity of the solution. The 3D Euler equation is

$$\partial_t u + u.\nabla u = -\frac{\nabla P}{\rho} + g, \quad \nabla.u = 0 \tag{4.3}$$

where u, the velocity field; P, the fluid pressure and ρ, fluid density.

Similarly, the 3D Navier Stokes equation is given by

$$\partial_t u + u.\nabla u = -\frac{\nabla P}{\rho} + \nu \nabla^2 u, \ \nabla.u = 0 \tag{4.4}$$

The fractional Laplacian $(-\Delta)^\alpha$ is defined through the Fourier transform as

$$\widehat{\Lambda^\alpha f(k)} = |k|^\alpha \widehat{f(k)}. \tag{4.5}$$

Using (4.5), the velocity u in the Fourier space can be expressed in terms of θ as

$$\hat{u} = \frac{i(-k_2\hat{\theta}, k_1\hat{\theta})}{|k|}. \tag{4.6}$$

The global existence and uniqueness of the solutions of the equations for $\alpha, \kappa \geq 0$ in time and space, both analytically and numerically are reviewed and explored by researchers in the field of fluid dynamics. The 2D inviscid SQG and 3D Euler equations share several common features which will be shown later. Similarly, there are some similarities between the 3D Euler Equation and 3D Navier-Stokes Equations. So, we pursue this review to reveal the current status of the SQG equation so that we may know about the possibility of further extension of the existing results to the 3D Euler and hence the Navier-Stokes equations.

The details of similarity with 3D Euler equation is expressed in the following subsection. The SQG equation is broadly discussed with its special cases and corresponding results in the following subsections.

4.1.1 Inviscid SQG Equation: Local Wellposedness and Regularity

In this section, we consider the inviscid SQG equation given by

$$\theta_t + u.\nabla\theta = 0 \tag{4.7}$$

with the velocity field u given by

$$u = (-R_2\theta, R_1\theta)$$

where R_1 and R_2 are the Riesz transforms. The following theorem guarantees the global existence of weak solution of (4.7). The main issue is: given an initial data, does the inviscid 2D SQG equation have a unique solution that is global in space and time.

Theorem 4.1. For a given $\theta_0 \in L^2(\mathbb{R}^2)$ or $L^2(T^2)$, there exist a global weak solution

$$\int_0^T \int [\theta(x,t)\phi_t(x,t) + \theta(x,t)u.\nabla\phi]dx\ dt + \int \theta_0(x)\phi(x,0)dx = 0$$

for smooth test function $\phi(x,t)$ and $T > 0$, where $\phi(x,T) = 0$.

This theorem was proved in the Ph.D. thesis of (Resnick, 1995).

The following theorem by (Beale, Kato, & Majda, 1984) gives a precise mathematical criterion in which smooth solutions of 3D Euler equation can become singular in finite time.

Theorem 4.2. Let u be the solution of the 3D Euler equation in the class $C([0,T_*);H^s) \cap C^1([0,T_*);H^{s-1})$ and if there is a time T_* beyond which the solution u cannot continue. Then $\int_0^{T_*} |\omega(t)|_{L^\infty} dt = \infty$, and in particular $\lim sup_{t\uparrow T*}|\omega(t)|_{L^\infty} = \infty$.

The local existence results can be applied to the solution of 2D SQG equation (4.7). More specifically , if the initial value $\theta(x,0) = \theta_0(x) \in H^s(\mathbb{R}^n)$ for some integer $s \geq 3$, then there is a smooth solution $\theta(x,t)$ of equation (4.7) that also belongs to $H^s(\mathbb{R}^n)$ for any time t in a sufficiently small time interval i.e., $t \in [0,T_*)$. Furthermore, if T_* is a maximal interval of existence of smooth solution and is finite,i.e., $T_* < \infty$, then

$$|\theta(.,t)|_s \to \infty \text{ as } t \to T_*.$$

The following theorem is analogous to theorem (4.2) in the case of inviscid 2D SQG equation.

Theorem 4.3. Let $\theta_0 \in H^s(\mathbb{R}^2)$ with $s > 2$ and $T_* < \infty$. Then the interval $0 \leq t < T_*$ is a maximal interval of existence of the solution of inviscid 2D SQG equation if and only if

$$\int_0^T |\nabla\theta|_{L^\infty}(s)\ ds \to \infty \ \text{ as } \ t \to T_*.$$

Theorem 4.4. Let $\theta_0 \in C^r \cap L^p$ with $r > 1$ and $p > 1$. Then there exists $T = T(||\theta_0||_{C^r \cap L^p}) > 0$ such that the inviscid 2D SQG equation has a unique solution θ on $[0,T]$. In addition, $\theta \in L^\infty([0,T];C^r \cap L^p)$.

4.1.2 Dissipative SQG Equation

In this section, we present some recent results for the dissipative SQG equation with $\kappa > 0$ and $\alpha > 0$. As in the inviscid case, the main issue is : given a smooth $\theta(x,0) = \theta_0(x)$, does the equation have global classical solution for all $\kappa > 0$ and

$\alpha > 0$? Recently, many important results have been obtained. For more details, we can see the works of (Caffarelli & Vasseur, 2010), (Carrillo & Ferreira, 2008), (Chen, Miao, & Zhang, 2007), (Chae, 2003, 2007), (P. Constantin, Cordoba, & Wu, 2001), (Córdoba, Córdoba, & Fontelos, 2005), (Cordoba, 1998; Cordoba & Fefferman, 2002b), (Cordoba & Fefferman, 2002b), (Deng, Hou, & Yu, 2005, 2006), (Kiselev, Nazarov, & Volberg, 2007), (Kiselev et al., 2007), (Ohkitani & Yamada, 1997), (Resnick, 1995), (P. Constantin, Majda, & Tabak, 1994), (Caffarelli & Vasseur, 2010), (P. Constantin & Wu, 2008; Wu, 2005), (Ohkitani & Yamada, 1997) and others.

The global regularity of the dissipative SQG equation depends on the parameter α. The index $\alpha = \frac{1}{2}$ is important for the theoretical point of view. If $\alpha > \frac{1}{2}$, the dissipative term $(-\Delta)^{\alpha}\theta$ is sufficient to control the nonlinear term. In this case, energy type estimates can be used to show that the solutions are global in time (P. Constantin & Wu, 1999). If $\alpha < \frac{1}{2}$, the dissipative term is not sufficient to control the nonlinear term. This phenomenon inturn makes the study of long time behavior of the solution much more difficult. So, to study the long time behavior of solution, we have to depend on the numerical results. Thus, we take the index $\alpha = \frac{1}{2}$ to be our starting value of parameter and let its value approach zero for our numerical study.

4.1.3 The Critical SQG Equation

When $\alpha = \frac{1}{2}$, the dissipative SQG equation becomes

$$\partial_t\theta + u.\nabla\theta + \kappa\,(-\Delta)^{\frac{1}{2}}\theta = 0, \nabla.u = 0 \tag{4.8}$$

Taking Fourier transform of (4.8), we have

$$\hat{\theta}_t + u_1\widehat{\frac{\partial\theta}{\partial x_1}} + u_2\widehat{\frac{\partial\theta}{\partial x_2}} + \kappa\widehat{(-\Delta)^{\frac{1}{2}}\theta} = 0$$

Using the definition of derivative in Fourier space and equation (4.5), we have

$$\frac{d}{dt}\widehat{\theta(l,t)} + il_1\widehat{u_1\theta(l,t)} + il_2\widehat{u_2\theta(l,t)} + \kappa|l|\widehat{\theta(l,t)} = 0,$$

where $l = (l_1, l_2)$ is the wave number and $i = \sqrt{-1}$. Using equation (4.6), it can be further written as

$$\frac{d}{dt}\widehat{\theta(l,t)} + \kappa|l|\widehat{\theta(l,t)} = -\sum_{j+k=l} i\frac{j^{\perp}.k}{|j|}\hat{\theta}(j,t)\hat{\theta}(k,t).$$

With these expression, we can say that nonlinearity and dissipation are comparable. This also gives the hint about the important role of κ in the long time behavior of the solution. There are some theorems which are about the global existence of solution of the equation (4.8) with small L^∞ initial data which was established by in (P. Constantin et al., 2001).

Theorem 4.5. Assume in Equation (4.8), $\theta_0 \in H^2$ and

$$||\theta_0||_{L^\infty} \leq C_\kappa \tag{4.9}$$

Then there exists a unique global solution θ of equation (4.8) satisfying

$$||\theta(.,t)||_{H^2} \leq ||\theta_0||_{H^2}.$$

Theorem 4.6. The critical SQG Equation (4.8) with periodic smooth initial data θ_0 has a unique global smooth solution. Moreover,

$$||\nabla\theta||_{L^\infty} \leq C||\nabla\theta_0||_{L^\infty} \text{exp exp } C||\theta_0||_{L^\infty}, \text{(Kiselev et al., 2007)}.$$

Theorem 4.7. Consider the SQG equation

$$\partial_t\theta + u.\nabla\theta + \kappa\,(-\Delta)^\alpha\theta = 0, x \in \mathbb{R}^n, t > 0 \tag{4.10}$$

with $\alpha = \frac{1}{2}$ and u satisfying $\nabla.u = 0, u = R(\theta)$ where R is a singular integral operator. Let $\theta_0 \in L^2(\mathbb{R}^n)$ and let θ be a Leray-Hopf weak solution, namely

$$\theta \in L^\infty([0,\infty); L^2(\mathbb{R}^n)) \cap L^2([0,\infty); \dot{H}^{\frac{1}{2}}(\mathbb{R}^n)).$$

Then θ is a classical solution for $t > 0$. In fact, for any $t > 0$

$$\theta \in C^\infty(\mathbb{R}^n \times [t,\infty)).$$

This theorem was established by (Caffarelli & Vasseur, 2010) and the proof is to improve the regularity of θ successively from L^2 to L^∞, from L^∞ to Höolder and from Höolder to C^∞.

4.1.4 Supercritical SQG Equation

We consider the equation

$$\partial_t\theta + u.\nabla\theta + \kappa\,(-\Delta)^\alpha\theta = 0, \kappa > 0, \alpha < \frac{1}{2} \tag{4.11}$$

The global regularity of equation (4.11) is extremely difficult as it is hard to apply the classical energy type estimates. (Chae & Lee, 2003), (Chen et al., 2007), (P. Constantin et al., 1994), (Córdoba & Córdoba, 2004), (Córdoba & Córdoba, 2004), (P. Constantin & Wu, 1999, 2008, 2009), (Zhifei, 2005) and others obtained the small data global regularity in various functional settings.

Definition 4.1. Let $f \in L^2_{loc}(\mathbb{R}^n)$ (i.e. for every set E of finite measure, $f \in L^2(E)$ and for every dyadic cube I

$$E_I(f) = \frac{1}{|I|}\int_I f dx, E_I^2(f) = E_I(|f - E_I(f)|^2).$$

Then $f \in \text{BMO}(\mathbb{R}^n)$ if $||f||_{BMO} = \sup_I (E_I^2(f))^{\frac{1}{2}} < \infty$.

Theorem 4.8. Let $\theta_0 \in L^2(\mathbb{R}^n)$ and θ be a Leray-Hopf weak solution of Equation (4.11). Then for any $t > 0$,

$$||\theta(.,t)||_{L^\infty} \leq C\frac{||\theta_0||_{L^2}}{t^{\frac{n}{4\alpha}}},\ ||u(.,t)||_{BMO}(\mathbb{R}^n) \leq C\frac{||\theta_0||_{L^2}}{t^{\frac{n}{4\alpha}}}.$$

Theorem 4.9. Let $\theta_0 \in L^2(\mathbb{R}^n)$ and θ be a Leray-Hopf weak solution of (4.11). Let u be associated velocity field. If we know that for some $0 < t_0 < t_1 < \infty$,

$$u \in L^\infty([t_0, t_1); C^{1-2\alpha}(\mathbb{R}^n)),$$

then

$$\theta \in L^\infty([t_0, t_1); C^\delta(\mathbb{R}^n))$$

for some $\delta > 0$.

Theorem 4.10. Let $\theta_0 \in L^2(\mathbb{R}^n)$ and θ be a Leray-Hopf weak solution of (4.11). If, for some $0 < t_0 < t_1 < \infty$,

$$\theta \in L^\infty([t_0, t_1); C^\delta(\mathbb{R}^n)),$$

with $\delta > 1 - 2\alpha$, then

$$\theta \in C^\infty((t_0, t_1] \times \mathbb{R}^n).$$

4.2 Analytical Results

The general 3D quasi geostrophic equations, first derived by J.G. Charney in 1940's, have been very successful in describing the major features of large-scale motions in the atmosphere and the oceans in the midlatitudes (Pedlosky, 1987). Also, the inviscid SQG is an important example of an active scalar and is an important testbed for turbulence due to some of its distinctive features (Blumen, 1978; Held et al., 1995).

In 1994, (P. Constantin et al., 1994) studied the formation of strong and potentially singular fronts in a two dimensional quasi geostrophic active scalar through the symbiotic interaction of mathematical theory and numerical experiments. They revealed the formation of geophysical flows in the atmosphere and the issues of frontogenesis, the formation of strong fronts between the masses of hot and cold air within quasi geostrophic approximations without explicitly incorporating ageostrophic effects. This studies developed the striking physical and mathematical analogies between the 2D inviscid SQG equation and the 3D Euler equation for the incompressible flow.

The quantities $\nabla^{\perp}\theta$ and $\omega = \nabla \times v$ play the similar role in two different situation, one in SQG equation and the second one in the 3D Euler equation respectively. The 3D incompressible Euler equation in vorticity-stream function form is given by

$$\frac{D\omega}{Dt} = (\nabla v)\omega \tag{4.12}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u.\nabla, v = (v_1, v_2, v_3)$ is the three dimensional velocity field with div $v = 0$ and $\omega = \nabla \times v$ is the vorticity vector.
The 2D QG active scalar is given by the equation

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + v.\nabla\theta = 0 \tag{4.13}$$

with the two dimensional velocity $v = (v_1, v_2)$ determined from θ by a stream function ψ is given by $(v_1, v_2) = (-\psi_{x_2}, \psi_{x_1})$. Differentiating equation (4.13), we have

$$\frac{D\nabla^{\perp}\theta}{Dt} = (\nabla v)\nabla^{\perp}\theta. \tag{4.14}$$

Here the velocity v in (4.12) is determined from the vorticity ω by the Biot-Savart law and the strain matrix S, the symmetric part of the velocity gradient. The right hand side of (4.12) can be written in terms of S and ω.

For the 2D SQG active scalar, the velocity field is given by

$$v = \nabla^{\perp}\psi = -\int_{\mathbb{R}^2} \frac{1}{|y|}\nabla^{\perp}\theta(x+y)dy. \tag{4.15}$$

The velocity in terms of ω is given by

$$v = \int_{R^d} K_d(y)\omega(x+y)dy \tag{4.16}$$

where $K_d(y)$ is homogeneous of degree $1-d$ in $\mathbb{R}^d$ for $d = 2, 3$.

Thus, with these $\nabla^{\perp}\theta$ and vorticity, the evolution equation for $\nabla^{\perp}\theta$ from (4.14) has completely parallel analytic structure in 2D as the equation of evolution of vorticity, ω, in (4.12) for the 3D incompressible flow.

Next, we present the geometric analogy between the level sets of θ for 2D active scalar and the vortex lines of 3D Euler equation. The infinitesimal length for vortex line, whose magnitude is ω, is given by

$$\frac{D|\omega|}{Dt} = \alpha|\omega| \tag{4.17}$$

with $\alpha(x,t) = S(x,t)\xi.\xi$, where S is the symmetric matrix. Again the infinitesimal length of level set for θ is given by $|\nabla^{\perp}\theta|$ and evolution equation for the infinitesimal arc length is given by

$$\frac{D|\nabla^{\perp}\theta|}{Dt} = \alpha|\nabla^{\perp}\theta| \tag{4.18}$$

with $\alpha(x,t) = S(x,t)\xi.\xi$ and $\xi = \frac{\nabla^{\perp}\theta}{|\nabla^{\perp}\theta|}$. Since the two level sets equations for the two situations in equations (4.17) and (4.18) have the similar structure, the level sets of the solutions of the quasi geostrophic active scalar seem to correspond to vortex lines in the 3D Euler equation.

The authors in (P. Constantin et al., 1994) developed the mathematical criterion to characterize how the smooth solution of the equation (4.13) can be singular. This is the simplest type which is analog to characterize the singular solution of 3D Euler equation in (Beale et al., 1984) which is stated as:

"*The time interval* $[0, T*]$ *with* $T* < \infty$ *is a maximal interval of a smooth solution for the* 2*D quasi geostrophic active scalar if and only if* $\int_0^T |\nabla\theta|_{L^\infty}(s)ds \to \infty$ *as* $T \to T*$ *with norm* $|f|_{L^\infty} = max_{x\in R^2}|f(x)|$" *(P. Constantin et al., 1994).*

Majda and Tabak, in 1996, found that the 2D Euler equation and SQG model have fronts with different behavior: the first one end up growing linearly due to the velocity field created nonlocally whereas the second gives a sustained nonlinear steepening of fronts. Also, their numerical studies showed that 2D Euler equation gives rise to fast-growing fronts switching nearly a linear regime whereas SQG fronts exhibit at a slower rate initially and then sustain a long nonlinear self-stretching process ending up with finite time collapse (A. J. Majda & Tabak, 1996).

In 1997, Wu considered inviscid limits for both the smooth and weak solutions for the 2D dissipative QGS equation and established that the classical solutions of dissipative equation with smooth initial data tend to the solutions of the corresponding non dissipative equations when the dissipative coefficient tends to zero. The convergence is in strong L^2 sense. The methods used by (Doering & Titi, 1995) and (Foias & Temam,

1989) for the NS equation was used to establish exponential decay of the spatial Fourier spectrum for the solutions of the dissipative quasi geostrophic equation with the consideration of the general norm and the different methods of treating nonlinear term (Wu, 1997).

In 2001, Wu established global regularity results for the regularized models with critical or subcritical indices. Also, the proof of Onsager's conjecture (Onsager, 1949) concerning weak solutions of 3D Euler equation and the notion of dissipative solution of (Duchon & Robert, 2000) were extended to the weak solution of the quasi geostrophic equation (Wu, 2002).

Constantin, Cordoba and Wu, in 2001, proved the existence and uniqueness of global classical solutions of the critical dissipative quasi geostrophic equation for the initial data that have small L^∞- norm. Here the importance of an L^∞ smallness condition is due to fact that L^∞ is a conserved norm for the non dissipative quasi geostrophic equation. The norm is non decreasing on all solutions of the dissipative quasi geostrophic equation, irrespective of size (P. Constantin et al., 2001).

Cordoba and Feffermen, in 2002, established that the distance between the two level curves cannot decrease faster than a double exponential time. This collapse assumption weakens the assumptions made in (P. Constantin et al., 1994) for the classical frontogenesis and the simple hyperbolic saddle in (Cordoba, 1998). They discussed two equations, Quasi-Geostrophic equation and two dimensional Euler equation, having common property that a scalar function is convected by flow. This implies that the level curves are transported by the flow (Cordoba & Fefferman, 2002a).

A. Cordoba and D. Cordoba, in 2004, studied the initial value problem for dissipative 2D Quasi Geostrophic equation. They proved the local existence and global results for small initial data in the supercritical case. Also, they studied decay of L^p-norms and asymptotic behavior of viscosity solution in the critical case. Their studies were based on the maximum principle (Córdoba & Córdoba, 2004).

Wu, in 2005, established existence and uniqueness results for the 2D dissipative quasi geostrophic(QG) equation with the initial data in the Besov space or the space created by him which is the generalization of the Besov space, and focused on the critical or supercritical fractional power of the Laplacian for which the dissipation is insufficient to balance the nonlinearity (Wu, 2005).

Dong and Du, in 2008, studied the critical dissipative quasi-geostrophic equation in $\mathbb{R}^2$ with arbitrary H^1 initial data and proved the global wellposedness result by adapting the method in (Kiselev et al., 2007) with a suitable modification and certain decay estimates. They also discussed the decay in time estimate for higher order homogenous Sobolev norms of the solutions (Dong & Du, 2007).

With the use of Little Paley Decomposition and Besov space techniques, the functions were represented in the Hölder space. They showed that if $\theta \in C^{\delta}$, then it also belongs to the Besov space $B_{p,\infty}^{\delta(1-\frac{2}{p})}$ for $p \geq 2$ where Besov space $B^{s,p,\theta}(\mathbb{R}^N)$ is the set of all functions $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ with $||u||_{B^{s,p,\theta}(\mathbb{R}^N)} = ||u||_{L^p(\mathbb{R}^N)} + |u|_{B^{s,p,\theta}(\mathbb{R}^N)} < \infty$ with $1 \leq p, \theta \leq \infty$ and $0 < s < 1$. For the sufficiently large value of p, they showed that the same solution belongs to the space $C^{\delta_1} \cap B^{\delta_1}_{p,\delta}$ for $\delta_1 > 1 - 2\alpha$ and extended the solution to the space $C^{\delta_2} \cap B^{\delta_2}_{p,\delta}$ with $\delta_2 > \delta_1$. Using iteration, they showed that the solution belongs to C^{γ} with $\gamma > 1$ and confirmed that the solution is a classical one (P. Constantin & Wu, 2008).

Dong and Pavlovic, in 2009, established a regularity criterion for weak solutions of the dissipative quasi geostrophic equation (with dissipation $(-\Delta)^{\frac{\gamma}{2}}, 0 < \gamma \leq 1$). More precisely, they proved the following result:

Theorem 4.11. If $\theta \in L_t^{r_o}((0,T); B^{\alpha}_{p,\infty}(\mathbb{R}^2))$ with $\alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r_o}$ is a weak solution of the 2D quasi geostrophic equation, then θ is a classical solution in $(0,T] \times \mathbb{R}^2$.

They extended the regularity result of (P. Constantin & Wu, 2008) to scaling invariant spaces (Dong & Pavlović, 2009).

Caffarelli and Vasseur, in 2010, showed that solution of quasi geostrophic equation with L^2 initial data and critical diffusion $(-\Delta)^2$ are locally smooth for any space dimension and they proved the global regularity for the general data for whole space. The proof is divided into 3 parts, i.e. from L^2 to L^∞, from L^∞ to Hölder and from Hölder to C^∞. In particular, they established the regularity of Leray- Hopf solution by proving the following claims:

1. Every Leray-Hopf weak solution corresponding to initial data $\theta_0 \in L^2$ is in $L^{\infty}_{\text{loc}}(\mathbb{R}^2 \times (0,\infty))$. Note that θ is a Leray Hopf weak solution to initial value problem for the SQG equation on $[0,T)$ if

    (a) $\theta \in L^{\infty}([0,T), L^2(\mathbb{R}^2)) \cap L^2([0,T), H^{\alpha}(\mathbb{R}^2))$, then
    $\int_{\mathbb{R}^2} \theta^2(x,t_2)dx + \int_{t_1}^{t_2} \int_{\mathbb{R}^2} (\Lambda^{\alpha}\theta(x,\tau))^2 dx\, d\tau \leq \int_{\mathbb{R}^2} \theta^2(x,t_1)dx$
    for $0 \leq t_1 < t_2 < T$.

    (b) θ and $u = R(\theta)$ satisfy the SQG equation in distribution sense.

2. The L^{∞} solutions are Hölder regular, i.e. they are in C^{γ} for some $\gamma > 0$, where the Hölder space with exponent γ is the function space

$$C^{k,\gamma}(\overline{\Omega}) = \left\{ u \in C^k(\overline{\Omega}) : ||u||_{C^{k,\gamma}(\overline{\Omega})} < \infty \right\}.$$

3. Every Hölder regular solution is a classical solution in $C^{1,\beta}$ (Caffarelli & Vasseur, 2010).

4.3 Numerical Results

Constantin, Majda,and Tabak, in 1994, performed the numerical experiments on a 2π-periodic box and predicted strong front formation and potential singular behavior of the solutions of the 2D inviscid SQG equation with smooth initial data. They used spectral collocation method with an exponential filter which was basically method developed by (E & Shu, 1993, 1994) for the incompressible flow with minor modifications. Their numerical method monitored two physical quantities, kinetic energy and the pseudo energy. They calculated $u(\theta)$ in the Fourier space and $u.\nabla\theta$ in the physical space and the time stepping through fourth order RK method. They used finer partitions ranging from 256^2 to 512^2 to 1024^2. The following three types of initial data were considered.

1. $\theta(x,0) = \sin x_1 \, \sin x_2 + \cos x_2$

2. $\theta(x,0) = -(\cos 2x_1 \cos x_2 + \sin x_1 \sin x_2)$

3. $\theta(x,0) = \cos 2x_1 \, \cos x_2 + \sin x_1 \, \sin x_2 + \cos 2x_1 \, \sin 3x_2$.

The first initial condition was considered as the simplest type of smooth initial data with nonlinear behavior and also the combination of two lowest eigenmodes.

The first data set involves a hyperbolic saddle in the initial level sets of temperature in the regime of strong nonlinear behavior. The numerical solutions indicate strong nonlinear front formation and potentially singular behavior. The second data set involves the elliptic level sets in θ and the numerical solutions asserts that the solution behaves nonlinearly as in the first set initially but self consistently saturates to exponential growths of gradients without singular behavior. In the third set, the more general initial condition is considered which also indicates the robust feature of strong front formation. The following conclusion is drawn:
"if the geometry of level sets of the active scalar is simple and does not contain a hyperbolic saddle in the region of strongly nonlinear behavior, then no singular behavior is possible" (P. Constantin et al., 1994). Figure 28 shows the evolution of level sets and 3D surface plot of the type (3) data.

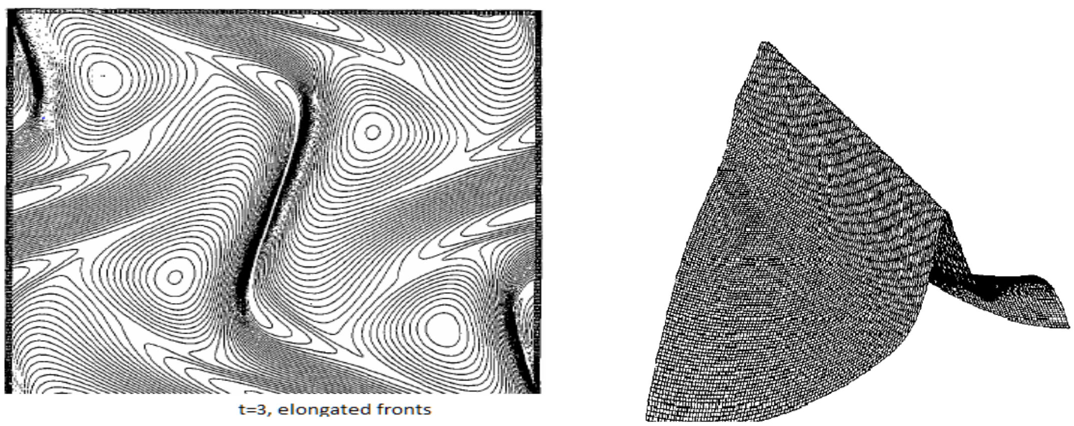

Figure 28: Contour plot and nature of front

Based on the simple initial condition used in (P. Constantin et al., 1994), the temperature gradient can be fitted equally well by a double-exponential function of time rather than an algebraic blow up which was proposed by Ohkitani and Yamada, in 1997. Also for the viscous case, a comparison was made between a series of computations with different Reynolds number. The critical time is found at which the temperature gradient attains the first local maximum depends double logarithmically on the Reynolds number, which suggests the global regularity of the inviscid flow (Ohkitani & Yamada, 1997).

The singular behavior of the solution of inviscid SQG equation when the level sets contain hyperbolic saddle is confirmed by D. Cordoba in 1998. He showed that simple hyperbolic saddle breakdown cannot occur in finite time. His other conclusions are that the angle of the saddle cannot come close in finite time and cannot be faster than a double exponential in time. The analogous results true for incompressible 2D and 3D Euler equations (Cordoba, 1998) can be implied.

Ohkitani and Sakajo, in 2012, studied numerically the long-time evolution of the surface quasi-geostrophic equation with generalized viscosity of the form $(-\Delta)^{\alpha}$, where global regularity has been proved mathematically for the subcritical parameter range $\alpha \geq \frac{1}{2}$. In the supercritical range, they found numerically that smooth evolution persists, but with a slow damping in the long run and also found that the index $\alpha = \frac{1}{2}$ is not the critical case for the numerical study (Ohkitani & Sakajo, 2010).

Authors in (P. Constantin, Lai, Sharma, Tseng, & Wu, 2012) used pseudospectral method with an improved exponential filter. They extended the work of (P. Constantin et al., 1994) and revealed the nature of solution for the longer time interval. The derivatives were calculated in the Fourier space and the products were calculated in the physical space. The time integration was carried out by the 4th order Runge Kutta method. The pseudospectral algorithm was parallelized by the slab decomposition for the purpose of parallel computation. To reduce the aliasing error, they used the exponential filter $f(x) = \exp(-\alpha x^m)$where $\alpha = 36, m = 19$ where $\alpha = -\log \epsilon$, ϵ being the order of machine precision. This helps to suppress the $1/3$ higher frequency modes and $2/3$ modes remains unchanged. The exponential filter they used is shown in Figure 29.

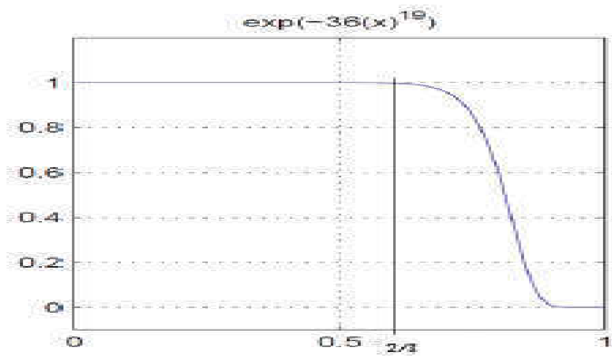

Figure 29: Exponential filter

The formation of strong hyperbolic saddle front is predicted about $t = 7.5$ as in previous calculation (P. Constantin et al., 1994) followed by a steep antiparallel double front while maximum gradient continues to grow up about $t = 13.5$ and then a decay in gradient afterwards with no regeneration of strong fronts. One of the graphs for the gradient growth is presented in Figure 30.

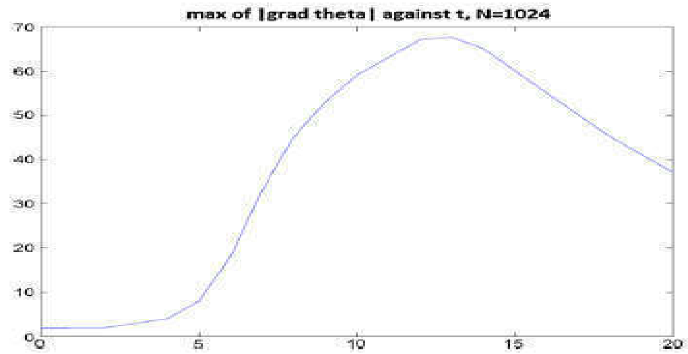

Figure 30: Gradient growth

Their study confirmed that there was no evidence of critical behavior at $\alpha = \frac{1}{2}$ which agreed with the study of (Ohkitani & Sakajo, 2010). Their numerical computation monitored the growth of L^2- norm and helicity. Larger value of N for more finer partition (P. Constantin et al., 2012) were used.

4.3.1 Current Status and Contributions

The numerical studies of SQG equation is discussed. In this regard, two issues in numerical computations are addressed. The first issue is whether there is a finite time blow up in the solution of inviscid SQG equation and the second issue is whether $\alpha = \frac{1}{2}$ is critical for the dissipative 2D SQG equation or not.

(P. Constantin et al., 1994) performed series of numerical computations for the SQG equation. Their experiments are carried in the periodic box $[0, 2\pi] \times [0, 2\pi]$ using the pseudospectral method with the maximum grid size of 1024. Lost of the resolution at $t = 7.5$ is observed during their computation. A possible finite time blow up is proposed and the blow up time was predicted to be $t = 8.25$. A series of computations with same initial data used in (P. Constantin et al., 1994) are performed by (Ohkitani & Yamada, 1997) and the following result is concluded "it is impossible to distinguish between whether there is blow up or not on the computation for the insviscid SQG equation".

With the work done by the pioneers, we are motivated to perform the numerical computation in the finer grids size with the same initial conditions using the pseudospectral method. In order to overcome the problems faced in the time of previous computations, dealiasing rule and also the exponential filter are used. Time integration is carried out by using fourth order Runge- Kutta method.

The conservation laws and others diagnostic test are included. Any singularity in the solutions of inviscid or dissipative SQG equations are not observed opposing the earlier predictions. In our computations $|\nabla^{\perp}\theta|$ is observed to be bounded. The index $\alpha = \frac{1}{2}$ is observed not to be the critical index for the numerical computations even though it was supposed to be the critical index for the analytical approach.

4.3.2 Numerical Simulation of SQG Equation

The numerical method is described which we going to use in our work. Due to the periodic boundary condition, the pseudospectral method appears to be the most appropriate method for this problem. The solution θ is approximated by $\widetilde{\theta}$ of the form

$$\widetilde{\theta}(x,t) = \sum_{k_1,k_2=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{\theta}(k_1,k_2)e^{ikx}$$

where $\hat{\theta}$ denotes the Fourier transform of θ and is given by

$$\hat{\theta}(k_1,k_2) = \frac{1}{(2\pi)^2}\int_{T^2} \theta(x_1,x_2)e^{-i(k_1x_1+k_2x_2)}dx_1dx_2$$

Here N is a fixed number. Also, taking the Fourier transform of SQG equation, the following relation is obtained

$$\partial_t\widehat{\theta(k)} = -ik_1\widehat{(u_1\theta)}(k) - ik_2\widehat{(u_2\theta)}(k) - 2\pi\kappa|k|^{2\alpha}\widehat{\theta}(k)$$

where $k = (k_1,k_2)$ is the wave number and $k_1,k_2 = -\frac{N}{2},\cdots,\frac{N}{2}-1$. The velocity field u computed in Fourier space is given by

$$\hat{u}(k) = i\frac{(-k_2,k_1)}{|k|}\hat{\theta}(k).$$

The product $u_1\theta$ and $u_2\theta$ are computed in the physical space. There is formation of $N\times N$ matrix for the modes θ, i.e. $\hat{\theta}(j_1,j_2,t)$ for $j_1,j_2 = -\frac{N}{2},\cdots,\frac{N}{2}-1$ and is given by $\partial_t\hat{\Theta} = A\hat{\Theta}$. This system of ordinary differential equation is solved by fourth order Runge Kutta method. In this calculation, the $2/3$ dealiasing rule is used.

With the above discussed numerical method, numerical solutions for the inviscid SQG equation and dissipative SQG equation especially in the neighborhood of $\alpha = \frac{1}{2}$ are computed.

The contour plots and power spectrum for different values of time are presented.

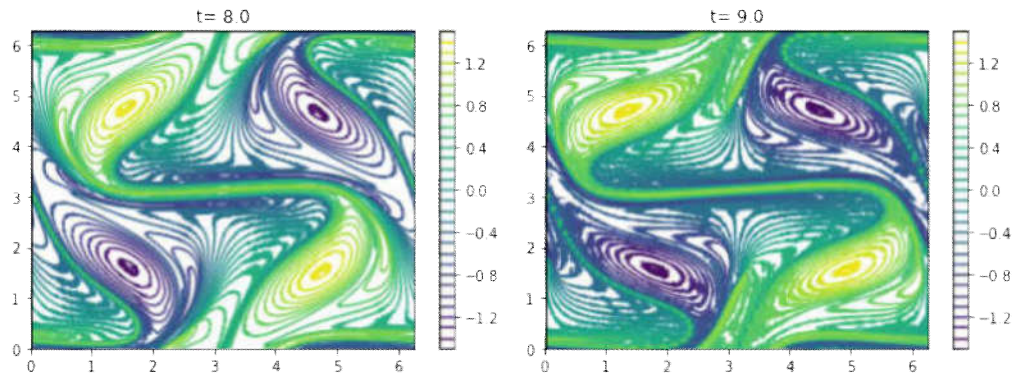

Figure 31: Plots at t = 8,9 inviscid

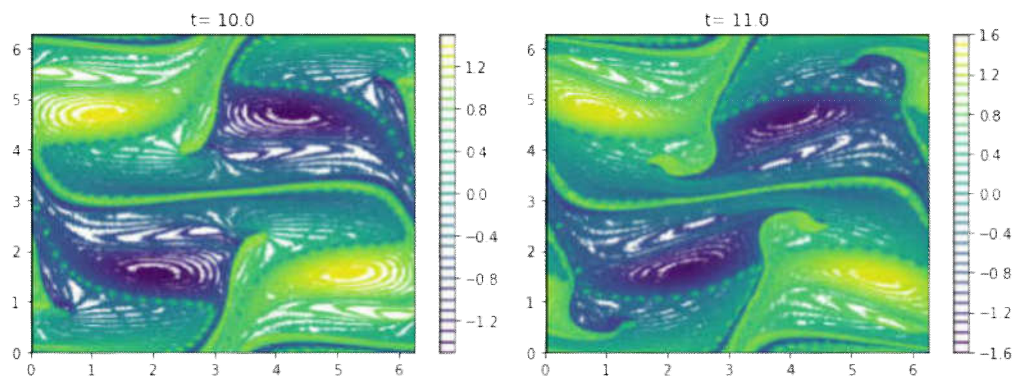

Figure 32: Plots at t = 10,11 inviscid

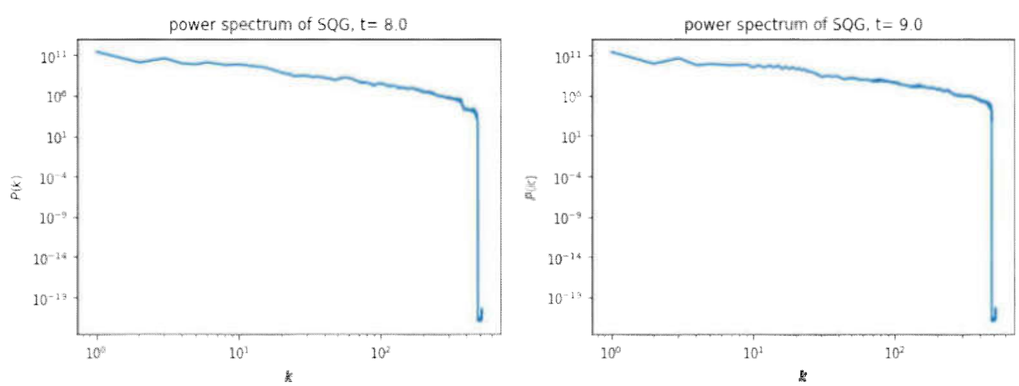

Figure 33: Power spectrum at t = 8,9

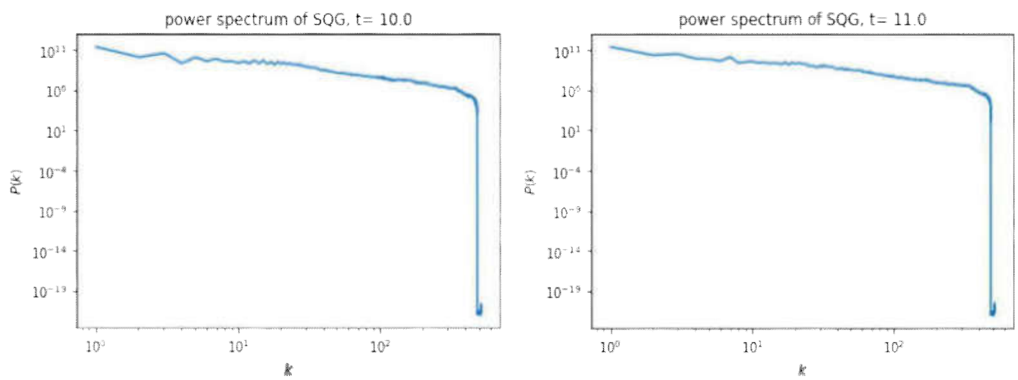

Figure 34: **Power spectrum at t = 10,11**

For value of $\alpha = 0.48$, few plots are presented.

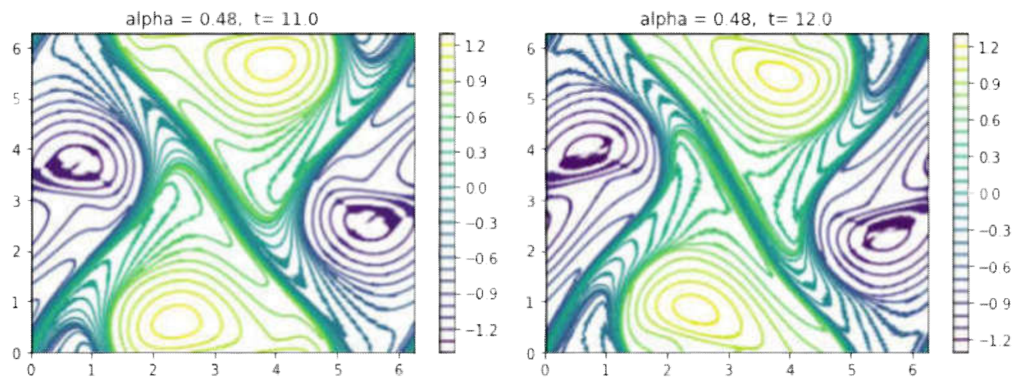

Figure 35: **Contour plot of viscid at t = 11,12**

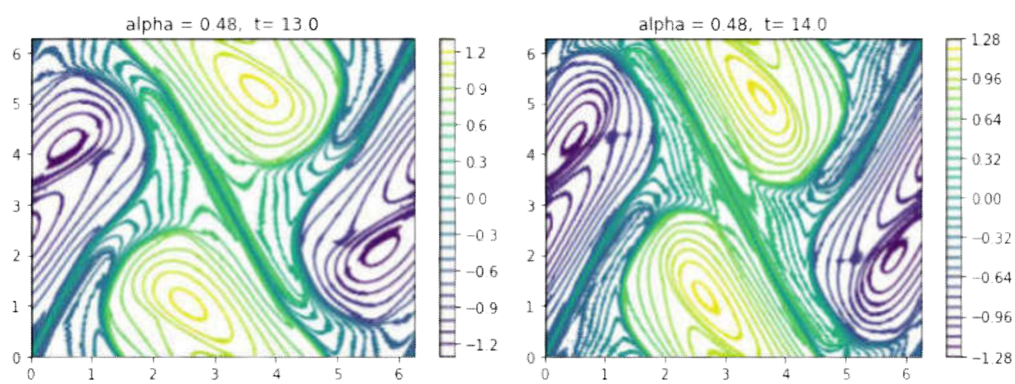

Figure 36: **Contour plot of viscid at t = 13,14**

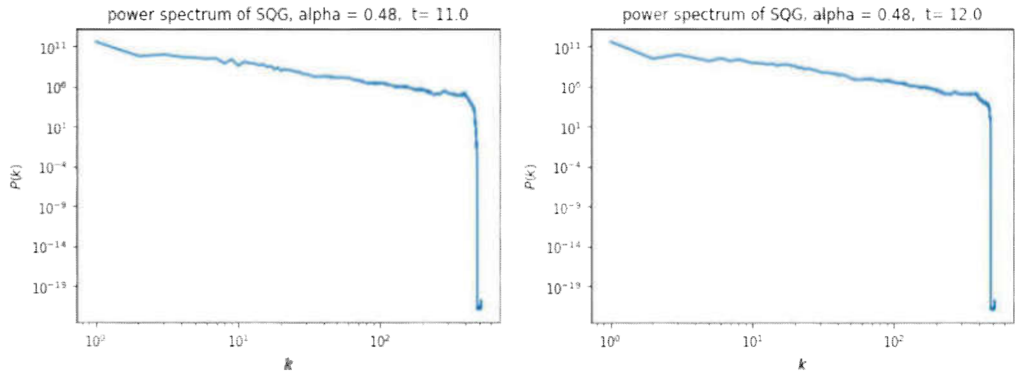

Figure 37: Spectrum at alpha=0.48 at t = 11,12

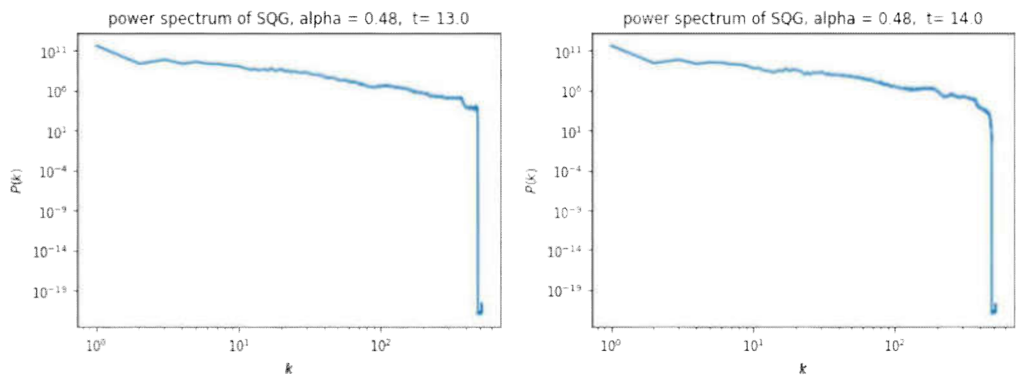

Figure 38: Spectrum at alpha=0.48 at t = 13,14

Some contour plots are presented for $\alpha = 0.51$ with different times.

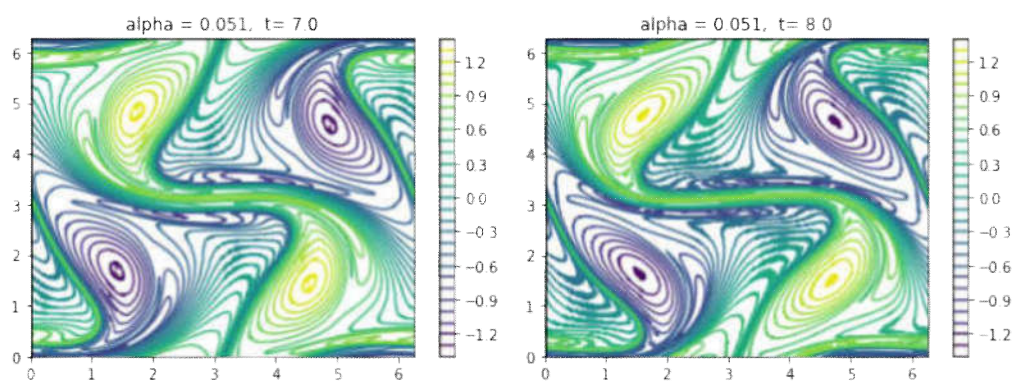

Figure 39: Contour plot of viscid at t = 7,8

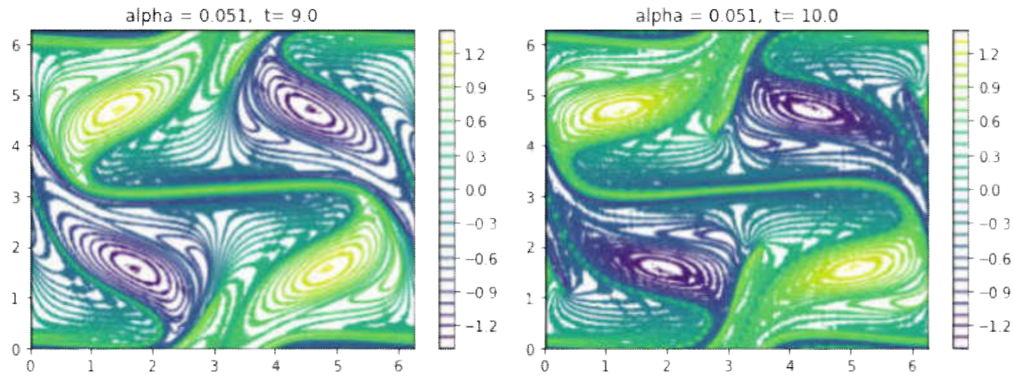

Figure 40: **Contour plot of viscid at t = 9,10**

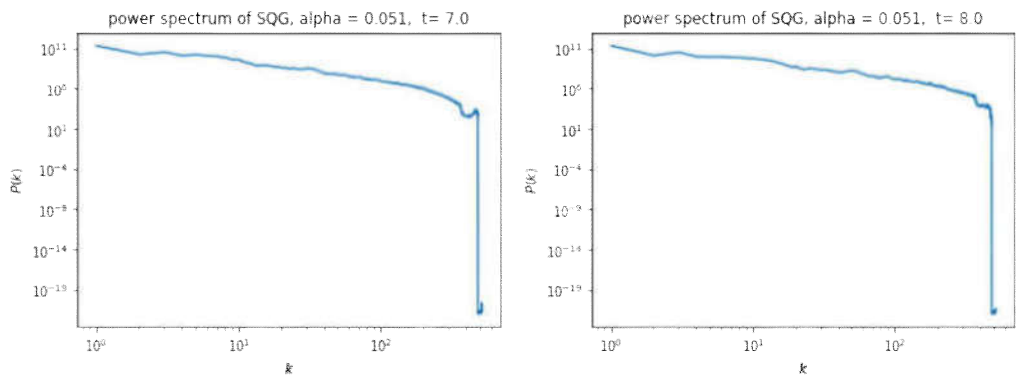

Figure 41: **Power spectrum at t = 7,8**

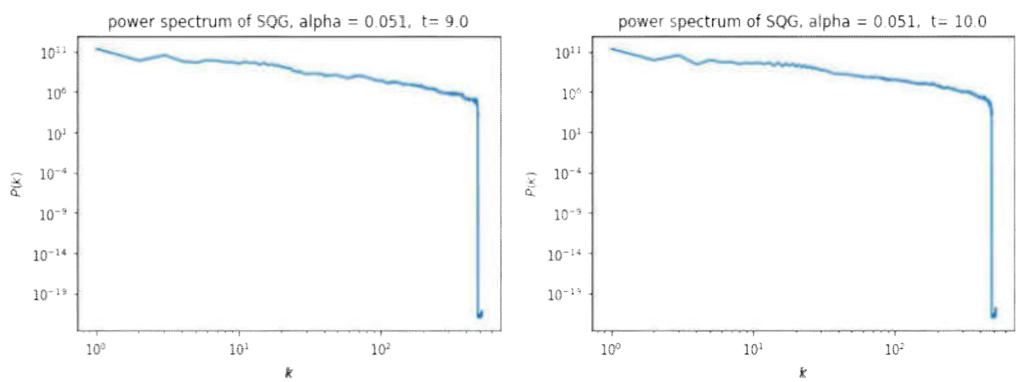

Figure 42: **Power spectrum at t = 9,10**

The numerical results obtained from the second initial condition with varying time and the parameters are presented in Figures. Some of these are presented here.

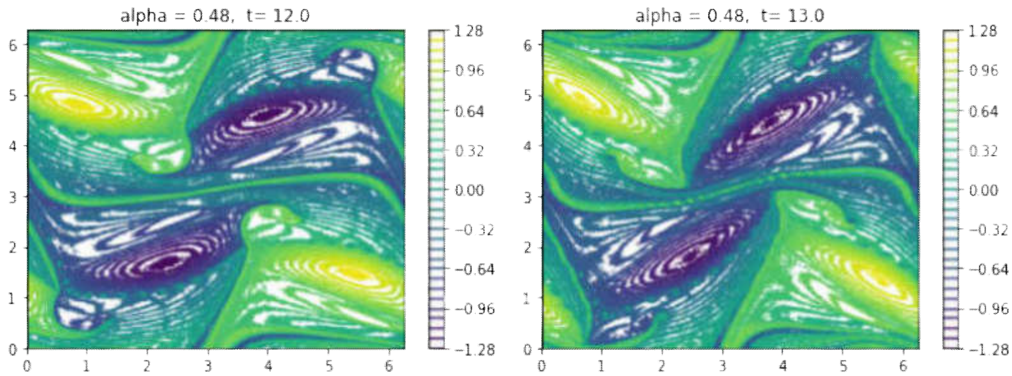

Figure 43: **Contour plot of t = 12 and 13**

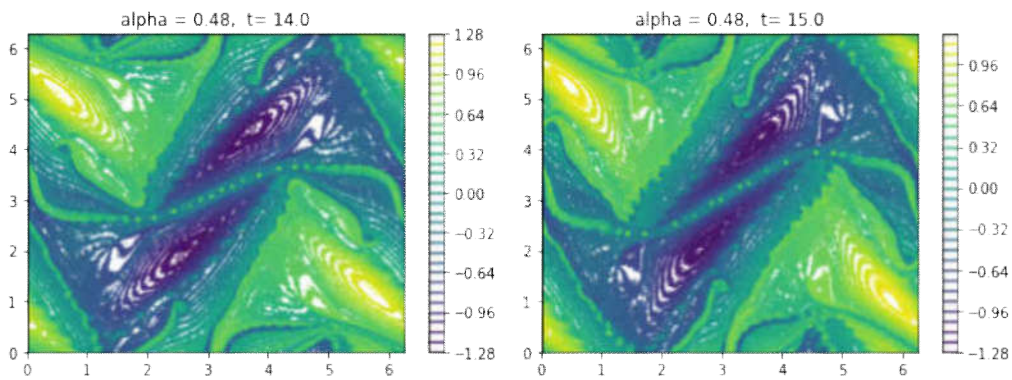

Figure 44: **Contour plot t = 14 and 15**

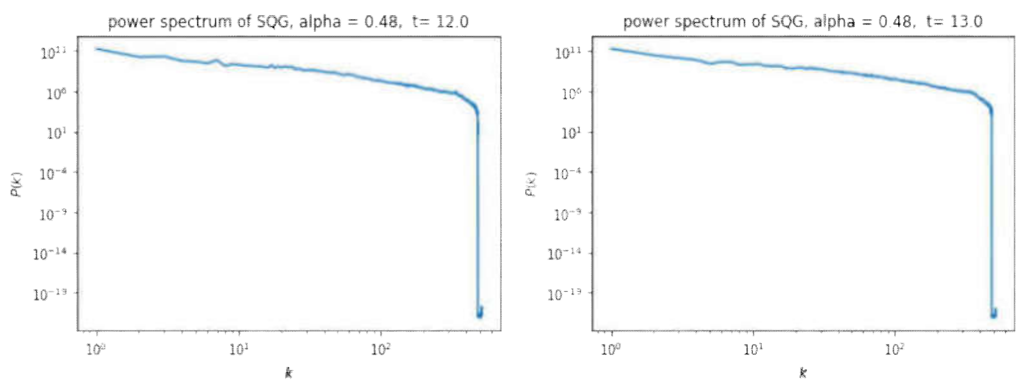

Figure 45: **Power spectrum of t = 12 and 13**

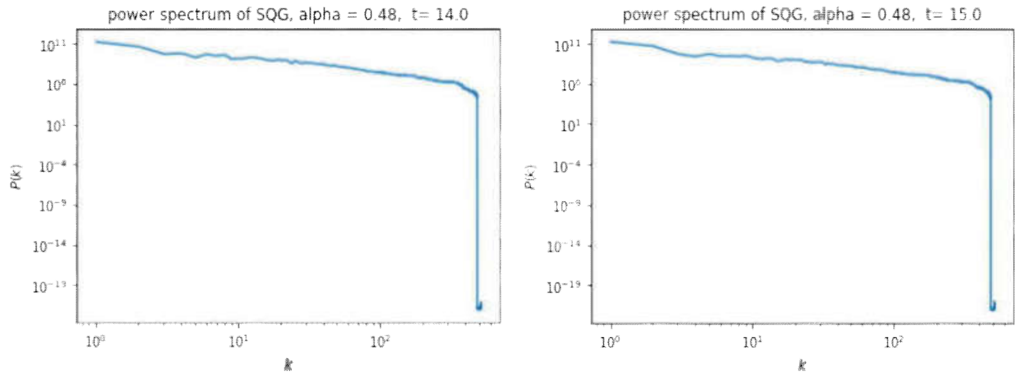

Figure 46: Power spectrum of t = 14 and 15

The following are the plots for $\alpha = 0.49$.

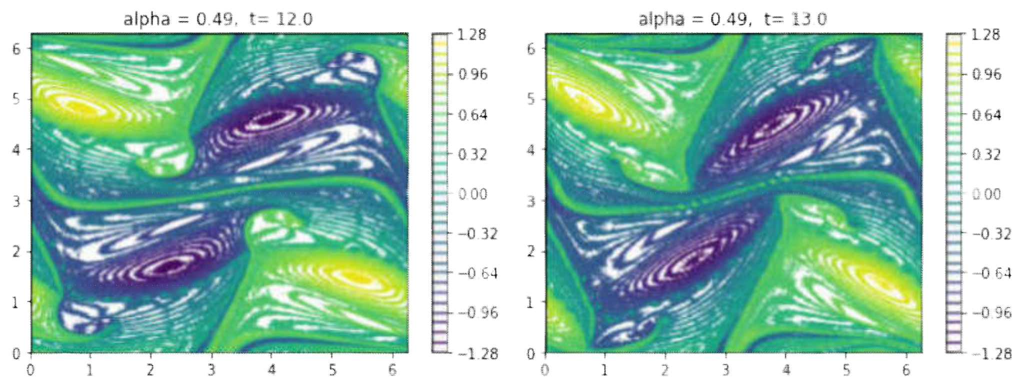

Figure 47: Contour plot of t = 12 and 13

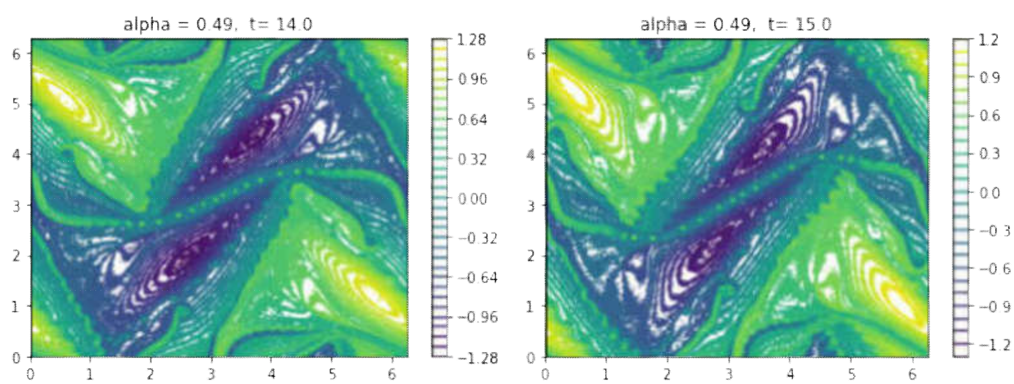

Figure 48: Contour plot of t=14 and 15

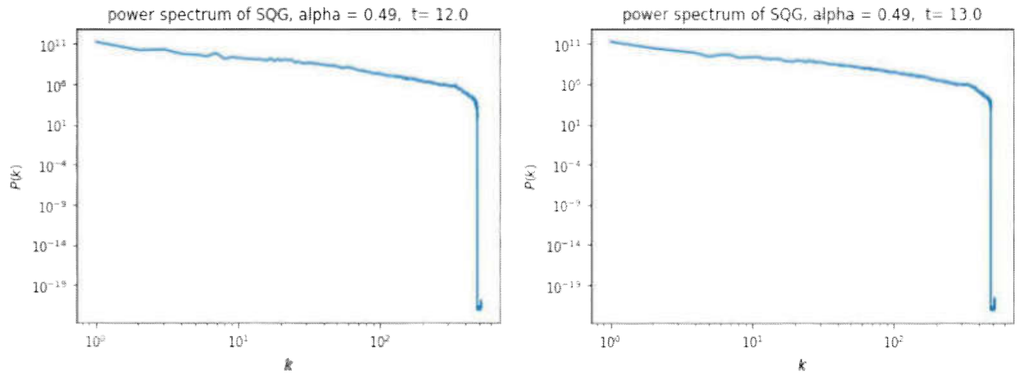

Figure 49: **Power spectrum of t = 12 and 13**

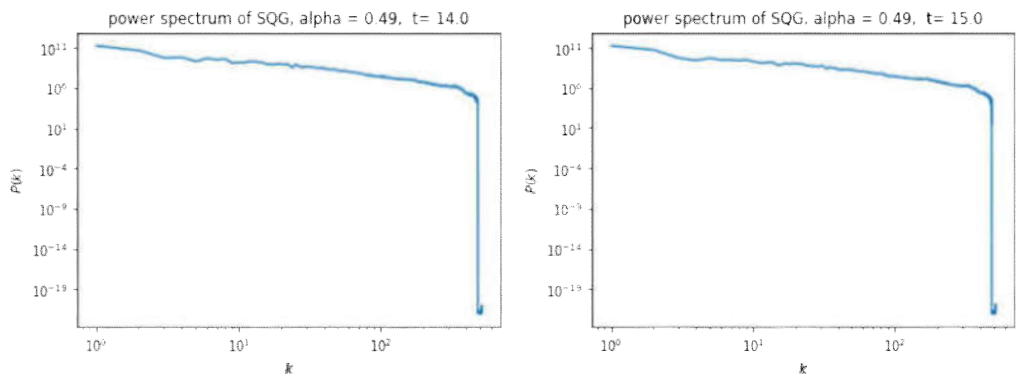

Figure 50: **Power spectrum of t = 14 and 15**

The following are the plots for $\alpha = 0.5$.

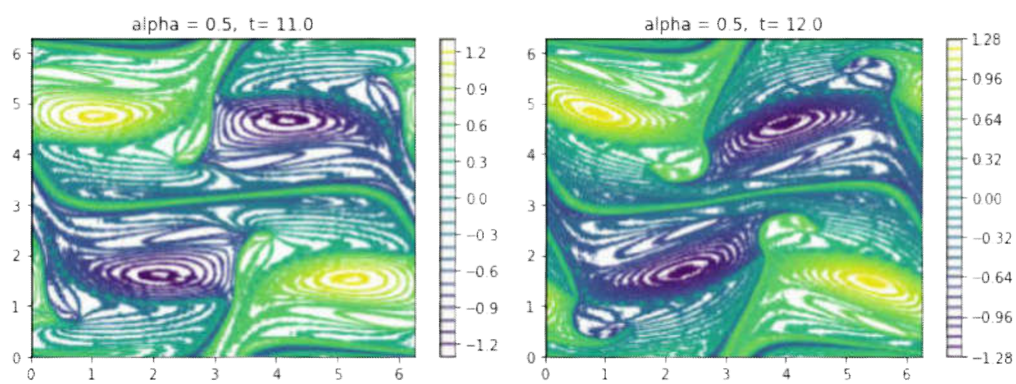

Figure 51: **Contour plot of t = 11 and 12**

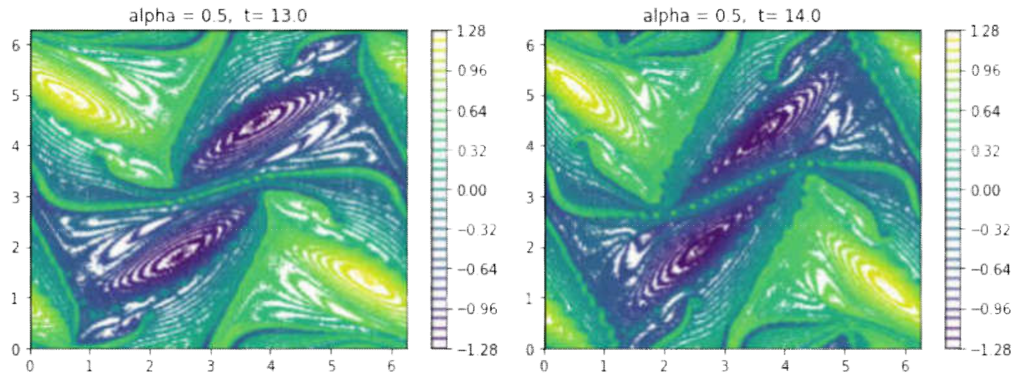

Figure 52: Contour plot of t = 13 and 14

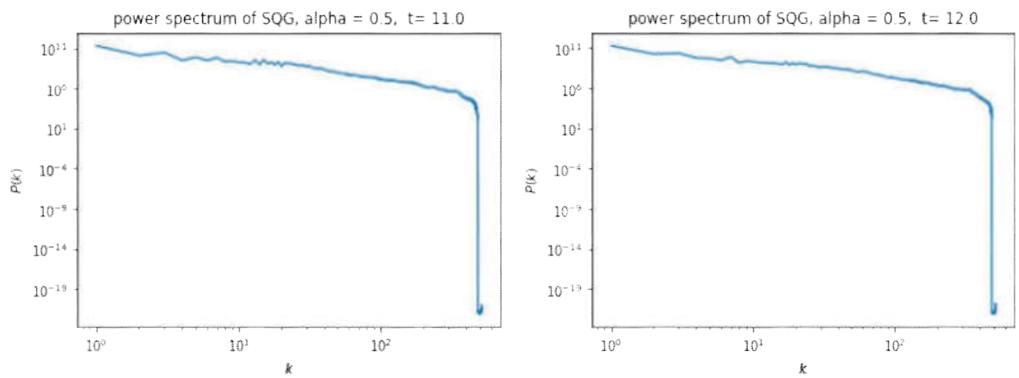

Figure 53: Power spectrum of t = 11 and 12

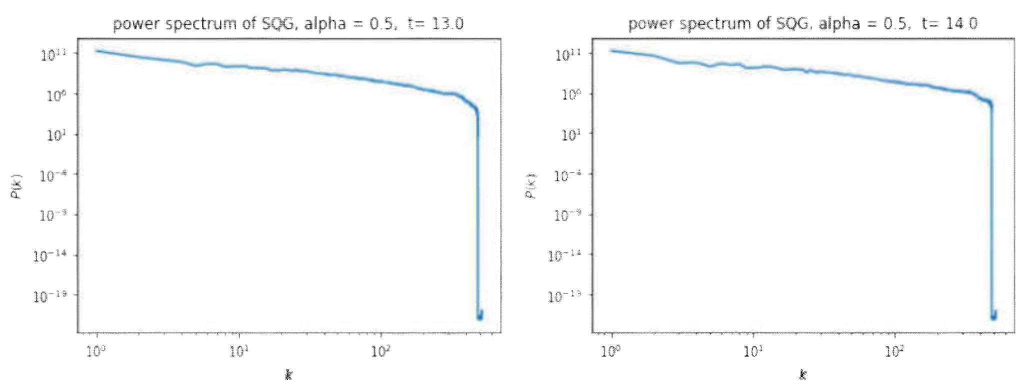

Figure 54: Power spectrum of t = 13 and 14

Table 1 and Table 2 show the values of L^2-norm in the inviscid and dissipative cases.

t	1	2	3	4	5
L2-norm	110.851252	110.851252	110.850575	110.839593	110.797750
t	6	7	8	9	10
L2-norm	110.730007	110.651406	110.594932	110.510071	110.466472

Table 1: Value of L^2-norm of Inviscid case

t	1	2	3	4	5
L2-norm	109.377354	107.916750	106.464917	105.009819	103.534279
t	6	7	8	9	10
L2-norm	102.039835	100.544520	99.060618	97.581038	96.097118

Table 2: Value of L^2-norm of Dissipative case for kappa=0.01

From the above tables, we see that the L^2 norm is conserved in the inviscid case as there is no viscosity. On the other hand, in the dissipative case the value of L^2- norm decreases constantly and energy dissipates continuously which is due to the viscosity.

Chapter 5

Water Wave Equation

5.1 Water Waves

The incompressible Euler equations with a free boundary are well recognized as the governing equations for water waves, and as such, they have been the focus of a variety of studies. The majority of theoretical studies on surface water waves make the assumption that the flow is zero vorticity (irrotational). Stokes investigated irrotational periodic traveling water waves and some of their nonlinear approximations in 1847, (Stokes, 1847). The boundary conditions of waves in uniformly deep water were satisfied by an approximation solution for waves of limited height. In the 1750s, Euler put forward the first thorough mathematical model of a fluid. Mathematicians and engineers continue to use Euler's equations and their variations as fundamental models. The most significant modification was made by Navier and Stokes to accommodate a viscous fluid.

The choice of the Cartesian coordinate system (x, y) places the origin at the mean water level, the horizontal x-axis pointing in the direction of wave propagation, the vertical y-axis pointing upward. In waves that are not disturbed, the flat surface has the equation $y = 0$, while the flat bottom has the equation $y = -d$ for some $d > 0$. When waves are present, the free surface is $y = eta(t, x)$ and the velocity field is $(u(t, x, y), v(t, x, y)$.

The equation of mass conservation for constant density fluid is given by

$$u_x + u_y = 0 \tag{5.1}$$

With the assumption that water is inviscid, the equation of motion is the Euler equation and is given by

$$u_t + uu_x + vu_y = -P_x \tag{5.2}$$

$$v_t + uv_x + vv_y = -P_y - g \tag{5.3}$$

where $P(t, x, y)$ denotes the pressure and g is the gravitational constant of acceleration.

The boundary conditions for the water wave problem are

$$P = P_{\text{atm}} \text{ on } y = \eta(t, x) \tag{5.4}$$

P_{atm} being the constant atmospheric pressure. The kinematic boundary conditions are

$$v = \eta_x + u\eta_y \text{ on } y = \eta(t, x) \tag{5.5}$$

and

$$v = 0 \text{ on } y = -d \tag{5.6}$$

The mathematical study of wave theory within the context of linear theory was initiated in the nineteenth century by (Airy, 1845) and (Stokes, 1847). On a fluid layer that is inviscid and has a constant mean depth, Stokes water waves are nonlinear and periodic surface waves. The progressive periodic waves of permanent form are these waves. These two-dimensional, periodic waves of water travel along the ocean or river's surface.

Stokes observed the actual properties of water waves and then thoroughly researched the nonlinear governing equations for water waves (Stokes, 1847). According to authors in (Benjamin & Olver, 1982; Craig, 1991; Craig & Groves, 1994), the governing equations for the two-dimensional irrotational gravity water waves have a Hamiltonian structure. The incompressible Euler equation with free boundary conditions is regarded as the governing equation for water waves in the majority of investigations. In (Amick, Fraenkel, & Toland, 1982; Buffoni & Toland, 2016; A. Constantin, 2006, 2016), the knowledge of dynamics of waves of enormous amplitude is provided with the aid of a qualitative understanding of periodic traveling waves in an irrotational flow over the flatbed. In (Compelli, Ivanov, & Todorov, 2018), a variety of features of the transmission of irrotational waves of the changeable bottom are covered. The authors of (Basu, 2018; Clamond, 2018; Kogelbauer, 2018) addressed the complete nonlinearity of the governing equations for the water waves and provided insightful quantitative data.

In linear periodic gravity water waves, there are no closed orbits for the water particles in the fluid and the paths are approximately elliptical which was confirmed by Constantin and Villari. They considered the flow up to the finite depth for the kinematic boundary condition (A. Constantin & Villari, 2008). Ehrnstrom and Villari showed that for the positive vorticity, the situation resembles that of Stokes waves. The particle trajectories are affected in particular for large enough vorticity. Also, for the negative vorticity, there is the appearance of internal waves and vortices and the trajectories are not closed ellipse (Ehrnström & Villari, 2008). A mathematical formulation of the water wave problem is presented by the author in (Wahlén, 2009). He gave the existence of result for small amplitude solution based on bifurcation theory. The particle motion in the physical frame is described and obtained the fluid in a moving frame, and confirmed the prediction from the linear theory (Ehrnström & Villari, 2008).

The authors in (A. Constantin & Escher, 2004) showed that the steady periodic deep-water waves are symmetric and propagate against a current with vorticity if their profile is monotone between crests and troughs. They discussed the traveling solution of water wave equation with zero and nonzero vorticity and showed that the symmetry is valid, particularly for the irrotational waves. They also showed that vorticity distribution vanishes at infinite depth. (A. Constantin & Strauss, 2004) constructed the two dimensional inviscid periodic traveling waves with vorticity with the consideration of a free surface under the effect of gravity over a flat bottom. These waves are symmetric and monotone between each crest and trough. They used the bifurcation theory and degree theory for the construction.

The authors in (A. Constantin & Strauss, 2002) constructed the periodic traveling waves with vorticity which are symmetric waves whose profiles are monotone between each crest and trough. For this, they used bifurcation theory and considered the water wave problem described by the Euler equation with a free surface over a flat bottom. Also, the authors in (A. Constantin & Escher, 2011) proved that the profile of a periodic traveling wave propagating on the surface of the water above a flatbed in a flow with real analytic vorticity must be real analytic with the assumption that wave speed exceeds the horizontal fluid velocity throughout the flow.

Recently with a degenerate diffusivity, Eyring-Powell viscosity term, and a Darcy-Forchheimer law in the porous medium, the authors in (Díaz Palencia, Rahman, Redondo, & Roa González, 2022) obtained the traveling wave profile and showed the existence of the asymptotic solution using the geometric perturbation theory. They also showed the existence of an exponential profile of the solution under an asymptotic approximation. Similarly, the authors in (J. Palencia & Rahman, 2022) explored the solutions in the traveling wave domain with the use of geometric perturbation theory. Furthermore, the authors in (J. L. D. Palencia, ur Rahman, & Naranjo, 2022) transformed the problem into the study of traveling wave kind solutions. With the use of geometric perturbation theory, they confirmed the existence of an exponentially decaying rate in the traveling wave profile. Both the authors in (J. Palencia & Rahman, 2022; J. L. D. Palencia et al., 2022) performed numerical simulations to validate their results.

The authors in (Díaz et al., 2022) explored the regularity, existence, and uniqueness of the solutions with the use of variational weak formulations. Then they transformed the Eyring-Powell equation into a traveling wave domain where they obtained the analytical solution with the use of geometric perturbation theory. Their main work was to show the existence of an exponential traveling wave tail together with a certain minimizing error critical speed.

All of these existing results apply only to the finite boundary, but this problem remains open for the infinite boundary. With the ideas in (A. Constantin & Escher, 2011; A. Constantin & Strauss, 2002), and (A. Constantin & Strauss, 2004), and using Crandall Rabinowitz theorem, we created the water wave profile by considering the kinematic boundary condition up to infinity. We studied this as an extension of the flatbed at a finite depth to an infinite depth. Our findings will be useful for the study of the extension of the solutions from finite boundary conditions to infinite boundary conditions. Similar concepts can be used in other situations while dealing with problems in fluid dynamics with infinite boundaries.

5.2 Equation of Motion

The equation of motion of the wave is given by

$$\partial_t \vec{u} + (\vec{u}.\nabla)\vec{u} + \nabla P = -g\vec{e_y}, \text{ div } \vec{u} = 0 \tag{5.7}$$

where u is the velocity, P is the pressure, and g is the acceleration due to gravity.

The related boundary conditions are :

1. Kinematic: The free surface with moving fluid is given by $\eta_t = \sqrt{1+(\nabla\eta)^2}\ \vec{u}\cdot\vec{n}$ where $\vec{n}$ is the unit normal to surface.

2. Dynamic: The dynamic boundary condition is the balance of forces at the free surface and is given by $P = P_{atm}$, the atmospheric pressure.

3. Bottom: At the bottom the second component of the fluid is zero. It just moves horizontally i.e.

$$\lim_{y\to-\infty} \vec{e_y}.\vec{u} = 0.$$

With these, the equation of motion is given by: $\Delta\phi = 0$ in D_η, $\phi = \psi$ at $y = \eta(t,x)$, and

$$\lim_{y\to-\infty} \partial_y\phi = 0.$$

Here ϕ is the velocity potential function, D_η is the domain of the fluid motion, $\eta(t,x)$ is the free surface which describes a wave on the bottom of the fluid dependent on time t and $\psi(t,x)$ is the value of potential at the free surface.

5.3 Traveling Waves for Water Waves

In this section, the water waves for the two cases with zero vorticity and constant vorticity are discussed.

5.3.1 Traveling Waves for Water Waves with Zero Vorticity

The water waves formulation (A. Constantin, Ivanov, & Prodanov, n.d.; Wahlén, 2007) is given by:

$$\eta_t = G(\eta)[\psi] \text{ and } \psi_t = -g\eta - \frac{{\psi_x}^2}{2} + \frac{1}{2}\frac{(G(\eta)\psi + \eta_x\psi_x)^2}{1+{\eta_x}^2}$$

where, $G(\eta)[\psi]$ is a Dirichlet Neumann operator (Nicholls & Reitich, 2003; Wilkening & Vasan, 2015) and is given by

$$G(\eta)\psi = \sqrt{1+{\eta_x}^2}\ \partial_{\vec{n}}\phi|_{y=\eta(x)}.$$

Here, we are looking for the small size traveling wave periodic in x. That means the solution is expected in the form : $\eta(t,x) = \breve{\eta}(x+ct)$ and $\psi(t,x) = \breve{\psi}(x+ct)$, where c

is the phase velocity. These solutions must satisfy the equations

$$
\begin{aligned}
c\eta_x &= G(\eta)[\psi] && (5.8)\\
c\psi_x &= -g\eta - \frac{{\psi_x}^2}{2} + \frac{1}{2}\frac{(G(\eta)\psi + \eta_x\psi_x)^2}{1+{\eta_x}^2}. && (5.9)
\end{aligned}
$$

For this, we look for the zeros of the equation which indicate the situation with zero vorticity is

$$
F(c,\eta,\psi) = \begin{pmatrix} -c\eta_x + G(\eta)[\psi] \\ c\psi_x + g\eta + \frac{{\psi_x}^2}{2} - \frac{1}{2}\frac{(G(\eta)\psi+\eta_x\psi_x)^2}{1+{\eta_x}^2} \end{pmatrix}. \tag{5.10}
$$

When $\eta = 0, \psi = 0$, equation (5.10) takes the form $F(c, 0, 0) = 0 \quad \forall c$, which is a bifurcation problem. To solve this type of bifurcation problem, the Crandall Rabinowitz Theorem (Crandall & Rabinowitz, 1971) is applied. For this, we must have:

1. $d_{(\eta,\psi)}F(c^*, 0, 0)$ has 1 dimensional kernel.
2. Im $d_{(\eta,\psi)}F(c^*, 0, 0)$ is closed and has codimension 1.
3. It must satisfy the transversability condition i.e. $\partial_{c,(\eta,\psi)}F(c^*, 0, 0)[(\eta^*, \psi^*)] \notin R$ where $(\eta^*, \psi^*) \in$ Ker$d_{(\eta,\psi)}F(c^*, 0, 0)$ where Ker$d$ is the kernel of d and R is the image set.

By linearizing equation (5.10) at $\eta = 0, \psi = 0$, we have

$$
d_{(\eta,\psi)}F(c, 0, 0)[(\hat{\eta}, \hat{\psi})] = \begin{pmatrix} -c\hat{\eta}_x + G(0)\hat{\psi} \\ c\hat{\psi}_x + g\hat{\eta} \end{pmatrix}. \tag{5.11}
$$

The series representation for $\hat{\eta}$ and $\hat{\psi}$ are assumed as

$$
\hat{\eta}(x) = \sum_{n\geq 0} \eta_n \cos nx
$$

and

$$
\hat{\psi}(x) = \sum_{n\geq 0} \psi_n \sin nx
$$

which are even and odd respectively together with the norms

$$
||\eta||_s^2 = \sum n^{2s}|\eta_n|^2
$$

and

$$
||\psi||_s^2 = \sum n^{2s}|\psi_n|^2
$$

which are the norms defined on the homogeneous Sobolev space H^s.

With these series representations, we have

$$d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \sum_{n\geq 1}\begin{pmatrix} (cn\eta_n + n\psi_n)\sin nx \\ (cn\psi_n + g\eta_n)\cos nx \end{pmatrix} + \begin{pmatrix} 0 \\ g\eta_0 \end{pmatrix}. \tag{5.12}$$

Define

$$f = \sum_{n\geq 0} f_n \cos nx, \quad g = \sum_{n\geq 0} g_n \sin nx.$$

For these g and f; we solve $d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \begin{pmatrix} g \\ f \end{pmatrix}$.

Now for $n \geq 1$, we have

$$\begin{pmatrix} cn & n \\ g & cn \end{pmatrix}\begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}.$$

We note that det$M_n(c) = 0$ iff $c = \pm\sqrt{\frac{g}{n}}$, where $M_n = \begin{pmatrix} cn & n \\ g & cn \end{pmatrix}$. For fix n_0 and choosing $c_{n_0} = \sqrt{\frac{g}{n_0}}$, the n_0-matrix has kernel. Now for c_{n_0}, we have

$$\begin{pmatrix} c_{n_0} n_0 & n_0 \\ g & c_{n_0} n_0 \end{pmatrix}\begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} c_{n_0} \\ -\frac{g}{n_0} \end{pmatrix}$$

Here we note that if $n_0 = 0$ then the inverse of that matrix $\begin{pmatrix} c_{n_0} n_0 & n_0 \\ g & c_{n_0} n_0 \end{pmatrix}$ does not exist. So the matrix becomes singular and hence the solution is not possible.

Therefore, $Kerd_{(\eta,\psi)}F(c_{n_0},0,0) = \begin{pmatrix} c_{n_0}\cos n_0 x \\ -\frac{g}{n_0}\sin n_0 x \end{pmatrix}$, which is of dimension 1.

Again

$$\begin{pmatrix} 0 \\ -g\eta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ f_0 \end{pmatrix} \text{ and } \begin{pmatrix} cn & n \\ g & cn \end{pmatrix}\begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}; n \geq 1.$$

This implies two possibilities, $n = n_0$ or $n \neq n_0$. In both cases, it can be shown that $\eta, \psi \in H^s$. Again for $n = n_0$, choosing $g_{n_0} = 1$, we have

$$R = Imd_{(\eta,\psi)}F(c_{n_0},0,0) = \sum_{n\neq n_0}\begin{pmatrix} g_n \sin nx \\ f_n \cos nx \end{pmatrix} + \begin{pmatrix} \sin n_0 x \\ c_{n_0}\cos n_0 x \end{pmatrix}.$$

But, this is perpendicular to $R^{\perp} = \begin{pmatrix} c_{n_0} \sin n_0 x \\ -\cos n_0 x \end{pmatrix}$. This shows that the range is closed and has codimension 1. Also, the transversability condition is satisfied as

$$\partial_{c,(\eta,\psi)} F(c_{n_0}, 0, 0)[(\eta^*, \psi^*)] = \begin{pmatrix} -c_{n_0} n_0 \sin n_0 x \\ -g \cos n_0 x \end{pmatrix} \notin R.$$

As all the conditions for the Crandall Rabinowitz theorem (Crandall & Rabinowitz, 1971) are satisfied, we have the following theorem:

Theorem 5.1. Let $s > \frac{5}{2}$ and $n_0 \in N : \exists \epsilon_0 > 0$ and C^1 functions $C_\epsilon : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$; $\begin{pmatrix} \breve{\eta}_\epsilon \\ \breve{\psi}_\epsilon \end{pmatrix} : (-\epsilon_0, \epsilon_0) \to H^s \times H^s$ such that $\begin{pmatrix} \breve{\eta}(x-ct) \\ \breve{\psi}(x-ct) \end{pmatrix}$ solves the water wave equation. Moreover, $C_\epsilon = C_{n_0} + 0(\epsilon)$, $\begin{pmatrix} \breve{\eta}_\epsilon(x) \\ \breve{\psi}_\epsilon(x) \end{pmatrix} = \epsilon \begin{pmatrix} c_{n_0} \cos n_0 x \\ -\frac{g}{n_0} \sin n_0 x \end{pmatrix} + O(\epsilon^2)$.

Thus, we have constructed traveling waves of the form

$$\eta(t,x) = \epsilon c_{n_0} \cos n_0 (x - c_\epsilon t) + O(\epsilon^2).$$

5.3.2 Traveling Waves for Water Waves with Constant Vorticity

The water waves formulation (A. Constantin et al., n.d.; Wahlén, 2007) is given by:

$\eta_t = G(\eta)[\psi] + \gamma\eta\eta_x$ and $\psi_t = -g\eta - \frac{{\psi_x}^2}{2} + \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x)^2} + \gamma\eta\psi_x + \gamma\partial_x^{-1} G(\eta)\psi$

where $G(\eta)[\psi]$ is a Dirichlet Neumann operator (Nicholls & Reitich, 2003; Wilkening & Vasan, 2015).

Here, we are aimed to find small size traveling waves periodic in x. That means we expect the solution of the form $\eta(t,x) = \breve{\eta}(x+ct)$ and $\psi(t,x) = \breve{\psi}(x+ct)$, where c is the phase velocity.

These solutions must satisfy the equations

$$\begin{aligned} c\eta_x &= G(\eta)[\psi] + \gamma\eta\eta_x && (5.13) \\ c\psi_x &= -g\eta - \frac{{\psi_x}^2}{2} + \frac{1}{2}\frac{G(\eta)\psi + \eta_x{\psi_x}^2}{1+{\eta_x}^2} + \gamma\eta\psi_x + \gamma\partial_x^{-1} G(\eta)\psi. && (5.14) \end{aligned}$$

For this, we look for the zeros of the equation which indicate the situation with constant vorticity

$$F(c,\eta,\psi) = \begin{pmatrix} -c\eta_x + G(\eta)[\psi] + \gamma\eta\eta_x \\ c\psi_x + g\eta + \frac{{\psi_x}^2}{2} - \frac{1}{2}\frac{G(\eta)\psi + \eta_x{\psi_x}^2}{1+{\eta_x}^2} - \gamma\eta\psi_x - \gamma\partial_x^{-1} G(\eta)\psi \end{pmatrix}. \quad (5.15)$$

When $\eta = 0,\ \psi = 0$, equation (5.15) takes the form $F(c, 0, 0) = 0;\ \forall\ c$, which is a bifurcation problem. Here also, Crandall Rabinowitz Theorem (Crandall & Rabinowitz, 1971) is applied to solve this type of bifurcation problem. For this, we must have:

1. $d_{(\eta,\psi)}F(c^*, 0, 0)$ has 1 dimensional kernel.
2. Im $d_{(\eta,\psi)}F(c^*, 0, 0)$ is closed and has codimension 1.
3. It must satisfy the transversability condition i.e. $\partial_{c,(\eta,\psi)}F(c^*, 0, 0)[(\eta*, \psi*)] \notin R$ where $(\eta^*, \psi^*) \in$ Ker$d_{(\eta,\psi)}F(c^*, 0, 0)$ where Ker$d$ is the kernel of d and R is image set.

By linearizing equation (5.15) at $\eta = 0, \psi = 0$, we have

$$d_{(\eta,\psi)}F(c, 0, 0)[(\hat{\eta}, \hat{\psi})] = \begin{pmatrix} -c\hat{\eta}_x + G(0)\hat{\psi} \\ c\hat{\psi}_x + g\hat{\eta} - \gamma\partial_x^{-1}|D|\psi \end{pmatrix} \tag{5.16}$$

We assume the series representation for $\hat{\eta}$ and $\hat{\psi}$ as

$$\hat{\eta}(x) = \sum_{n\geq 0} \eta_n \cos nx \ \text{ and } \ \hat{\psi}(x) = \sum_{n\geq 0} \psi_n \sin nx,$$

which are even and odd respectively together with the norms

$$||\eta||_s^2 = \sum n^{2s}|\eta_n|^2$$

and

$$||\psi||_s^2 = \sum n^{2s}|\psi_n|^2.$$

These are the norms defined on homogeneous Sobolev space H^s.

With these series representation, we have

$$d_{(\eta,\psi)}F(c, 0, 0)[(\hat{\eta}, \hat{\psi})] = \sum_{n\geq 1} \begin{pmatrix} (cn\eta_n + n\psi_n)\sin nx \\ ((cn + \gamma)\psi_n + g\eta_n)\cos nx \end{pmatrix} + \begin{pmatrix} 0 \\ (cn_0 + \gamma)\psi_0 + g\eta_0 \end{pmatrix} \tag{5.17}$$

Assuming

$$g = \sum_{n\geq 0} g_n \sin nx \text{ and } f = \sum_{n\geq 0} f_n \cos nx\ ;$$

we solve $d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \begin{pmatrix} g \\ f \end{pmatrix}$.

Then we have $\begin{pmatrix} cn & n \\ g & cn+\gamma \end{pmatrix} \begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}; n \geq 1.$

We note that det$M_n(c) = 0$ iff $c = -\frac{\gamma}{2n} \pm \sqrt{\frac{\gamma^2}{4n^2} + \frac{g}{n}}$, where $M_n = \begin{pmatrix} cn & n \\ g & cn+\gamma \end{pmatrix}$.

For fixed n_0 and choosing $c_{n_0} = -\frac{\gamma}{2n_0} \pm \sqrt{\frac{\gamma^2}{4n_0^2} + \frac{g}{n_0}}$, the n_0-matrix has kernel.

Now

$$\begin{pmatrix} c_{n_0}n_0 & n_0 \\ g & c_{n_0}n_0+\gamma \end{pmatrix} \begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} c_{n_0} + \frac{\gamma}{n_0} \\ -\frac{g}{n_0} \end{pmatrix}.$$

Therefore, Ker$d_{(\eta,\psi)}F(c_{n_0},0,0) = \begin{pmatrix} (c_{n_0} + \frac{\gamma}{n_0}) \cos n_0 x \\ -\frac{g}{n_0} \sin n_0 x \end{pmatrix}$, which is of dimension 1.

Again,

$$\begin{pmatrix} 0 \\ -g\eta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ f_0 \end{pmatrix} \text{and} \begin{pmatrix} cn & n \\ g & cn+\gamma \end{pmatrix} \begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}; n \geq 1.$$

This implies two possibilities $n = n_0$ or $n \neq n_0$. In both the cases, it can be shown that $\eta, \psi \in H^s$. Again for $n = n_0$ and choosing $g_{n_0} = 1$, we have

$$R = \text{Imd}_{(\eta,\psi)}F(c_{n_0},0,0) = \sum_{n \neq n_0} \begin{pmatrix} g_n \sin nx \\ f_n \cos nx \end{pmatrix} + \begin{pmatrix} \sin n_0 x \\ (c_{n_0} + \frac{\gamma}{n_0}) \cos n_0 x \end{pmatrix},$$

which is perpendicular to $R^{\perp} = \begin{pmatrix} -(c_{n_0} + \frac{\gamma}{n_0}) \sin n_0 x \\ \cos n_0 x \end{pmatrix}$.

This shows that the range is closed and has codimension 1. Also, the transversability condition is satisfied as

$$\partial_{c,(\eta,\psi)}F(c_{n_0},0,0)[(\eta^*,\psi^*)] = \begin{bmatrix} -(c_{n_0} + \frac{\gamma}{n_0})n_0 \sin n_0 x \\ -g \cos n_0 x \end{bmatrix} \notin R.$$

With the help of the Crandall Rabinowitz theorem (Crandall & Rabinowitz, 1971), we have obtained the following theorem:

Theorem 5.2. Let $s > \frac{5}{2}$ and $n_0 \in N$: $\exists \epsilon_0 > 0$ and C^1 functions $C_\epsilon : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$; $\begin{pmatrix} \breve{\eta}_\epsilon \\ \breve{\psi}_\epsilon \end{pmatrix}$: $(-\epsilon_0, \epsilon_0) \to H^s \times H^s$ such that $\begin{pmatrix} \breve{\eta}(x-ct) \\ \breve{\psi}(x-ct) \end{pmatrix}$ solves the water wave equation.

Moreover, $C_\epsilon = C_{n_0} + 0(\epsilon)$, $\begin{pmatrix} \breve{\eta}_\epsilon(x) \\ \breve{\psi}_\epsilon(x) \end{pmatrix} = \epsilon \begin{pmatrix} (c_{n_0} + \frac{\gamma}{n_0}) \cos n_0 x \\ -\frac{g}{n_0} \sin n_0 x \end{pmatrix} + O(\epsilon^2)$.
Thus, we have constructed traveling waves of the form

$$\eta(t,x) = \epsilon(c_{n_0} + \frac{\gamma}{n_0}) \cos n_0(x - c_\epsilon t) + O(\epsilon^2).$$

Chapter 6

Summary and Conclusions

6.1 Summary

Our thesis is organized in the following ways. Introduction of our research work together with objective and rationale included in Chapter 1. Basic terminologies, definition of different function spaces, weak solution of PDEs are given in the first part of Chapter 2. Following this, short descriptions of Navier Stokes Equation, 3D Quasi Geostrophic Equation, Surface Quasi Geostrophic(SQG) equation, Euler Equation, KdV equation and Water Wave equation, respectively are given. The many numerical techniques are finally explored, including the Discrete Fourier Transform, Fast Fourier Transform, Spectral Method, Pseudospectral Method, and numerical solution of partial differential equations (PDEs) utilizing spectral techniques.

The KdV equation is discussed in chapter 3 in detail up to a current status. An analytical solution of the KdV equation is discussed. Following it, the spectral method to solve generalized KdV type equation is discussed. After this, numerical computations by varying the coefficients are given in which there is a comparison of two different cases, one when the coefficients are exactly zero and the other coefficients are zero in the sense of limit. With our study, it is found that there is no significant difference between these two different situations.

The Surface Quasi-Geostrophic (SQG) equation is discussed in detail in chapter 4. The inviscid and dissipative cases of SQG equations are discussed. After that, subcritical,

critical, and supercritical cases are discussed. Some theorems regarding the regularity of solution of the equation are also presented for those particular cases. The numerical method which is used to solve the SQG equation is explained after this. The numerical computations are performed on the basis of (P. Constantin et al., 2012, 1994) and the index $\alpha = \frac{1}{2}$ is observed not to be the critical index from the numerical point of view, even though it was supposed to be critical index in the analytical study. Some plots of level sets and power spectrum at various times and for different values of parameters are presented at the last part of this chapter.

The water flow with the help of the Euler equation with different boundary conditions is described in Chapter 5. The finite boundary condition is extended to infinite, and the traveling wave solutions are created for the case of zero and constant vorticity for the Newtonian fluid through the application of Crandall Rabinowitz theorem. Chapter 6 concludes the dissertation.

6.2 Conclusions

The general type KdV equation is studied thoroughly. The analytical solution of KdV equation is also discussed. The particular values of constants are chosen so that the generalized KdV equation is converted into transport, burger and KdV equations. The pseudospectral method is used to observe the nature of the solution numerically when the value of the constant is exactly zero and when the constant is zero in the sense of limit. Similar results and structures of the solution are observed in both the cases.

The current status of the SQG equation is reviewed to know about the possibility of further extension of the existing result to the 3D Euler equation and hence the Navier Stokes equation. The analytical and numerical results for the viscid and inviscid cases of SQG equation are discussed. The pseudospectral method is used for numerical computations. The velocity field is calculated in the Fourier space, whereas the product of velocity and potential temperature are computed in the physical space. The time integration is carried out through the fourth order Runge-Kutta method. Our computation is performed to the time $t = 15$, which was earlier done for $t = 13.5$. The value of L^2-norm and helicity are monitored continuously for the validity of our result. The $2/3$

dealiasing rule and exponential filter are used to avoid the aliasing error. The same initial conditions as in (P. Constantin et al., 1994, 2012) are used. The index $\alpha = \frac{1}{2}$ is confirmed not to be the critical index for the numerical study even though it was supposed to be the critical index in the analytical study.

The various development of the water waves and related facts such as the wave profile, and path of the wave in the finite bottom conditions are studied. The finite depth is extended to the infinite depth condition in which there is construction of the water wave profile for the zero and constant vorticity conditions for the Newtonian fluid in an analytical way with the use of the Crandall Rabinowitz theorem.

6.3 Recommendations for further work

The numerical study of inviscid SQG and dissipative SQG equations are done up to the time $t = 15$ in two dimensional case monitoring two special values L^2 - norm and helicity and the time where the level curves come closer is observed. The further work can be done for the same equation to longer interval of time as well as in three dimensional case.

Likewise, we have created traveling water waves for the kinematic boundary condition up to infinite depth using Crandall Rabinowitz theorem for the Newtonian fluid. The same work may be extended to Non-Newtonian fluid.

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Appendix A

A.1 Certificates of Course Work

Tribhuvan University
Institute of Science and Technology
Dean's Office

SEMESTER EXAMINATION 2075

Name of Student: Pawan Shrestha

Level: Ph.D.

Department: Central Dept. of Mathematics

Exam Roll No.: 100023

Ph.D. Enrolment No.: 77/074

T.U. Regd. No.: 5-1-2-81-99

Semester: I

Grade Sheet

Code No.	Course Title	Cr. Hrs.	Grade Point	Grade
PHS 911	Philosophy of Science	3	3.7	A-
RM 912	Research Methodology	3	3.3	B+
Sem 913	Seminar	3	3.7	A-

SGPA: 3.6

Institute of Science & Technology
Dean's Office
Kirtipur 2045

Verified By: [signature]

Date: Oct. 9, 2018

Asst. Dean

Institute of Science & Technology
Dean's Office
Kirtipur 2045

Tribhuvan University
Institute of Science and Technology
Dean's Office

SEMESTER EXAMINATION-2075

Name of Student: Pawan Shrestha

Level: Ph.D.

Department: Central Dept. of Math

Exam Roll No.: 200022

Ph.D. Enrolment No.: 77/074

T.U. Regd. No.: 5-1-2-81-99

Semester: 2

Grade Sheet

Code No.	Course Title	Cr. Hrs.	Grade Point	Grade
MATH 951	Advanced Research Methodology	3	4	A
MATH 954	Fluid Mechanics and PDEs	3	4	A
MATH 952	Seminar	3	4	A

SGPA: 4.00

Verified By: ....

Date:- Sept. 16, 2019

Asst. Dean

A.2 List of Publications

1. Shrestha, P., KC, D. J., and Sharma, R: On the Construction of Traveling Water Waves With Constant Vorticity and Infinite Boundary, International Journal of Mathematics and Mathematical Sciences, Hindawi (MathSciNet, zbMATH, Scopus,...)

2. Shrestha, P., KC, D. J., and Sharma, R: 2D Surface Quasi Geostrophic Equations and its Regularity, A Numerical Study, The Nepali Mathematical Sciences Report, (MathSciNet, zbMATH, NepJOL,...) (accepted)

3. Shrestha, P., KC, D. J., and Sharma, R: 2D Surface Quasi Geostrophic Equations and its Regularity (under review)

4. Shrestha, P., KC, D. J., and Sharma, R: KdV Type Equations and Vanishing Viscosity (under review)

A.3 Articles

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Research Article

On the Construction of Traveling Water Waves with Constant Vorticity and Infinite Boundary

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The issue of whether there is a closed orbit in the water waves in an infinite boundary condition is an outstanding open problem. In this work, we first discuss the various developments on the structure of water waves in the context of finite bottom conditions. We then focus on the behavior of water for the kinematic boundary for the infinite depth. We present some findings to address this issue by creating a water wave profile for the zero and constant vorticity conditions through the application of the Crandall–Rabinowitz theorem.

1. Introduction

In the nineteenth century, Airy and Stokes started the mathematical study of wave theory within the framework of linear theory. Stokes water waves are nonlinear and periodic surface waves on an inviscid fluid layer of constant mean depth, which were introduced in the mid-nineteenth century. These waves are the progressive periodic waves of permanent form. These water waves, which propagate on the water surface of a sea or river, are periodic two-dimensional waves.

Stokes noticed the actual water wave characteristics and then studied the nonlinear governing equations extensively for water waves [1]. The governing equations for the two-dimensional irrotational gravity water waves have the Hamiltonian structure [2–4]. The irrotational flow (zero vorticity) was considered in most of the theoretical works of surface water waves. In most of the studies, incompressible Euler equations with free boundary conditions are considered governing equations for water waves. The insight into the dynamics of waves of large amplitude is given in [5–8] with the help of a qualitative understanding of periodic traveling waves in an irrotational flow over the flatbed and also the aspects of propagation of irrotational waves of the variable bottom are discussed [9]. The authors in [10–12] dealt with the full nonlinearity of the governing equations for the water waves providing valuable quantitative information.

Constantin and Villari proved that in linear periodic gravity water waves, there are no closed orbits for the water particles in the fluid and the paths are approximately elliptical, and they considered the flow up to the finite depth for the kinematic boundary condition [13].

Ehrnstrom and Villari showed that for the positive vorticity, the situation resembles that of Stokes waves. For large enough vorticity, the particle trajectories are affected. Also, for the negative vorticity, there is the appearance of internal waves, and vortices and the trajectories are not closed ellipses [14]. The author in [15] presented a mathematical formulation of the water wave problem. He gave the existence result for a small amplitude solution based on the bifurcation theory. Also, he described the particle motion in the physical frame and obtained the fluid in a moving frame, and confirmed the prediction from the linear theory [14].

The authors in [16] showed that the steady periodic deep-water waves are symmetric and propagate against a current with vorticity if their profile is monotone between crests and troughs. They discussed the traveling waves for water waves with zero and nonzero vorticity and showed that symmetry is valid, particularly for the irrotational waves. They also showed that vorticity distribution vanishes at the infinite depth. Constantin and Strauss [17] constructed the two-dimensional inviscid periodic traveling waves with vorticity with the consideration of a free surface under the effect of gravity over a flat bottom. These waves are symmetric and monotone between each crest and trough. They used the bifurcation theory and degree theory for the construction.

The authors in [18] constructed the periodic traveling waves with vorticity, which are symmetric waves whose profiles are monotone between each crest and trough. For this, they used bifurcation theory and considered the water wave problem described by the Euler equation with a free surface over a flat bottom. Also, the authors in [19] proved that the profile of a periodic traveling wave propagating on the surface of the water above a flatbed in a flow with real analytic vorticity must be real analytic with the assumption that wave speed exceeds the horizontal fluid velocity throughout the flow.

Recently with a degenerate diffusivity, Eyring–Powell viscosity term, and a Darcy– Forchheimer law in the porous medium, the authors in [20] obtained the traveling wave profile and showed the existence of the asymptotic solution using the geometric perturbation theory. They also showed the existence of an exponential profile of the solution under an asymptotic approximation. Similarly, the authors in [21] explored the solutions in the traveling wave domain with the use of geometric perturbation theory. Furthermore, the authors in [22] transformed the problem into the study of traveling wave-kind solutions. With the use of geometric perturbation theory, they confirmed the existence of an exponentially decaying rate in the traveling wave profile. Both the authors [21, 22] performed numerical simulations to validate their results.

The authors in [23] explored the regularity, existence, and uniqueness of the solutions with the use of variational weak formulations. Then they transformed the Eyring –Powell equation into a traveling wave domain where they obtained the analytical solution with the use of geometric perturbation theory. Their main work was to show the existence of an exponential traveling wave tail together with a certain minimizing error critical speed.

All of these existing results apply only to the finite boundary, but this problem remains open for the infinite boundary. With the ideas in [18, 19], and [17], and using the Crandall–Rabinowitz theorem, we created the water waves profile by considering the kinematic boundary condition up to infinity. We studied this as an extension of the flat bed at a finite depth to an infinite depth. Our findings will be useful for the study of the extension of the solutions from finite boundary conditions to infinite boundary conditions. Similar concepts can be used in other situations while dealing with problems in fluid dynamics with infinite boundaries.

In Section 2, we present the equation of motion with boundary conditions. In Section 3, the use of the Crandall–Rabinowitz theorem is proposed to guarantee the existence of traveling waves for water waves with zero and nonzero vorticity for the kinematic boundary condition up to the infinite depth. Section 4 concludes the paper.

2. Equation of Motion

The equation of motion of the wave is given by the following:

$$\partial_t \vec{u} + (\vec{u}.\nabla)\vec{u} + \nabla P = -g\vec{e}_y, \operatorname{div}\vec{u} = 0, \tag{1}$$

where u is the velocity, P is the pressure, and g is the acceleration due to gravity.

The related boundary conditions are as follows:

(1) Kinematic: The free surface with moving fluid is given by $\eta_t = \sqrt{1 + (\nabla\eta)^2}\vec{u} \cdot \vec{n}$ where $\vec{n}$ is the unit normal to the surface

(2) Dynamic: The dynamic boundary condition is the balance of forces at the free surface and is given by $P = P_{\text{atm}}$, the atmospheric pressure

(3) Bottom: At the bottom, the second component of the fluid is zero. It just moves horizontally i.e.,

$$\lim_{y\to-\infty} \vec{e_y}.\vec{u} = 0. \tag{2}$$

With these, the equation of motion is given by $\Delta\phi = 0$ in D_η, $\phi = \psi$ at $y = \eta(t,x)$, and

$$\lim_{y\to-\infty} \partial_y\phi = 0. \tag{3}$$

Here, ϕ is the velocity potential function, D_η is the domain of the fluid motion, $\eta(t,x)$ is the free surface which describes a wave on the bottom of the fluid dependent on time t and $\psi(t,x)$ is the value of the potential at the free surface.

3. Traveling Waves for Water Waves

In this section, the water waves for the two cases with zero vorticity and constant vorticity are discussed.

3.1. Traveling Waves for Water Waves with Zero Vorticity. The water waves formulation [24, 25] is given by the following:

$$\begin{aligned} \eta_t &= G(\eta)[\psi], \\ \psi_t &= -g\eta - \frac{\psi_x^2}{2} + \frac{1}{2}\frac{(G(\eta)\psi + \eta_x\psi_x)^2}{1+\eta_x^2}, \end{aligned} \tag{4}$$

where $G(\eta)[\psi]$ is a Dirichlet–Neumann operator [26, 27] and is given by

$$G(\eta)\psi = \sqrt{1+\eta_x^2}\partial_{\vec{n}}\phi|_{y=\eta(x)}. \tag{5}$$

Here, we are looking for the small size traveling wave periodic in x. That means we expect the solution of the form $\eta(t,x) = \breve{\eta}(x+ct)$ and $\psi(t,x) = \breve{\psi}(x+ct)$, where c is the phase velocity. These solutions must satisfy the equations as follows:

$$c\eta_x = G(\eta)[\psi],$$
$$c\psi_x = -g\eta - \frac{\psi_x^2}{2} + \frac{1}{2}\frac{(G(\eta)\psi + \eta_x\psi_x)^2}{1+\eta_x{}^2}. \tag{6}$$

For this, we look for the zeros of the equation, which indicate the situation with zero vorticity

$$F(c,\eta,\psi) = \begin{pmatrix} -c\eta_x + G(\eta)[\psi] \\ c\psi_x + g\eta + \dfrac{\psi_x^2}{2} - \dfrac{1}{2}\dfrac{(G(\eta)\psi + \eta_x\psi_x)^2}{1+\eta_x^2} \end{pmatrix}. \tag{7}$$

When $\eta = 0, \psi = 0$, (7) takes the form $F(c,0,0) = 0, \forall c$, which is a bifurcation problem. To solve this type of bifurcation problem, the Crandall-Rabinowiz theorem [28] is applied. For this, we must have the following:

(1) $d_{(\eta,\psi)}F(c^*,0,0)$ has the 1-dimensional kernel
(2) Im $d_{(\eta,\psi)}F(c^*,0,0)$ is closed and has codimension 1
(3) It must satisfy the traversability condition i.e., $\partial_{c,(\eta,\psi)}F(c^*,0,0)[(\eta^*,\psi^*)] \notin R$ where $(\eta^*,\psi^*) \in \mathrm{Ker} d_{(\eta,\psi)}F(c^*,0,0)$ where Ker d is the kernel of d and R is the image set.

By linearizing (7) at $\eta = 0, \psi = 0$, we have

$$d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \begin{pmatrix} -c\hat{\eta}_x + G(0)\hat{\psi} \\ c\hat{\psi}_x + g\hat{\eta} \end{pmatrix}. \tag{8}$$

We assume the series representation for $\hat{\eta}$ and $\hat{\psi}$ as

$$\hat{\eta}(x) = \sum_{n\geq 0}\eta_n \cos nx,$$
$$\hat{\psi}(x) = \sum_{n\geq 0}\psi_n \sin nx. \tag{9}$$

Which are even and odd, respectively, together with the norms

$$\|\eta\|_s^2 = \sum n^{2s}|\eta_n|^2,$$
$$\|\psi\|_s^2 = \sum n^{2s}|\psi_n|^2, \tag{10}$$

which are the norms defined in the homogeneous Sobolev space H^s.

With these series representations, we have

$$d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \sum_{n\geq 1}\begin{pmatrix} (cn\eta_n + n\psi_n)\sin nx \\ (cn\psi_n + g\eta_n)\cos nx \end{pmatrix} + \begin{pmatrix} 0 \\ g\eta_0 \end{pmatrix}. \tag{11}$$

We define

$$f = \sum_{n\geq 0} f_n \cos nx,$$
$$g = \sum_{n\geq 0} g_n \sin nx. \tag{12}$$

For these g and f; we solve, $d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \begin{pmatrix} g \\ f \end{pmatrix}$

For $n \geq 1$, we have

$$\begin{pmatrix} cn & n \\ g & cn \end{pmatrix}\begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}. \tag{13}$$

We note that det $M_n(c) = 0$ iff $c = \pm(\sqrt{g/n})$, where $M_n = \begin{pmatrix} cn & n \\ g & cn \end{pmatrix}$. For fixed n_0 and choosing $c_{n_0} = (\sqrt{g/n_0})$, the n_0-matrix has kernel. Now for c_{n_0}, we have

$$\begin{pmatrix} c_{n_0}n_0 & n_0 \\ g & c_{n_0}n_0 \end{pmatrix}\begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} c_{n_0} \\ -\dfrac{g}{n_0} \end{pmatrix}. \tag{14}$$

Here, we note that if $n_0 = 0$, then the inverse of that matrix $\begin{pmatrix} c_{n_0}n_0 & n_0 \\ g & c_{n_0}n_0 \end{pmatrix}$ does not exist. So, the matrix becomes singular, and hence the solution is not possible.

Therefore, $\mathrm{Ker} d_{(\eta,\psi)}F(c_{n_0},0,0) = \begin{pmatrix} c_{n_0}\cos n_0 x \\ -(g/n_0)\sin n_0 x \end{pmatrix}$ which is of dimension 1.

Again

$$\begin{pmatrix} 0 \\ -g\eta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ f_0 \end{pmatrix},$$
$$\begin{pmatrix} cn & n \\ g & cn \end{pmatrix}\begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}; n\geq 1. \tag{15}$$

This implies two possibilities, $n = n_0$ or $n \neq n_0$. In both cases, it can be shown that $\eta, \psi \in H^s$. Again for $n = n_0$, choosing $g_{n_0} = 1$, we have

$$R = \mathrm{Im} d_{(\eta,\psi)}F\left(c_{n_0},0,0\right) = \sum_{n\neq n_0}\begin{pmatrix} g_n \sin nx \\ f_n \cos nx \end{pmatrix} + \begin{pmatrix} \sin n_0 x \\ c_{n_0}\cos n_0 x \end{pmatrix}. \tag{16}$$

But, this is perpendicular to $R^{\perp} = \begin{pmatrix} c_{n_0}\sin n_0 x \\ -\cos n_0 x \end{pmatrix}$. This shows that the range is closed and has codimension 1. Also, the traversability condition is satisfied as

$$\partial_{c,(\eta,\psi)}F\left(c_{n_0},0,0\right)[(\eta^*,\psi^*)] = \begin{pmatrix} -c_{n_0}n_0 \sin n_0 x \\ -g\cos n_0 x \end{pmatrix} \notin R. \tag{17}$$

As all the conditions for the Crandall–Rabinowitz theorem [28] are satisfied, we have the following theorem:

Theorem 1. *Let $s > (5/2)$ and $n_0 \in N$: $\exists \epsilon_0 > 0$ and C^1 functions C_ϵ: $(-\epsilon_0,\epsilon_0) \longrightarrow R$; $\begin{pmatrix} \breve{\eta}_\epsilon \\ \breve{\psi}_\epsilon \end{pmatrix}(-\epsilon_0,\epsilon_0) \longrightarrow H^s \times H^s$ such that $\begin{pmatrix} \breve{\eta}(x-ct) \\ \breve{\psi}(x-ct) \end{pmatrix}$ solves the water wave equation. Moreover, $C_\epsilon = C_{n_0} + 0(\epsilon)$, $\begin{pmatrix} \breve{\eta}_\epsilon(x) \\ \breve{\psi}_\epsilon(x) \end{pmatrix} = \epsilon\begin{pmatrix} c_{n_0}\cos n_0 x \\ -g/n_0 \sin n_0 x \end{pmatrix} + O(\epsilon^2)$.*

Thus, we have constructed traveling waves of the form

$$\eta(t,x) = \epsilon c_{n_0}\cos n_0\left(x - c_\epsilon t\right) + O\left(\epsilon^2\right). \quad (18)$$

3.2. Traveling Waves for Water Waves with Constant Vorticity. The water waves formulation [24, 25] is given by

$$\eta_t = G(\eta)[\psi] + \gamma\eta\eta_x,$$
$$\psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{\left(\eta_x\psi_x + G(\eta)\psi\right)^2}{2\left(1+\eta_x\right)^2} + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi, \quad (19)$$

where $G(\eta)[\psi]$ is a Dirichlet–Neumann operator [26, 27].

Here, we are looking for the small size traveling waves periodic in x. That means we expect the solution of the form $\eta(t,x) = \breve{\eta}(x+ct)$ and $\psi(t,x) = \breve{\psi}(x+ct)$, where c is the phase velocity.

These solutions must satisfy the equations:

$$c\eta_x = G(\eta)[\psi] + \gamma\eta\eta_x,$$
$$c\psi_x = -g\eta - \frac{\psi_x^2}{2} + \frac{1}{2}\frac{G(\eta)\psi + \eta_x\psi_x^2}{1+\eta_x^2} + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi. \quad (20)$$

For this, we look for the zeros of the equation, which indicate the situation with constant vorticity

$$F(c,\eta,\psi) = \begin{pmatrix} -c\eta_x + G(\eta)[\psi] + \gamma\eta\eta_x \\ c\psi_x + g\eta + \frac{\psi_x^2}{2} - \frac{1}{2}\frac{G(\eta)\psi + \eta_x\psi_x^2}{1+\eta_x^2} - \gamma\eta\psi_x - \gamma\partial_x^{-1}G(\eta)\psi \end{pmatrix}. \quad (21)$$

When $\eta = 0$, $\psi = 0$, (21) takes the form $F(c,0,0) = 0$; $\forall c$, which is a bifurcation problem. Here also, the CrandallRabinowitz theorem [28] is applied to solve this type of bifurcation problem. For this, we must have the following:

(1) $d_{(\eta,\psi)}F(c^*,0,0)$ has 1 dimensional kernel
(2) Im $d_{(\eta,\psi)}F(c^*,0,0)$ is closed and has codimension 1
(3) It must satisfy the traversability condition i.e. $\partial_{c,(\eta,\psi)}F(c^*,0,0)[(\eta*,\psi*)] \notin R$ where $(\eta^*,\psi^*) \in \mathrm{Ker}d_{(\eta,\psi)}F(c^*,0,0)$ where Ker d is the kernel of d and R is image set.

By linearizing equation (21) at $\eta = 0$, $\psi = 0$, we have

$$d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \begin{pmatrix} -c\hat{\eta}_x + G(0)\hat{\psi} \\ c\hat{\psi}_x + g\hat{\eta} - \gamma\partial_x^{-1}|D|\psi \end{pmatrix}. \quad (22)$$

We assume the series representation for $\hat{\eta}$ and $\hat{\psi}$ as

$$\hat{\eta}(x) = \sum_{n\geq 0}\eta_n\cos nx,$$
$$\hat{\psi}(x) = \sum_{n\geq 0}\psi_n\sin nx, \quad (23)$$

which are even and odd, respectively, together with the norms

$$\|\eta\|_s^2 = \sum n^{2s}\left|\eta_n\right|^2,$$
$$\|\psi\|_s^2 = \sum n^{2s}\left|\psi_n\right|^2. \quad (24)$$

These are the norms defined on homogeneous Sobolev space H^s.

With these series representation, we have

$$d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \sum_{n\geq 1}\begin{pmatrix} (cn\eta_n + n\psi_n)\sin nx \\ ((cn+\gamma)\psi_n + g\eta_n)\cos nx \end{pmatrix} + \begin{pmatrix} 0 \\ (cn_0+\gamma)\psi_0 + g\eta_0 \end{pmatrix}. \quad (25)$$

Assuming

$$g = \sum_{n\geq 0} g_n\sin nx,$$
$$f = \sum_{n\geq 0} f_n\cos nx. \quad (26)$$

For these g and f; we solve $d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \begin{pmatrix} g \\ f \end{pmatrix}$

.

Then, we have $\begin{pmatrix} cn & n \\ g & cn+\gamma \end{pmatrix}\begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}$.

We note that $\det M_n(c) = 0$ if $c = -(\gamma/2n) \pm \sqrt{(\gamma^2/4n^2) + (g/n)}$, where $M_n = \begin{pmatrix} cn & n \\ g & cn+\gamma \end{pmatrix}$. For fixed n_0 and choosing $c_{n_0} = -(\gamma/2n_0) \pm \sqrt{(\gamma^2/4n_0^2) + (g/n_0)}$, the n_0 -matrix has kernel.

Now,

$$\begin{pmatrix} c_{n_0}n_0 & n_0 \\ g & c_{n_0}n_0+\gamma \end{pmatrix}\begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} c_{n_0} + \dfrac{\gamma}{n_0} \\ -\dfrac{g}{n_0} \end{pmatrix}. \quad (27)$$

Therefore, $\mathrm{Ker}d_{(\eta,\psi)}F(c_{n_0},0,0) = \begin{pmatrix} (c_{n_0} + (\gamma/n_0))\cos n_0x \\ -(g/n_0)\sin n_0x \end{pmatrix}$, which is of dimension 1.

Again

$$\begin{pmatrix} 0 \\ -g\eta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ f_0 \end{pmatrix},$$
$$\begin{pmatrix} cn & n \\ g & cn+\gamma \end{pmatrix}\begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}; n\geq 1. \quad (28)$$

This implies two possibilities $n=n_0$ or $n\neq n_0$. In both the cases, it can be shown that η, $\psi \in H^s$. Again for $n=n_0$ and choosing $g_{n_0} = 1$, we have,

$$R = \mathrm{Im} d_{(\eta,\psi)}F\left(c_{n_0}, 0, 0\right) = \sum_{n \neq n_0} \begin{pmatrix} g_n \sin nx \\ f_n \cos nx \end{pmatrix} + \begin{pmatrix} \sin n_0 x \\ \left(c_{n_0} + \frac{\gamma}{n_0}\right)\cos n_0 x \end{pmatrix}, \tag{29}$$

which is perpendicular to $R^{\perp} = \begin{pmatrix} -(c_{n_0} + (\gamma/n_0))\sin n_0 x \\ \cos n_0 x \end{pmatrix}$.

This shows that the range is closed and has codimension 1. Also the traversability condition is satisfied as

$$\partial_{c,(\eta,\psi)}F\left(c_{n_0}, 0, 0\right)\left[(\eta^*, \psi^*)\right] = \begin{bmatrix} -\left(c_{n_0} + (\gamma/n_0)\right)n_0 \sin n_0 x \\ -g \cos n_0 x \end{bmatrix} \notin R. \tag{30}$$

With the help of the Crandall–Rabinowitz theorem [28], we have obtained the following theorem:

Theorem 2. *Let* $s > (5/2)$ *and* $n_0 \in N$: $\exists \epsilon_0 > 0$ *and* C^1 *functions* C_ϵ: $(-\epsilon_0, \epsilon_0) \longrightarrow R$; $\begin{pmatrix} \breve{\eta}_\epsilon \\ \breve{\psi}_\epsilon \end{pmatrix} (-\epsilon_0, \epsilon_0) \longrightarrow H^s \times H^s$ *such that* $\begin{pmatrix} \breve{\eta}(x - ct) \\ \breve{\psi}(x - ct) \end{pmatrix}$ *solves the water wave equation.*

Moreover, $C_\epsilon = C_{n_0} + 0(\epsilon)$, $\begin{pmatrix} \breve{\eta}_\epsilon(x) \\ \breve{\psi}_\epsilon(x) \end{pmatrix} = \epsilon \begin{pmatrix} (c_{n_0} + (\gamma/n_0))\cos n_0 x \\ -g/n_0 \sin n_0 x \end{pmatrix} + O(\epsilon^2)$.

Thus, we have constructed traveling waves of the form

$$\eta(t,x) = \epsilon\left(c_{n_0} + \frac{\gamma}{n_0}\right)\cos n_0\left(x - c_\epsilon t\right) + O\left(\epsilon^2\right). \tag{31}$$

Here, we have created the water profile for the zero and constant vorticity cases. We use Newtonian fluid. A similar result may be useful for the non-Newtonian fluid. Also, a similar result may be extended to the positive or negative, or variable viscosity cases.

4. Conclusion

In this work, we studied the various development of the water waves and related facts such as the wave profile and path of the wave in the finite bottom conditions. We have then extended the finite depth condition to the infinite depth condition. In infinite depth conditions, we have constructed the water wave profile for the zero and constant vorticity conditions with the use of the Crandall–Rabinowitz theorem by using an analytical approach. While we focused on the zero and constant vorticity cases, similar work may be established on the positive and negative vorticity cases for the construction of wave profiles. Here we did our work for the Newtonian fluid. This work may be extended for the non-Newtonian fluid.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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CENTRAL DEPARTMENT OF MATHEMATICS

Institute of Science and Technology

CENTRAL DEPARTMENT OF MATHEMATICS TRIBHUVAN UNIVERSITY KIRTIPUR

KIRTIPUR, KATHMANDU
NEPAL

Date: Feb 05, 2023

Ref.

Subject: Letter of Acceptance

Dear Pawan Shrestha,

Thank you for a submission of Manuscript entitled "2D Surface Quasi Geostrophic (SQG) Equations and its Regularity, a Numerical Study" for a possible publication in "The Nepali Mathematical Sciences Report". Based in the reviewer's comments and your careful responses, I am pleased to inform you that the submitted paper has been accepted for publication. The paper will be published on the coming issue in 2023.

Dr.Tanka Nath Dhamala
Professor,
Editor- in- Chief
The Nepali Mathematical Sciences Report

A.4 List of Presentations

1. participated in three days workshop on "Numerical Linear Algebra, Modeling, and Simulations of Evolution Equations", October 17-19, 2022 held at Central Department of Mathematics, TU, Kathmandu

2. presented a paper entitled "KdV Type Equations and Various Limits" in the national conference organized by the Nepal Academy of Science and Technology held at Khumaltar, Lalitpur, Ashar 12-14, 2079

3. presented a paper entitled "KdV Type Equations and Vanishing Viscosity" at the National Conference on Mathematics and Its Applications (NCMA-2022) organized by Nepal Mathematical Society held at Ilam, Jestha 28-30, 2079

4. visited SISSA, Italy for three months supported by TWAS from March 19 to June 16, 2021

5. participated in a short-term course on "Computational Methods in Engineering Sciences", October 24-28, 2020 organized by Women Engineering College, Ajmer, India

6. presented a paper entitled "Regularity of SQG including bounds for the solution" in the International Conference on Applied Mathematics and Computational Sciences held in Dehradun India organized by Dehradun Institute of Technology (DIT) university, October 17-19, 2019

7. presented a paper on "Regularity of 2 D Surface Quasi Geostrophic Equations including Geometric Criteria" in the Second International Conference on Applications of Mathematics to Nonlinear Sciences (AMNS-2019) held at Pokhara organized by AANMA, CDM, and NMS, June 27-30, 2019

8. presented a paper entitled "Regularity of 2D Surface Quasi Geostrophic Equations" in the 7th national conference on Mathematics and Its Applications (NCMA-2019) held at Butwal organized by Nepal Mathematical Society, January 12-15, 2019

9. participated in the CIMPA school on dynamical system held in Kathmandu on October 25 to November 5, 2018

A.5 Certificates of Participations

International Centre of Pure and Applied Mathematics (CIMPA), France

and

Central Department of Mathematics, Tribhuvan University, Nepal

in cooperation with

The International Centre of Theoretical Physics (ICTP),
International Mathematical Union (IMU)

and

University Grants Commission (UGC), Nepal

present this

Certificate of Participation

To

Pawan Shrestha

Central Department of Mathematics, Tribhuvan University, Nepal

for actively participating
at the CIMPA Research School on Dynamical Systems, Kathmandu, Nepal
October 25 – November 5, 2018

CIMPA

Prof. Dr. Sergei B Kuksin
International Coordinator

Dr. Shree Ram Khadka
National Coordinator

Date: November 5, 2018, Kathmandu, Nepal

The Abdus Salam International Centre for Theoretical Physics

"Developing the Society Through Scientific Research & Mathematical Activities"

7th National Conference on **Mathematics and Its Applications** (NCMA-2019)

Certificate

This certificate is awarded to

Pawan Shrestha

Central Department of Mathematics, Tribhuvan University

for participating and presenting the paper entitled

Regularity of 2D Surface Quasi Geostrophic Equation

in the National Conference on Mathematics and Its Applications

organized by

Nepal Mathematical Society.

Prof. Dr. Chet Raj Bhatta
President
Nepal Mathematical Society

Sudarshan Baral
Chief Guest
Minister for Social Development
Province-5, Nepal

January 12-15, 2019
Butwal, Nepal

AMNS 2019

Second International Conference on Applications of Mathematics to Nonlinear Sciences (AMNS-2019)

June 27-30, 2019, Pokhara, Nepal

http://anmaweb.org/AMNS-2019

This certificate is awarded to

Pawan Shrestha

for presenting a talk

Regularity of 2D Surface Quasi Geostrophic Equations

in an invited session of

Second International Conference on Applications of Mathematics to Nonlinear Sciences (AMNS-2019)

on June 27-30, 2019, Pokhara, Nepal.

Dr. Naveen Vaidya
AMNS-2019 Conference Convener

Conveners

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Ref: GWECA/TEQIP-III/2020-21/330

Women Engineering College, Ajmer

(An Autonomous Institute of Govt. of Rajasthan)

SHORT TERM COURSE

ON

"Computational Methods in Engineering Science"

(October 24-28, 2020)

Certificate of Participation

This is to certify that **Pawan Shrestha** from **Tribhuwan University, Nepal** has participated in One Week Online Short Term Course (under TEQIP-III) on "**Computational Methods in Engineering Science**", jointly organized by **Department of Humanities and Sciences (Mathematics), Women Engineering College, Ajmer** and **Department of Chemical Engineering, Dr. B. R. Ambedkar National Institute of Technology, Jalandhar** during October 24-28, 2020 at **Women Engineering College, Ajmer.**

(Dr. A K Tiwari) CONVENER	(Ms. Shalini Agarwal) COORDINATOR	(Dr. Mahesh Bohra) COORDINATOR	(Dr. J.K. Deegwal) CONVENER

INTERNATIONAL CONFERENCE
ON
APPLIED MATHEMATICS & COMPUTATIONAL SCIENCES

Certificate No. 046

Certificate of Participation

This is to Certify that Prof./Dr./Mr./Ms. Pawan Shrestha

From Tribhuvan University ~~Participated~~ / presented an

oral paper / poster paper entitled Regularity of 2D Surface Quasi Geostrophic Equation. in ICAMCS - 2019, Organized by Department of Mathematics, DIT University, Dehradun & Sponsored by UCOST, Govt. of Uttarakhand from October 17-19, 2019.

ICAMCS 2019

(Dr. Jogendra Kumar)
Convener

(Prof. K.K. Raina)
Vice Chancellor

NATIONAL CONFERENCE ON MATHEMATICS AND ITS APPLICATIONS

(NCMA-2022), ILAM, Jestha 28-30, 2079 (June 11-13, 2022)

Certificate of Participation / Presentation

This is to certify that

Prof./Dr./Mr./Ms. Pawan Shrestha

Central Department of Mathematics (T.U.)

has participated / presented the paper entitled

KdV Type Equations and vanishing viscosity

in NCMA-2022 at ILAM held on June 11-13, 2022.

Prof. Dr. Narayan Prasad Pahari
President
Nepal Mathematical Society

June 13, 2022

Chief Guest
Hon. Jayram Yadav
Health Minister, Province-1, Nepal

D.A.O. Kathmandu Registration No.: 723/053/054
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नेपाल गणित समाज
Nepal Mathematical Society
Estd.: 1979

Nepal Academy of Science and Technology (NAST)

CERTIFICATE OF PARTICIPATION

Awarded to

Pawan Shrestha

for Presentation in Oral / Poster / Participation in the

9th National Conference on Science and Technology

June 26-28,2022 (Asar 12-14, 2079)

Khumaltar, Lalitpur, Nepal

Ms. Luna Vajra Chief, Promotion Division	Prof. Dr. Mahesh K. Adhikari Secretary	Dr. Sunil Babu Shrestha Vice Chancellor

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A Three Day Workshop on Numerical Linear Algebra, Modeling and Simulation of Evolution Equations

Certificate of Participation

This is to certify that

Mr. Pawan Shreshtha

of **Central Department of Mathematics, TU, Nepal** has actively participated in
A Three Day Workshop on Numerical Linear Algebra, Modeling and Simulation of Evolution Equations
jointly organized by Central Department of Mathematics, TU, Nepal and
TU Kaiserslautern, Germany.
October 17-19, 2022 (Ashwin 31 - Kartik 2, 2079)

Prof. Dr. Axel Klar
Facilitator
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