SOME SEQUENCE SPACES AND MATRIX TRANSFORMATION WITH VEDIC RELATIONS

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BY

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CERTIFICATE OF THE SUPERVISOR

This is to certify that the dissertation entitled "Some sequence spaces and matrix transformation with Vedic relations" which is being submitted by Mr. Suresh Ray in fulfillment of the requirements for the award of the degree of Doctor of Philosophy (Ph. D.) in mathematics from Nepal Sanskrit University, is a record of his own work carried out by him under my guidance and supervision.

The matter embodied in his dissertation has not been submitted for the award of any degree.

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DECLARATION BY CANDIDATE

I certify that the work embodied in this Ph. D. dissertation is my own bonafide work carried out me under the supervision of Prof. Dr. Kanhaiya Jha, Kathmandu University and subject experts Prof. Dr. Shailendra Kumar Mishra, Tribhuvan University and Dr. Dinesh Panthi, Nepal Sanskrit University for a period of time from June 1, 2016 to August 15, 2019 at Nepal Sanskrit University. The matter embodied in this Ph.D. dissertation has not been submitted elsewhere for the award of any other degree / diploma. I declare that I have faithfully acknowledged, given credit to an referred to the research workers wherever their works have been cited in the text and the body of the dissertation.

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ABSTRACT

The general theory of matrix transformation was special and classical results in summability theory "The most general linear operator to transform from new sequence space into another sequence space is actually given by an infinite matrix" to preserve the limit of the convergent sequences. The subspace of the family of all sequences of complex numbers is sequence space. In this connection, We have constructed new matrices by formula

$$S^n = \lambda = (\lambda_{nk}) = \begin{cases} n - k + 1 & n \geq k \\ 0 & \text{otherwise} \end{cases}$$

for Lower unitraingular matrix $\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

and operator sparse band matrix $\lambda_j = \begin{bmatrix} \frac{1}{t_1} & -\frac{1}{t_1} & 0 & 0 & \cdots \\ 0 & \frac{1}{t_2} & -\frac{1}{t_2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ to introduce new sequence spaces and then matrix transformation between sequence spaces is done. The set of inclusion relations gives practical application of sequence spaces. Matrix transformation relates between sequence space and Vedic notion.

LIST OF SOME SYMBOLS

These symbols are used in the dissertation:

$\mathbb{N}$: {1, 2, 3,...},the set of natural numbers.

$\mathbb{R}$: the set of all real numbers.

$\mathbb{R}^n$: n-dimensional Euclidean space.

$\mathbb{C}$: the set of complex numbers.

$\mathbb{Q}$: the set of rational numbers.

$\overline{E}$: closure of the set E.

e_o : {0 0, 0, 0,…}.

e_1 : {1, 0, 0, 0,…}.

e_2 : {0, 1, 0, 0,…}.

e_k : {0, 0, 0,0,…, 1 (k^{th} place), 0, 0, 0, 0,…}.

e : {1, 1, 1, 1,…}.

$\emptyset$: the space of all finite scalar sequences.

ϵ : a member of.

$\notin$: not a member of.

$>$: greater than.

$\leq$: less and equal to.

$<$: less than.

$\geq$: greater and equal to.

$=$: equal to.

$\neq$: not equal to.

$\rightarrow$, : tends to.

$\leftrightarrow$, : implies and if implies to.

$\subseteq$: subset of.

$\cap$: intersection.

$\cup$: union.

Σ : summation.

$\Rightarrow$: implies to.

$\nRightarrow$: does not imply to.

Sup : supremum.

Inf : infimum.

(x_n) : sequence.

O(1) : Order of one.

l_∞ = the space family of all bounded sequences

= $\{x = (x_k) \in \omega : \sup|x_k| < \infty\}$.(where ω is the set of all complex sequences).

c = The space of all convergent sequences

=$\{x = (x_k) \in \omega : |x_k - l\,| \to 0$ as $k\to\infty$ for some $l \in C\}$.

c_0= The space of all null sequences

=$\{x = (x_k) \in \omega : |x_k| \to 0$ as $k\to\infty\}$.

If p = (p_k) is a bounded sequence of strictly positive real numbers, then

$l(p)$ = The space of sequences x = (x_k) with absolutely p−summable series k.

=$\{x = (x_k) \in \omega : \sum_{k=1}^{\infty}|x_k|^{p_k} < \infty\}$.

If p_k = p for every k, then, $\bar{l}_p = \overline{l(p)}$ (because $l(p) = l_{(p)}$ for all $x \in X$).

$$\|x\| = \sum_{k=0}^{\infty} |x_k - x_{x+1}| + \lim_{k\to\infty} |x_k| \ (x \in b_v).$$

$b_{v_0} = b_v \cap c_0$.

$l_\infty(p) = \{x = \{x_k\} \in \omega : \sup_k |x_k - x_{k-1}|^{p_k} < \infty\}$.

$c(p) = \{x = (x_k) \in \omega : |x_k - l|^{p_k} \to 0 \ (k \to \infty)$ for some $l \in C\}$.

$c_0(p) = \{x = (x_k) \in \omega : |x_k|^{p_k} \to 0 \ (k \to \infty)$.

$x^\alpha = \{a = (x_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty$ for some $x \in X\}$.

$x^{\beta} = \{a = (\{a_k) : \sum_{k=1}^{\infty} a_k\, x_k$ is convergent for some $x \in X\}$.

$x^{\gamma} = \{a = (a_k) : \sup_n |\sum_{k=1}^{\infty} a_k| < \infty$ for some $x \in X\}$.

$X_t = \{x = (x_k)\ \omega : (t_k\, x_k) \in X\}$, where X = $\{l_\infty$, c, c_0, $c_s\}$ and t = (t_k) be any sequence of non zero complex numbers satisfying

$\lim_k inf\ (t_k)^{1/k} = r \quad (0 < r \leq \infty\}$.

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SUMMARY

CHAPTER ONE

INTRODUCTION

This chapter includes introduction, quotations, classification, historical developments, literature reviews, some important results and applications.

Introduction

Mathematical-research is a scientific and systematic search for pertinent information on a specific topic of mathematics. Mathematics is the study of relations and quantities through the use of numbers. Set of ordered numbers is called sequence and subspace of the family of all sequences of complex numbers is called sequence space.

Example: If term $a_n = \frac{1}{n}$, then $\{a_n\} = \{\frac{1}{n}\}$ is a sequence space in $[0 , 1] \subset \mathbb{R}$.

The array of numbers in rows and columns enclosed in square bracket is matrix and the relation between sequence spaces is matrix transformation.

Example: If X and Y be any two non empty subsets of space of all complex sequences and $A = (a_{nk})$ be an infinite matrix of complex numbers, then

sequence $Ax = A_n(x)$ if $A_n(x) = \sum_k a_{nk}x_k$ converges to each n.

If $x = (x_n) \in X \Rightarrow Ax = A_n(x) \in Y$, then matrix A transforms from X into Y.

Example: If $2x + 3y = 4$

$3x - 2y = 7$, then it can be transformed to matrix form $A\bar{x} = \bar{y}$,

where $A = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\bar{y} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ if $|A| \neq 0$.

It is an important tool for mathematical modeling.

[Choudhary B. and Nanda S., 1989].

We have studied sequence spaces by practical application:

When the data received from the reservoir to obtain some information have lower dimension and samples have larger size, the statistical methods such as that the covariance matrix, dot matrix and position weight matrix can deal with the cases promptly in a simplified way. However, when data have multidimensional character and the sample size is smaller, the statistical methods may lead to errors [Ray S., 2018].

Number Theory is the Queen of Mathematics and Mathematics is the Queen of sciences due to its foundational place in the discipline in modern period. The same sentiment is expressed in the ancient Sanskrit verse Shloka [Vedang Jyotish – 5].

यथा शिखा मयूराणां नागानां मणयो यथा ।

तद्व द्वेदांग शास्त्राणां गणितं मूर्ध्नि संस्थितम् ।।

In the Roman translation, we have

Yatha Sikha Mayuranaam, Naganam Manayo Yatha.

Tatvd Vedanga Shastranam, Ganitham Murdhani Sthitham.

This means like the comb of peacocks and the crest jewels of the serpents, so does the love of Jyotish stands as the head of all the lore's forming the auxiliaries of Vedas.

In Vedang period, Shulvasutras were the significant body of mathematical literature

In modern period, Baudhyana Shulvasutra is Pythagoras Theorem:

The sum of the areas produced by the length and the breadth is equal to the area produced by the diagonal of a rectangle.

[Sen S.N. and Bag A.K. 1983].

Bharata Krishna Tirthaji reconstructed sixteen mathematical formulae from sixteen Sutras, thirteen Subsutras from the Parisista of Atharvaveda and named Vedic mathematics. It helps to solve problems at a much faster speed.

The first Vedic Sutra is **"Ekādhikena Pūrvena."**

It's meaning is "By one more than the previous one."

Example: The square of a number 25, multiply the number of ten's place 2 by one more than itself 3 is 6 and write 25 ahead of that, then $(25)^2$ = 625.

[Tirthaji Vedic Mathematics 1965].

Among several interesting and worthy quotations on mathematics:

The great mathematician Pythagoras (570 BC-495BC) quoted mathematics as, "Numbers rule the universe."

The Greek Neoplatonist philosopher Proclus (412-485 A.D.) quoted about mathematics as, "Wherever there is number, there is beauty."

The great physicist, philosopher and Nobel Prize winner Albert Einstein (1879-1955) quoted about mathematics as, "Everything that can be counted does not necessarily count; everything that counts cannot necessarily be counted."

The English physicist Paul Dirac (1902-1984) Nobel Prize winner in physics quoted about mathematics as, "God used beautiful mathematics in creating the world."

The Polish mathematician Stefan Banach (1892-1945) quoted Mathematics as, "Mathematics is the most beautiful and most powerful creation of the human spirit."

The Italian Astronomer, physicist and Mathematician Galileo(1564-1642) quoted mathematics as, "Mathematics is the language with which God wrote the universe."

The English Mathematician James Joseph Sylvester (1814-1897) quoted mathematics as, “Mathematics is the music of reason.”

The German mathematician, physicist and philosopher David Hilbert (1862- 1943) quoted mathematics as, “Mathematics is a game played according to certain simple rules with meaningless marks on paper.”

The English Mathematician and poet Lewis Carroll (1832-1898) quoted about mathematics as, "The different branches of mathematics: ambition, distraction, uglification and derision."

The English and American Mathematician Keith Devlin (1947) quoted about mathematics as, "Just as music comes alive in the performance of it, the same is true of mathematics. The symbols on the page have no more to do with mathematics than the notes on a page of music. They simply represent the experience."

The English Mathematician and Metaphysician Bertrand Russell (1872-1970) quoted about mathematics as, "Mathematics possesses not only truth, but supreme beauty. What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought and brought again and again before the mind with ever-renewed encouragement."

The American automobile executive Lee Iacocca (1924) quoted about mathematics as, "In a completely rational society, the best of us would aspire to be teachers and the rest of us would have to settle for something less, because passing civilization along from one generation to the next ought to be the highest honor and highest responsibility anyone could have."

Mathematics has been classified into following five major branches:

Algebra is one of the broad parts of mathematics, together with number theory, geometry and analysis. As such, it includes everything from elementary equation solving to the study of abstractions such as groups, rings, and fields. The more basic parts of algebra are called elementary algebra, the more abstract parts are called abstract algebra or modern algebra. Elementary algebra is essential for any study of mathematics, science, or engineering, as well as such applications as medicine and economics. Abstract algebra is a major area in advanced mathematics, studied primarily by professional mathematicians. Much early work in algebra, as the origin of its name suggests, was done in the Near East by such mathematicians as Omar Khayyam (1050-1123).

Geometry is a branch of mathematics concerned with questions of shape, size, relative position of figures and the properties of space. A mathematician who works in the field of geometry is called a geometer. Geometry arose independently in a number of early cultures as a body of practical knowledge concerning lengths, areas, and volumes,with elements of formal mathematical science emerging in the West as early as Thales (6th Century BC). By the 3rd century BC, geometry was put into an axiomatic form by Euclid whose treatment (Euclidean geometry) set a standard for many centuries to follow. Archimedes developed ingenious techniques for calculating areas and volumes in many ways anticipating modern integral calculus. The field of astronomy especially as it relates to mapping the positions of stars and planets on the celestial sphere and describing the relationship between movements of celestial bodies, served as an important source of geometric problems during the next one and a half millennia. In the classical world, both geometry and astronomy were onsidered to be part of the Quadrivium, a subset of the seven liberal arts considered essential for a free citizen to master.

Topology is the mathematical study of shapes and topological spaces. It is an area of mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending but not tearing or gluing. This includes such properties as connectedness, continuity and boundary.

Number Theory is a branch of pure mathematics devoted primarily to the study of the integers, sometimes called *"The Queen of Mathematics"* because of its foundational place in the discipline. Number theorists study prime numbers as well as the properties of objects made out of integers (e.g., rational numbers) or defined as generalizations of the integer.

Analysis is the process of breaking a complex topic or substance into smaller parts to gain a better understanding of it. The technique has been applied in the study of mathematics and logic since before Aristotle (384–322 B.C.), though analysis as a formal concept is a relatively recent development [Boyer, C.B., 1991].

In Functional Analysis, topological space is the study of mathematical shape of sequence space. Any subspace of the family of all sequences of complex numbers is called sequence space. The set of ordered numbers is sequence. A set X over the field K is vector space with functions

$(x, y) \rightarrow x + y$ of $X \times X$ into X, called addition and

$(\lambda, x) \rightarrow \lambda x$ of $K \times X$ into X, called scalar multiplication,

if for all $x, y, z \in X$,

(i) $x + y = y + x$,

(ii) $(x + y) + z = x + (y + z)$,

X contains the element 0 (zero vector) such that

(iii) $x + 0 = 0 + x$,

(iv) $x + (-x) = 0$,

(v) $1.x = x$,

(vi) $\alpha(\beta x) = \alpha \beta(x)$, $(\alpha, \beta) \in \varphi$, $x \in X$,

(vii) $\alpha(x + y) = \alpha x + \alpha y$ and

(viii) $(\alpha + \beta)\, x = \alpha\, x + \beta\, x$.

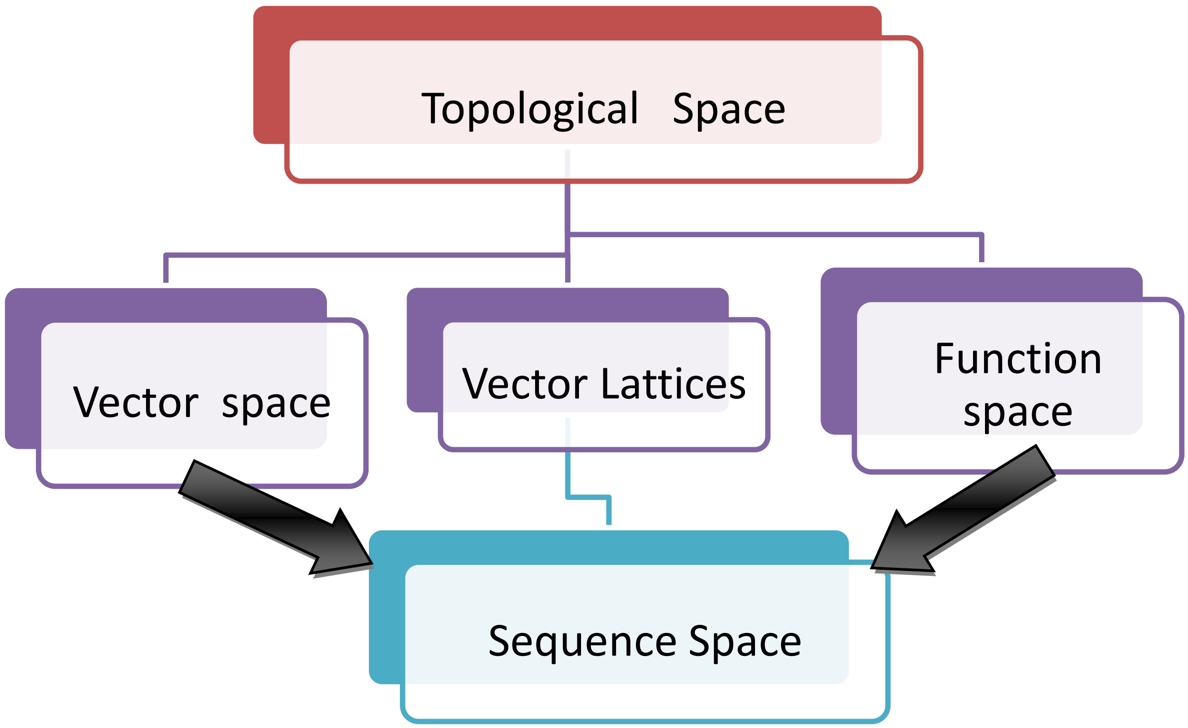
Vector lattices are frame work of crossed laths.

Function space is a rule which associates each element of set S into an unique element of field K and given by $f : S \to K$.

If f, g are two functions and $c \in K$ at an element x of S, then

$(f + g)(x) = f(x) + g(x)$ and $(c\, f)(x) = c\, f(x)$.

Thus, the sequence space is also the set of scalar sequence which is closed under coordinate wise addition and scalar multiplication. When it is closed under co-ordinate wise multiplication, then it is called the sequence algebra.



Sequence space formation

Problem statements

The problem statements of research work are

(i) What is ment by sequence space and its matrix transformation?

(ii) What is interpretation and practical application of sequence space?

(iii) Is any relationship between sequence and Vedic notion?

Objectives of Research Work

The objectives of research work are

(i) to study existing sequence spaces and establish new sequence spaces for their matrix transformation,

(ii) to show its practical application and

(iii) to show interrelation between sequence space and Vedic notion through examples in the Mahabharata.

Methodology

The methodology has been defined as research plan. This dissertation has been used Content-analysis (qualitative analysis) method which consists of analyzing the contents of documentary materials and source of data:

The primary source of data has been obtained through personal interviews and the secondary source of data has been obtained through various publications, books, newspapers.

An example approach has been followed personal interview with related experts such as Prof. Dr. Shankar Raj Panta, Dr. Vishnu Dahal, Dr. Rishi Raj Regmi, Dr. Rishi Ram Paudel.

Limitation

The work is concentrated in matrix transformation between sequence spaces with Vedic relations to get updated documents, are the major limitations with budgets.

1.2. Historical Developments

The study of the sequence spaces was motivated by the classical results of Summability theory in Functional Analysis. The results obtained by Cesaro, Borel, Norlund, Toeplitz and others.

In 1911, Toeplitz used to characterize matrix transformation. Toeplitz matrix transformations (linear operators) to transform one sequence space into another sequence space is given by

$$a_{nk} = \frac{1}{n} \ (1 \leq k \leq n)$$

$$= 0 \ (\text{for } k > n).$$

The result which characterizes the class (c , c , p) is the famous Silverman - Toeplitz theorem. Silverman proved the sufficiency of the conditions and necessity part was due to Toeplitz. The necessary and sufficient conditions for the matrices to be in (c ,c ,p) are known as “Toeplitz conditions” and the matrices are called the “Toeplitz matrices.”

In 1921, Schur obtained class (∞, c). These matrices are also called the 'regular matrices,' characterization of the class (c, c) is known as Kojima-Schur theorem and the matrices are known as convergence preserving matrices. Kojima –Schur characterized the matrices between the classes (c, c), Schur characterized (l_p, c).

Many characterizations has been observed between the classes (l_p , l_∞), (l_p, c), (l_∞(p), c), (c(p), c), (c_0(p), c), ($1 < p_k \leq \sup p_k < \infty$).

In 1927, Mazur gave the first application of functional analysis to the summability theory as Mazur Stain's law.

In 1932, he proved his famous Mazur`s consistency theorem from which he won the prize of the University LWOW.

In 1932, Banach used the Banach - Steinhaus theorem to give proof of Kojima – Schur theorem and hence that of the Toeplitz theorem. Toeplitz did not have such a short and sweet approach as presented by Banach.

In 1950, Robinson A., considered the action of the infinite matrices of the linear operators from one Banach space to that space . The classical Toeplitz theorem, Kojima - Schur theorem and many more results could be extended to those general settings, can be found in Maddox.

In 1969, Maddox I. J. has defined the sequence space

$l(\mathrm{p}) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$ and the Kothe –Toeplitz duals of sequence space $l(\mathrm{p})$ is found to observed so far.

If $\mathrm{p} = (p_k)$ is a bounded sequence of strictly positive real numbers, then

$l_\infty(\mathrm{p}) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\}$,

$(\mathrm{p}) = \{x = (x_k) : |x_k - l|^{p_k} \to 0$ for some $l \in \mathbb{C}\}$ and

$c_o(\mathrm{p}) = \{x = (x_k) : |x_k|^{p_k} \to 0 \ (k \to \infty)\}$.

The space $c_o(\mathrm{p})$ is metric linear space paranormed by $\|x\| = g(x) = \sup_k |x_k|^{\frac{p_k}{M}}$

The spaces $l_\infty(\mathrm{p})$ and $c(\mathrm{p})$ are paranormed by $g(x) = \sup_k |x_k|^{\frac{p_k}{M}}$

if inf $p_k > 0$.

Let t = (t_k) be any fixed sequence of nonzero complex numbers satisfying

$\lim_k \inf (t_k)^{\frac{1}{k}}$ = r $(0 < r \leq \infty)$.

In 1981, Kizmaz has introduced the concept of difference sequence

spaces. The work of Kizmaz was further generalized by Et and Cloak, Esi, Tripathy and Sharma and others. The infinite matrices

$$G(u, v) = (g_{nk}) = \begin{cases} u_n \; v_n, & 0 \leq k \leq n \\ \\ 0, & k > n \end{cases}$$

called generalized weighted mean;

$$\Delta = (\delta_{nk}) = \begin{cases} (-1)^{n-k}, & n-1 \leq k \leq n \\ \\ 0, & 0 \leq k \, n \text{ or } k > n \end{cases}$$

called the difference operator matrix;

$$S = (s_{nk}) = \begin{cases} 1, & 0 \leq k \leq n \\ \\ 0, & k > n \end{cases}$$

The operator matrix Δ_i which can be expressed as a sequential double band matrix is given by

$$\Delta_i = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and combination of them have been considered to represent difference operator.

In this connection, we have constructed new matrices by formula

$$S^n = \lambda = (\lambda_{nk}) = \begin{cases} n - k + 1 & n \geq k \\ 0 & \text{otherwise} \end{cases}$$

which is a lower unitraingular matrix and an operator sparse band matrix λ_i which can be expressed as a sequential double band matrix given by

$$\lambda_i = \begin{bmatrix} \frac{1}{t_1} & -\frac{1}{t_1} & 0 & 0 & \cdots \\ 0 & \frac{1}{t_2} & -\frac{1}{t_2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

to introduce the new sequence spaces.

In 1983, Dutta Srivastava, Guru Singh and others introduced a new general sequence spaces

$$X_t = \{x = (x_k) : (t_k x_k) \in X\},$$

where t = (t_k) is a fixed sequence of non zero complex numbers satisfying

$\lim_k \inf(t_k)^{\frac{1}{k}} = r \quad (0 < r \leq \infty) \quad (k \to \infty)$,

where X is any sequence space that is X = $\{l_{\infty},\ c,\ c_{0},\ c_s\}$.

In 1990, Fricke and Fridy, has characterized the classes

$(l_\infty(p),\ \Omega(t))$, $(c_0(p),\ \Omega(t))$, $(c(p),\ \Omega(t))$,

$(S\,l_\infty(p),\ \Omega(t))$, $(Sc_0(p),\ \Omega(t))$ and $(Sc(p),\ \Omega(t))$.

If $t = t_k$ be a positive number and let $\Omega(t) = \{x = (x_k) : x_k = 0(t_k)\}$.

S. C. Hu has established a fixed point theorem for discontinuous

quasi – monotone increasing maps R^n.

Schmidt has extended the results to c_0 and l_∞ $(1 \leq p \leq \infty)$.

In 1993, Chaudhary B. and Mishra S. K. have defined the sequence spaces $Sl_\infty(p)$, $Sc(p)$ and $Sc_0(p)$ and determined the Kothe –Toeplitz dual of $Sl_\infty(p)$. They have obtained the necessary and sufficient conditions for

$(Sl_\infty(p),\ l_\infty)$.

Further, S. K. Mishra has introduced some new sequence spaces as

follows:

if p = (p_k) be a bounded sequence of strictly positive real numbers, then we have

$Sl_\infty(p) = \{x = (x_k) : \sup_k |x_k - x_{k-1}|^{p_k} < \infty)\}$,

$Sc(p) = \{x = (x_k) : |x_k - x_{k-1}|^{p_k} \to 1 \ (k \to \infty)\}$, and

$Sc_o(p) = \{x = (x_k) : |x_k - x_{k-1}|^{p_k} \to 0 \ (k \to \infty) \ (x_o = 0)\}$.

If $p = (p_k)$ be a bounded sequence of strictly positive real numbers, then

$\bar{l}(p) = \{x = (x_k) : |t_k(x)|^{p_k} < \infty\}$, where

$$t_k(x) = \sum_{i=1}^{k} x_i \quad (k = 1,2,3,\ldots).$$

S.K. Mishra has determined the Kothe –Toeplitz duals too. He has obtained necessary and sufficient conditions for $(\bar{l}(p), l_\infty)$ and $(\bar{l}(p), c)$. Also, S.K. Mishra has characterized the BK (Banach –Coordinate) space which contains subspace isomorphic to

$c_0(\Delta) = \{x = (x_k) : |x_k - x_{k+1}| \to 0 \ (k \to \infty), \ x_1 = 0\}$

in terms of matrix maps and obtain the sufficient conditions for a matrix map from $Sl_\infty(\Delta) = \{x = (x_k) : \sup_k |x_k - x_{k+1}| < \infty), \ x_1 = 0\}$ into a BK space to be a constant operator. It is also shown that any matrix map from $Sl_\infty(\Delta)$ into a BK – space which does contain any subspace isomorphic to $Sl_\infty(\Delta)$ is constant. Here, $S = (s_{nk})$ is the matrix given by

$$s_{nk} = \begin{cases} 1 & 0 \leq k \leq n \\ 0 & k > n \text{ for all } n,\ k \in \mathbb{N} \end{cases}$$

For $p = \{p_k\}$ a bounded sequence of strictly positive real numbers, then the sequence space $\overline{l(p)}$ is given by

$$\overline{l(p)} = \{x = (x_k) : \sum_{k=1}^{\infty} |t_k(x)|^{p_k} < \infty\}, \text{ where } t_k(x) = \sum_{i=1}^{k} x_i.$$

When $p_k = p$ for every k, the sequence space $\overline{l(p)}$ is reduced to the sequence space

$$\overline{l_p} = \{x = (x_k) : \sum_{k=1}^{\infty} |t_k(x)|^{p} < \infty\}.$$

In 1998, Mursaleen A. K. and Saifi A. H. have introduced a new sequence space $l_\infty(\Delta_r p)$, where

$$l_\infty(\Delta_r p) = \{x = (x_k) : \Delta_r,\ x \in l_\infty(p)\}$$

$$= \{x = (x_k) : \sup_k |\ k^r (x_k - x_{k+1})\ |^{p_k}| < \infty < r < 1\}$$

$$l_\infty(p) = \{x\ (x_k) : \sup_k |x_k|^{p_k} < \infty\}$$

$$\Delta_r(x) = (k^r \Delta x_k)^{\infty}_{k=1},\ r < 1 \text{ and } \Delta x_k = (x_k - x_{k+1}).$$

These authors have determined the α, β – duals of $l_\infty(\Delta_r p)$ and characterized the matrix transformations of the class $(l_\infty(\Delta_r p),\ c_0)$.

In 2002, Altay and Basar have studied the space $r^t(p)$ which consists of all sequences whose Riesz transforms are in

$X \in (l_\infty,\ c,\ c_0,\ l_p)$.

The matrix $G(u, v) = (g_{nk})$ called generalized weighted mean or factorable matrix is given by

$$g_{nk} = \begin{cases} u_n v_k, & 0 \le k \le n \\ 0 & k > n \end{cases}$$

For all $n,\ k \in \mathbb{N}$; where u_n depends only on n and v_k depends only on k. The spaces $\overline{l(p)}$, $Z(u,\ v;\ p)$ and $r^t(p)$ may be represented as

$Z(u,\ v;\ p) = [X]_{G(U,\ V)}$,

$\overline{l(p)} = [l(p)]_s$ and $r^t(p) = [l(p)]_{R^t}$, here the matrix $R^t = (r^t_{nk})$ of the Riesz mean $(R,\ t_n)$ is given by

$$(r^t_{nk}) = \begin{cases} \dfrac{t_k}{\sum_{k=0}^{n} t_k} & 0 \le k \le n \\ 0 & k > n \end{cases}$$

with the sequence of real (t_k)

In 2005, Baral K.M., Mishra S.K. and Pant S.R. defined the sequence space $c_0(\Delta_r p)$, determined the Kothe –Toeplitz dual of $c_0(\Delta_r p)$

and characterized the class $(l_\infty(\Delta_r p), c_0)$, where

$$l_\infty(\Delta_r p) = \{x = (x_k) : \Delta_r x \in l_\infty(p)\}$$

$$= \{ x = (x_k) : \sup_k | k^r(x_k - x_{k+1}) |^{p_k} < \infty\}$$

If $P = (P_k)$ is a bounded sequence of strictly positive reals

$$l_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\}$$

$$\Delta_r(x) = \{x = (k^r \Delta x_k)_{k=1}^{\infty}, \ r < 1 \text{ where}$$

$$\Delta x_k = (x_k - x_{k+1}).$$

In 2014, Parajuli V. and Mishra S.K. introduced and studied sequence space $l(p, \lambda)$ as

$$l(p, \lambda) = \{x = (x_k) \in \omega : \lambda x \in l(p)\}, \quad \text{where}$$

$$S^n = \lambda = (\lambda_{nk}) = \begin{cases} n - k + 1 & n \geq k \\ \\ 0 & \text{otherwise} \end{cases}$$

where $l(p, \lambda)$ is the set of all sequences $\{u_k\}$

whose $S^n = \lambda -$ transforms are in the sequence space $l(p)$.

Thus, we have $l(p, \lambda) = [l(p)]_\lambda$, where the sequences

$\{\Delta u_k - u_{k-1}\} \in \overline{l(p)}$ with $u_0 = 0$ and

$$\{u_k\} = \{\textstyle\sum_{i=1}^{k}(k - i + 1)\, x_i\}.$$

In 2015, Mishra, S.K., Parajuli, V. and Ray, S. introduce and studied new Sequence Space $\overline{Sl(p)}$ generated by an Infinite Diagonal Matrix , found $\beta -$ duals of X as $X^\beta = \{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k$ is convergent for each $x \in X \}$ and characterization for the classes $(\overline{sl(p)}, l_\infty)$ and $\overline{(sl(p)}, c)$, where

$\overline{l(p)} = \{ x = (x_k) : \sum_{k=1}^{\infty} |f_i(x)|^{p_k} < \infty\}$, and $\qquad t_k(x) = \sum_{i=1}^{k} x_i$.

Infinite diagonal matrix $A = (a_{nk})$ is given by

$$a_{nk} = \begin{cases} 2^{-n} & n = k \\ 0 & \text{otherwise } 1 \end{cases}$$

In 2017, Ray S., Panthi D., Jha K. and Mishra S.K. defined and studied new sequence space

$X(p, \lambda)_t = \{x = (x_k) : (t_k x_k) \in X(p, \lambda)\}$ according to existing

sequence space $(p, \lambda) = \{x = (x_k) : \lambda x \in X\}$,

$$\lambda = \begin{bmatrix} 1 & 0 & \ldots \\ 2 & 1 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \text{ and } X = \{l_\infty, c, c_0\}.$$

This space is paranormed by $g^*(x) = g(t\,x)$ where g is paranormed in $X(p, \lambda)$.

We found $\beta -$ dual of $X(p, \lambda)_t$ and characterized class $(l_\infty(p, \lambda)_t, l_\infty)$.

We also defined $X(\Delta_r p, \lambda)_t = \{x = (x_k) : (t_k x_k) \in X(\Delta_r p, \lambda)\}$.

This space is paranormed by $g^*(x) = g(t\ x)$ where g is paranormed in $X(\Delta_r p, \lambda)$. We found $\beta -$ dual of $X(\Delta_r p, \lambda)_t$ and characterized class

$(l_\infty(\Delta_r p, \lambda)_t, c_0)$.

In 2018, Ray S. presented a practical application on the sequence spaces and function spaces on interval [0, 1] for DNA sequencing:

DNA (Deoxyribonucleic acid) sequence which is a specific sequence of all little bases of nucleotide A(Adenine), C(Cytosine), T(Thymine), G(Guanine) and are linked in different orders in extremely long DNA molecules. In these area of interest dimensional and complex, the sample size is relevantly small, they posses finitely many non zero elements in the sequence and some elements in the sequence repeat many times. When the data received from the reservoir to obtain some information have lower dimension and samples have larger size, the statistical methods such as that the covariance matrix, dot matrix and position weight matrix can deal with the cases promptly in a simplified way. However, when data have multidimensional character and the sample size is smaller, the statistical methods may lead to errors.

We examine the behaviors of sequences generated by DNA nucleotides. It has been aimed to extend the results of authors by introducing new function space in [0, 1], extending the basis function $\frac{x^n}{n!}$, introducing a new sequence b = (b_n) = $(\sum_{v=n}^{\infty} a_v)$ which can characterize DNA sequence. For the completion of the space (p[0, 1], $\|.\|_{\emptyset}$), authors have defined the following spaces on [0, 1]:

$$C_{\emptyset,\ 0}[0,\ 1] = \{\ f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} : \lim_{n\to\infty} a_n = 0\},$$

$$C_{\emptyset,\ p}\,[0,\ 1] = \{\ f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} : \sum_{n=0}^{\infty} |a_n|^p < \infty\} \text{ for } p \geq 1 \text{ and}$$

$$C_{\emptyset,\ \infty}\,[0,\ 1] = \{\ f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} : \sup_{n\geq 0} |a_n| < \infty\ \}.$$

These spaces are isomorphic to $c_0,\ l_p\ and\ l_\infty$ respectively .

Obviously, $P[0,1] \subset C_{\emptyset,0}\,[0,1] \subset C_M^{\infty}[0,\ 1]$ and authors have shown the following set inclusion relations

$P[0,1] \subset C_{\emptyset,1}\,[0,1] \subset C_{\emptyset,0}\,[0,1] \subset C_{\emptyset,\mathrm{p}}\,[0,1] \subset C_{\emptyset,\infty}\,[0,1] = C_M^{\infty}[0,\ 1],$

$1 \leq p < \infty.$

In 2019, Ray S., Panthi D., Jha K. and Mishra S.K. studied and introduced Vedic relations:

The first sutra is " Ekādhikena Pūrvena".

Its meaning is “By one more than previous one.”

Examples

(i)1+1, 2+1, 3+1, 4+1, 5+1, 6+1, 7+1, 8+1

i.e. 2, 3, 4, 5, 6, 7, 8, 9.

(ii) Vedic matrix is a nine by nine square array of numbers formed by taking a multiplication table and replacing each number by digit sum as follows:

10 becomes 1, 11 becomes 2, 12 becomes 3, 13 becomes 4, 14 becomes 5,

15 becomes 6, 16 becomes 7, 17 becomes 8, 18 becomes 9,

i.e., 1, 2, 3, 4, 5, 6, 7, 8, 9 and

1+8 = 9; 2+7 = 9; 3+6 = 9; 4+5 = 9, then

ordered pairs (1, 8), (2, 7), (3, 6) , (4, 5) are sequential matrices.

(iii) An Akshauhini contains 21,870 elephants, 21,870 chariots, 65,610 horses and 109,350 foot soldiers. They were arranged in sequences and the arrangement of arrays of battle field was as sequence space.

1.3. Literature reviews

We have following definitions and classical properties from the book of Choudhary B. and Nanda S., 1989.

Definition

Let X be a non-empty set, then metric 'd' (distance) on X is a function $d : X \times X \to R$, denoted by (X, d), with the four properties:

For $x, y, z \in X$,

(i) $d(x, x) = 0$,

(ii) $d(x, y) = 0 \Leftrightarrow x = y$,

(iii) d(x, y) = d(y, x),

(iv) d(x, z) $\leq$ d(x, y) + d(y, z).

Examples

(a) For any n ∈ N, the set

R^n = {x = $(x_1,\ x_2, \dots, x_n) : x_i \in R,$ i = 1, 2, ... ,n}

is a metric space with the metric d defined by

d(x, y) = $\{\sum_{i=1}^{n}(x_i - y_i)^2\}^{\frac{1}{2}}$, for x , $y \in R^n$.

The triangle inequality is a consequence of the Minkowski's inequality.

(b) For any n ∈ N, the set

$\mathbb{C}^n$ = {x = $(x_1,\ x_2, \dots, x_n) : x_i \in \mathbb{C},$ i = 1, 2,… ,n}

is a metric space with the metric d defined by

d(x, y) = $\{\sum_{i=1}^{n}(x_i - y_i)^2\}^{\frac{1}{2}}$, for x , $y \in \mathbb{C}^n$.

(c) Let S denotes the set of all sequences of real or complex numbers, then S is a metric space with the metric

$$d(x,\ y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)} ,$$

where x = (x_n) and y = (y_n) are in S.

The factor 2^{-n} ensure the convergence of the infinite series which defines d.

(d) Let C[0, 1] denote the set of all continuous real valued functions on the closed interval [0, 1]. Any f $\in$ C[0, 1] is bounded stains its bounds.

If we define d(f, g) = sup$\{|f(x) - g(x)| : x \in C[0, 1]\}$, where d is metric and $C[0, 1]$ is a metric space with this metric.

Another metric on $C[0, 1]$ is given by

$$p(f, g) = \int_0^1 |f(x) - g(x)| dx,$$

where the above integral is in the sense of Riemann.

(e) Let A denote the set of all complex functions f analytic on $\{z \in \mathbb{C} : |z| \leq 1\}$ and continuous on $\{z \in \mathbb{C} : |z| \leq 1\}$.

By the maximum modulus principle, it follows that its maximum on the boundary $\{z : |z| = 1\}$.

Therefore, we can define

$$d(f, g) = \max_{|z| \leq 1} |f(x) - g(x)|$$

$= \max_{|z|=1} |f(x) - g(x)|$ for f, g $\in$ A.

This is a metric on A and A is a metric space with this metric.

(f) The space $l(p)$.

Let $\{p_k\}$ be a bounded sequence of strictly positive real numbers so that $0 < p_k \leq \sup p_k = H < \infty$.

We define the set $l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$.

Also, we consider a map d : $l(p) \times l(p) \to R$ by

$$d(x, y) = \left(\sum_k |x_k - y_k|^{p_k}\right)^{\frac{1}{M}},$$

where M = max (1, H).

Since $t_k = \frac{p_k}{M} \leq 1$, if a_k and b_k are complex numbers, then

we have $|a_k - b_k|^{l_k} \leq |a_k|^{l_k} + |b_k|^{l_k}$.

Since M ≥ 1, the above inequality and Minkowski's inequality

imply that $\left(\sum_k |a_k - b_k|^{p_k}\right)^{\frac{1}{M}} = \left(\sum_k |a_k|^{p_k}\right)^{\frac{1}{M}} + \left(\sum_k |b_k|^{p_k}\right)^{\frac{1}{M}}$

Putting $a_k = x_k - y_k$, $b_k = y_k - z_k$ in the

above inequality, we get $d(x, y) \leq d(x, y) + d(y, z)$

Other axioms of a metric are trivially satisfied by d.

Thus, $l(p)$ is a metric space with this metric d.

If $p_k = p$ for all k, then we write l_p for $l(p)$.

Observe that if $1 \leq p < \infty$, then

l_p is a metric space with metric

$$d(x, y) = \left(\sum_k |x_k - y_k|^{p}\right)^{\frac{1}{p}},$$

and if $0 < p < 1$, then the metric for l_p is given by

$$d(x, y) = \sum_k |x_k - y_k|^{p} .$$

The case p = 1 and p = 2 are of special importance.

It is noted that the metrics for l_1 and l_2 are given by

$$d(x,\ y)\ =\ \sum_k |x_k - y_k| \text{ and } d(x,\ y)\ =\ \sum_k |x_k - y_k|^{\frac{1}{2}}.$$

(g) We define l_∞(p) = $\{x\ =\ (x_k) : \sup_k |x_k|^{p_k} < \infty\}$.

For $x,\ y \in\ l_\infty$(p), then we define

$$d(x,\ y)\ =\ \sup_k |x_k - y_k|^{\frac{p_k}{M}}, \text{ where } M\ =\ \max(1,\ \sup p_k).$$

So, l_∞(p) is a metric space with this metric.

For $p_k = p$ for all k, then we write l_∞ for $l_\infty(p)$,

l_∞ is the set of all bounded sequences,

$x\ =\ \{x_k\}$ of real or complex numbers and is a metric space with the natural metric

$$d(x,\ y)\ =\ \sup_k |x_k - y_k|.$$

(h) The spaces c(p) and c_0(p) with $\{p_k\}$ as above, we define

c(p) =$\{x\ =\ (x_k) :\ |x_k - l|^{p_k} \to 0$ as $k \to \infty$ for some $l \in \mathbb{C}\}$

and c_0(p) = $\{x\ =\ (x_k) :\ |x_k|^{p_k} \to 0$ as $k \to \infty\}$.

c(p) and c_0(p) are metric spaces with the metric

$$d(x,\ y)\ =\ \sup_k |x_k - y_k|^{\frac{p_k}{M}}, \text{ where } M\ =\ \max(1,\ \sup p_k).$$

(i) The spaces **c** and $\boldsymbol{c_0}$. If $p_k = p$ for all k, then we write c and c_0 for c(p) and c_0(p) respectively. Also, c and c_0 are the sets of all convergent sequences and null.

We note that c and c_0 are metric spaces with the metric

$d(x, y) = \sup_k |x_k - y_k|$.

In c, if we define $\rho(x, y) = |\lim(x_n - y_n)|$, then

$\rho(x, x) = 0$, $\rho(x, y)$ does not always imply that $x = y$.

For example, take $x_k = \frac{1}{k}$ and $y_k = 0$ for all k,

we observe that the other two axioms of a metric are satisfied by ρ.

Thus, ρ is not a metric c, but is a semi metric.

(j) Let $\Gamma = \{f : \mathbb{C} \rightarrow \mathbb{C},\ f(z) = \sum_k a_k z^k,\ |a_k|^{\frac{1}{k}} \rightarrow 0\}$ and

$\Gamma^* = \{f : \mathbb{C} \rightarrow \mathbb{C},\ f(z) = \sum_k a_k z^k,\ \sup_k |a_k|^{\frac{1}{k}} < \infty\}$,

Γ is the set of all entire functions, where Γ^* contains of analytic functions.

For f, g $\in \Gamma$, $f(z) = \sum_k a_k z^k$ and $g(z) = \sum_k b_k z^k$, so we define

$d(f, g) = \sup\{|a_0 - b_0|,\ |a_k - b_k|^{\frac{1}{k}} : k \geq 1\}$.This is a metric on Γ. Γ^* *is* also a metric space with the metric d. Given a semi-metric space, we can always construct a metric space out of it.

Remarks

(a) A metric d is always non-negative .

For $x, y \in X$ follows that

$d(x, y) + d(y, x) \geq d(x, x)$

i.e. , $2\,d(x, y) \geq 0$ and hence $d(x, y) \geq 0$.

(b) If $x, y, x', y' \in X$, it should be noted that

$|d(x, y) - d(x', y')| \leq d(x, x) - d(y', y')$.

Every nonempty set X can be made into a metric space in a trivial way. Define $d : X \times X \to R$ by $d(x, x) = 0$ and $d(x, x) = 1$ for $x \neq y$, called trivial metric.

Definition

Let (X, d) be a metric space and let sequence (x_n) in normed linear space is a Cauchy sequence if for every $\epsilon > 0$, there exists $n_0 \in N$ such that

$|x_m - x_n) | < \epsilon$ for N, $m \geq n_0$.

The sequence (x_n) is convergent to $x \in X$ if for every $\epsilon > 0$ there exists $n_0 \in N$ such that $\| x_n - x \| < \epsilon$ for $n \geq n_0$.

In other words, if (x_n) be a sequence in normed linear space X such that

$|x_m - x_n| \to 0$ as $m, n \to \infty$ and

there exists $(x_n) = x$ such that $\|x_n - x\| \to 0$ as $n \to \infty$, such a sequence (x) is called Cauchy sequence.

A normed linear space (nls) X is said to be complete if every Cauchy sequence in X converges to an element x of X.

A complete normed linear space is called a Banach space, i.e., a Banach space is a nls which is complete with the metric defined by its norm which means that every Cauchy sequence is required to converge.

It is clear that $\mathbb{R}^n$, $\mathbb{C}^n$, c_s, l_∞, c, c_0 are all Banach spaces.

A normed linear space X is complete if and only if every absolutely convergent series in X is convergent.

Examples

(a) Let X be any nonempty set and d the trivial metric on X, then (X, d) is a complete metric space.

(b) The set $\mathbb{C}$ of all complex numbers is complete with the usual metric defined by absolute value.

(c) $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathfrak{s}$, Λ, $C[0,1]$, c_s, l_∞(p), c(p), c_0(p), c, c_0, Γ, Γ* are complete metric spaces. Also, C[0, 1] is complete with d but not with p.

(d) Let S be a compact Hausdorff topological space and let C(S) denote the set of all real (complex) valued continuous functions defined on S. Note that C(S) is a metric space under the metric defined by d(f, g) = $\sup\{|f(x) - g(x)| : x \in S\}$.

Remarks

The real line $\mathbb{R}$ with the usual metric defined by the absolute value is a complete metric space. The set of all rational numbers Q is not complete in the metric defined the absolute value.

For example, the sequence (1.4, 1.41, …) is Cauchy, but does not converge in Q. When the above sequence is considered as a sequence in ℝ, then it converges to $\sqrt{2}$ (in ℝ). By the usual axiomatic approach to the real line, the completeness of ℝ is an axiom and known as the completeness axiom for the real numbers. There are of course other axioms which are equivalent to this.

For example, the axiom that every nonempty set of real numbers which is bounded above has a real supremum and bounded below infimum, is equivalent to and also known as the completeness axiom. On the other hand, if a constructive approach to the reals is taken, starting with the incomplete metric space Q, a new metric space is constructed and its completeness is proved. The complete metric space so obtained is called the completion of Q and is defined as the real line ℝ.

Similarly if a metric space is incomplete there is a definite procedure by which a complete metric space can be constructed and this new metric space is called the completion of the given one. However, in this text we shall not go into details of construction of ℝ starting with G or completion of an incomplete metric space. Also, with regard to the axiomatic approach, we shall not go into details of various axioms equivalent to the completeness axiom.

Definition: Let (X_1, d_1) and (X_2, d_2) be metric spaces, then function $f : X_1 \to X_2$ is said to be continuous at $a \in X_1$ if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon, a) > 0$ such that $d_1(f(x_1), f(a)) < \epsilon$ whenever $d_1(x, a) < \delta$.

Definition

The function f is said to be uniformly continuous if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $d_2(f(x_1), f(x_2)) < \epsilon$ whenever $d_1(x_1, x_2) < \delta$.

Definition

Let X and Y be two topological spaces, then $f : X \to Y$ is said to be continuous on X if the inverse image of every open set in Y is open in X and f is said to be an open mapping if the image of every open set in X is open in Y.

Example

Let X has the discrete topology and Y be any topological space, then $f : X \to Y$ is continuous.

Similarly, let X be any topological space and Y has the discrete topology, then every function $f : X \to Y$ is an open mapping.

Definition

Let X and Y be topological spaces and $f : X \to Y$, then f is said to be a homeomorphism if it is bijective (one to one and onto), continuous and open and X and Y are said to be homeomorphic.

Remarks

If two topological spaces are homeomorphic, they are identical to each other and regarded as indistinguishable. To be precise, we define a relation $'\sim'$ in the family of all topological spaces as follows: $X \sim Y$ if X and Y are homeomorphic , then $'\sim'$ is an equivalence relation and each equivalence class consists of all the topological spaces which are homeomorphic to each other.

As an example, we note that the open interval (0, 1) and the real line R are homeomorphic.

Definition

Let (X_1, d_1) and (X_2, d_2) be metric spaces, then function $f : X_1 \rightarrow X_2$, is said to be an isometry if it is one to one and onto and for all $x, y \in X_1$, then $d_2(f(x), f(y)) = d_1(x, y)$.

We note that an isometric is a homeomorphism and if two metric spaces are isometry, then they are regarded as indistinguishable as metric spaces.

Example

Let $\mathbb{R}$ have the usual metric defined by the absolute value. Let $\mathbb{R}^*$ denote the set of all functions then, $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$f(x) = \alpha x, \quad \alpha \in \mathbb{R}$.

We define a metric d in R^* by $d(f, g) = |\alpha - \beta|$, where

$f(x) = \alpha x$ and $g(x) = \beta x$.

Now we define a mapping

$F : \mathbb{R}^* \rightarrow \mathbb{R}$ by $F(f) = \alpha$, where $f(x) = \alpha x$, then F said an isometry and $\mathbb{R}$ and $\mathbb{R}^*$ are isometric as metric spaces.

Definition

Let X be a topological space, then a sequence $\{x_n\}$ in X is said to be convergent to $x \in X$, if for each neighborhood U of x, there exists $n_0 \in N$ such that $n_0 \in U$ whenever $n \geq n_0$.

If $\{x_n\}$ converges to x, we write $x_n \rightarrow x$ or $\lim_{n\rightarrow\infty} x_n = x$.

If $\{x_n\}$ is a sequence in X and $\{n_1 < n_2 < \ldots < n_k < \cdots\}$ is an infinite subset of the set of all positive integers, then the sequence $\{x_{n_k}\}$ is said to be subsequence of $\{x_n\}$.

Definition: A topological linear space (X, τ) is a linear space with topology τ on X such that the addition and scalar multiplication are continuous in (X, τ). If the topology τ on X is given by a metric (respectively, semi metric), then we speak of a metric linear space (respectively, semi metric linear space).

In other words if the topology τ on X is given by a metric in the Topological vector space (TVS), then the space is called a metric linear space. A vector space (or linear space) over a field K is a set with mappings. The elements of X are called vectors and those of the field K called scalars. We denote the set of all natural numbers, integers, rational numbers, real numbers and complex numbers by N, Z, Q, R and ℂ respectively. A vector space X is called a real or complex vector space according as the field K is ℝ or C.

Example

(a) For $n \in N$, the set

$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, \ i = 1, 2, \ldots, n\}$ is a real vector space with respect to addition and scalar multiplication defined as follows:

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \text{ where } \lambda \in R$$

Similarly, we have

$\mathbb{C}^i = \{(z_1, z_2, \ldots, z_n) : z_i \in R, \ i = 1, 2, \ldots, n\}$ is a complex vector space.

(b) Let S denote the set of all sequences $x = \{x_n\}$ of real (complex) numbers, then S is a real (complex) vector space under coordinate wise linear operations defined as follows;

$$\{x_n\} + \{y_n\} = \{x_n + y_n\}, \quad \lambda\{x_n\} = \{\lambda x_n\}$$

(c) Let X be any non-empty set and let $\mathcal{F}(X)$ denote the set of all real valued (complex valued) functions defined on X, then $\mathcal{F}(X)$ is a real (complex) vector space with respect to pointwise linear operations defined as follows:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)\},$$

$$(\lambda f)(x) = \lambda f(x) \text{ for all } x \in X, \text{ where } f_1, f_2 \in \mathcal{F}(X) \text{ and}$$

λ is real or complex.

(d) C(S), the set of all continuous real (complex) functions defined on a compact Hausdorff space, is a real (complex) vector spaces with respect to pointwise linear operations.

(e) The sequence spaces $l(p)$, $l_\infty(p)$, $c_0(p)$ and c(p) are vector spaces with respect to point wise linear operations defined as follows:

$$\{x_n\} + \{y_n\} = \{x_n + y_n\},$$

$$\lambda\{x_n\} = \{\lambda x_n\}.$$

Definition

Let X be a vector space over a field K and $\lambda_1, \lambda_2, \ldots, \lambda_n \in$ K.

An element $\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n$ is called a linear combination of the elements $x_1, x_2, \ldots, x_n \in$ X.

It is also called a linear combination of the set

A $= \{x_1, x_2, \ldots, x_n\}$. We write this element as $\sum_{i=1}^{n} \lambda_i x_i$.

It should be noted that the linear span of a set A is the set of all linear combination of A. A subset A of X is called linearly independent if for every nonempty finite subset $\{x_1, x_2, \dots, x_n\}$ of A, the relation

$\sum_{i=1}^{n} \lambda_i x_i = 0$ implies $\lambda_i = 0$ for i =1, 2, …,n.

We note that by this definition, the empty set is linearly independent. A subset A of X which is not linearly independent is called linearly dependent. We observe that any subset of X containing the zero vector, in particular the set {0}, is linearly dependent. A linearly independent subset of X which is maximal (with respect to set inclusion) is called a basis (Hamel basis) of X. It is a basis question whether every vector space has a basis and we need Zorn's lemma to answer this question.

Definition: Let X be any nonempty set. A partial order relation (pore in short) in X is a relation $\leq$ such that for all x, y, $z \in$ X, we have

(i) $x \leq x$ (reflexive),

(ii) $x \leq y$ and $y \leq x$ implies $x = z$ (anti symmetric) , and

(iii) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitive).

A nonempty a set X with pore defined on it is called a partially ordered set (poset in short). Two elements x and y in poset are comparable if either $x \leq y$ and $y \leq x$ holds.

A poset in which any two elements are comparable is called a linearly ordered set or totally ordered set or a chain.

Let $(X, \leq)$ be a poset and $A \subset X$.

An element $x \in X$ is called an upper bound for λ.

If $a \leq x$ for every $a \in A$; is said to be least upper bound (l.u.b.) or supremum (if it exists) for A. If x is an upper bound for A and if y is any other upper bound for A, then we have $x \leq y$.

Similarly, we define a lower bound and greatest lower bound (g.l.b.) or Infimum for a subset of a poset. An element x in a poset X is said to be maximal if $x \leq y$ implies $x = y$.

Definition

Let (X, d) be a metric space . X is said to be totally bounded if for $\epsilon > 0$, there exists a finite number of points $x_1, x_2, \dots, x_n$ such that

$$X = \bigcup_{i=1}^{n} B_i(x_i).$$

The set $\{x_1, x_2, \dots, x_n\}$ is called the $\epsilon -$net for X.

Note that every totally bounded metric space is bounded, but the converse is not true in general.

Observe that a totally bounded space need not be compact metric space is totally bounded.

Definition

A vector space X is said to have dimension n if the number of elements of any basis for X is n. If the dimension of X equals n for some n ∈ N, then X is said to be finite-dimensional.

If X = {0}, its dimension is defined to be zero and in this case also , X is said to be finite-dimensional. If X is not finite-dimensional, then it is called infinite-dimensional.

Examples

(a) Let X = R^3, the 3-space.

Let e_1 = (1, 0, 0) = i,

e_2 = (0, 1, 0) = j,

e_3 = (0, 0, 1) = k,

then the set { $e_{3,}$ e_3, e_3} is linearly independent. Moreover, it is a basis for R^3 and R^3 has dimension 3. This basis is called the standard basis for R^3. Observe that {(1, 1, 0), (1, 0, 1), (0, 1, 1)} is also a basis for R^3. On the other hand, {(1, 0, 1), (1, 1, 0), (-1, 0, -1)}is linearly dependent and hence is not a basis.

(b) Let X = $\mathbb{R}^n$. Let e_i be the element $\mathbb{R}^n$ whose i_{th} coordinate is 1 and all others are zeros, then the set $\{e_1, e_2, \dots, e_n\}$ is linearly independent and forms a basis for R^n. This is known as the standard basis for $\mathbb{R}^n$.

Definition

Let X and Y be any two vector spaces over a field K. A mapping f : X $\rightarrow$ Y is called a linear mapping (linear transformation or homomorphism)

if $f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2)$ for all $x, y \in X$ and $\lambda_1, \lambda_2 \in K$.

Further, if f is one to one; then it is called an isomorphism and two vectors spaces X and Y are said to be isomorphic if there exists an isomorphism from X onto Y.

Two important subspaces are associated with a linear mapping $f : X \rightarrow Y$; the null space of f and the range space of f. The null space of f, $\{x \in X : f(x) = 0\}$ is denoted by N(f) and the range space $\{f(x) \in Y : x \in X\}$ by R(f) if and only if $N(f) = \{0\}$.

Example

(a) The zero mapping $0 : X \rightarrow Y$ defined by $0(x) = 0$ for all $x \in X$ is a linear mapping.

(b) The identity map $I : X \rightarrow X$ defined by $I(x) = x$ is also a linear map.

(c) If $X = \mathbb{R}$, then the function $f : R \rightarrow R$ defined by $f(x) = x + \alpha$ (where $\alpha \in R$ is fixed) is usually called a linear map. But, it is not really as $f(x + y) = x + y + \alpha \neq f(x) + f(y) = x + \alpha + y + \alpha$.

(d) A constant map $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \alpha$ (where $\alpha \in \mathbb{R}$ is fixed) is not a linear map,

since $f(x + y) = \alpha \neq f(x) + f(y) = \alpha + \alpha$.

Remarks

Let L(X, Y) denote the set of all linear maps from X into Y. Define addition and scalar multiplication in L(X, Y) by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

and $(\lambda f)(x) = \lambda(f(x))$, then L(X, Y) forms a vector space.

Let X, Y and Z be any three vector spaces and let $f_1 : X \rightarrow Y$, $f_2 : Y \rightarrow Z$ be two linear maps. The composition $f_2 o f_1 : X \rightarrow Z$ is defined by $(f_2 o f_1)(x) = (f_2(f_1(x))$, for all $x \in X$.

It is clear that f_2 t o f_1 is a linear map. Thus, composition of two linear maps is also a linear map.

Definition

Let X be a complex (real respectively) vector space. A linear functional on X is a linear map $f : X \rightarrow C$ ($F_1 : X \rightarrow R$, respectively) that is, a function f defined on X whose values are complex (real respectively) numbers such that

$(c_1 x_1 + c_1 x_2)(x) = c_1 f(x_1) + c_1 f(x_1)$ if $c_{1,} c_2 \in C$(or $c_{1,} c_2 \in R$) respectively.

f is said to be a real or complex linear functional according as it is real or complex - valued.

Example

Let $V^n = C^n$ (or R^n) be a n-dimensional complex (or real)vector space If $b_1, b_2, \dots, b_n$ are n fixed complex (or real)numbers and if for every $x = (\xi_1, \xi_2, \dots, \xi_n) \in V^n$, we define

$f(x) = b_1\xi_1 + \dots + b_n\xi_n$, then f is linear functional on V^n and $f(e_i) = b_i$, $i = 1, 2,..,n$, where

$e_1 = (1, 0,\dots,0),\dots, e_n$

$= (0, 0, \dots,1)$ is the basis of V^n.

Conversely, given any linear functional f on V^n,

let $b_i = f(e_i)$

We shall have, for any vector $x = \xi_1 e_1 + \dots, + \xi_n e_n \in V,^n$

$$f(x) = \sum_{i=1}^{n} \xi_i f(e_i) = \sum_{i=1}^{n} b_i \xi_i.$$

Thus, the linear functional f on V^n are in one to one correspondence with the systems $(b_1, b_2, \dots, b_n)$ of n complex (or real) numbers and that $f(x) = \sum_{i=1}^{n} b_i \xi_i$ is the general form of such a functional.

More generally, let X be any vector space and $B = (z_j)$ a fixed basis of X. In order to define a function f on X, we have to give the values f(x) for all $x \in X$. However, if f is linear, then it is enough to give only the values $f_s = f(z_s)$ for $z_s \in B$. In fact, since every x is of the form $x = c_1 z_{s1} + \ldots + c_n z_{sn}$, then

$f(x) = \sum c_i f(z_{si})$, so that it is sufficient to know the values $f(z_{si})$.

In other words, let $B = \{z_s\}$ be a fixed basis of X and let f be a linear functional on X. We know for each $x \in X$, there is a function

$$c_s = c(z_s) \text{ defined on B such that } x = \textstyle\sum_i c_s z_s = \sum_i c(z_n) z_s.$$

Putting $f_s = f(z_s)$, $f(x) = \sum_i c_s f_s$ (where $x \to \{c_s\}$).

In this way, to each functional f, there corresponds a function $f_s = f(z_s)$ on B, which determines completely the functional.

Conversely, assigning to each $z_s \in B$ an arbitrary number $f_s = f(z_s)$ and defining f by the above formula (for each x), we obtain a linear functional f on X.

Remarks

The set of all linear functionals on X form a vector space X*in which the operations f, h and cf are defined pointwise. X* is called the algebraic dual of X. Observe that the zero of X* is the functional f such that $f(x) = 0$ for all $x \in X$.

If X is the space of all linear supported functions defined on B implies that X* is the set of all functions defined on the basis B.

If X is finite - dimensional, then B has a finite number of elements and all functions defined on B are finitely supported. In this case, X is isomorphic with X* (but this is not longer valid for infinite dimensional vector spaces). This fact may be also expressed as follows :

Let dim. $X = n < \infty$ and let $\{e_1, e_2, \ldots, e_n\}$ be a basis X.

In such a way that every $x \in X$ can be written as $x = \sum \xi_i e_i$, where scalars ξ_i. We determine $x = \xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_n e_n$.

Also, we define for each $i = 1, 2, \ldots, n$,

the functional f_i by the formula $f_i(x) = \xi_i$,

for $x = \xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_n e_n$.

It follows immediately that the f_i defined is a linear functional, that is, $f_i \in X^*$ and that f_i is characterized by $f_i(e_j) = 0$, if $i \neq j$, $f_i(e_i) = 1$.

From this characterization, we infer that the functionals f_i are linearly independent, because if $c_1 f_1(x) + \ldots + c_n f_n(x) = 0$,

putting $x = e_i$, then $c_i f_i(e_i) = e_i = 0$, $i = 1, 2, \ldots, n$.

Moreover, every other linear functional f is a linear combination of these functionals f_i.

Indeed, putting $c_i = f(e_i)$, for all $x = \xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_n e_n$

Therefore $\{f_1, \ldots, f_n\}$ is a linear basis of the vector space X*,

called a dual basis of $\{e_1, \ldots, e_n\}$.

In particular, this shows that dim X^* = n.

Remarks

If X is a real vector space, then every linear functional of X is real and there are no other linear functionals.If X is a complex space, then it will be necessary to distinguish between real linear functionals and complex linear functionals.We now show the relation between real and complex linear functionals of the complex space X. If f is a complex - valued linear functional, then for each x,

$$f(x) = f_1(x) + f_2(x), \text{ where } f_1(x), f_2(x) \text{ are real numbers.}$$

We show that f_1 and f_2 are linear functionals.

Indeed, if $c_1, c_2 \in R$, then

$$f(c_1x_1 + c_2x_2) = f_1(c_1x_1 + c_2x_2) + if_2(c_1x_1 + c_2x_2)$$

$$= \{c_1f_1(x_1) + c_2f_1(x_2)\} + i\{c_1f_2(x_1) + c_2f_2(x_2)\}$$

and therefore $f_1(c_1x_1 + c_2x_2) = c_1f_1(x_1) + c_2f_1(x_2)$.

However, in general, f_1 is not a complex linear functional.

Since $f(ix) = f_1(ix) + i\, f_2(ix)$

$$= -f_2(x) + i\, f_1(x),$$

we obtain $f_1(ix) = -f_2(x)$,

$f(x) = f_1(x) - i\, f_1(ix)$ (f_1 is the real part of f).

Conversely, if $f_1(x)$ is a real linear functional, defining

$f(x) = f_1(x) - i\, f_1(ix)$, it is easy to verify that f is a (complex) linear functional and f_1 is the real part of f.

Examples

(a) $l(p)$ is a linear metric space with the total paranorm g defined by

$$g(x) = \left(\sum_k |x_k|^{p_k}\right)^{\frac{1}{M}}, \text{ where } M = \max(1, \sup p_k).$$

We shall only prove the continuity of scalar multiplication.

We note that if $\lambda \in \mathbb{C}$, then $|\lambda|^{p_k} \leq \max(1, |\lambda|^M)$.

Therefore, for $x \in l(p)$ and $\lambda \in \mathbb{C}$,

$$g(\lambda x) \leq \max(1, |\lambda|)\, g(x).$$

So the function $(\lambda, x) \rightarrow \lambda x$ is continuous at $\lambda = 0$. X = 0

and when x is fixed, the function $x \rightarrow \lambda x$ is continuous at $x = 0$.

If $x \in l(p)$ is fixed and $\epsilon > 0$ is given, we can

choose K such that $\sum_{k > K} |x_k|^{p_k} < \frac{\epsilon}{2}$ and $\delta > 0$,

so that $|\lambda| < \min(1, \delta)$ gives $\sum_{k \leq K} |\lambda x_k|^{p_k} < \frac{\epsilon}{2}$.

If $|\lambda| < \min(1, \delta)$, then $g(\lambda x) < \epsilon$.

Thus, the function $\lambda \to \lambda x$ is continuous at $\lambda = 0$ for fixed x.

(b) $c_0(p)$ is a linear metric space paranormed by g

denoted by $g(x) = \sup |x_k|^{\frac{p_k}{M}}$

(c) The spaces $l_\infty(p)$ and c(p) fail to be linear metric spaces.

In each case the continuity of scalar multiplication fails, although all other although all other axioms are satisfied. However, if $\inf p_k > 0$, then they turn out to be linear metric spaces.

In particular, l_∞ and c are linear metric spaces. We

shall prove the assertion only for $l_\infty(p)$.

Let $p_k = \frac{1}{k}$ and $x_k = 1$ for every k, then $x \in l_\infty(p)$.

Let $0 < |\lambda| < 1$, then $|\lambda|^{\frac{1}{k}} < 1$ for all k and $|\lambda|^{\frac{1}{k}} \to 1$ as $k \to \infty$,

so that $g(\lambda x) = \sup |x_k|^{\frac{p_k}{M}} = \sup |x_k|^{\frac{1}{k}} = 1$

Hence $\lambda x \nrightarrow 0$ as $\lambda \to 0$ and thus $l_\infty(p)$ is not a linear

metric space. Now assume that $\inf p_k = \theta > 0$ and $x \in l_\infty(p)$, then

we have $g(\lambda x) \leq \max (|\lambda|, |\lambda|^{\frac{1}{k}})\, g(x)$.

Thus, $\lambda x \nrightarrow 0$ as $\lambda \to 0$ and thus $l_\infty(p)$ is not a linear metric space.

Conversely, if $\inf p_k = 0$, then the above example shoes that there is an $x \in l_\infty(p)$ for which $\lambda x \to 0$ as $\lambda \nrightarrow 0$.

(d) Γ is a linear metric space with the paranorm defined by

$\sup\{|a_0|, |a_k|^{\frac{1}{k}} : k \geq 1\}$, but Γ^* is not so. Γ^* becomes a linear metric space for $\inf p_k > 0$. We have already observed that every vector space has a Hamel basis. We shall now introduce the concept of a basis in a linear metric space. This concept was introduced by J. Schauder and is known as Schauder basis.

Remarks

Let (X, g) be a paranormed (total paranormed) space, then it is easy to verify that X is a semi-metric (metric) space defined by $d(x, y) = g(x - y)$. It is also clear that every paranormed space is a semi-metric (linear metric) space. It can also be shown that on every linear semi-metric space, it is possible to define a paranorm which gives rise to the same semi-metric topology. Thus, paranormed spaces and linear semi-metric spaces are essentially the same thing.

Definition

Let (X, g) be a paranormed space.

A set $\{x_k : x_k \in X,\ k \in N\}$ is called a Schauder basis for X.

If for every x ∈ X, there exists unique scalars λ_k, k ∈ N such that

$x = \sum_{k=1}^{\infty} \lambda_k x_k$;

i.e. , $g(x - \sum_{k=1}^{n} \lambda_k x_k) \to 0$ as $n \to \infty$.

We note that a Hamel basis is free from topology, where as Schauder basis involves convergence and topology. It should be noted that for finite - dimensional spaces both the concepts coincide.

Examples

Let e_k be sequence with 1 in the k^{th} place and zeros elsewhere and let $e = \{1, 1, \ldots\}$.

(a) The sequence $\{e_k\}$ is a Schauder basis for $l(p)$, c_0 and s.

Note that $\{e_k\}$ is certainly not a Hamel base for c_0.

(b) $\{e, e_1, e_2, \ldots\}$ is a Schauder basis for c.

Remark

Let X be a real or complex vector space of finite or infinite dimension. Let k be the field R of real numbers or the field C of complex numbers. When the scalar field is not specified, then it is understood that the results are valid for both cases.

Definition

Two nlss X and Y are said to be linearly homeomorphic if there exists a linear map $F : X \rightarrow Y$ which is also a homeomorphism, X and Y are said to be linearly isometric or isometrically isomorphic if there exists a linear map

$F : X \rightarrow Y$ which is also an isometry, X and Y are said to be equivalent nlss if and only if they are isometrically isomorphic. We observe that $(X, \|.\|_1)$ and $(X, \|.\|_2)$ are equivalent if there exist

$\alpha, \beta > 0$ such that $\alpha \|.\|_1 \leq \|.\|_2 \leq \beta \|.\|_1$.

Definition Let K denote the real field R or the complex field C and let X and Y be normed linear spaces over K. A linear transformation T from X into Y is said to be a bounded linear transformation if there exists a constant $M > 0$ such that $\|Tx\| \leq M\|x\|$ for every $x \in X$. In the above inequality $\|x\|$ is the norm of x in X and $\|Tx\|$ is the norm of T x in Y. It will frequently happen that several norms occur together; but we will use the same symbol for all the norms. If T is a bounded linear transformation, then the norm of T is defined by

$\|T\| = \sup \{\frac{\|Tx\|}{\|x\|} : x \in X, \ x \neq 0\}$.

It may be noted that we can restrict ourselves only to $x \in X$ with

$\|x\| = 1$ without changing the supermum, since for $\alpha \in K$,

$\|T(\alpha x)\| = |\alpha| \, \|T(x)\|$.

Therefore, the norm of T can also be defined by

$$\|T\| = \sup \{\|Tx\| : x \in X, \ \|x\| = 1\}.$$

Also, it may be noted that

$$\|T\| = \inf \{M : M > 0, \ \|Tx\| \leq M\|x\|\} \text{ for } x \in X.$$

In other words, $\|Tx\| \leq \|T\| \, \|x\|$ for $x \in X$.

If T is a bounded linear transformation from a normed linear space X into itself, then we call T a bounded linear operator.

A bounded linear transformation from a normed linear space X into the field K called a real or complex bounded linear functional according as K is the real field R or complex field C.

Examples

(a) The identity operator $I : X \to X$ on a normed linear space $X \neq \{0\}$ is a bounded linear operator with the norm $\|I\| = 1$.

(b) The zero transformation $0 : X \to Y$ on a normed linear space X is bounded linear operator with the norm $\|0\| = 0$.

(c) The norm $\|.\| : X \to R$ on a linear space X is not a linear functional; it is a sub linear functional.

Remarks

B(X), the set of all bounded linear operators defined on a normed linear space X is a normed linear space and that B(X) is a Banach space if X is a Banach space. Also, X* the set of all bounded linear functionals defined on a normed linear space X is a Banach space. It should be noted that X* is always a Banach space even if X is not complete .

X* is called the conjugate or dual space of X. Since X* is always a Banach space, we talk of the dual space of X*, called the second conjugate space of X as X**.

Remarks

(1) X_t need not to be a sequence algebra even if X is so. Indeed, it is known that c_0 is a sequence algebra, but $(c_0)_t$ is not a sequence algebra for (t_k) = ($\frac{1}{k}$).

For let x = ($\sqrt{k}$) and y = (α$\sqrt{k}$), where α ϵ C is a constant, then

(t_k) = ($\frac{1}{k}$)

(x_k) = ($\sqrt{k}$)

$(y_k$) = (α$\sqrt{k}$) $, where$ α is a constant.

Thus, $(t_k x_k)$ = ($\frac{1}{k}$) ($\sqrt{k}$) = ($\frac{1}{\sqrt{k}}$) ϵ c_0 and x ϵ $(c_0)_t$

and $(t_k y_k)$ = ($\frac{1}{k}$) (α$\sqrt{k}$) = ($\frac{\alpha}{\sqrt{k}}$) ϵ c_0 and y ϵ $(c_0)_t$

Now, x y = z

= $(x_k y_k)$ = ($\sqrt{k}$) (α$\sqrt{k}$) = (αk) $\notin$ c_0, then $(t_k z_k)$ = $(\frac{1}{\sqrt{k}})$ $(\alpha k) = \alpha\sqrt{k} \notin (c_0)_t$.

Thus, z $\notin (c_0)_t$

(2) X_t need not to be a sequence algebra even if X is so. Indeed, it is known

that c_0 is a sequence algebra, but $(c_0)_t$ is not a sequence algebra for

(t_k) = $(\frac{1}{k})$.

For let x = $\sqrt{k}$ and y = $\alpha\sqrt{k}$, where $\alpha \epsilon C$ is a constant.

Now c_0(x) = {x = (x_k) : λ x ϵ c_0},

{c_0(x) }$_t$ = {x = (x_k) : u_k ϵ c_0},

λ x ϵ c_0, then $(t_k x_k)$ ϵ $c_0(\lambda)$ and

λ y ϵ c_0, then $(t_k y_k)$ ϵ $c_0(\lambda)$

but x y = z $\notin c_0(\lambda)$, therefor x y $\notin \{c_0(\lambda)\}_t$.

We set x = (x_k)

$$= (\sum_{k=1}^{\infty}(k - i + 1)^{\infty}{}_{k=1}(\sqrt{k}), \text{ putting k = 1, 2, 3, …}$$

$$= (\sqrt{1},\ \sqrt{2} - 2,\ \sqrt{3} - 2\sqrt{2} + 1,\ \ldots)$$

$y = (\alpha x_k)$

$t = (\frac{1}{k})$

$tx = (\frac{1}{\sqrt{k}}) \in c_0$ and

$t\,y = (\frac{\alpha}{\sqrt{k}}) \in c_0$

But z = x y

$= (\sum_{k=1}^{\infty}(k - i + 1)^{\infty}{}_{k=1}(\sqrt{k}\,)\ \{\alpha(\sum_{k=1}^{\infty}(k - i + 1)^{\infty}{}_{k=1}(\sqrt{k}\,)\}.$

Hence $t\,z = (\sum_{k=1}^{\infty}(k - i + 1)^{\infty}{}_{k=1}(\sqrt{k}\,) \notin c_0$

Also $y_k = \begin{bmatrix} 1 & 0 & \ldots \\ 2 & 1 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & \ldots \\ 2 & 1 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \sqrt{1} \\ \sqrt{2} - 2 \\ \vdots \end{bmatrix}$

$= \begin{bmatrix} 1\sqrt{1} + & 0(\sqrt{2} - 2) + & 0(\sqrt{3} - 2\sqrt{2} + 1) + \cdots \\ 2\sqrt{1} + & 1(\sqrt{2} - 2) + & 0(\sqrt{3} - 2\sqrt{2} + 1) + \cdots \\ \ldots & \ldots & \ldots \end{bmatrix}$

$= \begin{bmatrix} \sqrt{1} \\ \sqrt{2} \\ \vdots \end{bmatrix}.$

Definition: If f is a complex function defined on [0, 1], then f is said to be of bounded variation on [0, 1] if V(f) = sup $\sum_{i=0}^{n}|f(x_{i+1}) - f(x_i)|$ is finite, where the supermum is taken over all partitions

0 = $x_0 < x_1 < \cdots < x_n$ = 1. We write [0, 1] to denote the set of all functions of bounded variation on [0, 1].

Definition: Let (x_n) be a sequence in a nls X, then the sequence (x_n) is said to converge weakly to x in X written $x_n \rightarrow$ x, weakly iff

f(x_n) → f(x) as n→∞ for every f ∈ X^* (where X^* is a continuous dual function of X).

We note that weak convergence is the convergence of sequence of numbers f(x_n) for every f ∈ X*.

Let {f_n} be a bounded linear functionals on X. The weak* converges of {f_n} means that there exists f ∈ X* such that $f_n(x)$ → f(x) for all x ∈ X and we state this as $f_n \rightarrow$ f.

For applying weak convergence we need to know certain basic properties.

Equality

Let p > 1 and q be defined by $\frac{1}{p} + \frac{1}{q} = 1$, a ≥ 0, b $\geq$ 0,

then a b $\leq \frac{a^p}{p} + \frac{b^q}{q}$

Equality holds if and only if $a^p = b^q$.

Holder's inequality

Let p > 1 and q be defined by $\frac{1}{p} + \frac{1}{q} = 1$, for any numbers $a_1, a_2, \ldots, a_n \geq 0$ and

$b_1, b_2, \ldots, b_n \geq 0$

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k\, p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k\; q\right)^{\frac{1}{q}}$$

Minkowski's inequality

Let p $\geq$ 0, for any numbers $a_1, a_2, \ldots, a_n \geq 0$ and $b_1, b, \ldots, b_n \geq 0$, then

$$\left(\sum_{k=1}^{n}(a_k + b_k)\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n}(a_k)\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}(b_k)\right)^{\frac{1}{p}}.$$

It is note that for any n ϵ N,

$R^N = \{x = (x_1, x_2, \ldots) : x_i \in R, i = 1, 2\ldots, n$ is a metric space with metric 'd'

defined by

$g(x, y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2}$, for x , y $\in$ R and for any n $\in$ N,

$C^n = \{x = (x_1, x_2, \ldots, x_{n)} : x_i \in C, i = 1, 2, \ldots, n\}$

is a metric space with the metric

$$d(x, y) = \sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}, \qquad x, y \in C^n.$$

Definition

A topological linear space (X, τ) is a linear space with topology τ on X such that that the addition and scalar multiplication are continuous in (X, τ). If the topology τ on X is given by a metric(respectively, semi metric) then we speak of a metric linear space(respectively, semi metric linear space). In other words if the topology τ on X is given by a metric in the Topological vector space (TVS) then the space is called a metric linear space.

Definition

A vector space V along with a function(called Inner Product or Scalar Product) whose domain is the set all ordered pairs of members of V and range consists of scalars (real or complex numbers), which satisfies the following axioms with (x, y) denoting the inner product of x and y:

(1) $\langle ax, y\rangle = \bar{a}\langle x, y\rangle$

(2) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(3)$\langle x, y\rangle = \langle \bar{y}, x\rangle$, $\langle x, y\rangle$ is a real and positive if $x \neq 0$ [sometimes the first of these axioms is replaced by $\langle ax, y\rangle = a\langle x, y\rangle$ in which case $\bar{a}\langle x, y\rangle$].

If a norm is defined by letting $x = \langle x, x\rangle$, then the space becomes a normed vector space. A finite dimensional inner product space with real scalars is an Euclidean space and finite dimensional inner product space and is a finite dimensional inner product space with complex scalars is an unitary space. An

inner product space is also called a generalized Euclidean space or Pre- Hilbert space.

Definition

A complete inner product space is called a Hilbert Space. Note that the set of all infinite sequencer = $\langle x_1, x_2, x_3, \ldots \rangle$ of complex number.

Definition

The strict inductive limit of an increasing sequence of Banach spaces will be

called LB-space. Let us mention just a few as examples:

(i) (R^1,$R^{2\cdots}$ R^n), (C^1, C^2,…C^n), (l_∞, c, c_o),

(ii) Spaces of continuous functions on compact spaces,

(iii) The L^P-spaces that occur in integration theory,

(iv) Hilbert spaces,

(v) Closed relative of Euclidean spaces etc.

Definition

A Banach space is a normed linear space which is complete in the matrix defined by its norms. This means that every Cauchy sequence is required to converge in the Babach Space. Many of the best known function spaces are the Banach spaces. Thus, A Banach Space $(X,\| \, . \, \|)$ is a complete normed space where completeness means that every sequence (x_n) in X with

$\| x_m - x_n\| \rightarrow 0$ as $m, n \rightarrow \infty$, such that $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

Definition

A sequence space X with a linear topology τ is called K- space provided each of the maps p: $X \rightarrow C$ defined by $P(x) = x$ is continuous for

(i =1, 2, 3,…).

Definition

A K- space X is called a Banach K- space (FK space) provided X is a

F-Space.

Definition

A K- space X is called a Banach K space (BK- space).

Definition

A K-space X is an LBK-space provided X is a LB-space.

Theorem (Kojima –Schur)

A ∈ (C, C) iff

$(i) \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} a_{nk} = a_k$ and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = \alpha$.

Here the matrices A in the class (c, c) are known as conservative or convergence preserving matrices.

Theorem (Duals)

If X is a sequence space, we define

$(i) x^{\alpha}$={ a = (a_k) : $\sum_{k=1}^{\infty} |a_k x_k| < \infty$, for each x ϵ X}

(i) x^{β} = {a = (a_k) : $\sum_{k=1}^{\infty} a_k x_k$ is convergent for each x ϵ X}

(ii) x^{γ}={ a = (a_k) : $\sup_n | \sum_{k=1}^{n} a_k x_k | < \infty$, for each x ϵX}, then

x^{α}, x^{β} and x^{γ} are called the α-(or Kothe-Toeplitz),

β-(or generalized Kothe- Toeplitz) and γ -dual spaces of X respectively.

These duals introduced by Garling.

Note that $\phi \subset x^{\alpha} \subset x^{\beta} \subset x^{\gamma}$.

Theorem (Schur, 1921) $A \in (l_{\infty}, lc)$ if

(i) $\sum_{k=1}^{\infty} |a_{nk}|$ Convergesuniformly for $n \in N$.

(ii) There exists $\lim_n a_{nk} = a_k$ for each $k \in N$.

Here the class (l_{∞}, c) was obtained by Schur in 1921. The characterization of this class is known as Schur theorem and the matrices in this class (l_{∞}, c) are known as Schur matrices.

Theorem (Cohen and Dunford, 1937)

$$A \in (l, l_p) \text{ iff } \sup_k \left(\sum_{n=1}^{\infty} |a_{nk}|^p\right) < \infty, \text{ where } (1 < p < \infty).$$

Theorem (Knopp and Lorentz, 1949)

$$A \in (l, l) \text{ iff } \sup_k \left(\sum_{k=1}^{\infty} |a_{nk}|\right) < \infty.$$

It is noted that $a_{nk}x_k$. The order of matrix becomes $A_n(x)$: Bounded linear operator. In every cone $A \in (X, Y)$ $A_x \in Y$ converges.

1.4. Some important results

Here, we introduce some important results:

Lemma (weak convergence)

Let (x_n) be a weakly convergent sequence in a normed space X, i.e. $x_n \xrightarrow{w} x$, then the weak limit of (x_n) is unique.

(a) Every subsequence of (x_n) converges weakly to x.

(b) The sequence $(\|x_n\|)$ is bounded .

Proof : Suppose that $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$, then

$f(x_n) \to f(x)$ and $f(x_n) \to f(y)$.

Since $(f(x_n))$ is a sequence of numbers $f(x_n)$, weak limit is unique.

For every $f \in X^*$, $f(x) \to f(y) = f(x - y) = 0$

implies $(x - y) = 0$ and shows that the weak limit is unique.

(b) This follows from the fact that $(f(x_n))$ is a convergent sequence of numbers so at every subsequence of $(f(x_n))$ converges and has the same limit as the sequence.

(c) Since $(f(x_n))$ is a convergent sequence of numbers, it is bounded say $|f(x_n)| \leq C_j$ for all n , where C_j is a constant depending on f but not on n .

Using the canonical mapping $X \to X^n$, we can define g_n on X* by

g_n (x) = f(x_n) , f ∈ X*, then

$|g_n(x)| = |f(x_n)| < C_j$

That is, the sequence ($|g_n(x)|$) is bounded for every f ∈ X*.

Since X* is complete, the uniform boundedness theorem is applicable and implies that ($\|g_n\|$) is bounded.

We know from an earlier result that for every fixed x in a normed space X, the functional g_x is defined by

$g_x(f) = f(x)$, where f ∈ X* is a bounded linear functional

on X*, so that f ∈ X** and has the norm $\|g_n\| = \|x\|$.

Theorem [Mishra S. K., Parajuli V. and Ray S., 1915]

$\overline{sl(p)}$ linearly is isomorphic to $\overline{sl_p}$.

Proof : For each x ∈ $\overline{sl(p)}$, we have Ax ∈ $\overline{l(p)}$, where

$$A = (a_{nk}) = \begin{cases} 2^{-n} & n = k \\ 0 & \text{otherwise} \end{cases}$$

Moreover, A is linear and bijective.

Also, the matrix B = (b_{nk}) defined by

$$b_{nk} = \begin{cases} 2^n & n=k \\ 0 & otherwise \end{cases}$$

is inverse of A. Thus $\overline{sl(p)}$ linearly is isomorphic to $\overline{sl_p}$.

Corollary

$\bar{s}l_p$ and $\bar{l}_p$ are linearly isomorphic.

Proof :

Using the same arguments in theorem (1.4.2.), it can be shown that $\bar{s}l_p$ and $\bar{l}_p$ are linearly isomorphic.

Theorem [Mishra S. K., Parajuli V. and Ray S., 1915]

$\overline{sl(p)}$ is complete paranormed space paranormed by

$$g(x) = \left(\sum_{k=1}^{\infty} \frac{1}{2^k} |t_r(x)|^{p_k}\right)^{\frac{1}{M}}, \text{ where } M = \max\left(\frac{1}{2}, \sup_k \frac{p_k}{2^k}\right)$$

Proof : $\overline{sl(p)}$ and $\overline{l(p)}$ linearly isomorphic and $\overline{l(p)}$ is completely paranormed space $\bar{s}l_p$ with paranorm

$g(x) = (\sum_{k=1}^{\infty} |t_r(x)|^{p_r})^{\frac{1}{M}}$, where $M = \max(1, p_k)$, then $\overline{sl(p)}$ is a completely paranormed space with paranorm

$g(x) = P(Ax)$; where p is usual paranorm on $\overline{l(p)}$.

Theorem [Mishra S. K., Parajuli V. and Ray S., 1915]

A Banach space for $1 \leq p < \infty$ and $t_0(x) = 0$; normed by

$$\|x\| = (\sum_{k=0}^{\infty} \frac{1}{2^k} |t_k(x)|^p)^{\frac{1}{p}}.$$

Proof : The proof follows immediately by using the fact that

$\|x\| = \|Ax\|_p$, where $\|.\|_p$ is the usual nor on $\bar{l}_p$.

Theorem [Mishra S. K., Parajuli V. and Ray S., 1915]

$\bar{sl}_2$ is a Hilbert space with inner product

$$\langle x, y\rangle = \sum_{k=1}^{\infty} 2^{2k} t_x(x)\overline{t_k(y)},$$ where bar denotes the conjugate.

Proof :

We have $\bar{l}_2$ is a Hilbert space with inner product

$$\langle x, y\rangle = \sum_{k=1}^{\infty} t_x(x)\overline{t_k(y)}.$$

Also, from theorem (1.4.2.) for $x \in \bar{s}l_2$, $Ax \in \bar{l}_2$; we can easily see that $\bar{s}l_2$ is also a Hilbert space.

Theorem [Mishra S. K., Parajuli V. and Ray S., 1915]

If z be a closed subset of $\overline{l(\mathrm{p})}$, then $\frac{z}{2^k}$ is a closed subset of $\overline{sl(\mathrm{p})}$.

Proof :

Since $z \in \overline{l(\mathrm{p})}$, $\frac{z}{2^k} \in sl(\mathrm{p})$.

Let x belongs to closure of $\frac{z}{2^k}$, then there exists a sequence

$(x^n) \subset \frac{z}{2^k}$ such that (x^n) converges to x . This implies that

$G(x^n - x) \to 0$ as $n \to \infty$.

Hence by definition, $t_k{}^n(x) - t_k(x) \to 0$

i.e., $t_k{}^n \to t_k(x)$. This completes the proof .

Theorem (Silverman –Toeplitz, 1911)

If $A = a_{nk}$ (n, k =1, 2, .., ∞) be an infinite matrix and

if A ∈ (c, c, p), where A satisfies the Toeplitz conditions, then A is a Toeplitz matrix, the conditions of the matrices which are in the classes(c, c, p) are called the Toeplitz matrices which preserve the limit of the convergent sequences.

Theorem (Banach – Steinhaus, 1955)

Let $\{T_i\}$ be a nonvoid family of bounded linear transformation from a Banach space X into a normed linear space Y.

If sup $\|T_i x\| < \infty$ for each $x \in X$, then $\sup_i \|T_i\| < \infty$.

Proof :

For each positive integer n, we define

$$F_n = \{x : x \in X \text{ and } \|T_i x\| \leq n \text{ for all } i\}.$$

F_n is clearly a closed subset of X and $X = \coprod_{n=i}^{\infty} F_n$.

Since X is complete by the Balre category theorem , then one of the F_n's say F_{n_0} has a nonempty integer, then

F_{n_0} contains a closed sphere S_0 with centre x_0 and radius $r_0 > 0$.

Thus, $\|T_i x\| \leq n_0$ for $x \in S_0$ and $i = 1, 2, \ldots$,then

$\|T_i(S)\| \leq n_0$ for $i = 1, 2, \ldots$

$\|T_i(S_0 - x_0)\| \leq 2n_0$, for $x_0 \in S_0$.

Hence $\|T_i\| \leq \frac{2n_0}{r_0}$ or every i.

Theorem [Ray S., 1917]

Sequence space $c_0(p, \lambda)_t$ is a linear space paranormed by g and defined by

g(x) = $\frac{sup}{k} |\lambda t_k x|^{\frac{p_k}{M}}$, where

M = max(1, $\frac{sup}{k} p_k$)

= $\frac{sup}{k} |v_k|^{\frac{p_k}{M}}$, where $\{v_k\} = \{\sum_{i=1}^{k}(k-i+1)t_i x_i$

Proof: From the definition of g,

g(x) = 0 ⇔ x = 0 and g(- x) = g(x) for all x ϵ $c_0(p, \lambda)_t$.

To show linearity of $c_0(p, \lambda)_t$ with respect to coordinate-wise addition and

scalar multiplication,

we take two sequences x, y ϵ $c_0(p, \lambda)_t$ and scalars α, β ϵ ℝ.

Since Λ is linear operator, then g(αx, βx) = $\frac{sup}{k} |\Lambda(\alpha tx, \beta ty)|^{\frac{p_k}{M}}$

$\leq$ max {1, |α|} $\frac{sup}{k} |\lambda t x|^{\frac{p_k}{M}}$ + max {1, |β|} $\frac{sup}{k} |\lambda t y|^{\frac{p_k}{M}}$

$= \max\{1, |\alpha|\}\, g(x) + \max\{1, |\beta|\}\, g(y).$

Thus, $g(x + y) \leq g(x) + g(y)$

Let $\{x^n\}$ be any sequence of the points in $c_0(p, \lambda)_t$ such that $g(x^n - x) \to 0$ and $\{x^n\}$ be any sequence of real scalars such that $\alpha_n \to \alpha$.

Now , we have $g(x^n) \leq g(x) + g(x^n - x)$

Further,

$$g(\alpha_n x^n - \alpha x) = \frac{sup}{k} |\Lambda(\alpha_n t^n x^n - \alpha t x)|^{\frac{p_k}{M}}$$

$$\leq \{ |(\alpha_n - \alpha)|^{\frac{p_k}{M}} g(x^n) + |(\alpha_n - \alpha)|^{\frac{p_k}{M}} g(t^n x^n - tx)\})$$

$$< \infty \text{ for all n.}$$

Since $\{g(x^n)\}$ is bounded, then $g(\alpha_n x^n - \alpha x) < \infty$, for all $n \in \mathbb{N}$

That is, the scalar multiplication for g is continuous and therefore g is a paranorm on the sequence space $c_0(p, \lambda)_t$.

Theorem [Ray S., 1917]

If $X(p, \lambda)$ is a complete paranormed space, then $X(p, \lambda)_t$ is also a complete paranormed space.

Proof:

Since $X(p,\ \lambda)_{\mathrm{t}}$ is a linear space, then

function g*(x) = g (t x), where g is the paranorm in X(p, λ), because

g*(0) = 0,

g*(x) = g*(-x) and

g*(x + y) $\leq\ g*(x) +\ g*(y)$ and

$\lambda_n\ \to \lambda$ in C and g*$(x^n - x) \to 0$ as n $\to \infty$

implies that g*($\lambda_n\, x^n$ - $\lambda\, x$) $\to$ 0 as n $\to \infty$, where

$x^n = (x_k^n)_k = (x_{1,}^n\, x_{2,}^n \ldots, x_{k,}^n \ldots)$ and x = (x_k).

Let (x^n) be a Cauchy sequence in $X(p,\ \lambda)_t$, where

$x^n = (x_{1,}^n\, x_{2,}^n \ldots) \in X(p,\ \lambda)_t$, then

(tx^n) = $((t_k x_k^1), (t_k x_k^2) \ldots)$ is a Cauchy sequence in X(p, λ).

Since X(p, λ) is complete, then it converges to (z_k) say.

Let $z_k =\ t_k x_k$, so that $x_k = t_k^{-1} z_{k,}$ then

(tx^n) converges to $(t_k x_k)$ in X(p, λ).

Hence, g$(t_k x_k^n)$ - $(t_k x_k)$ = g{t(x^n - x)} → 0(g$(t_k x_k^n)$ - $(t_k x_k)$

= g{t(x^n - x)} → 0 as n→ ∞,

which implies that g*(x^n – x) → 0 as n → ∞.

Therefore (x^n) is convergent,

consequently $X(p, \lambda)_t$ is a complete paranormed space.

Theorem [Ray S., 1917]

The sequence spaces $X(p, \lambda)_t$ for X ∊ $\{l_\infty, c, c_0\}$ are complete metric

spaces paranormed by g.

Proof:

We take a Cauchy sequence (x^n) in the space $c_0(p, \lambda)_t$, where

$x^n = \{x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \ldots\}$, then for given ε > 0, there exists a positive integer

$c_0(\epsilon)$ such that g$(t^n x^n - t^m x^m) < \varepsilon$ for all $\geq n_0(\varepsilon)$.

Also, from the definition of g for each fixed k ∊ ℕ,

$$|\{\lambda\, t^n x^n\}_k - \{\lambda\, t^m x^m\}_k)|^{\frac{p_k}{M}}$$

$$\leq \frac{sup}{k} |\{\lambda t^n x^n\}_k - \{\lambda t^m x^m\}_k)|^{\frac{p_k}{M}}$$

$< \varepsilon$ for all m, n $\geq n_0(\varepsilon)$

implies that $\{(\lambda t^0 x^0)_k,\ (\lambda\, t^1 x^1)_k,\ (\lambda t^2 x^2)_k, \ldots\}$ is a Cauchy sequence in $\mathbb{R}$ for each fixed k $\in \mathbb{N}$.

Since $\mathbb{R}$ is complete, then sequence t^n converges and

let $\{\lambda t^n x^n\}_k \to \{\lambda\, tx\}_k$ as n $\to \infty$.

For each fixed space k $\in \mathbb{N}$, n $\to \infty$ and n $\geq n_0(\varepsilon)$,

it is clear that $|\{\lambda t^n x^n\}_k - \{\lambda tx\}_k)|^{\frac{p_k}{M}} \leq \frac{\varepsilon}{2}$, then

$x^n = \{x_k^{(n)}\} \in c_0(p,\ \lambda)_t$.

Also, $|\{\lambda\, t^n x^n\}_k|^{\frac{p_k}{M}} \leq \frac{\varepsilon}{2}$ for each fixed space k $\in \mathbb{N}$

Combing ,

$|\{\lambda tx\}_k)|^{\frac{p_k}{M}} \leq \left|\{\lambda\, t^n x^n\}_k - \{\lambda\, tx\}_k)\right|^{\frac{p_k}{M}} + |\{\lambda t^n x^n\}_k|^{\frac{p_k}{M}}$

$\leq \varepsilon$ for all n $\geq n_0(\varepsilon)$.

Hence, the sequence $\{\lambda tx\} \in n_0(\varepsilon)$

Since $\{x^n\}$ was an arbitrary Cauchy sequence in $c_0(p,\ \lambda)_t$, then space $c_0(p,\ \lambda)_t$ is complete. Hence the theorem is proved.

Theorem [Ray S., 1917]

$X(p, \lambda)_t$ is an $\eta -$ space if and only if $X(p, \lambda)$ is an $\eta -$ space, where

$\eta = \alpha, \beta$ or γ.

Proof: Let $X(p, \lambda)$ be an $\eta -$ space, then $\{X(p, \lambda)\}^{\eta\eta} = X(p, \lambda)$.

Now $[\{X(p, \lambda)\}_t]^{\eta\eta} = \{X(p, \lambda)^{\eta\eta}\}_t$

$$= X(p, \lambda)_t.$$

Thus $X(p, \lambda)_t$ is an $\eta -$ space.

Conversely, if $X(p, \lambda)_t$ is an $\eta -$ space, then

$[\{X(p, \lambda)\}_t]^{\eta\eta} = \{X(p, \lambda)\}_t$ which implies that

$[\{X(p, \lambda)\}^{\eta\eta}]_t = X(p, \lambda)_t$, then it follows that

$\{X(p, \lambda)\}^{\eta\eta} = X(p, \lambda)$, is an $\eta -$ space.

Theorem [Ray S., 1917]

Let η denotes α, β or γ, then

(i) If the $\eta -$ doal $\{X(p, \lambda)\}^{\eta}$ exists, then $[\{X(p, \lambda)\}_t]^{\eta}$

exists and

$[\{X(p,\ \lambda)\}_t]^\eta = \{a = < a_k > : < \frac{a_k}{x_k} > \epsilon\ X^\eta\} = < X^\eta >_t$

(ii) If $\{X(p,\ \lambda)\}^{\eta\eta}$ exists, then $[\{X(p,\ \lambda)\}_t]^{\eta\eta}$ exists and

$[\{X(p,\ \lambda)\}_t]^{\eta\eta} = \{x = (x_k) : (t_k x_k) \epsilon \{X\ (p,\ \lambda)\}^{\eta\eta}$

$= [\{X\ (p,\ \lambda)\}^{\eta\eta}]_t$

Proof: Let $\eta = \alpha$ and D = $\{a = (a_k) : (\frac{a_k}{t_k})\ \epsilon\ \{X\ (p,\ \lambda)\}^\alpha\}$.

We show that $[\{X\ (p,\ \lambda)\}_t]^\alpha = D$.

Let a $\in$ $[\{X(p,\ \lambda)\}_t]^\alpha$, then

$\sum_k |a_k x_k| < \infty$ for every $x \in \{X(p,\ \lambda)\}_t$ so that

$\sum_k |\ \frac{a_k}{t_k} t_k x_k\ | = \sum_k |a_k x_k| < \infty$.

Since $(t_k x_k)\ \epsilon\ X\ (p,\ \lambda)$ implies that $\alpha \in D$.

Hence $[\{X(p,\ \lambda)\}_t]^\alpha \subset D$.

Conversely,

if $\alpha\ \epsilon$ D and x $\{X(p,\ \lambda)\}_t$, then $(t_k x_k) \in X(p,\ \lambda)$

$$\text{so that } \sum_k |a_k x_k| = \sum_k |\ \frac{a_k}{t_k} t_k x_k\ | < \infty.$$

As $x \epsilon \{X(p, \lambda)\}_t$, then a $\in$ $[\{X(p, \lambda)\}_t]^\alpha$.

Hence D $\subset$ $[\{X(p, \lambda)\}_t]^\alpha$.

Consequently $[\{X(p, \lambda)\}_t]^\alpha = [\{X(p, \lambda)\}^\alpha]_t$.

For $\eta = \beta$ and $\eta = \gamma$ the proofs are similar; therefore we omit them.

(i)Let $\eta = \alpha$ and let $\{X(p, \lambda)\}^{\alpha\alpha}$ exists, then

$[\{X(p, \lambda)\}_t]^{\alpha\alpha} = [[\{X(p, \lambda)\}_t]^\alpha]^\alpha = [[\{X(p, \lambda)\}^\alpha]_t]^\alpha = \{X(p, \lambda)^{\alpha\alpha}\}_t$.

For $\eta = \beta$ and $\eta = \gamma$ the proof follows as in (i).

1.5. Some Applications

The theory of sequence space occupies a very important position in Analysis which has applications in several branches of mathematics. The study of sequence spaces is being the subject of wonderful interest due to its beauty, sweetness and the aestheticism as well. The idea of the sequence spaces was motivated through the classical results in summability theory which were first introduced Emesto Cesaro, Emile Borel, Niels Erik Norlund and others. The general theory of matrix transformation was special and classical results in Summability "The most general Linear operator to transform from new sequence space into another sequence space is actually given by an infinite matrix" to preserve the limit of the convergent sequences.

Otto Toeplitz used to characterize matrix transformation. Toeplitz matrix transformations (linear operators) to transform one sequence space into another sequence.

The first application of functional analysis to the summability theory was done by Mazur Stain's law in 1927, when he proved his famous Mazur`s consistency theorem from which he won the prize of the university LWOW in 1932. Later Banach used the Banach-Steinhaus theorem to give proof of Kojima – Schur theorem and hence that of the Toeplitz theorem.

We are mainly concerned with matrix transformation which is a method where the most general linear operator to transform from new sequence space into another sequence space is actually given by an infinite matrix.Matrix transformation theory deals with the characterization of matrix mappings between sequence spaces by giving necessary and sufficient conditions of the infinite matrices.

The matrix transformation plays the very important roles in the study of sequence spaces. It has highly significant applications in the several branches of mathematics like engineering fields and mathematical analysis and without the knowledge of matrix transformation, the study of sequence space becomes incomplete. From mathematical knowledge, skills and attitudes, we can able to solve all problems and reach in the truth, beauty and goodness.

Research Gap

In 1969, Maddox I. J. has defined the sequence space

l(p) ={x = (x_k) : $\sum_k |x_k|^{p_k} < \infty$} and the Kothe –Toeplitz duals of sequence space l(p) is found to observed so far.

If p = (p_k) is a bounded sequence of strictly positive real numbers, then

l_∞(p) = {x = (x_k) : $\sup_k |x_k|^{p_k} < \infty$},

c(p) = {x = (x_k) : $|x_k - l|^{p_k} \to 0$ for some $l \in$ C} and

c_o(p) = {x = (x_k) : $|x_k|^{p_k} \to 0$ (k→ ∞)} .

The space c_o(p) is metric linear space paranormed by $\|x\| =$ g(x) = $\sup_k |x_k|^{\frac{p_k}{M}}$

The spaces l_∞(p) and c(p) are paranormed by g(x) = $\sup_k |x_k|^{\frac{p_k}{M}}$

if inf $p_k > 0$.

Let t = (t_k) be any fixed sequence of nonzero complex numbers satisfying $\lim_k \inf(t_k)^{\frac{1}{k}}$ = r $(0 < r \leq \infty)$.

In 1998, Mursaleen A. K. and Saifi A. H. have introduced a new sequence space $l_\infty(\Delta_r p)$, where

$$l_\infty(\Delta_r p) = \{x = (x_k) : \Delta_r,\ x \in l_\infty(p)\}$$

$$= \{x = (x_k) : \sup_k |\ k^r (x_k - x_{k+1})\ |^{p_k}| < \infty < r < 1\}$$

$$l_\infty(p) = \{x\ (x_k) : \sup_k |x_k\ |^{p_k} < \infty\}$$

$$\Delta_r(x) = (k^r \Delta x_k)^\infty_{k=1},\ r < 1 \text{ and } \Delta x_k = (x_k - x_{k+1}).$$

In 2014, Parajuli V. and Mishra S.K. introduced and studied sequence space $l(p,\ \lambda)$ as

$$l(p,\ \lambda) = \{x = (x_k) \in \omega : \lambda\, x \in l(p)\}, \quad \text{where}$$

$$S^n = \lambda = (\lambda_{nk}) = \begin{cases} n - k + 1 & n \geq k \\ \\ 0 & \text{otherwise} \end{cases}$$

where $l(p,\ \lambda)$ is the set of all sequences $\{u_k\}$

whose $S^n = \lambda -$ transforms are in the sequence space $l(p)$.

Thus, we have $l(p,\ \lambda) = [l(p)]_\lambda$, where the sequences

$\{\Delta u_k - u_{k-1}\} \in \overline{l(p)}$ with $u_0 = 0$ and

$$\{u_k\} = \{\sum_{i=1}^{k} (k - i + 1)\, x_i\}.$$

As a motivation, these sequence spaces are the major sequence spaces to establish the new sequence spaces like

$X(p, \lambda)_t = \{x = (x_k) : (t_k x_k) \in X(p, \lambda)$ and

$X(\Delta_r p, \lambda)_t = \{x = (x_k) : (t_k x_k) \in X(\Delta_r p, \lambda)\}$ in this work. For higher dimensional classical properties, matrix transformation is one of the dominational tool. Also, as a sequence space is a mathematical structure, it is necessary to study its connection with Vedic notion relative to compatable structure. These are the important factors for the current research work.

CHAPTER TWO

SOME PARANORMED SPACES AND MATRIX TRANSFORMATION

This chapter introduces the ideas of some important definitions, new sequence spaces which will be used to study existing sequence spaces and establish new sequence spaces for their matrix transformation in determining the main results on matrix transformations between sequence spaces with applications.

2.1. Introduction

In this unit, we have the following useful definitions and classical results from the book of Maddox I. J. and Michael A. L., 1974.

Definition

A paranormed space is a linear space X together with a paranorm g over the field R if there is a subadditive function $g : X \rightarrow R$ such that

(i) $g(x) = 0$, where zero is element in X,

(ii) $g(x) = g(-x)$ for all $x \in X$ and

(iii) $\lambda_n \rightarrow \lambda$ and $g(x_n - x) \rightarrow 0 \Longrightarrow g(\lambda_n x_n - \lambda x) \rightarrow 0$ for all $\lambda \in C$ and for $x \in X$, the linear space.

A paranorm is said to be total if $g(x) = 0$ implies $x = 0$.

Every paranormed space (total paranormed space) is a semi metric linear space.

Conversely, a semi metric linear space can be made into a paranormed space. So, the total paranormed space and semi metric linear space are essentially the same.

The sequence spaces are typically equipped with a norm.

Definition

A norm on space X is a real function $\|.\| : X \to R$ defined on such that for any $x, y \in X$ and for all $\lambda \in K$ as

(1) $\|x\| > 0$,

(2) $\|x + y\| \leq \|x\| + \|y\|$,

(3) $\|\lambda x\| = |\lambda| \, \|x\|$ and

(4) $\|x\| = 0$ implies $x = 0$.

Definition

A p-norm on space X is a real function $\|.\| : X \to R$ defined on such that for any x, y $\in$ X and for all $\lambda \in$ K as

(1) $\|x\| > 0$,

(2) $\|x + y\| \leq \|x\| + \|y\|$ and

(3) $\|x\| = 0$ implies x = 0.

We note that a p-norm with p = 1 is just a norm.

Definition

A semi norm on space X is a real function $\|.\| : X \to R$ defined on such that for any x, y $\in$ X and for all $\lambda \in$ K as

(1) $\|x\| > 0$,

(2) $\|x + y\| \leq \|x\| + \|y\|$ and

(3) $\|\lambda x\| = |\lambda| \, \|x\|$

This concept is important in the theory of topological vector spaces.

Every semi-norm is a paranormed (total paranormed) but not conversely.

Definition

A normed linear space is a vector space with a norm. We shall abbreviate normed linear space as nls. The space c[a, b] is a normed linear space with the norm $\|f\| = \sup_{x \in (a,b)} |f(x)|$, where c[a, b] is a set of continuous functions on [a, b].

Every normed linear spaces may be regarded as a metric space together with metric d(x, y), that is distance between x and y is $\|x - y\| = d(x, y)$.

Since l_∞, c, c_o are the normed linear spaces with norm $\|x\| = \sup_{x \in k} |x_n|$ and they are bounded, convergent and null sequence space respectively.

Since x = (x_n) is called bounded iff there exists $M \geq 0$ such that

$M \geq |x_n|$ for all n ∈ N, x = (x_n) is called convergent iff for every $\epsilon > 0$, there exists $N_0 = N_0(\epsilon)$ such that $|x - l| < \epsilon$ for all $n \geq N_0$ and the set of all null sequence is c_0.

We shall assume $\{p_k\}$ be a bounded sequence of strictly positive real numbers with sup p_k = H and M = max.{1, H}.

The linear space $l(p)$ was defined by

$$l(p) = \{x = (x_k) \in \textstyle\sum_k |x_k|^{p_k} < \infty\}$$

which is a complete space paranormed by

$$g(x) = (\textstyle\sum_k |x_k|^{p_k})^{\frac{1}{M}}.$$

If p = (p_k) is a bounded sequence of strictly positive real numbers, then

$$l_\infty(\text{p}) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\},$$

$$c\,(\text{p}) = \{x = (x_k) : |x_k - l|^{p_k} \to 0 \text{ for some } l \in \mathbb{C}\} \text{ and}$$

$$c_o(\text{p}) = \{x = (x_k) : |x_k|^{p_k} \to 0 \ (k \to \infty)\}, \text{ where}$$

the spaces l_∞(p) and c(p) are paranormed by g(x) = $\sup_k |x_k|^{\frac{p_k}{M}}$

if inf $p_k > 0$ and the space c_o(p) is metric linear space paranormed by

$$\|x\| = g(x) = \sup_k |x_k|^{\frac{p_k}{M}}.$$

We defined new sequence space X(p, λ)$_t$ by

X(p, λ)$_t$ = {x = (x_k) : ($t_k x_k$) ∈ X(p, λ)}, where

X(p, λ) ={x = (x_k) : λx ∈ X}, $\lambda = \begin{bmatrix} 1 & 0 & \dots \\ 2 & 1 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$ and X = $\{l_\infty, c, c_0\}$.

This space is paranormed by g*(x) = g(t x), where g is paranorm in $X(p, \lambda)$.

2.2. Sequence space $X(p, \lambda)_t$

Definition (Dual Space)

If X is a sequence space, then β − dual of X defined by

X^{β} = {a = {a_k}: $\sum_{k=1}^{\infty} a_k x_k$ is convergent for each x ∈ X}.

We call X^{β} - (or generalized Kothe -Toeplitz) dual space of X.

Theorem

Let $p_k > 0$ for every k ∈ ℕ, then $l_{\infty}^{\beta}(p, \lambda)_t = \overline{M_{\infty}(p,\ \lambda)_t}$, where

$\overline{M_{\infty}(p,\ \lambda)_t} = \bigcap_2^{\infty}\{a = (a_k): \sum_{k=1}^{\infty} |\Delta^2 a_k| N^{1/p_k} < \infty\}$ and

$\Delta^2 a_k = \Delta a_k - \Delta a_{k+1}$

Proof: Let a ∈ $\overline{M_{\infty}(p,\ \lambda)_t}$ and x ∈ $l_{\infty}(p,\ \lambda)_t$,

choosing an integer N > max (1, $\sup_k |u_k|^{p_k}$), then

$|\sum_{k=1}^{m} a_k x_k| = |\sum_{k=1}^{m} (\Delta a_k - \Delta a_{k+1}) v_k|$,

where $v_k = \sum_i^k (k - i + 1)\ t_i x_i$

$= \sum_{k=1}^{\infty} |\Delta^2\ a_k v_k|$

$\leq \sum_{k=1}^{\infty} |\Delta^2\ a_k|\ |v_k|$

$\leq \sum_{k=1}^{\infty} |\Delta^2\ a_k| N^{1/p_k}$

$< \infty$, then

$l_0^{\beta}(p,\ \lambda\,)_t = \overline{M_0(p,\ \lambda)}_t.$

Matrix transformation

The array of numbers in rows and columns enclosed in square bracket is matrix and matrix transformation is relation between sequence spaces.

Example: If X and Y be any two non empty subsets of space of all complex sequences and A= (a_{nk}) be an infinite matrix of complex numbers, then sequence Ax = A_n(x) if $A_n(x) = \sum_k a_{nk}x_k$ converges to each n.

If x = (x_n) ∈ X ⇒ Ax = A_n(x) ∈Y, then matrix A transforms from X into Y.

It is an important tool for mathematical modeling.

Example: If 3x + 4y = 5

4x – 3y = 8, then it can be transformed to matrix form

A$\overline{x}$ = $\overline{y}$, where A = $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\bar{y} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ if $|A| \neq 0$.

Theorem

Let p_k > 0 for every k ∈ ℕ, then A ∈ $((l_\infty(p,\ \lambda)_t,\ l_\infty)$

iff $\sup_n \sum_{k=1}^{\infty} |\Delta^2\, a_{nk}|\, N^{\frac{1}{p_k}} < \infty$ for every integer N > 1.

Proof:

Let the condition holds, then $\sup_n \sum_{k=1}^{\infty} |\Delta^2\, a_{nk}|\, N^{\frac{1}{p_k}} < \infty$.

We take x ∈ $l_\infty(p,\ \lambda\,)$, then λx ∈ l_∞(p), hence $\lambda tx \ \in\ l_\infty(p)_t$.

Hence $\sup_k |\lambda tx|^{p_k} < \infty$.

So there exists an integer N ≥ 1 such that $|\lambda tx| \ \leq\ N^{\frac{1}{p_k}}$, then

$|\sum_{k=1}^{\infty} a_{nk}x_k|$ = $|\sum_{k=1}^{\infty}\Delta^2\ a_{nk}v_k|$, where v_k = $\sum_{i=1}^{k}(k-i+1)\, x_i$

$$\leq \sum_{k=1}^{\infty} |\Delta^2\, a_{nk}|\, |v_k|$$

$$\leq \sup_n \sum_{k=1}^{\infty} |\Delta^2\, a_{nk}|\, N^{\frac{1}{p_k}}$$

$$< \infty.$$

Hence, $\sum_{k=1}^{\infty} a_{nk}x_k$ converges for each n ∈ ℕ and Ax ∈ l_∞.

On the other hand, let A ∈ $((l_\infty(p,\ \lambda)_t,\ l_\infty)$.

As a contrary, let us assume that there exists an integer such that $\sup_n \sum_{k=1}^{\infty} |\Delta^2\, a_{nk}|\, N^{\frac{1}{p_k}} < \infty$, then

the matrix $(\Delta^2\, a_{nk}) \notin (l_\infty(p,\ \lambda)_t,\ l_\infty)$ and so there exists

$y = (y_k) \in l_\infty$ with $\sup_k |y_k| = 1$ such that $\sum_k \Delta^2 \, a_{nk} y_k \neq 0(1)$.

Although, if we define the sequence $\mu = \{\mu_k\}$ by

$\mu_k = y_{k-2} - 2y_{k-1} + y_k$ with $y_j = 0$ for $j \leq 0$,

$= t_{k-2}y_{k-2} - 2t_{k-1}y_{k-1} + t_k y_k$, putting $k = 1, 2, 3, \ldots$

$= t_1 y_1 + t_2(y_2 - 2y_1) + \cdots$, then $\mu = l_\infty(p, \lambda)_t$

and therefore $\sum_{k=1}^{\infty} a_{nk} \mu_k = \sum_{k=1}^{\infty} \Delta^2 \, a_{nk} y_k$.

It follows that the sequence $\{A_n(\mu)\} \notin l_\infty$ is contradiction to our assumption.

Hence $\sup_n \sum_{k=1}^{\infty} |\Delta^2 \, a_{nk}| \, N^{\frac{1}{p_k}} < \infty$, then $A \in (l_\infty (p, \lambda)_t, l_\infty)$.

[Ray, S., Panthi, D., Jha, K. and Mishra, S. K., 2017]

2.3. Sequence space $X(\Delta_r p, \lambda)_t$

We also studied sequence space

$X(\Delta_r p, \lambda)_t = \{x = (x_k) : (t_k x_k) \in X(\Delta_r p, \lambda)\}$.

This space is paranormed by $g^*(x) = g(t\,x)$ where g is paranorm in $X(\Delta_r p, \lambda)$.

Some Classical Results in space $X(\Delta_r p, \lambda)_t$

Theorem: Let $p_k > 0$ for every k, then $A \in (l_\infty(\Delta_r p, \lambda)_t, c_0)$ iff

(i) R $\in(l_\infty(\Delta_r \text{p}, \lambda)_t, c_0)$ where R = $r_{n,k}$ = $[\sum_{v=k}^{\infty} a_{n,v}]$

(n, k = 1, 2, 3,…).

(ii) $A_n[\sum_{i=1}^{k} N^{1/p_i} \in c_0$ (n, k = 1, 2, 3,…), for all integers

N > 1 and

(iii) $\lim_{n\to\infty} a_{n,k} = \alpha_k$ (k = 1, 2, 3,…)

Proof: Let us first prove the sufficiency part.

We consider any x $\in(l_\infty(\Delta_r p, \lambda)_t)$ for N > 1

so that $\sup_k |\Delta x_k|^{p_k} < \text{N}$.

We write $\sum_{k=1}^{\infty} a_{nk}\, \alpha_k = \sum_{k=1}^{\infty} r_{nk}\, \Delta x_k - r_{m+1}\text{m}\Delta x_k$ (m = 1, 2, 3,…)

By condition (ii) $\sum_{k=1}^{\infty} a_{nk}\, [\sum_{i=1}^{k} N^{\frac{1}{p_i}}]$ is convergent by (i) for each n

(n = 1, 2, 3,…).

Hence, it follows that $\lim_{m\to\infty}$ $\text{m}\sum_{i=1}^{m} N^{\frac{1}{p_i}} = 0$.

By condition (i) R $\in l_\infty((\Delta_r \text{p}, \lambda)_t, c_0)$ and x $\in l_\infty(\Delta_r \text{p}, \lambda)_t$

iff $\Delta x = l_\infty(\Delta_r \text{p}, \lambda)_t$.

Hence by corollary (ii) in it follows that $\sum_{k=1}^{\infty}\left|r_{n,k}\right| N^{\frac{1}{p_k}}$ is uniformly convergent in n and $\lim_{n\to\infty} r_{nk}$ exist for each k = 1, 2, 3,...

Since $\sum_{k=1}^{\infty}\left|r_{n,k}\right|\left|\Delta x_k\right| \leq \sum_{k=1}^{\infty}\left|r_{n,k}\right| N^{\frac{1}{p_k}}$,

we find that $\sum_{k=1}^{\infty} a_{nk}\, x_k$ is absolutely and uniformly convergent in n.

Finally, we have $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}\, x_k = \sum_{k=1}^{\infty} a_k\, x_k$.

The necessities of (iii) and (ii) are respectively obtained by taking

x = e = (1, 1, 1,…) ∈ $l_\infty(\Delta_r \text{p},\ \lambda)_t$ (is bounded) and X = $[\sum_{k=1}^{\infty} N^{\frac{1}{p_i}}]$

(k = 1, 2, 3,…).

Now, we consider the necessity of (i). If it is not true, then there exists

x = $\{x_k\}$ ∈ $l_\infty(\Delta_r \text{p},\ \lambda)_t$ with $\sup_v |x_v|^{p/v}$ = 1 such that

$[\sum r_{nv} x_v]^\infty \notin$ c.

Although, if we define a sequence y = $\{y_k\}$ by $y_v = \sum_{i=1}^{v} x_i$

(v = 1, 2, 3,…), then y ∈$l_\infty(\Delta_r \text{p},\ \lambda)_t$.

But that $[\sum_{v=1}^{\infty} a_{nv}\, y_v = \sum_{v=1}^{\infty} a_{nv}\, x_v] \notin$ c.

This contradicts the fact that A ∈ $l_\infty((\Delta_r \text{p},\ \lambda)_t,\ c_0)$ and therefore (i) must hold.

2.4. Some Applications

The matrix transformation plays the very important roles in the study of sequence spaces. It has highly significant applications in the several branches

of Mathematics and without the knowledge of matrix transformation the study of sequence space becomes incomplete. The theory of sequence space occupies a very important role for matrix transformation in Analysis which has many applications in mathematics and engineering field. The study of sequence spaces is a special case of the general study of function space which is a branch of functional analysis, where we are concerned mainly on the matrix transformation problems. Matrix transformation deals with the characterization of matrix mappings between sequence spaces by giving necessary and sufficient conditions of the infinite matrices. A fine account of these results can be found in infinite matrices of operators. The study of Orlicz and Lorentz spaces was initiated with certain specific purposes in Banach space theory. Indeed Linderberg got an interested result on Orlicz spaces in connection. With finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_o or l_p ($1 \leq p \leq \infty$).

As conclusion, we introduce matrix transformation which deals with the characterization of matrix mappings between sequence spaces by giving necessary and sufficient conditions of the infinite matrices.

CHAPTER –THREE

SOME SEQUENCE SPACES, FUNCTIO NSPACES ON [0, 1] FOR DNA – SEQUENCING

This chapter introduces the ideas of some sequence spaces, function spaces, main results and practical applications to find the real application of sequence space related to matrix transformation for DNA – sequencing [Ray S., 2018].

3.1. Introduction

John Maynard Smith in 1970, first introduced the notion of sequence space for protein evolution. He proposed a "sequence space" where all possible proteins are arranged in a protein space in which neighbors can be interconnected by single mutation. These problems are not only unique to protein structures but relevant to many other areas such as DNA (Deoxyribonucleic acid) sequence which is a specific sequence of all little bases of nucleotide A(Adenine), C(Cytosine), T(Thymine), G(Guanine) and are linked in different orders in extremely long DNA molecules. In these area of interest, dimensional and complex, the sample size is relevantly small, they posses finitely many non zero elements in the sequence and some elements in the sequence repeat many times. When the data received from the reservoir to obtain some information have lower dimension and samples have larger size, the statistical methods such as that the covariance matrix, dot matrix and position weight matrix can deal with the cases promptly in a simplified way. However, when data have multidimensional character and the sample size is smaller, the statistical methods may lead to errors.

The necessity of the new definition of norm to fit a given data ' α ' in a set some of class samples S as follows:

Set S as a set of some class samples and α as a given data.

A simpler approach is to consider problem $\inf_{s \in S} \|\alpha - s\|_p$, where p

denotes the norm in l_p space. In most cases, there is at least one $s_0 \in S$ such that $\|\alpha - s_0\|_p = \inf_{s \in S} \|\alpha - s\|_p$.

To see disadvantage, we divide sequence $s \in S$ into three segments

(s_1, s_2, s_3) where the first segment s_1 is composed of the first n_1 elements, the second segment s_2 is composed of the second n_2 elements and the third segment s_3 is composed of the third n_3 elements.

Similarly, we also divide α into three parts $(\alpha_1, \alpha_2, \alpha_3)$.

We consider, $\inf_{s_1} \|\alpha_1 - s_1\|_p$, $\inf_{s_2} \|\alpha_2 - s_2\|_p$,

$\inf_{s_3} \|\alpha_3 - s_3\|_p$.

Perhaps, we would find that $F(\alpha_1) \cap F(\alpha_2) \cap F(\alpha_3) = 0$.

From this example, we see that we need a new definition of the norm to fit application. Motivated by this questions, we revisit the sequence spaces and function spaces defined on [0, 1]. we examine the behaviors of sequences generated by DNA nucleotides. It has been aimed to extend the results of authors by introducing new function space in [0, 1], extending the basis function $\frac{x^n}{n!}$, introducing a new sequence b = (b_n) = $(\sum_{v=n}^{\infty} a_v)$ which can characterize DNA sequence.

Here, we give some definitions and results:

Definition

DNA sequence is a specific sequence of all little bases each base is either Adenine, Cytosine, Thymine or Guanine.

Definition

DNA stands for Deoxyribonucleic acid which is the chemical stuff.

Structurally, DNA is polymer – a larger structure that is made up of repeating parts of smaller structure – like a brick wall is made up not just one brick but of many similar bricks all closely joined.

Definition

In the DNA polymer, the tiny repeating structure are called Nucleotides. In other words, nucleotides are organic molecules that serve as the monomers or subunits of DNA. The millions of tiny unit nucleotides together form the entire DNA polymer which is called a DNA stand having double helix structure. There are four types of nucleotides A = Adenine, C = Cytosine, G = Guanine and T = Thymine.

Definition

Sequence alignment is the procedure of comparing two (pair – wise alignment) or more multiple sequences by searching for a series of individual characters or patters that are in the same order in the sequences. There are two types of alignment: local and global. Local alignment concentrates on the finding stretches of sequences with high level of matches.

Definition

DNA sequencing is the process of determining the precise order of nucleotides within a DNA molecule. It includes any method or technology that is used to determine the four bases: adenine, cytosine, thymine and guanine in a strand of DNA.

Sequence Space and Function Space on [0, 1] for DNA – Sequencing:

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n, \dots)$ be a DNA sequence, where

$\alpha_n \in$ {A, C, T, G} and

$\alpha(x) = Ap_1(x) + Cp_2(x) + Tp_3(x) + Gp_4(x)$.

Clearly, for different DNA sequence, we have different polynomials $p_j(x)$. It is a simple reverse form. To extend it into a sequence of infinitely many non zero terms, we take $x \in [0, 1]$. Here, $\alpha(x)$ is a generation function in the classical queuing theory. We remark that the generation function is not continuous function defined in [0, 1]. Hence in order to find out a feasible form of $\alpha(x)$, we integrate first and then differentiate:

$$L^1 (1)(x) = \int_0^x 1dx = x$$

$$L^2 (1)(x) = \int_0^x L^1(1)xdx = \frac{x^2}{2!}$$

$$\ldots \quad L^n (1)(x) = \frac{x^n}{n!} \quad \text{for all } n \in \mathbb{N}.$$

For any polynomial

$$p_n (x) = a_0 1 + a_1 x + a_2 \frac{x^2}{2!} + \cdots + a_n \frac{x^n}{n!}$$

$$= [\sum_{k=0}^{n} a_k L^k] (1)(x)$$

Next, we consider the differential operator D for the function $\frac{\mathrm{x}^{\mathrm{n}}}{\mathrm{n}!}$ which yields

$$D^1\left(\frac{x^n}{n!}\right) = D^2\frac{x^{n-1}}{(n-1)!}, \ldots, D^k\left(\frac{x^n}{n!}\right) = \frac{x^{n-k}}{(n-k)!}$$

Therefore, the coefficient sequence is given by

$$(\alpha_0, \alpha_1, \ldots, \alpha_n) = (\mathrm{D}^0, \mathrm{D}^1, \ldots, D^n)\, p_{n\ \mathrm{I}}\ \mathrm{x} = 0$$

Moreover, the polynomial space over [0, 1], denoted by

p[0, 1], is a normed space with the norm

$$\|p\|_{\emptyset} = \sup_{n \geq 0}\{\|D^n p\|_{\infty}\}, \text{ where } \|f\|_{\infty} = \max_{0 \leq x \leq 1}|f(x)|$$

In this space, the integral and differential operations are bounded linear operators. To extend to an infinite sequences, we take a subset

$\mathrm{C}_{\mathrm{M}}^{\infty}$ [0, 1] of C^{∞} [0, 1] defined by

$$\mathrm{C}_{\mathrm{M}}^{\infty} = \{ \mathrm{f} \in C^{\infty}\ [0, 1] : \sup_{n \geq 0}\{\|D^n p\|_{\infty}\} < \infty .$$

Thus, $\mathrm{P}[0, 1] \subset C_M^{\infty}[0, 1] \subset C^{\infty}[0, 1] \subset \mathrm{C}^{\mathrm{k}}[0, 1] \subset \mathrm{C}[0, 1] \subset \mathrm{L}^{\infty}[0, 1]$

$\subset \mathrm{L}^{\mathrm{P}}[0, 1] \subset \mathrm{L}^1[0, 1]$.

But the completion of (p[0, 1], $\|.\|_{\emptyset}$) is not the space (C_M^{∞} [0, 1], $\|.\|_{\emptyset}$).

For the completion of the space (p[0, 1], $\|.\|_{\emptyset}$), authors have defined the following spaces on [0, 1]

$$C_{\emptyset,\ 0}[0,\ 1] = \{f(x) = \sum_{n=0}^{\infty} \alpha_n \frac{x^n}{n!} : \lim_{n \to \infty} \alpha_n = 0\},$$

$$C_{\emptyset,\ p}[0,\ 1] = \{f(x) = \sum_{n=0}^{\infty} \alpha_n \frac{x^n}{n!} : \sum_{n=0}^{\infty} |\alpha_n|^p < \infty\} \text{ for } p \geq 1 \text{ and}$$

$$C_{\emptyset,\ \infty}[0,\ 1] = \{f(x) = \sum_{n=0}^{\infty} \alpha_n \frac{x^n}{n!} : \sup_{n \geq 0} |\alpha_n| < \infty\}.$$

These spaces are isomorphic to c_0, l_p and l_∞ respectively . Thus,

$$P[0, 1] \subset C_{\emptyset,\ 1}[0,\ 1] \subset C_{\emptyset,\ p}[0,\ 1] \subset C_{\emptyset,\ 0}[0,\ 1] \subset C_{\emptyset,\ \infty}[0\ \ 1] = C_M^{\infty}[0, 1],$$

$1 \leq p < \infty$.

We have the following new sequence space and function space on [0, 1] for DNA – sequencing and new set of inclusions:

we define for $x \in [0, 1]$, a polynomial function of order n

$$p_n(x) = \sum_{v=1}^{n} \alpha_v \left[\sum_{k=1}^{v} L^k\right](1)(x), \quad a_0 = 0$$

$$= \sum_{v=1}^{n} \alpha_v \left(\sum_{k=1}^{v} \frac{x^k}{k!}\right),$$ where L is integral operator and $\sum_{k=1}^{v} \frac{x^k}{k!}$ for $v = 1, 2, \ldots, n$.

Further, by using differential operator for the basis function $\sum_{k=1}^{v} \frac{x^k}{k!}$ for v = 1, 2,…, n, We fined that

$$D^k \left[\sum_{k=1}^{v} \frac{x^k}{k!}\right] = \frac{x^{v-k}}{(v-k)!}, \quad 1 \leq k \leq v.$$

Obviously,

$$D^1 p_n(0) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

$$D^2 p_n(0) = \alpha_2 + \cdots + \alpha_n$$

…

$$D^n p_n(0) = \alpha_n.$$

Therefore, the coefficient sequence $b = (b_n)$ is given by

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_n, \ \alpha_2 + \cdots + \alpha_n, \ldots, \ \alpha_n)$$

$$= (D^1, \ D^2, \ldots, D^n)\, p_n(x)\, I_{x=0}$$

Thus, we obtain new coefficient sequence to characterize DNA sequence. With the coefficient sequence $b = (b_k)$ defined by

$b_k = \sum_{v=k}^{n} \alpha_v$ for all k ; we can characterize DNA sequence and the result is helpful to explore for the possible application in DNA

sequencing. Table 1, distribution of the coefficient sequence $b = (b_k)$

$b_1 = \alpha_1\alpha_2\alpha_3 \ldots \alpha_n$

$b_2 = \alpha_2\alpha_3 \ldots \alpha_n$

…

$b_n = \alpha_n$,

where $\alpha_n \in \{A, C, T, G\}$.

In computational process, if we input a DNA sequence, BLAST (Basic Local Alignment Search Tool) will display all possible gene matches with closure similarities between the existing DNA sequence in Gene Bank and the input sequence. The most likely matches will be displayed from top to bottom sequence alignments. The polynomial space p[0, 1] is now a normed space normed by

$$\|p\|_{\psi} = \sup_{n \geq 1} \left\| \sum_{k=1}^{n} (D^k p - D^{k-1} p) \right\|_{\infty}$$

To extend the case to an infinite dimension, we consider a subset of function space C^{∞} [0, 1] defined by

$$C_M^{\infty}\ [0,\ 1] = \{f \in C^{\infty}\ [0, 1] : \sup_{n \geq 1} \left\| \sum_{k=1}^{n} (D^k p \right\|_{\infty} < \infty\},$$

which is a linear space.

Thus, P[0, 1] $\subset C_{\emptyset,\ 1}[0,\ 1] \subset C_{\emptyset,\ p}\ [0,\ 1] \subset C_{\emptyset,\ 0}[0,\ 1] \subset \mathrm{C}_{\emptyset,\ \infty}\ [0,1]\ =$ $C_M^{\infty}\ [0, 1]$, $1 \leq$ p $<\ \infty$.

3.2. Some Classical Results

Here, we introduce some theorems which gives classical results of DNA – sequencing:

Theorem

The space $C_{\psi,\ 0}[0,\ 1]$ = {g(x) = $\sum_{k=1}^{\infty} \alpha_k \left(\sum_{k=1}^{v} \frac{x^n}{v!}\right) : \underset{n \to \infty}{lin}\ b_n = 0$}

is isomorphic to the space $C_{\phi,\ 0}[0,\ 1]$, where b = ($b_n = \sum_{v=n}^{\infty} \alpha_v$).

Proof:

We define an operator T : $C_{\psi,\ 0}[0,\ 1] \rightarrow C_{\phi,\ 0}[0,\ 1]$ by

(b_n) = (α_n) = T((b_n)).

The linearity of T is obvious.

Now, T((b_n)) = g(x)

$$= \sum_{n=1}^{\infty} \alpha_n \frac{x^n}{n!}$$

$$= b_1 \frac{x}{1!} + b_2 \frac{x^2}{2!} +$$

$$= (\alpha_1 + \alpha_2 + \cdots)\frac{x}{1!} + (\alpha_1 + \alpha_2 + \cdots)\frac{x^2}{2!} + \cdots$$

$$= \alpha_1 \frac{x}{1!} + \alpha_2 \left(\frac{x}{1!} + \frac{x^2}{2!}\right) + \alpha_3 \left(\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right) + \cdots$$

$$= \sum_{n=1}^{\infty} \alpha_n \left(\sum_{k=1}^{n} \frac{x^k}{k!}\right)$$

Here T is bijective.

Thus, T is isomorphism mapping and

$C_{\psi,\ 0}[0,\ 1]$ is isomorphic to $C_{\phi,\ 0}[0,\ 1]$.

Now, for $p \geq 1$, we define new norm on the space p[0, 1] by

$$\|p\|_{\psi,\ p} = \{\sum_{n=1}^{\infty} \|\sum_{k=1}^{n} (D^v p - D^{v-1} p)\|_{\infty}^{p}\}^{\frac{1}{p}}.$$

Let $C_{\psi,\ 0}[0,\ 1]$ be the completion of the space $C_{\phi,\ 0}[0,\ 1]$, then we have the following representation theorem.

Theorem

The space $C_{\psi,\ p}[0,\ 1] = \{g(x) = \sum_{n=1}^{\infty} \alpha_n \left(\sum_{k=1}^{n} \frac{x^v}{v!}\right) :\ \sum_{n=0}^{\infty} |b_n|^p < \infty\}$

is isomorphic to the space $C_{\phi,\ p}[0,\ 1]$.

The proof of the theorem follows immediately by using isomorphism operator defined as the proof of first theorem.

Further, let $p \rightarrow \infty$, we define new norm on the space p[0, 1] by

$\|p\|_{\psi,\ \infty} = \sup_{n \geq 1} \left\| \sum_{k=1}^{n} (D^k p - D^{k-1} p) \right\|_{\infty}$, then

we have the following theorem.

Theorem

The space

$$C_{\psi,\ \infty}[0,\ 1] = \{g(x) = \sum_{k=1}^{\infty} \alpha_k \left(\sum_{v=1}^{k} \frac{x^v}{v!} \right) : \sup_{n \geq 1} |b_n| < \infty\}$$

is isomorphic to the space $C_{\phi,\ \infty}[0,\ 1]$.

The proof is similar to the proof of the first theorem. Therefore, the following sets inclusion relations:

$P[0,1] \subset C_{\emptyset,\ p}[0,\ 1] \subset C_{\psi,\ p}[0,\ 1] \subset C_{\emptyset,\ 0}[0,\ 1] \subset C_{\psi,\ 0}[0,\ 1] \subset$

$\left[C_{\emptyset,\ \infty}[0,\ 1] \subset C_{\psi,\ \infty}[0,\ 1] \right] = C_M^{\infty}[0, 1]$, $1 \leq p < \infty$.

The spaces $C_{\psi,\ 0}[0,\ 1]$, $C_{\psi,\ p}[0,\ 1]$ and $C_{\psi,\ \infty}[0,\ 1]$ are equivalent to

$C_{\phi,\ 0}[0,\ 1]$, $C_{\phi,\ p}[0,\ 1]$ and $C_{\phi,\ \infty}[0,\ 1]$ respectively

3.3. Some Applications

DNA – Sequencing plays important roles in the notion of sequence space for protein evolution. "Sequence space" where all possible proteins are arranged in a protein space in which neighbors can be interconnected by single mutation. These problems are not only unique to protein structures but relevant to many other areas such as DNA (Deoxyribonucleic acid) sequence which is a specific sequence of all little bases of nucleotide A(Adenine), C(Cytosine), T(Thymine), G(Guanine) and are linked in different orders in extremely long DNA molecules. In these areas of interest dimensional and complex, the sample size is relevantly small they posses finitely many non zero elements in the sequence and some elements in the sequence repeat many times.

When the data received from the reservoir to obtain some information have lower dimension and samples have larger size, the statistical methods such as that the covariance matrix, dot matrix and position weight matrix can deal with the cases promptly in a simplified way. However, when data have multidimensional character and the sample size is smaller, the statistical methods may lead to errors.

As conclusion, we introduces the ideas of some sequence spaces, function spaces, some main results and practical applications to find the real application of sequence space related to matrix transformation for DNA – sequencing.

CHAPTER – FOUR

VEDIC MATHEMATIS WITH VEDIC RELATIONS

This chapter includes introduction, Sutras and Sub-sutras to interrelate sequence space with Vedic notion through examples in Vedic and Mahabharata Era Epic.

4.1. Introduction

The ‘Vedas’ are considered ‘divine’ in origin and are assumed to be direct revelations from God. The Vedas are four in number: Rig-Veda, Saamaveda, Yajurveda and Atharvaveda.

The famous dictum of Rg Veda, ‘ekam sad vipra bahuda vadanti’ uses ‘ekam’ meaning one is the cardinal number.

The Sama Veda “ The song of knowledge”. It was written down in 1200 to 1000 BCE at about the same time as Yajur Veda and Atherva Veda.

During Yajur Veda Samhita 17^{th} chapter, 2^{nd} mantra describes the numerical value in a **sequence** like Ayuta, Laksha, Niyuta, Koti, Arbud, Vrinda, Kharav, Nikharav, Shankha, Padma, Sagar, Anta, Madhya, Paradha etc. Parardha’s value is equal to 10 raced to the power 12. Vedic times, decimal system was very much in vogue in Indian continant.

A list of these main 16 Sutras of formulae and of 13 Sub-sutras from the Parisist of Atherva Veda had been reconstructed by Tirthaji B. K.

First Sutra is *Ekādhikena Pūrvena* which means by one more than prevish one. The Vedas are full of mathematical concepts, the Indian prowess in the field of mathematics is immense. The work by scholars like Aapastamba, Baudhayana, Bhaskaracharya, Brahmagupta, Arya Bhatta and Bharati Krishna Tirthaji in the field of Mathematics remains unparalleled.

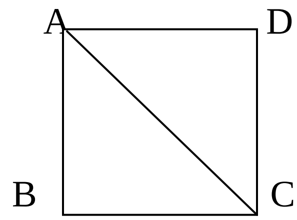
In Vedang period, Shulvasutras were the significant body of mathematical literature. 'Śulba' means 'rope' and 'Sūtras' means 'formula'. Thus 'Sulva sutra' literally means 'formulas of measurements' which basically means 'geometry'. Their origin can be traced to Vedas and scholars consider that these must have been known are least eight centuries B.C. The ideas presented therein seem to be motivated by considerations of forming yajna alters. Śulba Sūtra contains what is called Pythagoras theorem but not the proof. Two main Śulbsūtra are that of Baudhayana and of Apastambha. Baudhayana lived around 800 BC and Apastambha 600 BC. Apastambha improved and expanded on the rules given by Baudhayana.

Examples

(i) Square Root of 2, Baudhayana in his Shulvasutras computed the value of $\sqrt{2}$ correct to seven decimal places; $\sqrt{2} = 1.4142156\ldots$

(ii) In modern period, Baudhyana Shulvasutra is Pythagoras Theorem:

The sum of the areas produced by the length and the breadth is equal to the area produced by the diagonal of a rectangle ABCD,



for right angled triangle ABC

i.e. $(AB)^2 + (BC)^2 = (AC)^2$

[Sen S. N. and Bag A. K., 1983].

For several centuries, the Vedas were not written down but passed from generation to generation through oral transmission. While religious significance is essential for maintaining Aryan supremacy and the caste system, the claims made about the Vedas were of the highest order of hyperbole. In traditional Hinduism, the Vedas as a body of knowledge were to be learnt only by the 'upper caste' (Hindus) and the 'lower castes' (Sudras) and so-called 'untouchables' (who were outside the Hindu social order) were forbidden from learning or even hearing to their recitation.

Religious extremism has been the root cause of most of the world problems since time immemorial. It has decided the fates of men and nations. In a small nation like Nepal, the imposition of religious dogma and discrimination upon the people has taken place after the upsurge of Hindu rightwing forces in the political arena. As a consequence of their political ascendancy in the northern states of India, they started to rewrite school textbooks in an extremely biased manner that was fundamentalist and revivalist. There was a plan to introduce Vedic Astrology in the school syllabus across the nation which was dropped after a major hue and cry from secular intellectuals.

In 1921, Jagadguru Swami Sri Bharati Krsna Tirthaji Maharaja was installed on the pontifical throne of Sharada Peetha Sankaracharya and in 1925, became the pontifical head of Sri Govardhan Math Puri where he served the remainder of his life spreading the holy spiritual teachings of Sanatana Dharma. In 1957, when he decided finally to undertake a tour of the USA, he rewrote from his memory the present volume of Vedic Mathematics giving an introductory account of the sixteen formulae reconstructed by him. One such popular topic of Hindutva imposition was Vedic Mathematics which gives multidimensional thinking in human brain. Much of the hype about this topic is based on one single book authored by the Sankaracharya (the highest Hindu pontiff) Jagadguru Swami Sri Bharati Krsna Tirthaji Maharaja titled Vedic Mathematics, published in the year 1965 and reprinted several times since the 1990s. This book was used as the foundation and the subject was systematically introduced in schools across India. It was introduced in the official curriculum in the school syllabus in the states of Uttar Pradesh and Madhya Pradesh. The entire field of Vedic Mathematics is supposedly based on 16 one - to three - word sutras (aphorisms) in Sanskrit, which they claim can solve all modern mathematical problems. But this work of Sri Sankaracharyaji deserves to be regarded as a new Parisista by itself and it is not surprising that the Sutras mentioned herein do not appear in the hitherto known Parisistas. A list of these main 16 Sutras and of their sub-sutras or corollaries is prefixed in the beginning of the text and the style of language also points to their discovery by Sri Swamiji himself. At any rate, it is needless to dwell longer on this point of origin since the vast merit of these rules should be a matter of discovery for each intelligent reader. There is an open admission that these sutras are the product of one man's imagination.

4.2. Sutras and Sub-Sutras

4.2.1. *Ekādhikena Pūrvena (एकाधिकेन पूर्वेण)*

It's meaning is "By one more than the previous one."

Examples:

(i) 1+1, 2+1, 3+1, 4+1, 5+1, 6+1, 7+1, 8+1

i.e. 2, 3, 4, 5, 6, 7, 8, 9 which is new sequence.

(ii) Vedic Matrix is a nine by nine square array of numbers formed by taking a multiplication table and replacing each number by digit sum as follows: 10 becomes 1, 11 becomes 2, 12 becomes 3, 13 becomes 4, 14 becomes 5, 15 becomes 6, 16 becomes 7, 17 becomes 8, 18 becomes 9

i.e. 1, 2, 3, 4, 5, 6, 7, 8, 9 which is another new sequence and ordered pairs

(1, 8), (2, 7), (3, 6) , (4, 5) are sequential matrices.

Its Vedic matrix relations are

(1, 8) : 1 + 8 = 9; 9×2 = 18 and 9×9 = 81,

(2, 7) : 2 + 7 = 9; 9×3 = 27 and 9×8 = 72,

(3, 6) : 3 + 6 = 9; 9×4 = 36 and 9×7 = 63 and

(4, 5) : 4 + 5 = 9; 9×5 = 45 and 9×6 = 54.

Its Sub-sutra *Ānurūpyena (आनुरूप्येण)*

4.2.2. *Nikhilam Navataścaramam Daśatah (निखिलं नवतश्चरमं दशतः)*

Its meaning is "All from 9 and the last from 10".

Example: We consider the product 98 × 95, then, we get

98 - 02

× 95 – 05

---

93 / 10

Its Sub-sutra is *Śisyate Śesasamjnah (शिष्यते शेषसंज्ञः)*

4.2.3. *Ūrdhva -Tiryagbhyām (ऊर्ध्वतिर्यग्भ्याम्)*

Its meaning is Vertically and cross-wise.

Example

We consider the product 56 × 74, Then,

56

× 74

———

$41_6\ 4_2\ 4$

Its sub-sutra is *Ādyamādyenantyamantyena (आद्यमाद्येनान्त्यमन्त्येन)*

4.2.4. *Parāvartya Yojayet (परावर्त्य योजयेत्)*

Its meaning is "Transpose and Apply."

Example

If we divide $x^3 + 7x^2 + 6x + 5$ by $x - 2$, then

$(i) x^3$ divided by *x* gives us which is therefore the first term of the quotient

$$\frac{x^3 + 7x^2 + 6x + 5}{x - 2}$$

$\therefore$ Q = $x^2 + \cdots$

(ii) $x^2 \times -2 = -2x^2$ but we have $7x^2$ in the dividend. This means that we have to get $9x^2$ more. This must result from the multiplication of *x* by 9*x*. Hence the 2nd term of the divisor must be 9*x*

$\therefore$ Q = $x^2 + 9x + \cdots$

(iii) As for the third term, we already have $-2 \times 9x = -18x$. But, we have 6*x* in the dividend. We must therefore get an additional 24*x*. Thus can only come in by the multiplication of x by 24. This is the third term of the quotient.

$\therefore$ Q $= x^2 + 9x + 24$

(iv) The last term of the quotient multiplied by – 2 gives us – 48. But the absolute term in the dividend is 5. We have therefore to get an additional 53 from some where. But there is no further term left in the dividend. This means that the 53 will remain as the remainder

Therefore, we get Q $= x^2 + 9x + 24$ and R = 53.

Its Sub-sutra is *Kevalaih Saptakam Gunỹat (केवलः सप्तकं गुण्यात्)*

4.2.5. *Sūnyam Samyasamuccaya (शून्यं साम्यसमुच्चये)*

Its meaning is "When the Samuccaya is the same that Samuccaya is zero, i.e. it should be equated to zero."

Example

We consider the equation $\frac{2x+9}{6x+7} = \frac{2x+7}{2x+9}$ then, since the total of the numerators and the total of the denominators is the same. So, according to the Sutra that total can be equated to zero : $4x + 16 = 0$ and so $x = -4$ as the solution.

Its Sub-sutra is *Vestanam (वेष्टनम्)*

4.2.6. *Ānurūpye Śūnyamanyat (आनुरूप्ये शून्यमन्यत्)*

Its meaning is " Whatever the extent of its deficiency".

Example

We take the product 46×44, then

46 - 04

$\times$ 44 - 06

---

2) 40 / 24

---

20 / 24

Its sub- sutra is *Yāvadūnam Tāvadūnam* (*यावदूनम् तावदूनम्*)

4.2.7. *Sankalana -Vyavakalanābhyām* (*संकलन-व्यवकलनाभ्याम्*)

Its meaning is "By addition and by subtraction."

Example

If we need to find the Highest Common factor (HCF) of

($x^2 + 7x + 6$) and $x^2 - 5x - 6$. Then, $x^2 + 7x + 6 = (x + 1)(x + 6)$ and $-5x - 6$

$= (x + 1)(x - 6)$.

So that the HCF is x + 1.

Its Sub-sutra is *Yāvadūnam Tāvadūnīkritya Vargaňca Yojayet* (*यावदूनंतावदूनीकृत्य वर्गंच योजयेत्*)

4.2.8. *Puranāpuranābhyām* (*पूरणापूरणाभ्याम्*)

Its meaning is “By the completion or not completion”.

Its Sub-sutra is Antyayordasakepi (*अनत्ययोर्दशकेऽपि*).

4.2.9. *Calana-Kalanabhyam* (*चलनकलनाभ्याम्*)

Its meaning is “The ultimate and twice the penultimate”

Its Sub-sutra is *Antyayoreva* (अन्त्ययोरेव)

4.2.10. *Yāvadūnam* (यावदूनम्)

Its meaning is " Whatever the extent of its deficiency".

Example

We consider 94^2, then we have $(94 - 6)(6^2) = 8836$.

Its Sub-sutra is *Samuccayagunitah* (समुच्चयगुणितः)

4.2.11. *Vyastisamastih* (व्यष्टिसमष्टिः)

Its meaning is "Part and whole".

Its Sub-sutra is *Lopanasthāpanabhyām* (लोपनस्थापनाभ्याम्)

4.2.12. *Śesānyankena Caramena* (शेषाण्यङ्केन चरमेण)

Its meaning is "The remainders by the last digit".

Example

We need to find the decimal value of 1/7. The remainders are 3, 2, 6, 4, 5 and 1. So, Multiplied by 7 these remainders give successively 21, 14, 42, 28, 35 and 7. So, ignoring the left hand side digits, we simply put down the last digit of each product and we get 1/7 = .14 28 57! Now, this 12th sutra has a subsutra *Vilokanam*. *Vilokanam* means "mere observation."

Its Sub-sutra is *Vilokanam* (विलोकनम्)

4.2.13. *Sopantyadvayamantyam* (सोपान्त्यद्वयमन्तयम्)

Its meaning is "The ultimate and twice the penultimate".

Its Sub-sutra is *Gunitasamuccayah Samuccayagunitah* (*गुणितसमुच्चयः समुच्चयगुणितः*)

4.2.14. *Ekanyūnena Pūrvena (एकन्यूनेन पूर्वेण)*

Its meaning is "By one less than the one before".

Example: We consider the product 763×999. Then, we get

$(763 - 1)(999 - 762) = 762237$.

4.2.15. *Gunitasamuccayah (गुणितसमुच्चयः)*

Its meaning is "The product of the sum is equal to the sum of the product."

4.2.16. Gunakasamuccayah (गुणकसमुच्चयः)

Its meaning is "The product of the sum of the coefficients in the factors is equal to the sum of the coefficients in the product."

Example

We consider the quadratic equation $(x + 7)(x + 9) = x^2 + 16x + 63$ and we observe that $(1 + 7)(1 + 9) = 1 + 16 + 63 = 80$. Also, we find that this rule is applicable to higher degree polynomials. Thus, if and when some factors are known this rule helps us to fill in the gaps [Tirthaji B. K., 1965].

4.3. Some interrelations with Vedic notions

4.3.1. Vedic Mathematics

Vedic Mathematics is an ancient system of Mathematics which gives multidimensional thinking in human brain. On the title "a trick in the name of Vedic Mathematics" though professors in mathematics praise the sutras, they argue that the title Vedic Mathematics is not well suited. It may help school students but only in certain problems where shortcut methods can be used. Because it gives answers very fast and so it can be called “speed maths”.

The views of Wing Commander Vishva Mohan Tiwari as “Vedic Mathematics mainly deals with various Vedic mathematical formulas and their applications of carrying out tedious and cumbersome arithmetical operations and to a very large extent executing them mentally. He feels that in this field of mental arithmetical operations the works of the famous mathematicians Trachtenberg and Lester Meyers (High speed mathematics) are elementary compared to that of Jagadguruji. Several people are also engaged in research in the new “Vedic Mathematics.” To top it all, when in the early nineties the Uttar Pradesh Government introduced “Vedic Mathematics” in school text books, the contents of the Swamiji’s book were treated as if they were genuinely from the Vedas; this also naturally seems to have led them to include a list of the Swamiji’s sutras on one of the opening pages and to accord the Swamiji a place of honour. The Swamiji tells us that it is a sutra for finding the digits in the decimal expansion of numbers such as 1/19, where the denominator is a number with 9 in the unit’s place; he goes on to give a page - long description of the procedure to be followed, whose only connection with the sutra is that it involves.

The phrase *Ekanyunena Purvena* which means “by one less than the previous one” is however given to mean something which has neither to do with decimal expansions nor with squaring of numbers but concerns multiplying together two numbers, one of which has 9 in all places (like 99,999, so on.)! The sutras and the sub sutras or corollaries as given in the book. Thus the lal trikon (red triangle) formula may be seen to be “from the Atharva Veda,” thanks to the Swamiji’s novel technique. The mathematical step is not arrived at by understanding or interpreting what are given as sutras; rather, sutras somewhat suggestive of the meaning of the steps are attached to them like names. It is like associating the **‘sutra’ VIBGYOR to the sequence of colours** in rainbow which make up the white light.

From the Vedanga period, there is in fact available a significant body of mathematical literature in the form of Shulvasutras, from the period between 800 bc and 500 bc, or perhaps even earlier, some of which contain expositions of various mathematical principles involved in construction of sacrificial ‘vedi’s needed in performing’ yajna’s. Baudhyana Shulvasutra, the earliest of the extant Shulvasutras, already contains, for instance, what is currently known as Pythagoras’ Theorem**.** It is the earliest known explicit statement of the theorem in the general form and precedes Pythagoras by at least a few hundred years. The texts also show a remarkable familiarity with many other facts from Euclidean Geometry and it is clear that considerable use was made of these, long before the Greeks formulated them. The Swamiji claims to give "sutras relevant to successive differentiation, covering the theorems of Leibnitz, Mclaurin, Taylor, etc. It should also be borne in mind that if calculus were to be known in India in the early times, it would have been acquired by foreigners as well, long before it actually came to be discovered as there was enough interaction between India and the outside world. We learn that analytic conics has an important and predominating place for itself in the Vedic system of mathematics. We find a whole list of subjects such as dynamics, statics, hydrostatics, pneumatics and applied mathematics listed alongside such elementary things as subtractions, ratios, proportions and such money matters as interest and annuities (!), discounts (!) to which we are assured without going into details that the Vedic sutras can be applied. Besides, equipment such as calculators and computers have made it unnecessary to tax one's mind with arithmetical computations.

Incidentally, the suggestion that this "Vedic Mathematics" of the Shankaracharya could lead to improvement in computers is totally fallacious, since the underlying mathematical principles involved in it were by no means unfamiliar in professional circles. An objective assessment of the methods from the point of view of overall use can only be made by comparing how many individual calculations are involved in working out various general examples, on an average and in this respect the methods of fast arithmetic do not show any marked advantage which would offset the inconvenience indicated earlier. The Swamiji's claim that "there is no part of mathematics, pure or applied which is beyond their jurisdiction" is ludicrous. Mathematics actually means a lot more than arithmetic of numbers and algebra of polynomials; in fact multiplying big numbers together which a lot of people take for mathematics, is hardly something a mathematician of today needs to engage himself in.

The mathematics of today concerns a great variety of objects beyond the high school level involving various kinds of abstract objects generalizing numbers, shapes, geometries, measures and so on and several combinations of such structures, various kinds of operations often involving infinitely many entities; this is not the case only about the frontiers of mathematics but a whole of it, including many topics applied in physics, engineering, medicine, finance and various other subjects. Despite all its pretentious verbiage page after page, the Swamiji's book offers nothing worthwhile in advanced mathematics whether concretely or by way of insight. Modern mathematics with its multitude of disciplines (group theory, topology, algebraic geometry, harmonic analysis, ergodic theory, combinatorial mathematics - to name just a few) would be a long way from the level of the Swamiji's book.

There are occasionally reports of some "researchers" applying the Swamiji's "Vedic Mathematics" to advanced problems such as Kepler's problem, but such work involves nothing more than tinkering superficially with the topic, in the manner of the Swamiji's treatment of calculus and offers nothing of interest to professionals in the area. Even at the school level "Vedic Mathematics" deals only with a small part and more importantly, there too it concerns itself with only one particular aspect that of faster computation. One of the main aims of mathematics education even at the elementary level consists of developing familiarity with a variety of concepts and their significance. Not only does the approach of "Vedic Mathematics" not contribute anything towards this crucial objective but in fact might work to its detriment because of the undue emphasis laid on faster computation.

It is often claimed that "Vedic Mathematics" is well appreciated in other countries and even taught in some schools in UK etc. In the normal course one would not have the means to examine such claims, especially since few details are generally supplied while making the claims. Thanks to certain special circumstances I came to know a few things about the St. James Independent School, London which I had seen quoted in this context. The School is run by the 'School of Economic Science' which is, according to a letter to me from Mr. James Glover, the Head of Mathematics at the School, "engaged in the practical study of Advaita philosophy". The people who run it had substantial involvement with religious groups in India over a long period. Thus, in essence their adopting "Vedic Mathematics" is much like a school in India run by a religious group adopting it; that school being in London is beside the point. It would be interesting to look into the background and motivation of other institutions about which similar claims are made. At any rate, adoption by institutions abroad is another propaganda feature, like being from ancient source, and should not sway us. It is not the contention here that the contents of the book are not of any value. Indeed, some of the observations could be used in teaching in schools. They are entertaining and could to some extent enable children to enjoy mathematics. It would however be more appropriate to use them as aids in teaching the related concepts rather than like a series of tricks of magic.

Ultimately, it is the understanding that is more important than the transient excitement, however, such pedagogical application has limited scope and needs to be made with adequate caution without being carried away by motivated propaganda. It is shocking to see the extent to which vested interests and persons driven by guided notions are able to exploit the urge for cultural self -

assertion felt by the Indian psyche. One would hardly have imagined that a book which is transparently not from any ancient source or of any great mathematical significance would one day be passed off as a storehouse of some ancient mathematical treasure. It is high time saner elements joined hands to educate people on the truth of this so called Vedic Mathematics and prevent the use of public money and energy on its propagation, beyond the limited extent that may be deserved, lest the intellectual and educational life in the country should get vitiated further and result in wrong attitudes to both history and mathematics, especially in the coming generation.

Bharati Krishna's Special Cases

Sri Bharati Krishna Tirthaji makes a clear and definite distinctions between general and special cases:

(i) With regard to every subject dealt with in the Vedic Mathematical Sutras , the rule generally holds good that the Sutras have always provided for what may be termed the General Case (by means of simple processes which can be easily and readily may, instantaneously applied to any and every question which can possibly arise under any particular heading.)

(ii) But at the same time, we often come across special cases which although classifiable under the general heading in question, yet present certain additional and typical characteristics which reader them still easier to solve, therefore special provision is found to have been made for such special cases by means of special Sutras, Subsutras, Corollaries etc., relating and applicable to those particular types alone.

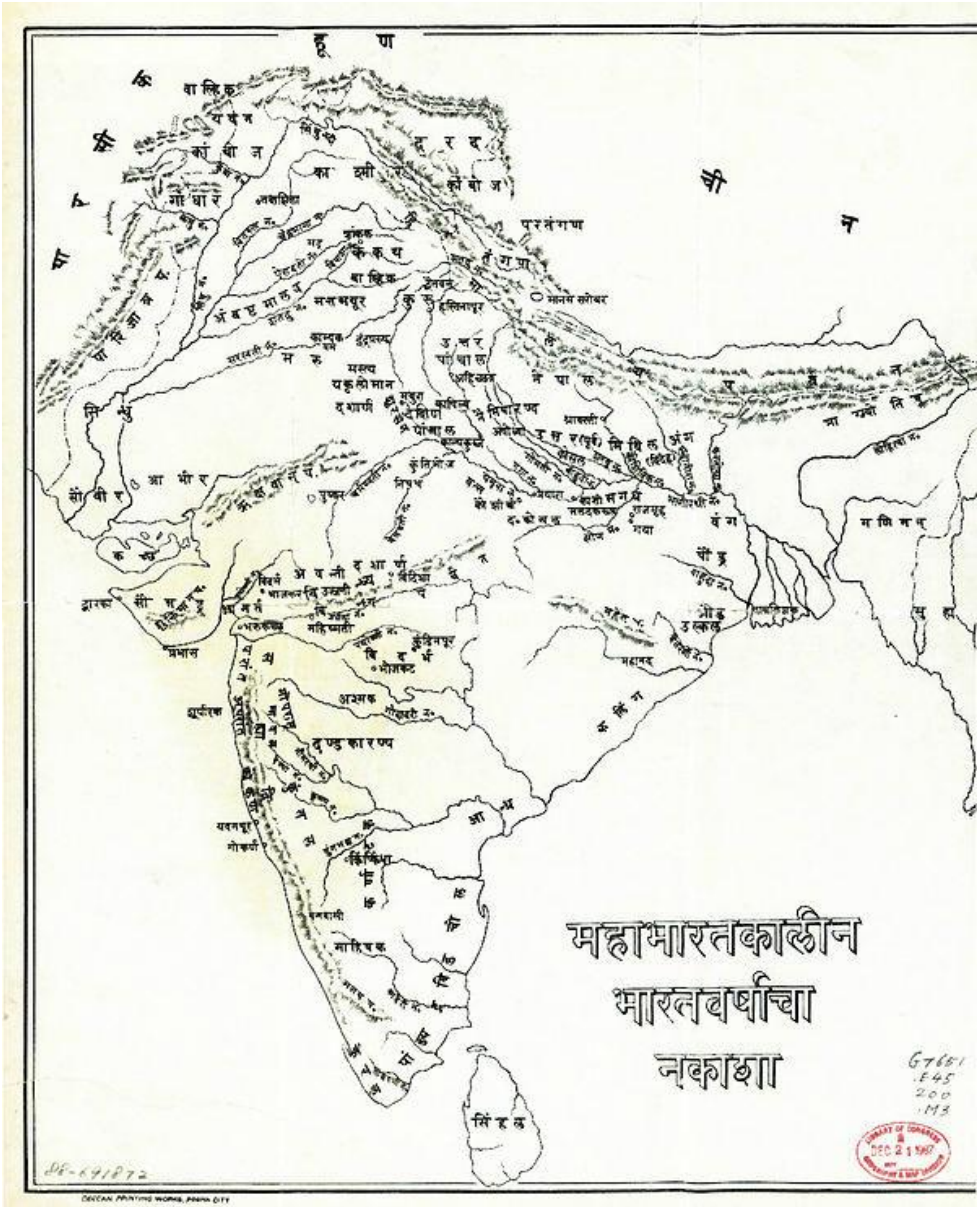
(iii) All that student of these Sutras has to do is to look for the special characteristics in question, recognize the particular type before him and determine and apply the special formula prescribed.

(iv) Generally speaking it is only in case no special case is involved that the general formula has to be resorted.

These special cases and methods that go with them and also the general idea of special methods are however not usually considered to be a part of modern mathematics teaching. Contemporary mathematics teaching relies mainly on general methods and they are fine and very powerful as they handle a whole class of problems but this can be like ‘using a sledgehammer to crack a nut’. Conventional mathematics does sometimes use special methods through but they are not seen as such [Tirthaji B. K., 1994].

4.3.2. The Mahabharata

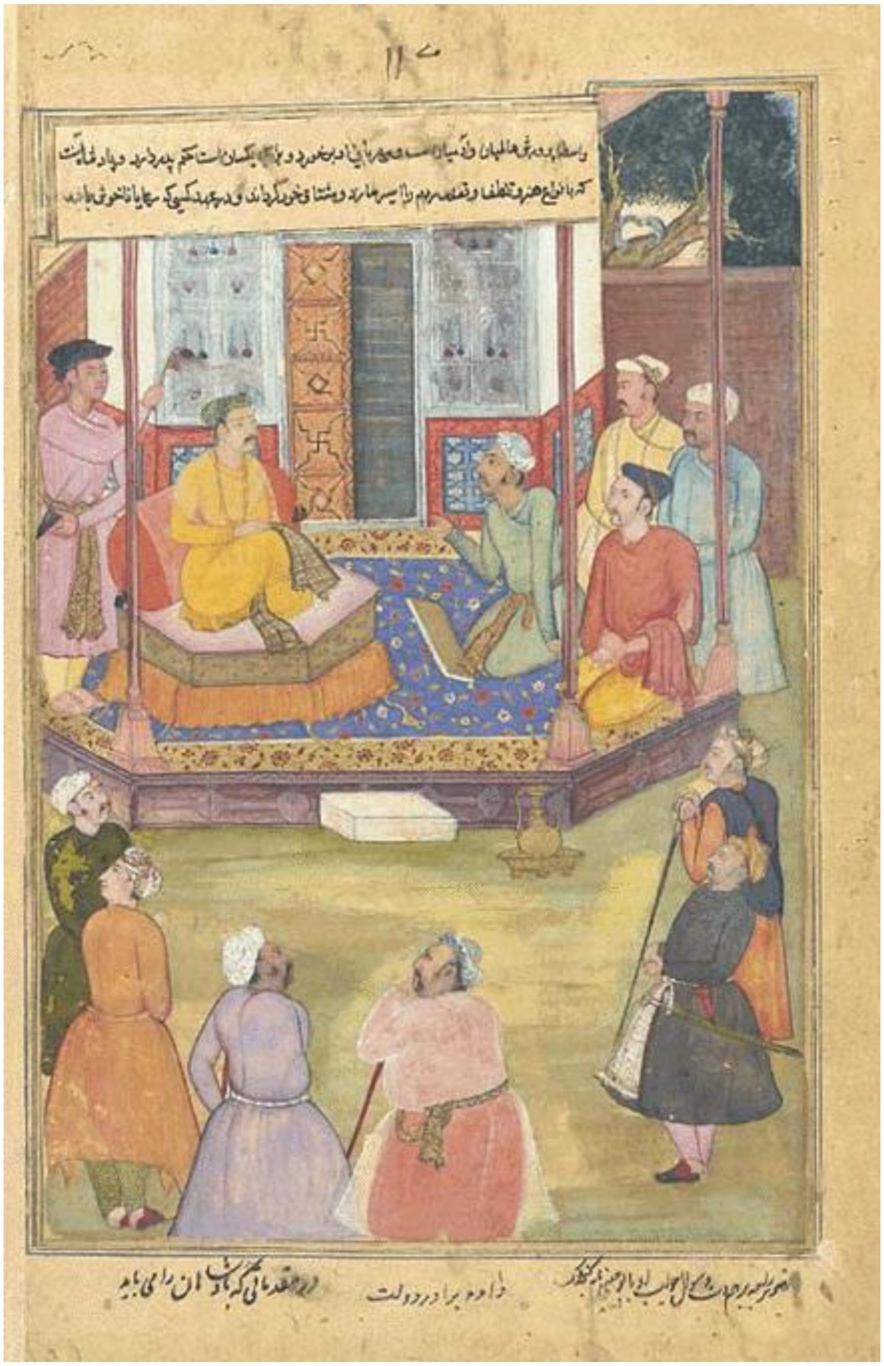
4.3.2.1. Introduction



The Hindu tradition has an amazingly large corpus of religious texts, spanning Vedas, Vedanta (Brahmanas[1], Aranyakas[2], Vedangas[3]), Smritis, Puranas, Dharmashastras and Itihasa. In the listing of the corpus above igures itihasa, translated into English as history. History doesn't entirely capture the nuance of itihasa, which better translated as 'this is indeed what happened'. Itihasa isn't myth or fiction. It is a chronicle of what happened; it is fact so runs the belief. Itihasa consists of Indian's two major epics: the Ramayana and the Mahabharata. The Mahabharata is far more unstructured than the Ramayana. The former is believed to have been composed as poetry and latter as prose. This isn't quite correct. The Ramayana has segments in prose and the Mahabharata has segments in poetry. Itihasa doesn't quite belong to the category of religious texts in a way that the Vedas and Vedanta are religious. However, the dividing line between what is religious and what is not fuzzy.

After all, itihasa is also about attaining the objectives of dharma,[4] artha[5], kama[6] and moksha[7] and the Mahabharata includes Hinduism's most important spiritual text – the Bhagavad Gita. The epics are part of the Smriti tradition. Smriti tradition had composers. Sections just kept getting added and it is no one's suggestion that Krishna Dvaipayana Vedavyasa composed the text of the Mahabharata as it stands today. The major sections of Mahabharata are known as parvas and while one meaning of the word parva is limb or member or joint.

The Mahabharata is about the lunar dynasty. As to be expected, the lunar dynasty begins with Soma (the moon) and goes down through **Pururava,** who married with Urvashi, Nahusha and Yayati. Yayati became old and asked sons to temporarily loan him their youth. Ones who refused were cursed never kings and called Yadavas (Yadu). The one who agreed was **Puru** and lunar dynasty continued through him. His son was **Duhshanta** famous by Kalidasa in Duhshanta - Shakuntala story. His son was Bharata contributing to the name of Bharatavarsha. His grandson was **Kuru**. Kuru's grandson was **Shantanu.** Shantanu had already fathered **Bhishma** through Ganga and also **Chitrangada and Vichitravirya** through Satyavati. **Dhritarasthra and Pandu** were fathered on Vichitravirya. Pandu's son was **Arjun**. Arjun's son was **Abhimanyu**, Abhimanyu's son was **Parishit** and Parishit's son was **Janamejaya** by Krishna Dvaipayana. Krishna Dvaipayana Vedavyasa's powers of composition were remarkable. Having classified the Vedas, he composed the Mahabharata in 100,000 shlokas or couplets. The original version was called **Jaya** having 8,800 shlokas. This was expanded to 24,000 shlokas, called **Bharata**. Finally, it was expanded to 100,000 shlokas, called **Mahabharata.** The Mahabharata was composed over a span of more than 1000 years, perhaps between 800 BCE and 400 BCE in Dvapara Yuga [Bibek Debroy, 2012].



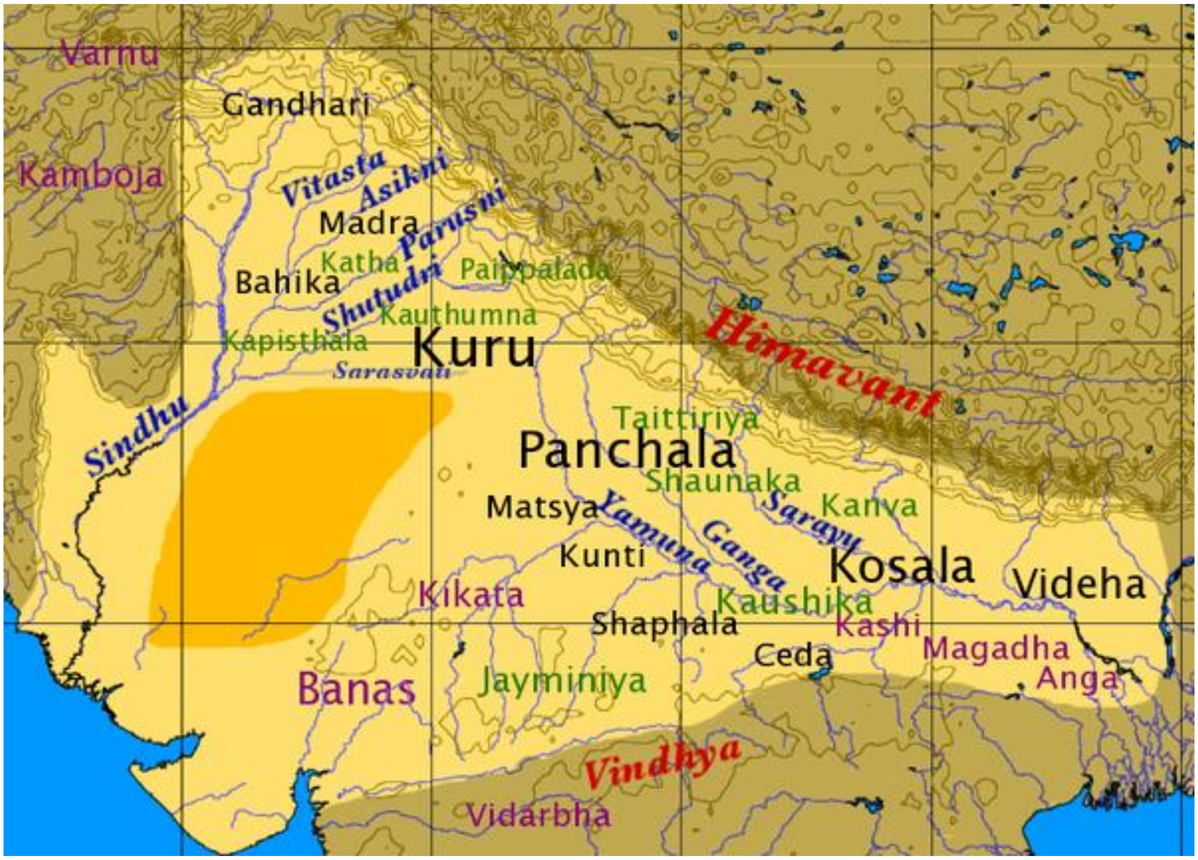
Before the Mahabharata war

Mahābhārata is an account of the life and deeds of several generations of a ruling dynasty called the Kuru clan. Central to the epic is an account of a war that took place between two rival families belonging to this clan. Kurukshetra (literally “field of the Kurus”), was the battleground on which this war, known as the Kurukshetra War, was fought. Kurukshetra was also known as “Dharmakshetra” (the “field of Dharma") or field of righteousness. Mahābhārata tells that this site was chosen for the war because a sin committed on this land was forgiven on account of the sanctity of this land.

The Kuru territories were divided into two and were ruled by Dhritarashtra (with his capital at Hastinapura) and Yudhishthira of the Pandavas (with his capital at Indraprastha). The immediate dispute between the Kauravas (sons of Dhritarashtra) and the Pandavas arose from a game of dice which Duryodhana won by deceit, forcing their Pandava cousins to transfer their entire territories to the Kauravas (to Hastinapura) and to “go into exile” for thirteen years. The dispute escalated into a full scale war when Prince Duryodhana, the eldest of the Kauravas, driven by jealousy, refused to restore the Pandavas their territories after the exile as earlier decided as Duryodhana objected that they were discovered while in exile and that no return of their kingdom was agreed upon. Prior to the war, the disinherited Pandavas in the kingdom of Matsya, advised by Krishna, tried to find a diplomatic and peaceful solution to the conflict. Balarama, Krishna’s older brother, advised the Pandavas to send an emissary to get the support of the elders of the family like Bhishma, Dhritarashtra, Drona, Kripa etc. with the message “Let us avoid armed conflict as much as possible. Only that which is accrued in peace is worthwhile. Out of war, nothing but wrong can issue”. [Rajagopalachari C., 1994].

After that, the Pandavas continued with war preparations. They sent messages requesting assistance to a number of neighbouring kingdoms. Their ambassador of peace was insulted and turned away by Duryodhana who was intent on war, defying the counsel of elders like Bhishma. After several failed attempts on peace, war seemed inevitable. The two sides to the war were the Pandavas and the Kauravas (the official Kuru tribe now ruling both Hastinapura and Indraprastha), both with their allies. Jaya, the core of Mahābhārata, is structured in the form of a dialogue between Kuru king Dhritarashtra and Sanjaya, his advisor and chariot driver. Sanjaya narrates each incident of the Kurukshetra War, fought in 18 days, as and when it happened. Dhritarāshtra sometimes asks questions and doubts and sometimes laments, knowing about the destruction caused by the war to his sons, friends and kinsmen. He also feels guilty due to his own role, that led to this war, destructive to the entire Indian subcontinent. In the beginning Sanjaya gives a description of the various continents of the Earth, the other planets and focuses on the Indian Subcontinent and gives an elaborate list of hundreds of kingdoms, tribes, provinces, cities, towns, villages, rivers, mountains, forests etc. of the (ancient) Indian Subcontinent (Bhārata Varsha). He also explains about the 'military formations adopted by each side on each day, the death of each hero and the details of each war-racing. Some 18 chapters of Vyasa’s Jaya constitutes the Bhagavad Gita, one of the sacred texts of the Hindus. Thus, this work of Vyasa, called Jaya, deals with diverse subjects like geography, history, warfare, religion and morality. According to Mahabharata itself, the Jaya was recited to the King Janamejaya who is the great-grandson of Arjuna by Vaisampayana, a disciple of Vyasa (then called the Bharata). The recitation of Vaisampayana to Janamejaya was then recited again by a professional storyteller named Ugrasrava Sauti, many years later, to an assemblage of sages performing the 12-year-long sacrifice for King Saunaka Kulapati in the Naimisha forest (then called the Mahabharata).

4.3.2.2. Combatants

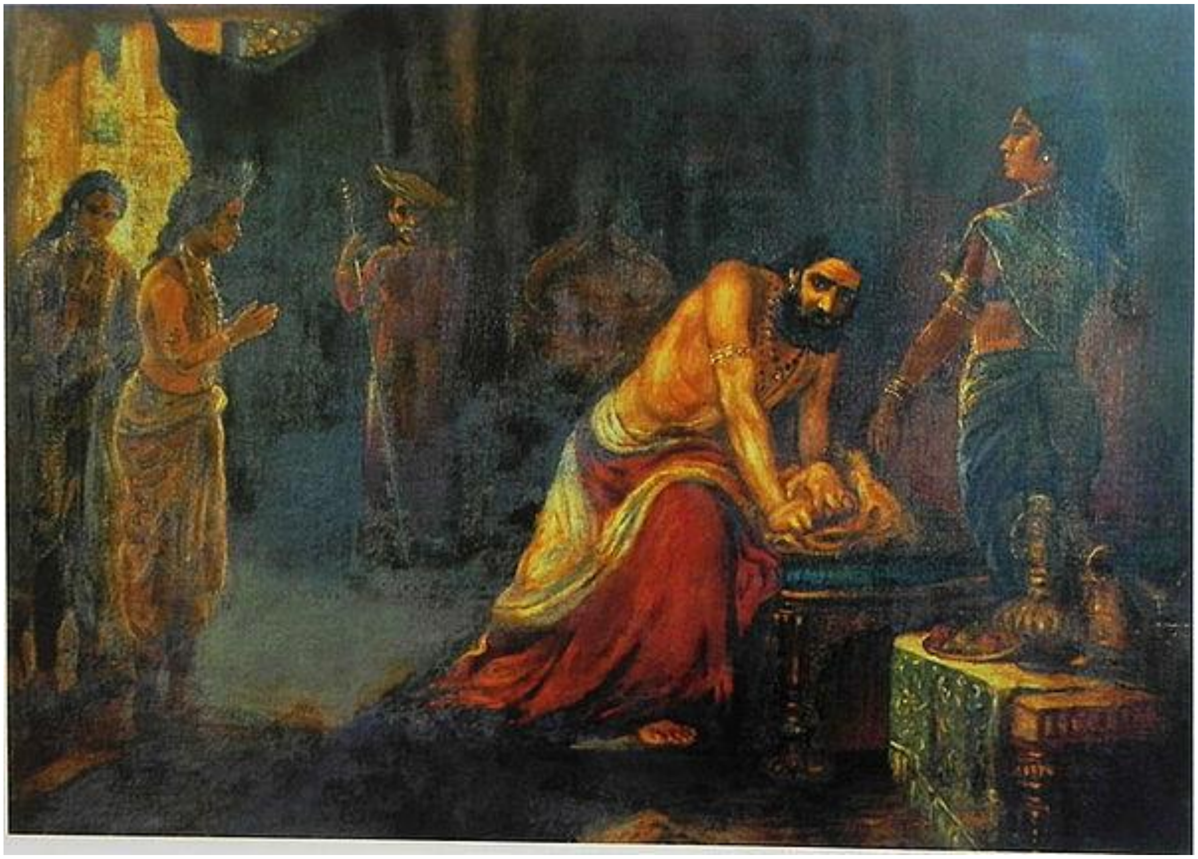


The position of the Kuru and Panchala kingdoms in Iron Age Vedic India

The Kurus formed a kingdom in the Vedic period of India. They formed the first political center after the Rigvedic period, after their emergence from the Punjab and it was there that the codification and redaction of the Vedic texts began. Archaeologically, they most likely correspond to the Black and Red Ware Culture of the 12th to 9th centuries BC, at the beginning of the Iron Age in western India. Panchala was the second “urban” center of Vedic civilization, as its focus moved east from the Punjab, after the focus of power had been with the Kurus in the early Iron Age. This period is associated with the Painted Grey Ware culture, arising around 1100 BCE, and declining from 600 BCE, with the end of the Vedic period. The ruling confederacy, the Panchalas, as their name suggests, probably consisted of five clans – the Krivis, the Turvashas, The Keshins, the Srinjayas and the Somakas. Drupada, whose daughter Draupadi was married to the Pandavas belonged to the Somaka clan. However, the Mahabharata and the Puranas consider the ruling clan of the northern Panchala as an offshoot of the Bharata clan and Divodasa, Sudas, Srinjaya, Somaka and Drupada (also called Yajnasena) were the most notable rulers of this clan.

4.3.2.3. Krishna’s peace mission

As a last attempt at peace, Krishna traveled to Hastinapur to persuade the Kauravas to embark upon a peaceful path with him. At Hastinapur, Krishna took his meals and stayed at the house of the minister, Vidura a religious man and a “devotee” of Krishna. Duryodhana was insulted that Krishna had turned down his invitation to dine with him and stay in his royal palace. Determined to stop the peace mission, Duryodhana plotted to arrest Krishna.

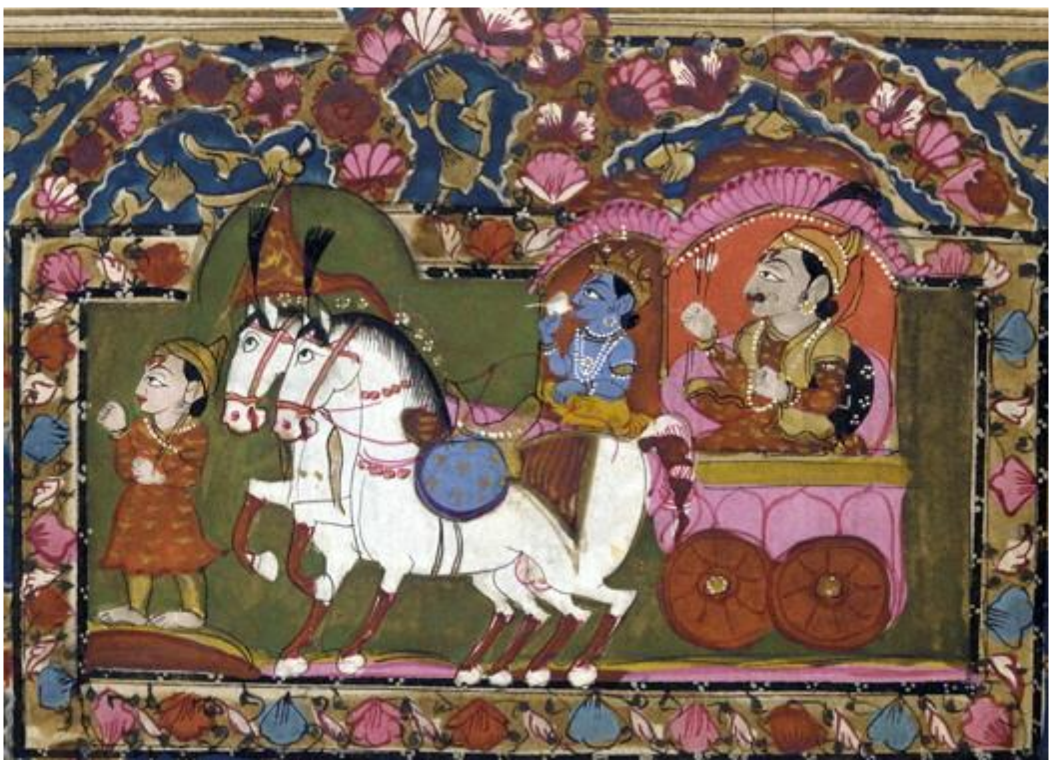


Krishna's peace mission at the court of Hastinapur

At the formal presentation of the peace proposal by Krishna at the court of Hastinapur, Krishna asked Duryodhana to give back Indraprastha or if not at least five villages, one for each of the pandavas but Duryodhana said he could not give land even as much as tip of a needle, Krishna's peace proposals were ignored and Duryodhana publicly ordered his soldiers to arrest Krishna. Krishna laughed and displayed his divine form radiating intense light. Furious at the insult inflicted upon Him, Lord Krishna cursed Duryodhana that his downfall was certain to the shock of Dhirtharastra who tried to pacify the Lord. His peace mission being rejected by Duryodhana, Krishna returned to

Upaplavya to inform the Pandavas that the only course left to uphold the principles of virtue and righteousness was inevitable - war. During the course of his return, Krishna met Karna, Kunti's firstborn (before Yudhisthira) and said to help his brothers but being helped by Duryodhana, Karna said to Krishna that he would battle against Pandavas.

4.3.2.4. War preparations

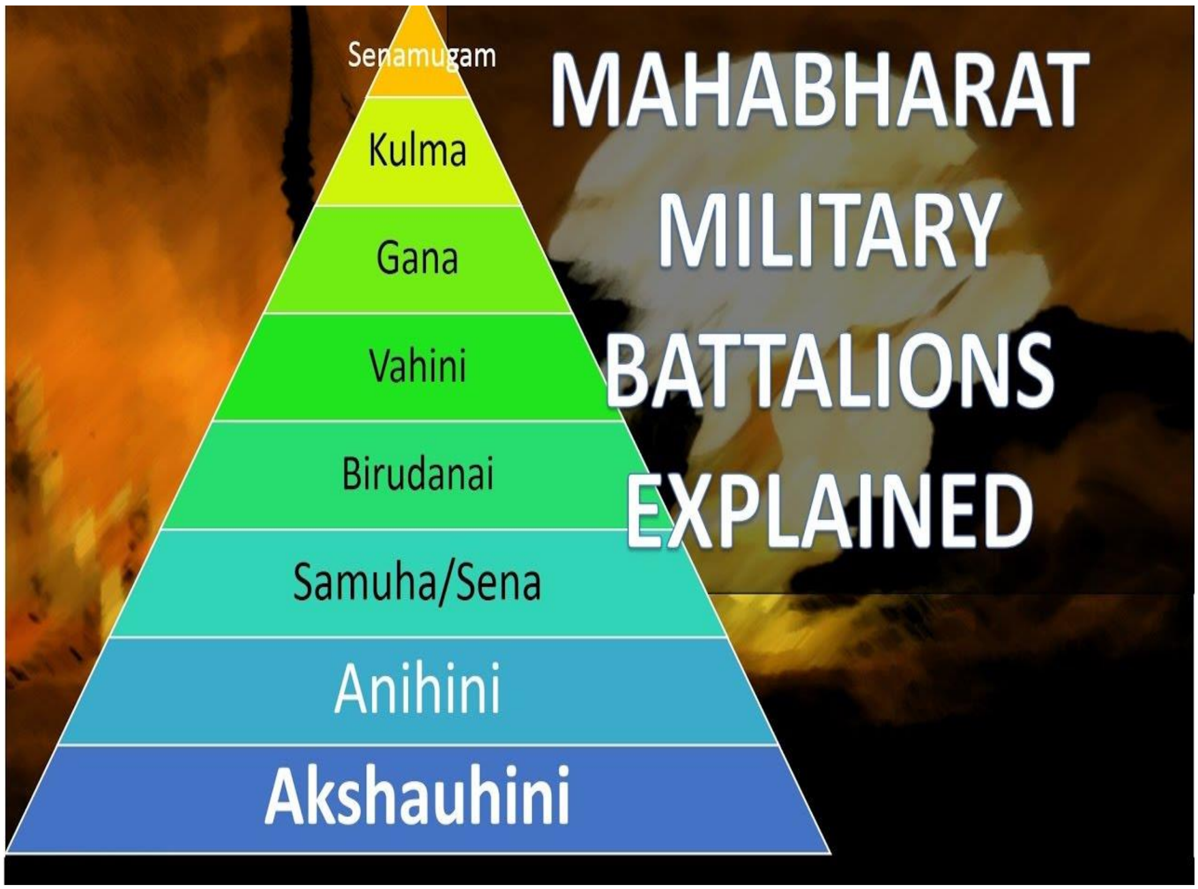


War preparations

Krishna had a large force called the Narayani Sena and was himself a great warrior. Duryodhana and Arjuna thus both went to Krishna at Dwarka to ask for his help. Duryodhana arrived first and found Krishna asleep. Being arrogant and viewing himself as equal to Krishna, Duryodhana chose a seat at Krishna's head and waited for him to rouse. Arjuna arrived later and being a humble devotee of Krishna, chose to sit and wait at Krishna's feet. When Krishna woke up, Krishna saw Arjuna first and gave him the first right to make his request. Krishna told Arjuna and Duryodhana that he would give the Narayani Sena to one side and himself as a non-combatant to the other. Since Arjuna was given the first opportunity to choose, Duryodhana was worried that Arjuna would choose the mighty army of Krishna. When given the choice of either Krishna's army or Krishna himself on their side, Arjuna on behalf of the Pandavas chose Krishna, unarmed on his own, relieving Duryodhana, who thought Arjuna to be the greatest fool. Later Arjuna requested Krishna to be his charioteer and Krishna, being an intimate friend of Arjuna, agreed wholeheartedly and hence received the name Parthasarthy or 'charioteer of the son of Pritha'. Both Duryodhana and Arjuna returned satisfied. While camping at a place called Upaplavya, in the territory of Virata, the Pandavas gathered their armies. Contingents arrived from all parts of the country and soon the Pandavas had a large force of seven divisions. The Kauravas managed to raise an even larger army of eleven divisions. Many kingdoms of ancient India such as Dwaraka, Kasi, Kekaya, Magadha, Chedi, Matsya, Pandya, and the Yadus of Mathura were allied with the Pandavas while the allies of the Kauravas comprised the kings of Pragjyotisha, Kalinga, Anga, Kekaya, Sindhudesa, Avanti in

Madhyadesa, Gandharas, Bahlikas, Mahishmati, Kambojas (with the Yavanas, Sakas, Trilinga, Tusharas) and many others.

4.3.2.5. MAHABHARATA MILITARY BATTALIONS EXPLAINED



Military Battalions

An Akshauhini is actually composed of smaller military units call Anikinis which is divided into smaller units. The systematic construction of an Akshauhini:

One elephant (Gaja), one chariot (Ratha), three horses (Ashwa), and five foot soldiers (Padhata) form a Patti; three Pattis form a Sena-Mukha; three Sena-Mukhas form a Gulma; three Gulmas form a Gana; three Ganas form a Vahini; three Vahinis form a Pruthana; three Pruthanas form a Chamu; three Chamus form a Anikini and ten Anikinis form an Akshauhini.

Thus, an Akshauhini contains 21,870 elephants, 21,870 chariots, 65,610 horses and 109,350 foot soldiers.

A vast army of kingdoms participated in the Great Mahabharata war. Many kings and princes were slain during the course of the war. In total 18 Akshauhini of soldier fought the war, 11 on the Kaurava side and 7 on the Pandava side and the list of the Kingdeons stretch all the way from Kashmir in the North to Tamil Nadu in the side, Assam in the east to Hindukush in the west. Of course not all the existent kingdoms in the India Sub continent participated [Sharma S. K. and Joshi Sourabh Krishna, 2017].

4.3.2.6. Kauravas Army

The Kaurava army consisted of 11 Akshauhinis. Duryodhana requested Bhishma to command the Kaurava army. Bhishma accepted on the condition that while he would fight the battle sincerely, he would not harm the five Pandava brothers. In addition, Bhishma said that Karna would not fight under him as long as he is in the battlefied. It is believed by many that Bhishma pushed Karna into taking this decision due to his affection towards the Pandavas - the Kauravas would be overwhelmingly powerful if both he and Karna appeared in battle simultaneously. However the excuse he used to prevent their simultaneous fighting was that his guru (Parshurama) was insulted by Karna. He also knew that Karna was a Kaunteya(Son of Kunti) from the day he met him in Ranakshetra when Karna offered Arjuna to fight against him and Bhishma wanted to keep Karna out, so that there will be someone to lead the army once he is unable to continue with the war. Regardless, Duryodhana agreed to Bhishma's conditions and made him the supreme commander of the Kaurava army, while Karna was debarred from fighting. But Karna enters the war later when Bhishma is wounded by Arjuna. Apart from the one hundred Kaurava brothers, headed by Duryodhana himself and his brother Dushasana, the second eldest son of Dhritarashtra, the Kauravas were assisted on the battlefield by Drona and his son Ashwathama, the Kaurava’s brother-in-law Jayadratha, the brahmin Kripa, Kritavarma, Shalya, Sudakshina, Bhurisravas, Bahlika, Shakuni, and many more who were bound by their loyalty towards either Hastinapura or Dhritarashtra.

Kuru Army of Akshauhini is formed by the kingdom of Hastinapura in army, alliance with the races like the Samsaptakas, Trigartas, the Narayana the Sindhu army and Shalya of Madra.

Commanders in Chief: Bhishma (10 days), Drona (5), Karna (2), Shalya (1), Ashwatthama (after Duryodhana loses the mace match with Bhima).

King / Prince / Commander	Kingdom	Size of the Army
Bhagadatta	Pragiyotisha	1 Akshauhini
Shalya	Madra	1 Akshauhini
Nila	Mahishmati	1 Akshauhini
Kritavrma	Yadavas	1 Akshauhini
Jayadratha	Saindhava	1 Akshauhini
Sudakshina	Kambhoja	1 Akshauhini
Vinda and Anuvinda	Avanti	1 Akshauhini

Srutayudha	Kalinga	1 Akshauhini
Shakuni	Gandhara	1 Akshauhini
Susharma	Trigarta	1 Akshauhini
Duryodhana	Kuru and Allies	1 Akshauhini

Skilled Units on Kaurava Side

* Bheema = Double Maharathi = 2x Maharathi = 2x12 Antirathis

= 2x12x12 Rathi = 288.

* Karna = Double Maharathi = 2x 144 = 288.

* Drona = Maharathi = 12 Antirathi = 12x12 Rathis = 144.

* Aswathamma = Maharathi = 144.

* Vrishasena = Maharathi = 144.

* Bhagdatta = Maharathi = 144.

* Kripacharya = Antirathi = 12x Rathis = 12.

* Kritvarma = Antirathi = 12.

* Bhurisrava = Antirathi = 12.

* Shalya = Antirathi = 12.

* Duryodhana = 8x Rathis = 8.

* Shakuni = Rathi = 1.

* Jayadratha = Rathi = 1.

* Sudakshhina = Rathi = 1.

* Jayadratha = Rathi = 1.

* 98 Kaurava brothers = Rathi = 98x1 = 98.

Total unit count = 1021.

4.3.2.7. Pandava army

Seeing that there was now no hope for peace, Yudhisthira, the eldest of the Pandavas, asked his brothers to organize their army. The Pandavas accumulated seven Akshauhinis army with the help of their allies. Each of these divisions were led by Drupada, Virata, Dhristadyumna, Shikhandi,

Pandava army

Satyaki, Chekitana and Bhima. After consulting his commanders, the Pandavas appointed Dhristadyumna as the supreme commander of the Pandava army. Mahābhārata says that kingdoms from all over ancient India supplied troops or provided logistic support on the Pandava side. Some of these were: Kekaya, Pandya, Cholas, Magadha, and many more.

Pandavas Army was a coalition of 7 Akshauhini, primarily the Panchala and Matsya forces, the Rakshasa forces of Bhima's son. The Chedi and Magadha armies Vrishni – Yadava he roes.

King / Prince / Commander	Kingdom	Size of the Army
Satyaki	Vrishini	1 Akshauhini
Kuntibhoja	Bhoja	1 Akshauhini
Dhrishaketu	Chedi	1 Akshauhini
Sahadeva (s/o Jarasandhha)	Magadha	1 Akshauhini
Drupada	Panchala	1 Akshauhini
Virata	Matsya	1 Akshauhini
	Panda, Chola, Rakshasas and other allies.	1 Akshauhini

Skilled Units on Pandava side

* Arjuna = Double Maharathi = 288

* Abhimanyu = Maharathi = 144

* Drupada = Maharathi = 144

* Virata = Maharathi = 144

* Dhristaketu = 144

* Yuyutsu = Atirathi = 12

* Dhrityadhumna = Atirathi = 12

* Satyaki = Atirathi = 12

* Kuntibhoja = Atirathi = 12

* Ghatotkacha = Atirathi = 12

* Prativindhya = Atirathi = 12

* Sutasoma = Atirathi = 12

* Satanika = Atirathi = 12

* Shrtsena = Atirathi = 12

* Shrutkarma = Atirathi = 12

* Bheema = 8 Rathis = 8

* Yudhisthir = 1

* Nakul = 1

* Sahadeva = 1

* Shikhandi = 1

* Uttar = 1

* Sons of Drupad (9) = 9

Total Unit Count = 998.

So we can see that skill wise, both the sides were nearly equal in terms of unit count.

Note:

There are other warriors whom I have missed, it is impossible to list each and every warrior because there were hundreds of other kingdoms who participated in the war. Most of these warriors were "Rathis" and not well known.

4.3.2.8. Military Units on both sides

The measure of the army that fought in Mahabharata war was measured in "Akshauhini." The ratio was 7:11

i.e. 7 Akshauhini units on Pandava side while 11 Akshauhini units on Kaurava side. One Akshauhini division contains 21,870 elephants, 21,870 chariots, 65,610 horses and 109,350 infantry men.

In total, there were 18 Akshauhini units which means the total combined strength of the army on both side is

* 1968300 Infantry men

* 393660 Elephants

* 1180980 Horses

* 393660 Chariots

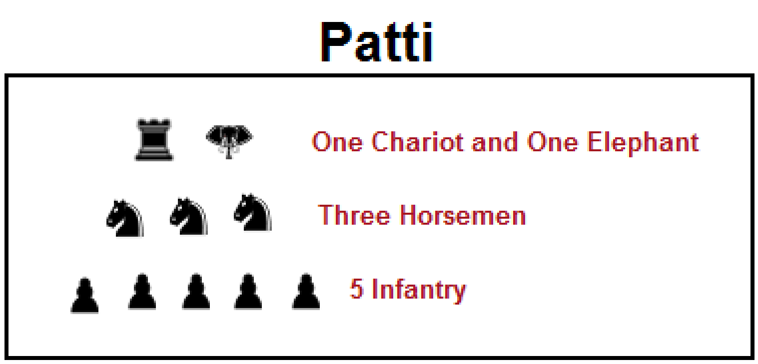
This has a sharp co-relation with the game of Chess (Chaturaga) which was invented in ancient India:

* Infantrymen = Pawn

* Elephant = Bishop

* Horse = Knight

* Chariot = Root.



4.3.2.9. Neutral parties

The kingdom of Vidarbha, with its King Rukmi, Vidura, the ex-prime minister of Hastinapur and younger brother to Dhritarashtra and Balarama were the only neutrals in this war. Rukmi wanted to join the war, but Duryodhana and Arjuna refused to allow him because he boasted about his war strength and army. Vidura did not want to see the bloodshed of the war, although he was a good battle strategist. The powerful Balarama refused to fight at Kurukshetra, because he was both Bhima and Duryodhana's wrestling coach**. [**Rukmini www. (2008)].

4.3.2.10. Army divisions and weaponry

Each army consisted of several divisions; the Kauravas had 11 while the Pandavas controlled 7. A division (akshauhini) includes 21,870 chariots and chariot-riders, 21,870 elephants and riders, 65,610 horses and riders, and 109,350 foot-soldiers (in a ratio of 1:1:3:5). The combined number of warriors and soldiers in both armies was approximately 3.94 million**.[**Rajagopalachari C. (1994)].

Each Akshauhini was under a commander or a general, apart from the Commander in chief or the generalissimo who was the head of the entire army. During the Kurukshetra War, the weapons used included:

The bow, the weapon of choice for Arjuna, Bhishma, Drona, Karna, Satyaki, Vikarna and Abhimanyu; the mace, chosen by Bhima and Duryodhana, the spear, chosen by Yudhistira and Shalya; the sword chosen by Nakula, Dushasana, Dhristadymna and other Kauravas; and the axe chosen by Sahadeva.

4.3.2.11. Rules of engagement

Shiva and Kali		MahaMahaRathi
Indra, Brahmna, Soorya, Pavan, Yama, Ganesha and Other Gods		Lower Maha Maharathi
Parashurama, Krishna, Indrajit		Ati Maharathi
Arjuna	Bheesma, Karna	Double Maharathi
Abhimanyu	Drona	Upper Maharathi
Drupada, Virata, Dhrishtaketu	Ashwathama, Vrishasena, Bhagdatta	Lower Maharathi
Yuyutsu, Drishtadyumna, Satyaki Kuntibhoja, Ghatotkacha, Prativindhya, Sutasoma, Satanika, Shrutasena, Shrutakarma	Kripacharya, Kritvarma, Bhurisrava, Shalya	Atirathi
Bheema	Duryodhana	Upper Rathi (8 Rathis)
Yudhisthir, Nakul, Sahadev, Shikhandi, Uttar, Sons of Drupad	Shakuni, 98 Kaurava Brothers, Jayadratha, Sudakshina,	Rathi

Warriors of both sides

In the war of Mahabharata, 18 Akshauhini army was destroyed.

* Padatik- Infantry Men.

* Ashvarohi – Horseman. One Ashvarohi = 12 Infantrymen.

* Gaja – Soldier on an Elephant. One Gaja = 12 Ashvarohi.

* ArdhaRathi – A soldier on a chariot who is considered equal to one Gaja.

* Rathi- A warrior who is considered equal to 12 Ardharathis

* AtiRathi – A warrior who is considered equal to 12 Rathis

* Maharathi – A warrior who is considered equal to 12 Atirathis

* Ati-Maharathi – A warrior who is considered equal to 12 Maharathis.

* Mahamaharathis – A warrior who is considered equal to 24 Ati-Maharathis.

The above ranks are not just a reflection of skill or mastery of a weapon. It also includes knowledge of battle formations, strategic skill set, tactical knowledge, past experience, ability to crack"battle traps" and so on. Simply because an archer rode a chariot did not make him a Rathi – the warrior needed to have some fearsome reputation and class battle experience too.

For Example Yudhisthir and Bheema both are considered Rathis but their respective sons – Prativindhya and Sutasoma are Atirathis. Everyone may wonder why is that when both Yudhisthir and Bheema are more experienced than their sons? The reason is – Prativindhya and Sutasoma were highly skilled in battle plans, strategies, forming bottlenecks and tackling different battle formations ("Vyuhas"). They also had mastery of all forms of weapons which includes bows and arrows, mace, sword and spear.

Again, Abhimanyu – the youngest son of Pandava family is a Maharathi. This is because he had knowledge of cracking Chakravyuha and knew how to move the fastest in the battlefield. Only Abhimanyu and Arjuna had the mettle of tackling Drona, Bheesma, Aswathhama and other Maharathis from Kaurava side.

So one Abhimanyu (Maharathi) is equal to 12 X 12 = 144 Yudhisthirs.

Please also note that just because a warrior has a higher rank, that doesn’t mean a warrior from a lower rank cannot beat or kill him. Example – Shikhandi was only a Rathi but he ended up killing Bheesma – a Maharathi.

Now coming to the warriors who were considered fit for each rank.

Maha-Maharathis

No Warrior in Mahabharata held this rank because of their Versetility. Shiva and Kali are said to be the ultimate Mahamaharathis who held the power of destroying the whole universe in one blink of an eye.

The only person who was capable of becoming a MahaMahaRathi was Aswathhama but there was a condition – he had to be extremely angry and excited to unleash his full potential (which he himself did not knew). Aswathhama had this potential because he descended as an avatar of Rudra (Shiva), which is Shiva himself.

Only Drona, Bheesma and Krishna knew the true potential of an angry Aswathhama but they never revealed this to anyone. Aswathhama was also one of the Chiranjivis and is supposed to live as an immortal, superceeding the

“Ichhamrityu” of Bheesma. Vyasa is of the opinion that Shiva was a silent observer of the whole ordeal (in the form of Aswathhama) and would have taken steps to destroy the whole humankind, if Krishna failed to preserve Dharma before Kali Yuga prevailed.

Ati-Maharathi

No warrior in Mahabharata war held this rank. Vyasa was of the opinion that Krishna is the only Atimaharathi after Parashurama, because of his Sudarshan Chakra which is deadlier than Bhrahmasthra, Pashupatashtra and all divine weapons combined.

Apart from Parashurama and Krishna, Ravana’s son Indrajit was considered to be an Ati-Maharathi.

Maharathi

Bheesma, Drona, Arjuna, Karna, Ashwathama, Abhimanyu, Drupada, Virata, Vrishasena (Karna’s son), Dhrishtaketu, JaraSandha, Bhagadatta, Narakasura, Balarama.

Acording to Vyasa – Bheesma, Karna and Arjuna were equal to 2 Maharathis but not Drona. This is because Drona is a Brahmin and he does not have Khastriya blood while the other three were princes of royal clan and had that legacy with them. Karna, although a “Soot putra” to the society is actually a prince and belongs to the prestigious “Khastriya” class, having obtained mastery of all weapons, battle formations from Parashurama, is considered equal to 2 Maharathis.

Atirathi

Kripacharya, Kritvarma, Yuyutsu, Drishtadyumna, Satyaki, Shishupala, Bhurisrava, All son's of Draupadi, Shalya, Kuntibhoja, Ghatotkacha.

Vyasa considers both Duryodhana and Bheema have the talent of an Atirathi but since both are weak in strategy and tactical thinking, they are slightly lesser than an Atirathi but higher than Rathis. Bheema and Duryodhana are considered equal to 8 Rathis.

Rathi

Duryodhana, Bheema, Nakul, Sahadev, Yudhisthir, Shikhandi, Drupad's son, 99 Kaurava brothers (except Yuyutsu who is an Atirathi), Uttar, Shakuni, Jayadratha.

Here is the hierarchical chart of different warriors. Ardharathi, Gaja, Ashvarohi and Padatik composed of ordinary soldiers.

Skilled Units on Both Sides

If we consider Rathi as one unit,

* 1 Rathi = 1 Unit

* 1 AtiRathi = 12 X Rathis = 12 Units

* 1 MahaRathi = 12 Atirathis = 12 X 12 = 144 Units

Kaurava side

* Bheesma = Double Maharathi = 2 X Maharathi = 2 X 12 Atirathis = 2 X 12 X 12 Rathis = 288

* Karna = Double Maharathi = 2 X 144 = 288

* Drona = Maharathi = 12 Atirathi = 12 X 12 Rathis = 144

* Aswathamma = Maharathi = 144

* Vrishasena = Maharathi = 144

* Bhagdatta = Maharathi= 144

* Kripacharya = Atirathi = 12 X Rathis = 12

* Kritvarma = Atirathi = 12

* Bhurisrava = Atirathi = 12

* Shalya = Atirathi = 12

* Duryodhana = 8 Rathis = 8

* Shakuni = Rathi = 1

* JayaDratha = Rathi = 1

* Sudakshina = Rathi = 1

* 98 Kaurava brothers = Rathi = 98 X 1 = 98

Total unit count = 1021 (I have counted Bheesma and Karna only once since they did not fight the war at a time, one replaced the other)

Pandava Side

* Arjuna = Double Maharathi = 288

* Abhimanyu = Maharathi = 144

* Drupada = Maharathi = 144

* Virata = Maharathi = 144

* Dhristaketu = 144

* Yuyutsu = Atirathi = 12

* Dhrityadhumna = Atirathi = 12

* Satyaki = Atirathi = 12

* Kuntibhoja = Atirathi = 12

* Ghatotkacha = Atirathi = 12

* Prativindhya = Atirathi = 12

* Sutasoma = Atirathi = 12

* Satanika = Atirathi = 12

* Shrtsena = Atirathi = 12

* Shrutkarma = Atirathi = 12

* Bheema = 8 Rathis = 8

* Yudhisthir = Rathi = 1

* Nakul = Rathi = 1

* Sahadeva = Rathi = 1

* Shikhandi = Rathi = 1

* Uttar = Rathi = 1

* Sons of Drupad (9) = 9

Total Unit Count = 994.

So we can see that skill wise, both the sides were nearly equal in terms of unit count .

Please note: There are other warriors whom I have missed, it is impossible to list each and every warrior because there were hundreds of other kingdoms who participated in the war. Most of these warriors were “Rathis” and not well known.

Military Units on Both Sides

From Kaurava side:-

Rathis

* Sudakshina, the ruler of the Kambojas

* Duryodhana's son Lakshmana and the son of Dussasana

* Jayadratha, the king of the Sindhu and brother in law of Kauravas

* All 99 brothers of Duryodhana including Dushasan are single Rathis

* Duryodhana an excellent mace fighter, was classified as a warrior equal to 8 Rathis.

Atirathis

* Bhoja chief Kritavarma

* The ruler of the Madra, Shalya

* Bhurisravas, the son of Somadatta

* Kripa, also known as Kripacharya, the son of Saradwata

Maharathis

* The ruler of Pragjyotisha, the brave Bhagadatta

* Guru Drona

* Ashwathama, the son of Guru Drona, one of the eight chiranjeevi

* Karna, is equal to 2 Maharathi warriors.

From Pandava side:

Rathis

* Uttamauja and Yudhamanyu Sons of Drupada

* Shikhandi, the son of the King Drupada of the Panchala

* Yudhishtira the son of Pandu and Kunti

* Nakul and Sahadeva are single Rathis

* Bhima is classified as a warrior equal to 8 Rathis

Atirathis

* Satyaki of the Vrishni race,

* Dhrishtadyumna the son of Drupada

* Kuntibhoja,the maternal uncle of Pandavas

* All the sons of Draupadi, the Upapandavas

* Ghatotkacha,prince of Rakshasas and master of all illusions, son Bhima and Hidimba.

Maharathis

* Virata King of Virata Kingdom

* Drupada, King of Panchalas

* Dhrishtaketu, the son of Shishupala, the king of the Chedis

* Abhimanyu,the son of Arjuna

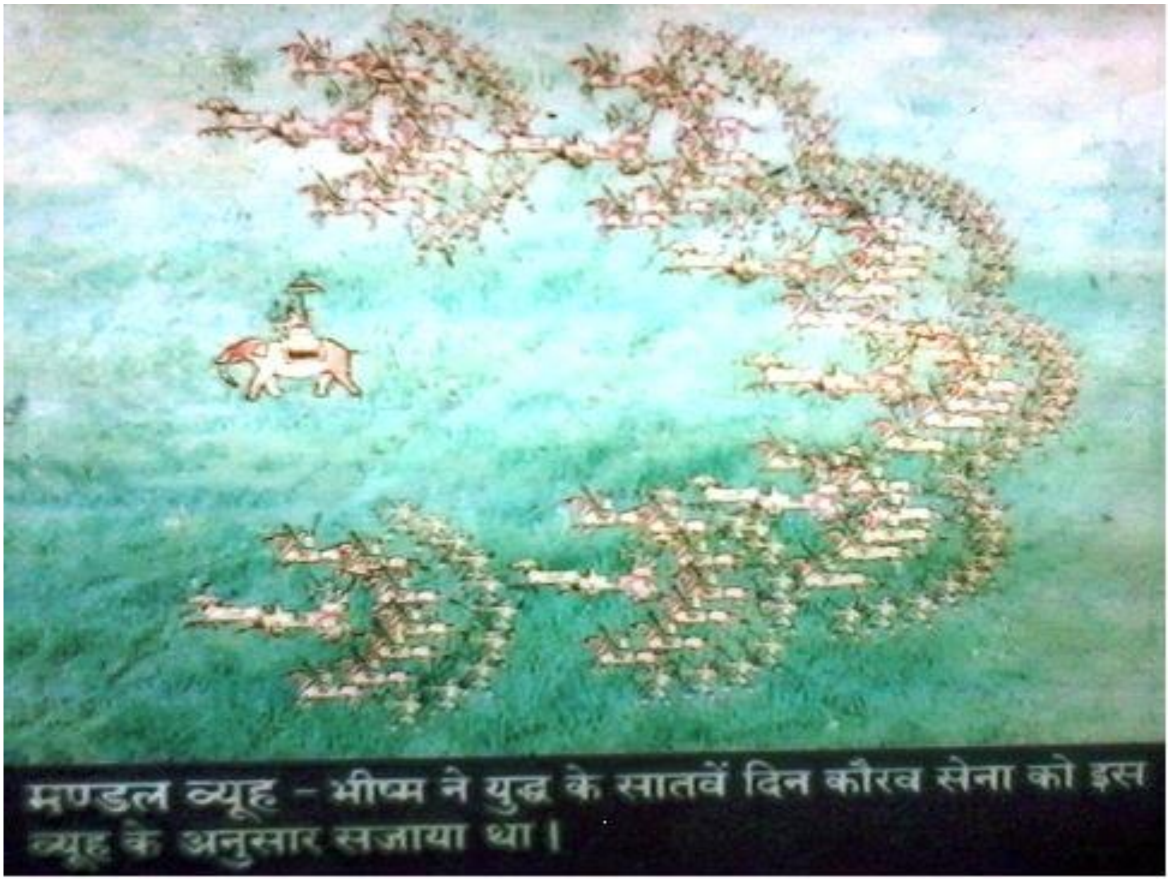
* Arjuna was equal to 2 maharathi warriors

The measure of the army that fought in Mahabharata war was measured in "Akshauhini". The ratio was 7:11 i.e 7 Akshauhini units on Pandava side while 11 Akshauhini units on Kaurava side.

By calculation, One Akshauni division contains 21,870 elephants, 21,870 chariots, 65,610 Horses, and 109,350 infantry men.

[JETIR, 2017]

4.3.2.12. Vyuhas (Battle Formations)



Mandala vyuha (galaxy formation)

A Vyuha is an arrangement of the army divisions to attain a specific objective at the end of the day's war. Every day the commander in chief of each army will arrange the army in a specific formation, so that his army can attain the objectives of that specific day. Depending on the objective, risks, threats, opportunities, the choice of "Vyuha" was made.

Please also note that a "Vyuha" is not a static position where in the army would simply arrange the soldiers and generals in that formation and go attack the enemy. A "Vyuha" is dynamic in nature and it continuously adapts itself to the position and situation of the war.

If one of the generals dies, the Vyuha is technically designed in such a way that every other general knows what needs to be done to restructure the Vyuha and maintain it's formation. If one of the wing is taking huge casualties, the other parts of the Vyuha would restructure itself and save the formation. If the goal of the Vyuha is to capture a key personnel of the enemy, the Vyuha would constantly change and adapt itself so as to ensure that the target is met. The opponent will counter with their own Vyuha and the position of the key has to designed in such a way that the final objective is met through continuous restructuring, adapting and re- in forcing.

This was done through the code of big drums, Trumphets, Sankhs and other horns which would instruct the army what to do. Nagaras were even bigger, mounted on the top of an elephant and drummed by long bamboo pipes. When the general wants to convey a command to his army or division to change positions, he would instruct his Nagara player to beat a specific tune. That tune is a code and the moment that tune is played, everyone in his division would know what to do and how to re-align the army.

There are other techniques as well. For example, when Arjuna fired "Varunastra", Bheesma knew that Arjuna had fired Varunastra and instruct his army to take protective measures. He blew his "Sankh" in a specific note and all the other generals were thus informed that Arjuna has fired "Varunastra". With this knowledge, the Vyuha would re-align itself to avoid causalities until Bhishma fired a counter weapon to stop the effects of Varunastra.

Once Varunastra was neutralized by Bhishma, then the Sankh was played one more time and the "Vyuha" re-aligned itself once again to attain it's objectives.

The position of a Maharathi, Atirathi and Rathi was determined by the commander in chief and as per the merit of the general, the position was given. Every counter measure was taken and analyzed, different plans were made to neutralize damages, if a Maharathi dies. A Maharathi or general simply could not choose his opponent as per his wish, everybody had a fixed target and task that had to be catered to.

The Mahābhārata lists the following vyuhas:

1. Krauncha vyuha (heron formation)
2. Makara vyuha (crocodile formation)
3. Kurma vyuha (tortoise or turtle formation)
4. Trishula vyuha (trident formation)
5. Chakrvyuha|Chakra vyuha (wheel or discus formation)
6. Kamala vyuha or Padma vyuha (lotus formation)
7. Garud vyuha (eagle formation)
8. Oormi vyuha (ocean formation)
9. Mandala vyuha (galaxy formation)
10. Vajra vyuha (diamond or thunderbolt formation)
11. Shakata vyuha (box or cart formation)
12. Asura vyuha (demon formation)
13. Deva vyuha (divine formation)

14. Soochi vyuha (needle formation)

15. Sringataka vyuha (horned formation)

16. Chandrakala vyuha (crescent or curved blade formation)

17. Mala vyuha (garland formation)

It is not clear what the formations actually indicate. They may be arrangements of bear resemblance to animals, or they may be names given to battle strategies [Rajagopalachari C. (1994)].

These Vyuhas were used in the Mahabharata war:

1st Day:

Kaurava – Bheeshma Sarvatomukhi Dand Vyuha.

Pandavas – Arjuna Vajra Vyuha.

2nd Day:

Kaurava – Bheeshma Garuda Vyuha.

Pandavas – Arjuna Krounch Vyuha.

3rd Day:

Kaurava – Bhishma Garuda Vyuha.

Pandavas – Arjuna Ardhchandra Vyuha.

4th Day:

Kaurava – Bhishma Mandal Vyuha.

Pandavas – Arjuna Sringataka Vyuha.

5th Day:

Kaurava – Bhishma Makar Vyuha.

Pandavas – Arjuna Shyen Vyuha.

6th Day:

Kaurava – Bhishma Krounch Vyuha.

Pandavas – Arjuna Makar Vyuha.

7th Day:

Kaurava – Bhishma Mandal Vyuha.

Pandavas – Arjuna Vajra Vyuha.

8th Day:

Kaurava – Bhishma Kurma Vyuha.

Pandavas – Arjuna Trishul Vyuha.

9th Day:

Kaurava – Bhishma Sarvatobhadra Vyuha.

Pandavas – Arjuna Nakshtra Mandal Vyuha.

10th Day:

Kaurava – Bhishma Asur Vyuha.

Pandavas – Arjuna Dev Vyuha.

11th Day:

Kaurava – Drona - Shakat Vyuha.

Pandavas – Arjuna Krounch Vyuha.

12th Day:

Kaurava – Drona - Garuda Vyuha.

Pandavas – Arjuna - Ardhchandra Vyuha.

13th Day:

Kaurava – Drona - Chakra Vyuha.

Pandava - Abhimanyu – No Vyuha was formed on this day because Abhimanyu was the only warrior that entered Enemy's formation while rest of the army was held back on gate 1.

14th Day:

Kaurava – Drona –Chakrashatak Vyuha.

Pandavas – Arjuna - Khaddag Sarp Vyuha.

15th Day:

Kaurava – Drona – Padma Vyuha.

Pandavas – Arjuna – Vajra Vyuha.

16th Day:

Kaurava – Karna – Makar Vyuha.

Pandavas – Arjuna – Ardhchandra Vyuha.

17th Day:

Kaurava – Karna – Surya Vyuha.

Pandavas – Arjuna – Mahish Vyuha.

18th Day:

Kaurava – Shalya – Sarvatobhadra Vyuha.

Pandavas – Arjuna - Krounch Vyuha.

4.3.2.13. The rules included

* Fighting must begin no earlier sunrise and end exactly at sunset.

* No more than one warriors may attack a single warrior.

* Two warriors may "duel" or engage in prolonged combat only if they carry the same weapons and they are on the same type of mount.

* A warrior needs to be challenged and warned before any duel.

* No warrior may kill or injure a warrior who has surrendered.

* One who surrenders becomes a prisoner of war and will then be subject to the protections of a prisoner of war.

* No warrior may kill or injure an unarmed warrior.

* No warrior may kill or injure an unconscious warrior.

* No warrior may kill or injure a person or animal not taking part in the war.

* No warrior may kill or injure a warrior whose back in turned away.

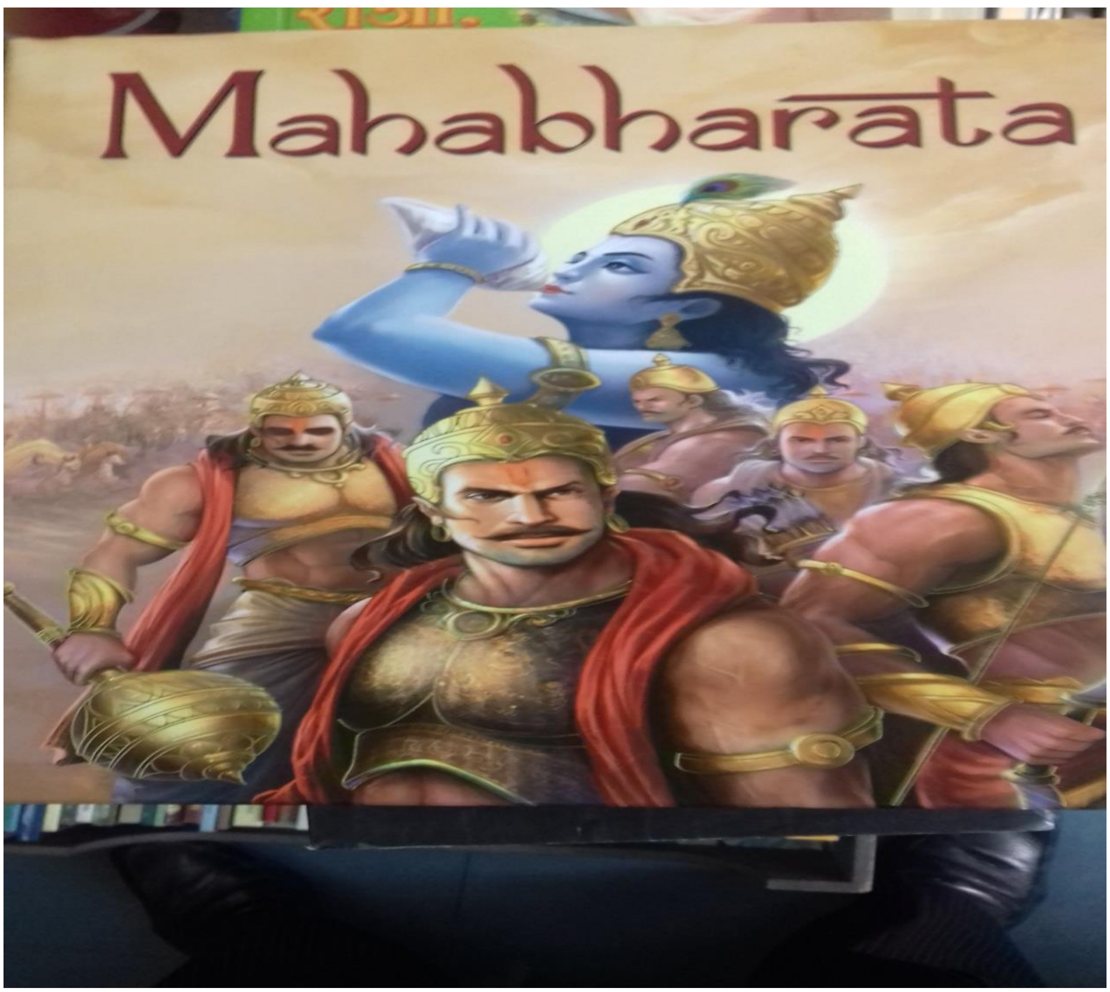
* No warrior may attack a woman.

* No warrior may strike an animal not considered a direct threat.

* The rules specific to each weapon must be followed. For example, it is prohibited to strike below the waist in mace warfare.

* Warriors may not engage in any unfair warfare.

4.3.2.14. The War



Warriers from Pandav side

The Mahabharata is one of the greatest stories ever told. It is much more than the core story of a war between the Kouravas and Pandavas, battle in Kurukshetra in which Bhishma Vadha Parva has 3947 shlokas and 77 chapters. Vadhya means the action of killing. This action is so named because it is about the killing of Bhishma. The battle have Bhishma as the commander-in-chief from Kouravas side for the ten days.

On seeing , the brave Pandavas, Somakas and their followers were delighted and blew on conch shells that had been generated from the ocean. Drums, Peshis, Krakachas and Trumpers made from the horns of cows were sounded together and there was a tumultuous sound. Together with Gandharvas, Ancestors, Siddhas and Masses of charanas came to witness. The immensely fortunate Rishis arrived, with Shatakratu at the forefront, desiring to see that great slaughter. On seeing that the two armies, resembling two oceans, were ready to fight and were repeatedly moving, the brave Yudhishthira removed his armour and cast aside his supreme weapons. He swiftly descended from his chariot, joined his hands in salutation and advanced on foot, glancing towards the grandfather Bhishma where the enemy forces were stationed. On seeing, his brothers and the illustrious Vasudeva also followed him with the foremost kings at the rear. Extremely anxious, the foremost kings also advanced. Arjuna said, why have you abandoned your brothers and are advancing on foot towards the east, where the enemy forces are stationed?"

"Bhimasena said, where are you going, having thrown your armour and weapons away?"…

Yudhishthira did not utter the single word, but continued to advance. The immensely wise and great-minded Vasudeva smiled and told, 'I know his intentions. He will fight with the enemy kings only after he respects to Bhishma, Drona, Goutama, Shalya and all the other seniors. It has been heard in the accounts of the earlier eras that who shows his respects towards his seniors and revered relatives in accordance with the sacred texts and fights with them is certain to be victorious in battle. That is my view.

Yudhishthira, together with his brothers, will seek shelter. All the warriors censured Yudhishthira and his brothers, together with Keshava. The Kourava soldiers cried 'Shame!' to Yudhishthira. Bhishma was ready for battle. The Pandava king grasped his feet with both his hands and spoke. "Yudhishthira asked, 'O father! We are inviting you to fight with us." "Bhishma replied, O great king! If you had not come to me before this battle, I would have cursed you so that you might be defeated. I am pleased with you. Fight and victorious. Whatever else you might desire, obtain all that in this battle. Ask for a boon. What is it that you desire? May it be such that you do not face defeat. A man is the servant of wealth. But wealth is never anyone's servant. This is truth. I am tied to the Kouravas because of wealth. The Kouravas have robbed me through wealth. Other than battle, what else do you wish for? " …

In that field of battle, Vasudeva went to Radheya, until Bhishma has been killed, come over to our side.

"Karna replied, 'O Keshava! I will not do anything that causes displeasure to Dhritarashtra's son. Know that I am engaged in Duryodhana's welfare and have given up my life."

Krishna then returned to the Pandavas, who had Yudhishthira in the forefront. Together with his younger brothers, king Yudhishthira donned his armour again, as resplendent as gold. They arranged themselves in battle formations as they had earlier. On seeing Pandavas, tigers among men, stationed on their chariots, all the kings, together with Dhrishtadyumna, were delighted and roared again. Yuyutsu also abandoned the Kouravyas and joined with Pandavas's side, was welcomed. On seeing, all thing were arranged in sequence. The kings spoke about the friendship, compassion and kindness those great – souled ones displayed towards their relatives, on the appropriate occasions. 'Excellent', 'superb'- these words of praise were heard everywhere.

There were clamorous sounds in the both armies, with krakachas, trumpers made out of the horns of cows, drums, the roars of horses and elephants on each side…

Delighted in their minds and wishing to fight with Bhishma, Yudhishthira with Droupadi's sons, Subhadra's son, Nakula, Shahdeva, Parshata Dhrishtadyumna, repulsed the Bhimasena at the forefront. In that great encounter and confrontation, the giant armies of the Pandavas and on the other side, Duryodhana advanced with Durmukha, Duhshasana, Durmarshana, placing Bhishma at the forefront of the army.

When that tumultuous sound arose and make the body hair stand up, the mighty armed Bhimasena roared like a bull. On hearing the roar of that brave one, enemies solders were frightened. When that mighty archer attacked them, all the enemies would be surrounded by all the brothers who are Dhritarashtra's sons-Duryodhana, Durmukha, Duhshasana, Durmarshana. They showered him with arrows, like clouds enveloping the sun …

On the first day, in the forefront of that battle, the spirited one, Dhananjaya grasped the bow Gandiva advanced against Ganga's son Bhishma. Those two tigers of the Kuru lineage wished to kill each other. Ganga's son pierced partha but he did not waver. Pandava did the same There was a tumultuous encounter between the two and it made the body hair stand up. Roaring and using sharp arrows, Satyaki pierced Kitavarma and Kitavarma pierced Satyaki with arrows in their bodies and the battle was started.

The great archer Abhimanyu fought with Brihadhala. The king of Kosala sliced down the standard of Subhadra's son and brought down his charioteer. When his charioteer was brought down from the chariot, Subhadra/s son was enrage. He pierced Brihadbala with nine arrows. With a sharp and yellow arrow, he sliced off standard. With another he brought down his parshani. With yet another, he brought down his charioteer. They were angry and continued to weaken each other with sharp arrows. In that battle, Bhimasena fought with proud maharatha son Duryodhana who had been the cause of enmity. Both of those tigers among men were immensely strong and foremost among the Kurus. In that field of battle, they enveloped each other with showers of arrows. On seeing the wonderful ways those great – souled and skilled ones fought, all the beings were astounded. Duhshasana advanced against maharatha Nakula and pierced his innermost organs with many sharp arrows. Smiling, Madri's son used sharp arrows to slice down his standard, bow and arrows. He then pierced him with twenty fives mall arrows. In that great battle, he pierced and brought down Nakula's horses, arrows and standard. Durmukha attacked the immensely strong Shahadeva. He fought him in that great battle and pierced him with a shower of arrows. The brave Shshadeva used an extremely sharp arrow to bring down Durmukha's charioteer. Both of them were invincible in battle and attacked each other. Desiring to each other, they used terrible arrows to create fight. King Yudhishthira himself advanced against king Madra. The king of Madra sliced the revered one's bow into two. When his bow sliced, Yudhishthira took up another bow that was stronger and more forceful. Angered, the king told the lord of Mard, 'Wait' and covered him with straight – tufted arrows.

In that battle, Dhrishtadyumna attacked Drona with a firm bow that was capable of destroying enemies. Thus angered, Drona sliced it into three and unleashed an extremely terrible arrow that was as the staff of death.

In that battle, elephants fought against elephants, chariots fought against chariots, horses fought against horses, foot soldiers fought against foot soldiers and infantry fought against infantry so the whole battle was looking in sequence.

After that, the hundreds and thousands of bouts that took place here, without showing any considerations of respect. The son did not recognize the father or the father the son born from his own lions, a brother did not recognize a brother there, nor did a friend recognize his friend there and so on. Some tigers among men used to bring down and shatter chariots, destroying their yokes. Some united against others who were united. They all wished to rob each other of their lives. Gigantic elephants had their temples shattered and fell down on the other elephants. They were angry and used their tusks to attack each other in many places and so on.

Many terrible cries were heard in every division of the armies. It was an extremely terrible encounter that destroyed horses. In that battle, the father killed the son and the son killed the father. The sister's son killed the maternal uncle and the maternal uncle killed the sister's son and so on. No mercy was shown in the fearful and terrible encounter. On encountering Bhishma armies of the Parthas trembled. Bhishma's bow and weapons then sliced off the heads with extremely forceful, straight–tufted and broad–headed arrows. Some elephants were pierced by him in their vital parts and screamed piteously. Abhimanyu was extremely enraged and rushed towards Bhishma's charior, stationed on his own chariot, which was yoked to supreme and tawny horses. His standard was embellished with pure gold and looked like a karnikara. He attacked Bhishma and those supreme charioteers. Striking the palm standard with sharp arrows, the brave one fought with Bhishma and his followers. He pierced Kritavarma with ten arrows, Shalya with five arrows and weakened his great–grandfather with nine sharp arrows. With a broad–headed and straight–tufted arrow that was capable of penetrating every kind of armour, he severed the head from the body of Durmukha's charioteer. With another broad–headed arrow, he sliced down Kripa's bow, decorated with gold. He wounded all of them. On seeing the success with which Krishna's son hit the targets, all the charioteers, with Bhishma at the forefront, thought that he possessed the spirit of Dhananjaya himself. His bow twanged like Gandiva and when it was stretched and stretched again in every direction, it seemed to whirl like a circle of fire. The brave one was surrounded by maharathas who were on the side of Dhritarashtra's son. On seeing that the standard had been brought down by the arrows of Subhadra's son, Bhima roared loudly, so that Subhadra's son might be encouraged.

"At his, ten maharatha archers from the Pandava side swiftly advanced on their chariots, so as to protect Subhadra's son. They were Virata and his son, Parshata Dhrshtadyumna, Bhima, the five from Kekaya and Satyaki. When they advanced towards him, Bhishma pierced Panchala with three and Satyaki with sharp arrows. Through the horses were killed, the lord of Madra remained on his chariot. So as to kill Uttara, he hurled a lance that was like a serpent. Virabahu fought with Virata's son Uttara and Virata's son, Uttra, is killed…

On the second day, when the troops were withdrawn on the first day, Duryodhana was delighted at having seen the enraged Bhishma in battle. With all brothers and kings on his side, Dharmaraja swiftly went to Janardana, O king! Having witnessed Bhishma's valour and overcome with great sorrow as he reflected on his defeat, he spoke to Varshneya. Behold the greater archer Bhishma, whose valour is terrible. He consumes my soldiers with arrows as dry grass. Because of me and my brothers, they have been dislodged from their kingdom and happiness. I will not bring about the destruction of my friends in battle. Swiftly tell me what should be done for my own welfare. He performs difficult deeds on elephants, chariots, horses, soldiers and infantry…

On learning that Pandava was oppressed by sorrow, with his senses robbed by unhappiness, Govinda spoke, delighting all the Pandavas, 'O foremost among the Bharata lineage! Do not sorrow. You should not sorrow when all your brothers are brave and archers who are famous in the worlds. I am engaged in ensuring your welfare and so are maharatha Satyaki, the aged Virata, Drupada and Parshata Drishtadyumna. So are all these kings and their soldiers. They are waiting for your favours and are devoted to you. The immensely strong Parshata Drishtadyumna has always been engaged in your welfare and doing that which pleases you. He has appointed as overall commander. The mighty – armed Shikhandi is certain to bring about Bhishma's death.' Having heard this, the king spoke to maharatha Drishtadyumna, in that assembly and in Vasudeva's hearing. 'O Drishtadyumna! O venerable one! Listen to what I am telling to you. You should not transgress the words that I will speak. With Vasudeva's approval, you are our supreme commander of Pandu, just as in earlier times, Kartikeya always was that of the gods. I will follow you, Bhima and Krishna, together with sons of Madri and the armoured sons of Droupadi and the foremost among all the other lords of the earth. Dhrishtadyumanu replied, 'O Partha! In earlier times, I have been ordained by Shambhu as the one who will kill Drona. I will now fight with Bhishma, Drona, Shalya, Jayadratha and all others.

Extremely terrible encounter in battle, Bhishma fought with Arjuna. He pierced Arjuna with seventy–seven iron arrows, Drona pierced him with twenty–five arrows, Kripa pierced him with fifty arrows, Duryodhana pierced him with sixty–five arrows, Shlya pierced him with nine arrows and Vikarna pierced him with ten broad–headed arrows. But though he was struck in every direction with sharp arrows, the mighty armed and great archer did not suffer and was like a mountain that has been pierced. In return, Arjun struck Bhishma with twenty–five arrows, Kripa with nine arrows, Drona with sixty arrows, Vikarna with three arrows, Artayani with three arrows and the king with five arrows. Satyaki, Virata, Parshata Dhrishtadyumna, Droupadi's sons and Abhimanyu surrounded Dhananjaya.

Thus instructed by Duryodhana, the immensely strong one was protected by a large army and advanced towards Bhima's chariot. That large army of Kalingas was full of chariots, elephants and horses and was armed with many mighty weapons. The warriors killed each other. The entire ground was strewn with flesh and blood. Driven by the desire to kill, the warriors could not distinguish between their own and those of the enemy. When Chedies retreated, Pandava did not retreat. Resorting to the strength of his own arms, he faced all the Kalingas. The immensely strong Bhimasena remained stationary on his chariot. He enveloped the Kalingas army with sharp arrows. The great archer who was the king of Kalinga and his maharahta son, famous by the name of Shakradeva, attacked Pandava with arrows. But the mighty armed Bhima brandished his beautiful bow. Resorting to the strength of his arms, he fought with Kalinga. Shakradeva killed Bhimasena's horses with arrows and showered down clouds arrows. But the immensely strong Bhimasena remained stationed on his chariot, despite his horses having been slain and hurled a club made completely of steel at Shakradeva. With his standard his charioteer, he fell down from the chariot onto the ground and was killed. On seeing that his own son had been killed, the king of Kalinga surrounded Bhima from every direction with many thousands of chariots. At this, the mighty armed Bhima discarded that giant club. He possessed a giant roar himself loudly at which the Kalinga soldiers were frightened. With that large sword, he sliced Bhanuman down the middle and killed. His sword was capable of bearing a great load and he made it descend on the neck of the leader of the elephants and fell down, like the summit of a mountain shattered by the battering of the sea.

Everywhere, he looked like a whirling circle of fire. The lord of slaughtered masses of elephants, horses, chariots, foot soldiers, infantry covered with blood. With great force, he sliced off their bodies and heads. He was like Yama at the time of destruction and confounded them. The great–souled Pandava sliced down some with his supreme sword. Then, the large army of Kalingas surrounded Bhishma attacked Bhimasena. On Seeing that, Bhimasena attacked Shrutayu (the head of the Kalinga army). He advanced towards Kalinga, exclaiming, 'Wait. Wait.' At this, the powerful Shrutayu was enraged and displaying the dexterity of his hands, despatched sharp arrows at Bhima. He was pierced by nine sharp arrows released from that supreme bow. Having been wounded with the force of Kalinga, Bhima was enraged and stretching his bow, killed the king of Kalinga with seven arrows. He killed mostly of the Kalinga soldiers. The day has the Pandavas victorious, the highlight being Bhima's destruction of the Kalingas,…

On the third day, Shantanu's son Bhishma, the grandfather of the Kurus, formed a great Vyuha known as Garuda. Your father Devavrata stationed himself on Garuda's mouth and Bharadvaja's son and Satvata Kritavarma were the eyes. Ashvathama and the famous Kripa were the head, supported Trigartas, Matyas, Kekayas and Vatadhanas. Bhurishrava, Shala, Shalya, Bhagadatta, Madrakas, Sindhus, Souviras and those from the land of the rivers, together with the Jayadratha, constituted the neck. King Duryodhana, together with his brothers and follers, constituted the back. Vinda and Anuvinda from Avanti, together with the Kamboja and the Shakas and the Shurasenas, constituted the tail. Magadhas and Kalingas, together with masses of Dasherakas, were armoured and stationed themselves on the right wing of the Vyuha. Kananas, Vikunjas, Muktas, Pandravishas, together with Brihadbala, were stationed with the left wing.

On seeing this battle formation of your soldiers, Savyasachi, the scorcher of enemies, together with Dhrishtadyumna, arranged a counter Vyuha for encounter. This Vyuha was in the form of a half–moon and this Vyuha was extremely terrible. Bhmasena stationed himself on the right horn. He was surrounded by the kings from many countries, wielding many different kinds of weapons. Maharatha Virata and Drupada were next to him. Next to him was Nila, accompanied by Nilayudha. Next to Nila was maharatha Dhrishtaketu.

He was surrounded by the Chedis, the Kaashis, the Karusha and the Pouravas. Dhrishtadyumna, Shikhandi, the Panchalas and the Prabhadrakas were stationed in the midst of the large army, ready for the battle. Dhamaraja was also there, surrounded by an army of elephants. Satyaki was also there, together with Droupadi's five sons. Abhimanyu was there and beyond him was Iravan. Bhimasena's son was there, together with the maharatha Kekayas. Next to him on left flank, was the foremost of men. His protector was Janardana, the protector of the entire universe. It was thus that the Pandavas formed a giant Vyuha as a counter Vyuha, for the death of your sons and those who have assembled on your side. The battle between those on your side and those of the enemy then commenced, seeking to kill each other in a melee of chariots and elephants. As they sought to kill each other in that tumultuous battle. It was like the extremely terrible battle that was earlier fought between the gods and the Asuras. Bhimasena and Ghatotkacha performed extremely great deeds. Ghatotkacha smiled and pierced the intolerant Duryodhana in the chest with an arrow. Duryodhana lost his senses from this bow. He sank down on his chariot and fainted. On seeing that he had lost his senses, the charioteer swiftly carried him away from the field of battle and his soldiers ran away. While the Kourava soldiers were running away in all directions, Bhima pursued them and killed them with sharp arrows. Many who fell down in the battle cried, 'O father! O brother! O relative! …

Having seen that the army had been killed in large numbers, King Duryodhana Said 'O grandfather! I do not think it is praiseworthy that my soldiers should flee in this way. I do not think that the Pandavas are a force capable of the withstanding you, Drona, Kripa. That is the reason you are pardoning them this act of killing my soldiers. Hearing this, he could be seen everywhere, like a circle of fire. Tormented by the shower of arrows, the large army trembled, while Vasudeva and the Partha looked on. Many soldiers on the side of the sons of Pandu were seen to run away,…

Unable to tolerate it, the protector of all the Dasharhas, spoke approvingly to Shani's descethendant. I will grasp the terrible chakra and will kill Bhishma and his followers and Drona and foremost among charioteers. I will kill the all sons of Dhristarashtra and the foremost among kings who are on their side. On seeing, the got, foremost among men, advance with the chakra, Bhishma remained fearlessly stationed on his chariot, with the bow and arrows in his hand. O lord of the gods! You have shown me great honour and my valour will be celebrated in the three worlds. But as he swiftly advancing towards Bhishma, Partha forcibly grasped him by the feet. When Krishna had stopped, bedecked with a beautiful and golden garland, Arjuna said, 'O Keshava! You are the refuge of the Pandavas. I swear in the names of my sons and brothers that I will not deviate from the acts that I have promised to carry out. I will certainly destroy the Kurus.' Hearing the promise and the pledge, Janardana was happy and pacified. there is a ding-doing battle with Bhishma triumphant initially (which is when Krishna decides to take up arms), followed by Arjun's victory,…

On the fourth day, on seeing fourteen of Duryodhana's brothers attacked Bhima. The immensely strong Bhimasena saw your sons and the brave one licked the corners of his mouth, like a wolf amidst smaller animals. With a kshurapra arrow, Pandava sliced off Senapati's head. He pierced Jalasandha and despatched him to Yama's abode. He killed Sushena and sent him to the land of death. With a bhalla arrow, he brought down Ugra's head, with the helmet and earrings and as the handsome as the moon, to the ground. He used seven arrows to dispatch Bhimabahu to the land of dead, together with his horses, his standard and his charioteer. He swiftly despatchhana to Bhima and Bhimaratha, to Yama's abode. He used kshurapra arrow to convey Sulochana to Yama's abode. On witnessing Bhimasena's terrible valour, remaining Dhritarasthra sons were struck with fear on account of Bhima. In this way, Pandavas triumph on and Bhima kills fourteen of the Duryodhana's brothers …

On the fifth day, on hearing about the defeat of his sons in every way, Dhritarashtra is overcome with grave thoughts about what will happen. Vidura's certain words are oppressing his heart. Because of destiny, it is seen that everything is occurring as he said it would. Pandavas are fighting in just a cause and they are fighting according to their powers. They have performed all their deeds and tasks in accordance with dharma. Where there is dharma, victory exists there. That is the reason the Pandavas can not be killed. On the other side, your evil – souled son has acted always acted according to evil. He is cruel and has performed inferior deeds. That is reason he is decaying in this battle.

Bhishma reminded to Duryodhana, I often spoke to you. But you did not act in accordance with my words. I asked you to act so that there might be peace with the Pandavas. I thought that this would have been beneficial for both of you and the world. You would have happily enjoyed the earth with your brothers. Because you dishonoured the Pandus and confronted with this calamity. There is no one in this world who can vanquish the Pandavas in battle. They are protected by the wielder of the Sharnga bow. You have been not to enter into a fight with the intelligent Vasudeva and also with the Pandavas.

There is a lot of fighting, the highlight is Bhurishrava's killing of the ten of Satyaki's sons,…

On the sixth day, having accomplished this terrible deed, the mighty armed Satyaki, grasped his bow and confronted Bhurishrava, the extender of the deeds of the Kuru lineage, in that battle. On seeing that the soldiers had been brought down Yuyudhana, he angrily attacked him. He stretched an extremely large bow that had the complexion of Indra's weapon. He displayed the dexterity of his hands and released thousands of arrows that were as virulent as serpents and like vajra. These arrows like touch of death to Satyaki's followers and they fled, leaving the invincible Satyaki in that fight.

After that, Yuyudhana's ten sons advanced against Bhurishrava. They were maharathas and immensely strong. All of them were extremely angry in that great battle and spoke to the one who had a mark of a sacrificial stake on his standard. Come and fight with us, either singly. You have spoken well. If you desire , fight with me now. I will endeavor to kill all of you. In that battle, there was one is on one side and many united ones on the ten arrows other. They enveloped that single –handed foremost warrior with arrows. He used Ten arrows to slice down their bows and used sharp, straight–tufted and broad – headed arrows to slice off their heads.

On seeing that his brave strong sons had been killed in battle, Varshneya roared and attacked Bhurishrava. When the sun is assumed to reddish tingle, Dhananjaya swiftly killed twenty–five thousands maharathas. While there is a great deal of fighting. On the balance, the Pandava side is more successful,…

On the seventh day, King Yudhishthira spoke to Dhrishtadyumna to construct makara vyuha in which Drupada and Dhanabjaya were at head, Sahadeva and Nakula were the eye's , Bhimasena was beak, Subhadra's son, Droupadi's sons, Ghatotkacha, Satyaki and Dharmaraja were stationed at the neck, Virata was at the back, surrounded by Dhrishtadyumna and a large army, five brothers from Kekaya were on the left flank, Dhrishtaketu and Karakarsha were stationed on the right flank, Kuntibhoja and Shatanika were stationed at the feet and Shikhandi, Somakas and Iravat were stationed at the tail of the vyuha.

On seeing this vyuha, Bhishma arranged a giant counter – vyuha in which Drona was at its beak, Ashvatthama and Kripa were the eyes, Kritavarma, with Kamboja and Bahlika was at the head, Shurasena and Duryodhana with soldiers were at the neck, kings of Pragiyotisha, Madras, Souviras and Kekayas with army, Susharma, stationed himself along the left wing, Tusharas, Yavanas, Shakas and Chuchupas stationed right wing and Shrutayu, Shatayu and Somadatta's son were stationed at the near of the vyuha.

King Duryodhana attacked Bhima desiring victory. Bhimasena was extremely enraged and spoke these words. I have desired this moment for many years and that time has came now. If you do not flee from the battle, I will kill you today. I will dispel the misery of Kunti today and the difficulties we faced during our exile in the forest. I will kill you and dispel Droupadi's woes. At the time of gambling with the dice you insulted the Pandavas. Witness the calamity that has befallen you because of your evil act. In earlier times, you relied on the views of Karna and Soubala. You did not think of Pandavas and did as you wished. When Dasharha came as a supplicant, you disregarded him because of your delusion. In delight, you gave Uluka a message to deliver to us. I will kill you today, with your relatives and your kin. I will avenge all the evil deeds you have committed earlier. Having said this, he repeatedly stretched his terrible bow. He took up terrible arrows that were as radiant as the great vajra. In great anger, he swiftly twenty – six arrows at Duryodhana. After that, he struck his bow with two and his charioteer with another two. With four arrows, he despatched his swift horses to Yama's abode. He used two arrows which sliced down the king's umbrella from his supreme chariot. With more three, he sliced down the flaming and supreme standard. All the kings saw that the standard of the lord of Kuru has been brought down. He used ten arrows to slay his mighty elephant. Then Jayadratha and Kripa saved Duryodhana. There is a nothing that merits a special mention,...

On the eighth day, Those brave ones were driven by enmity toward each other. They retired to their respective camps, drenched in blood. Having rested for some time, they honored each other in accordance with the proper forms. They were then seen armoured again, desiring to do battle. Your son was overwhelmed with anxiety and blood was trickling from his limbs. He told the grandfather, our soldiers are terrible and fierce. They are arrived and bear many standards. But the brave Pandava rathas have swiftly shattered, slain and oppressed us. Having confounded all our warriors, they have obtained fame. The makara vyuha like the vajra. But Bhima penetrated it and wounded me with arrows that was like the staff of death. On seeing him enraged, I was overcome with fear and lost my senses. I cannot find peace even now. You are truthful in your vows. Through your favours, I wish to obtain victory and slay the Pandavas. When he was addressed, Ganga's great-souled son, foremost among the wielders of weapons, knew that Duryodhana was overcome by grief. Through his mind was distracted, the intelligent one replied, I will make supreme efforts to penetrate their army, as much as I can. I wish to grant you victory and joy. But I will not hide anything for your sake. These maharathas are terrible and are skilled in the use of weapons. They have become the aides of vomiting the venom of their wrath. Those brave ones are firm in their enmity towards you. They are incapable of being vanquished easily. But for your sake, I will strive against them to the best of my ability, giving up my life. In this battle, for your sake, I will no longer attempt to remain alive today. For your sake, I will take on the gods, the daityas and all the worlds, not to speak of the enemies here. To bring you pleasure, I will fight with Pandavas and do everything that you desire. On hearing these words, Duryodhana was supremely content and delighted.

He cheerfully instructed all the soldiers and all the kings to advance. On hearing his instructions, the chariots, elephants, horses and infantry began to advance. They were happy and were armed with a large number and many kinds of weapons. With the chariots, elephants, horses and infantry, your army was extremely resplendent. There masses of tuskers, stationed in arrays and commanded well. The warriors, gods among men and skilled in the use of weapons, stationed themselves amidst the masses of soldiers. The arrays of chariots, elephants, horses and infantry advanced, proceeding along the proper formations (sequences). They raised a dust that was tinged like the morning sun and shrouded the sun's rays. There were bright standards on chariots and elephants. In every direction, they fluttered in the air. In that battle, their different colours looked like clouds tinged with lighting in the sky. The kings twanged their bows and a tumultuous and terrible sound arose. This was like the roar of the ocean, when it was churned by the gods and the great asuras in the first yuga. With that great roar and with many forms and colours, the army of your sons was greatly agitated. The soldiers were ready to kill the soldiers of the enemy and looked like masses of clouds at the end of a yuga.

Sanjaya said, Gangeya saw that your son was still immersed in thought. He then again spoke these pleasing words to him. Drona, Shalya, Satvata, Kritavarma, Ashvathama, Vikarna, Somadatta's son, Saindhava, Vinda and Anuvinda from Avanti, Bahlika and the Bahlika forces, the powerful king of Trigarta, the invincible king of Magadha, Brihadhala from Kosala, Chitrasena, Vivimshati, many thousand rathas with radiant and giant standards, the horses from many countries and horse-riders astride them, crazy kings of elephants with must armed with many different kinds of weapons, warriors with raised weapons who have assembled in your cause from many countries-these and many others have assembled in your cause, ready to give up their lives. It is my view that they are capable of determining even the gods in battle. But I must always tell you words that are for your welfare. The Pandavas are incapable of being vanquished, even by the gods with Vasava. They have Vasudeva as their side and are like the great Indra in their valour. In every way, I will work according to your words. I will defeat the Pandavas in battle or you will defeat me. Having thus spoken, he gave him the sacred vishalyakarrani. This herb possessed great efficacy and he used it to heal his wounds.

When it was morning and the sky was clear, the valiant Bhishma, skilled about vyuhas, himself arranged his soldiers in an array tha was in the form of the mandala vyuha. It abounded in many different kinds of weapons. It was full of the foremost warriors, tuskers and infantry. There were many thousands of chariot in every direction. There were large numbers of horse-riders, wielding swords and lances. There were seven chariots near every elephant and horse. There were ten archers near every horse-rider and there were seven with shields near every archer. Such was the vyuha in which your maharatha soldiers were arrayed. Protected by Bhishma, they were stationed, ready for the great battle. Ten thousands horses, many elephants, ten thousands chariots and armoured sons, the brave Chitrasena and the others, protected the grandfather. He was seen to be protected by those brave ones and those immensely strong kings were themselves armoured. In that battle, Duryodhana was armoured and stationed on his chariot. He blazed in prosperity, like Shakra in heaven. A great roar arose from the army of your sons. There was the tumultuous sound of the chariot wheels and the noise made by musical instruments. Arrayed by Bhishma, the battle formation of the sons of Dhritarashtra advanced towards the west. It was in the form of a giant vyuha known as mandala, impenetrable and the destroyer of the enemy. It was beautiful in every direction and was incapable of being assailed by the enemy.

On seeing the extremely terrible vyuha known as mandala, King Duryodhana himself created the vyuha known as vajra. The different divisions were stationed in the form of this array. The charioteers and horse-riders roared like lions. Desiring to do battle, the warriors wished to break each other vyuhas. Here and there, with their soldiers, brave ones began to strike. Bharadvaja's son advanced against Matsya and Drona's son against Shikhandi. King Duryodhana attacked Parshata himself. Nakula and Sahadeva advanced against the lord of Madra. Vinda and Anuvinda from Avanti attacked Iravat. In that encounter, all the kings fought against Dhananjaya. Bhimasena stove in that battle and countered Hardikya. In that battle, Arjuna's illustrious son fought with sons Chitrasena, Vikarna Durmarshana. Hidimba's son, supreme among rakshasas, advanced forcefully against the great archer from Pragiyotisha and it was like a crazy elephant encountering another crazy elephant. The rakshasa Alambusa was enraged in that war. He attacked Satyaki, invincible in battle, together with his soldiers. Bhurishrava make every effort in that battle and fought against Dhrishtaketu. Yudhishthira, Dharma's son, confronted king Shrutayu. In that battle, Chekitana fought against Kripa. The remaining ones fought with Bhima.

Thousands of kings surrounded Dhananjaya, with spears, lances, iron arrows, maces and clubs in their hands. Arjuna became extremely angry and told

Varshneya, O Madhava! Look at the soldiers of the sons of Dhrishtarsshtra, arrived from battle. They have been arranged in this formation by the great – souled Gangeya, knowledgeable about vyuhas. Look at armoured and brave ones, wishing to do battle. Behold the king of Trigarta, together with brothers. While you look on, I will kill all of them today. They are wishing to fight against me in that field of battle. Kounteya touched the string of his bow and showered arrows towards the masses of kings. Those supreme archers also showered back arrows in return and it was like clouds pouring down onto a lake during the monsoon season. In that great battle, the two Krishnas were seen to be completely covered through those arrows and a great lamentation arose admidst the soldiers. The gods, the devarshis, the gandharvas and the giant serpents were struck with great wonder, when they two Krishnas in that state. At this, Arjuna was enraged and unleashed aindra weapon. We witnessed Vijaya's extraordinary valour. The showers of weapons released by enemies were repulsed by his innumerable arrows. Among those thousands of kings, horses and elephants, there was not a single one who was not wounded. Partha pierced others with two or three arrows each. Having been thus killed by Partha, they sought refuge with Bhishma. At that time, they seemed to be immersed in fathomless waters and Bhishma became their protector. Your soldiers confronted a calamity there and were scattered. They were agitated, like the great ocean in a storm.

Sanjaya said, Susharma retreated from the battle. The brave ones were routed by the great-souled Pandava. However, the battle continue. Your army, which was like the ocean, had been agitated. Gangeya swiftly advanced towards Bijaya. On witnessing Parth's valour in battle, Duryodhana hastened towards all those kings and spoke to them. The brave and the immensely strong Susharma was at the forefront and was stationed in the midst of all the soldiers. These words delighted them. This Bhishma wishes to fight with Dhananjaya with all his heart. He is the best of Kurus and is willing to give up his own life. With all the soldiers of the Bharata army, he will advance against the army of the enemy. All of you unite in the battle and protect the grandfather. Having been thus urged, all the division of all those kings of men followed the grandfather. Bhishma swiftly went to where Arjuna was and the immensely strong one of the Bharata lineage had also been advancing towards him. He was resplendent on a great chariot that roared like the clouds. Large white horses were yoked to it and the terrible ape was on the standard. On seeing Dhananjaya Kiriti advance in battle, all the soldiers in your army were frightened and let out a tumultuous roar. Krishna held the reins in that battle and looked as dazzling as the sun in midday. They were unable of glancing at him. Like that, the Pandavas were incapable of glancing at Bhishma. His horses were white and he held a white bow. He looked like the white planet when it has risen. He was surrounded on every side by the extremely great-souled Trigarta and his brothers, your sons and many other maharathas.

Bharadvaja's son pierced Matsya with an arrow in that encounter. He brought his standard down with one arrow and sliced down his bow with another. Discarding his brother bow, Virata, the leader of an army, quickly took up another bow that was powerful and could bear a great burden. He used venomous arrows those were like flaming serpents. He pierced Drona with three of these and his horses with four. He pierced his standard with one and his charioteer with five. With a single arrow, he pierced his bow. Drona, bull among brahmanas, became angry. Using eight straight-tufted arrows, he killed his horses and his charioteer with a single one. With his horses slain and his charioteer also slain, the best of charioteers jumped down from his chariot and swiftly ascended Shankha's chariot. The father and the son were on the same chariot and powerfully countered Bharadvaja's son with a great shower of arrows. Bharadvaja's son became enraged in that battle. He despatched an arrow that was like a venomous serpent towards Shankha. In that encounter, it Pierced his heart and drank up fell down his blood. Then the arrow on the ground, smeared in blood. Killed by the arrow released by Bharadvaja's son, he fell down from the chariot. While his father looked on, the bow and arrows dropped from his grasp. On seeing that his son had been killed, Virata gave up the fight and fled in fear. Drona was like death with a gaping mouth. Bharadvaja's son swiftly attacked the great army of the Pandavas. In that battle, he scattered hundreds and thousands of them.

Shikhandi confronted Drona's son in battle and struck him between the brows with three swift and iron arrows. With three adhering to his forehead, that tiger among men looked like Mount Meru with three golden peaks. Ashvatthama became angry. In that encounter, in an instant, he showered down many arrows at Shikhandi's charioteer, standard, horses and weapon and brought them down. With his horses slain, the supreme of rathas descended from his chariot. He grasped a sharp sword and a polished shield. Drona's son did not find an opportunity to strike him and it was extraordinary. He unleashed many thousand arrows in that battle.

Bhishma arranged his soldiers in the form of mandala vyuha in which many different kinds of weapons, warriors, tusters, infantry, horses, elephants, chariots etc. So, all things were in sequence.

On seeing the terrible vyuha, Yudhishthira himself created vajra vyuha in which the different divisions were stationed in the form of sequence.

Desiring to do battle, the warriors wished to break each other vyuhas. Here and there, with their soldiers, the brave ones began to strike. Drona advance against Matsya and Drona's son against Shikhandi. King Duryodhana attacked Parshata himself. Vinda and Anuvinda from Avanti attacked Iravat. The two great-souled and great archers from Avanti, immensely strong, saw Iravat in that battle and encountered him with ferocity. The battle that took place between them was tumultuous and made the body hair stand up. Iravat was extremely enraged. He quickly pierced those brothers, who were like gods, with sharp and straight–tufted arrows. Bhima kills eight of Duryodhana's brothers. Iravat, Arjun's son, kills several of Shakuni's brothers and is himself killed by the rakshasa Alambusa. Bhima again kills nine of Duryodhana's brothers,…

On the ninth day, Your father, Bhishma, constructed a great vyuha as terrible as the ocean, with mounts as its waves. Bhishma advanced at front of all soldiers. He was supported by the Malavas and Avanti. Drona was next to him. Bhagadatta was next to Drona, together with Magadhas, the Kalingas and the Pishachas,…

On seeing the great vyuha, the brave Parshata constructed Shringataka and it was destructive of the vyuhas of enemies. Bhimasena and Satyaki were at the two horns, with many thounds of soldiers, horses, chariots and infantry. King Yudhishthira was in the canter, with Pandavas who were Madri's sons.

Abhimanyu was at the rear, with Virat, Droupadi's sons, Ghatotkacha. The brave ones stationed themselves in the battle, wising to fight and desiring victory. Armpits were slapped and a terrible noise in all directions.

In that battle, the brave Bhishma sliced off the arms and heads of the those rathas.

Also, the Pandavas eventually get the worst of it, Krishna decides to kill Bhishma and is restrained by Arjuna. Arjuna replied, O Krishna! It is certain that Shikhandi will be the cause of Bhishma's death. As soon as he sees Panchala, Bhishma will withdraw. Therefore, we will place Shikhandi ahead of us. It is my view that this is the means for bringing about Gangeya's downfall. I will restrain the other great arches with my arrows. Shikhandi will fight with Bhishma, the best warriors. I have heard from the child of Kurus that he will not kill Shikhandi. He was born as a maiden earlier and became a man later. The Pandavas consult Bhishma about how he may be killed and are advised to use Shikhandi,…

On the tenth day, Bhishma consoled your son, O Duryodhana! In the earlier times, I had taken a pledge that I would kill ten thousand great-souled Kshatriyas every day and would then return from the battle. For the sake of your welfare, I have carried out the pledge, I made to you. I will perform an even greater task in this great battle today. I will sleep after being slain or I will kill the Pandavas today. I will today free myself from the great debt I owe you…

On witnessing his valour, Madhusudana spoke to Dhananjaya that Bhishma is stationed between the two armies. Kill him with your power and become victorious. He is shattering our soldiers there. Go and use your strength to repulse him there. Olord! No one other than you is capable of withstanding Bhishma's arrows. O king! The one with the monkey on his banner was incited at that moment. He used his arrows to make Bhishma, his standard, his chariot and his horses disappear. But the bull, the foremost among the Kurus countered Pandava's arrows with his own torrent of arrows and dispersed the many showers of arrows that had been targeted towards him. O great king! The valiant Dhrishtaketu, the king of Panchala, Pandava Bhimasena, Parshata Dhrishtadyumna, the twins, Chekitana, the five from Kekaya, Satyaki, Subhadra's son, Ghatotkacha, Droupadi's sons, Shikhandi, the brave Kuntibhoja, Virata and many other immensely strong ones among the Pandaveyas were oppressed by Bhishma's arrows and were immersed in an ocean of grief.

Phalguna rescued them. With great force, Shikhandi grasped a supreme weapon. Protected Kiriti, he dashed towards Bhishma. Knowing what must he done in the battle, the victorious Bibhatsu killed all the followers and himself rushed against Bhishma. Satyaki, Chekitana, Parshata Dhrishtadyumma, Virata, Drupada and the Pandavas who were Madri’s sons also attacked Bhishma, protected by the one whose bow was firm. In that battle, Abhimanyu and Droupadi’s five sons also attacked Bhishma in the encounter, holding up great weapons. All of them were firm in wielding the bow and never ran away from the field of battle. They pierced Bhishma with well-armed arrows all over his body. However, the one whose soul was never distressed disregarded all those arrows released by the best of kings. He penetrated the Pandava army. As if smiling, the grandfather repulsed all those arrows. Bhishma smiled repeatedly at Panchala Shikhand and remembering that he had been a woman, did not target a single arrow at him. But he killed seven maharathas from Dropada’s army of rathas. Cries of lamentation then arose among the Matsyas, the Pandavas and the Chedis, all of whom had attacked the solitary one. With supreme horses, a cluster of chariots, elephants and foot soldiers, they enveloped the solitary one, like clouds around the sun. There was a battle between Bhishma and Kiriti, who placed Shikhandi at the forefront, like that between the gods and asuras.

All the Pandavas placed Shikhandi at the head. O descendant of the Bharata lineage! In that encounter, they surrounded Bhishma from all sides and wounded him, using extremely terrible shataghnis, javelins, battleaxes, clubs, maces, spears, many types of catapults, goid-tufted arrows, spikes, lances, kampanas, iron arrows, vatsadantas and slings. Together with all the Srinjayas, they assailed Bhishma. His armour was shattered and he was oppressed everywhere, in many ways. But despite having been pierced in his inner organs, Gangeya was not distressed. The radiant bow and arrows and weapons seemed to be like the flames of a fire, fanned by the wind. The roar of the wheels of itself. His colourful bow was extremely resplendent and the one with the great bow was the destroyer of brave ones. Bhishma was like the fire at the end of a yuga, traversing through the enemy. He passed and brought down masses of chariots in the battle. He was again seen, roaming around the midst of those kings among men. He ignored the king of Panchala and Dhrishtaketu and forcibly penetrated into the midst of the Pandava army. He and Parshata Dhrishtadyumna with extremely forceful arrows that could penetrate the armour of enemies. These six were struck with arrows that made a terrible roar and were as radiant as the sun. However, the maharathas repulsed those sharp arrows. Each of them struck Bhishma with great energy, using ten arrows each. In this battle, Shikhandi released arrows towards the one who was great in his vows. These were gold-tufted and sharpened on stone and swiftly penetrated Bhishma. Placing Shikhandi at the forefront, Kiriti impetuously attacked Bhishma and served his bow. When Bhishma's bow was sliced down, the maharathas-Drona, Kritavarma, Saindhava, Jayadratha, Bhurishrava, Shala, Shalya and Bhagadatta could not tolerate this. Extremely enraged, these seven attacked Kiriti. The maharathas displayed supreme and divine weapons. They attacked in great anger and enveloped Pandavas. As they advanced towards Phaguna, sounds could be heard. It was like the sound being raised by the oceans at the time of the destruction of a yuga. Bring forward, Grasp, fight, slice off, such were the tumultuous sounds as they advanced towards Phaguna's chariot. O bull among the Bharata lineage! On hearing that dreadful sound, the Pandava maharathas attacked, so as to protect Phaguna. Satyaki, Bhimasena, Parshata Dhrishtadyumna, Virata, Drupada, rakshasa Ghatotkacha and Abhimanyu-these seven were enraged and became senseless with anger. They wielded colourful bows and swiftly advanced. The battle that commenced was dreadful and the body hair stand up. O best of the Bharata lineage! It was like the battle between the gods and the danavas.

Kiriti, best among rathas, was protected by Krishna and in that battle, after Bhishma's bow had been severed, pierced him with the ten arrows. He struck down his charioteer with ten and his standard with one. Gangeya grasped a bow that was more powerful. However, Phalguna sliced that down with sharp and broad- heated arrow. Pandava was enraged and one after another, Savyasachi, the scorcher of enemies, severed every bow that Bhishma took up. When the bows were severed, he became wrathful and licked the corners of his mouth. In the great wrath, he grasped a javelin that was capable of shattering mountains. In anger, he hurled this towards Phalguna's chariot. On seeing it descend, of the like the flaming vajra, the descendant of the Pandava lineage brought the javelin down with five sharp and broad-headed arrows. O best of the Bharata lineage! When that javelin, hurled angrily by Bhishma's powerful arms, was severed with five arrows by the enraged Kiriti, it was shattered and fell down on the ground, like lighting dislodged from a mass of clouds. On seeing that the javelin had fallen down, Bhishma was overcome with anger. In the battle, the brave and the intelligent one, the destroyer of the cities of enemies, began to think. I am capable of slaying all the Pandavas with a single bow, had the immensely strong Vishvaksena not been their protector now.

There are two reasons for me not to fight with the Pandavas-the Pandus can not be killed and Shikhandi's feminity. In earlier times, when my father married Kali, my father was satisfied and granted me the boon that I would be invincible in battle, except when I decided to die myself. I think the time has come for me to decide on my death. On learning that was the decision of the infinitely energetic Bhishma, the rishis and the Vasus, who were stationed in the sky, spoke these words to Bhishma. O brave one! We are extremely delighted with the decision you have taken. O great archer! Act in accordance with your decision and withdraw from the battle. When those words were spoken, an auspicious and fragrant breeze began to blow. In all the directions, it was moistened with drops of water that smelt nice. The drums of the gods were sounded with a great roar. O king! A shower of flowers fell down on Bhishma. O king! But the words mighty alarmed Bhishma and me, because of the energetic sage. Olord of the earth! There was great agitation among the thirty gods, at the prospect of Bhishma, beloved of all the worlds, falling down from the chariot.

Bhishma smiled and said, 'Whether you desire it or not, I will never fight with you. You are still the Shikhandini.[408] that the creator made.' On hearing these words, Shikhandi became senseless with anger. Arjun told Shikhandi towards Bhishma, Bhishma determined that he would not fight with Bibhatsu any more and die. While he was speaking thus, Phalguna placed Shikhandi in the forefront of the battle and pierced Bhishma with sharp arrows. Bhishsma was severely pierced by the sharp arrows released by the wielder of Gandiva of Arjuna and fell down from his charior, great sounds of lamentation were heard from the gods in heaven and the kings in every direction. On seeing that the great-souled grandfather had fallen down, together with Bhishma, all our hearts also fell down. When the mighty-armed one fell down, the earth seemed to roar. The great archer fell down, like an uprooted pole that has been erected in Indra's honour. Because he was covered with a large number of arrows, he did not touch the ground. The great archer, bull among men, was supine on a bed of arrows. When he fell down from the chariot, a divine essence permeated him. The clouds showered rain and the earth trembled. When he fell down, it was seen that the sun was diminished. He heard divine voices from everywhere in the sky. Why should the great-souled Gangeya, tiger among men and the supreme among those who wield all weapons, decide on a time that is dakshinayana? On hearing these words, Gangeya replied, I am still here.

On seeing that Bhishma had fallen, Duhshasana, used great speed and proceeded towards Drona's army. Kourava informed Drona that Bhishma had been brought down. Hearing this unpleasant news, Drona suddenly fell down from his chariot. On seeing the Kurus withdraw, the Pandavas and their soldiers also retreated. Hundreds and thousands of warriors withdrew from the battle. The Kurus and Pandavas showed him their obeisance and stood there. Shantanu's son, spoke to them, welcome I am delighted to see you. You are equals of the immortals. My head is hanging down, please give me a pillow. The kings brought many soft and delicate pillows but the grandfather did not accept them. The grandfather laughed and told to Dhananjaya to give me appropriate pillow. With tears full eyes, Phalguna! grasped Gandiva and shot three extremely forceful and sharp arrows for grandfather's appropriate pillow. Bhishma, the best of the Bharata lineage and learned about dharma and artha, was satisfied and praised Dhananjaya for having given him that pillow. Kunti's was the best of warriors and brought delight to his well-wishers. He spoke to him, 'O Pandavas! You have done well by giving me something that is appropriate for this bed. Had you done otherwise, I would have cursed you in rage. Having spoken thus to Bbhatsu, he spoke to all the kings and princes. See what Pandava has given me. I will sleep on this bed until the sun changes its path. All maharatha including Pandavas and Kurus went to their own camps in the evening and reflected in great misery.This was destiny. He could kill with his eyes.

The Kurus and Pandavas were showing Bhishma their obeisance.

On the eleventh day

With Bhishma unable to continue, Karna entered the battlefield, much to Duryodhna's joy. He made Drona the supreme commander of the Kaurava forces. Karna and Duryodhana wanted to capture Yudhisthira alive. Killing Yudhisthira in battle would only enrage the Pandavas more, whereas holding him as hostage would be strategically useful. Drona formulated his battle plans for the eleventh day to this aim. He cut down Yudhisthira's bow and the Pandava army feared that their leader would be taken prisoner. Arjuna rushed to the scene, however, and with a flood of arrows made Drona retreat.

On the twelvth day

With his attempts to capture Yudhisthira thwarted, Drona confided to Duryodhna that it would be difficult as long as Arjuna was around. He summoned King Bhagadatta, the monarch of Pragjyotisha (modern day Assam, India). Bhagadatta had thousands of gigantic elephants in his stable and was considered the strongest warrior on this planet in elephant warfare. Bhagadatta attacked Arjuna with his gigantic elephant named Suprateeka. It was a fierce battle in which Bhagadatta matched Arjuna astra for astra but Arjuna slew him. Drona continued to try and capture Yudhisthira. The Pandavas however fought hard anddelivered severe blows to the Kaurava army.

On the thirteenth day

Abhimanyu was able to just break in and not out of the Vyuha

Chakra Vyuha is a battle formation which was used during ancient Warfare. In that formation, the soldiers arrange themselves in concentric circles forming a labyrinth of circles.

The king of Trigartadesa, Susharma along with his 3 brothers and 35 sons who were fighting on the Kaurava side made a pact that they would make sure that Arjuna comes and doesn't break the Chakra Vuyh and kill him or die. They went into the battlefield on the twelfth day and challenged Arjuna. Arjuna gave them a fierce fight in which the brothers fell dead after fighting a brave fight. Drona continued to try and capture Yudhisthira. On the other side of the battlefield, the remaining four Pandavas and their allies were finding it impossible to break Dronacharya's "Chakra Vyuh" formation. As Arjuna was busy fighting with the Trigartadesa princes and the Prajayogastha monarch on the other side of the battlefield, he could not be summoned to break the Chakra vyuha formation, which could only be broken by entering and exiting the formation. Yudhisthira instructed, Abhimanyu, one of Arjuna's sons to break the Chakra vyuha formation. Abhimanyu knew the secret of entering the Chakra vyuh formation, but did not know how to exit it. Abhimanyu slew tens of thousands of warriors. He also killed Dhuryodhana's son. Dhuryodhana got enraged and ordered Durmashana who was Dushasana's son to attack

Abhimanyu but he died under the hands of Abhimanyu. Next he ordered his men to attack Abhimanyu all at once. Abhimanyu fought but was surrounded and killed by many warriors in a combined attack. Upon learning of the death of his son, Arjuna vowed to kill Jayadratha on the morrow before the battle ended at sunset, otherwise he would throw himself into the fire.

On the fourteenth day

Arjuna Kills Jayadrath

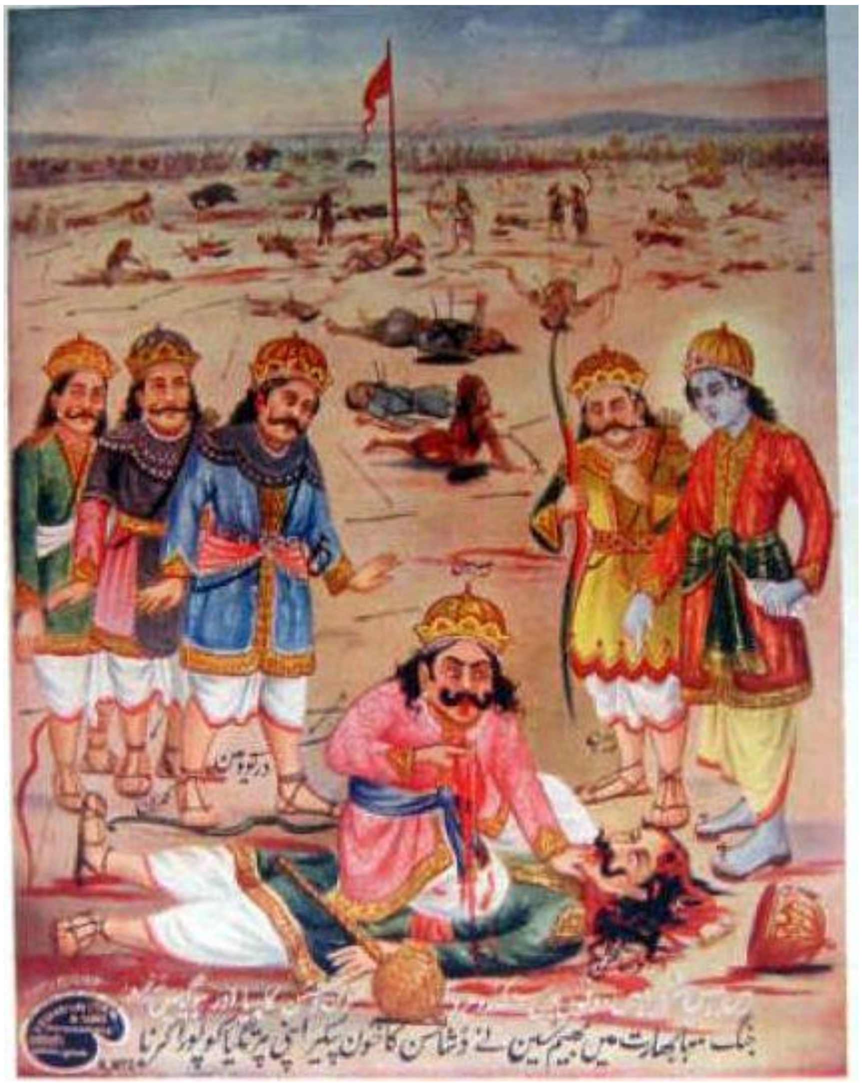
While searching for Jayadrath on the battlefield, Arjuna slew an akshauhini (battle formation that consisted of 21,870 chariots (Sanskrit ratha); 21,870 elephants; 65,610 cavalry and 109,350 infantry) of Kaurav soldiers. The Shakatavuyha Kaurav army tightly protected Jayadratha, however, preventing Arjuna from attacking him. Finally, in late afternoon, Arjuna found Jayadrath guarded by the mighty kaurav army. Seeing his friend’s plight, Lord Krishna raised his Sudarshan Chakra to cover the sun, faking a sunset. Arjun fought a powerful battle with Jayadrath and finally defeated him. Then, Arjuna shot a powerful arrow decapitating Jayadrath.While Arjuna destroying the rest of the Shakatavuyha, Vikarna, the third eldest Kaurava, challenged Arjuna to an archery fight. Arjuna asks Bhima to decimate Vikarna, but Bhima refused to, because Vikarna defended the Pandavas during the Vastranam. Bhima and Vikarna shower arrows at each other. Later Bhima throws his mace at Vikarna, killing him. The muscular Pandava was devastated and mourned his death saying he was a man of Dharma and it was a pity how he lived his life.

The battle continued past sunset. When the bright moon rose, Ghatotkach, son of Bhima slaughtered numerous warriors, attacking while flying in the air. Karna stood against him and both fought fiercely until Karna released the Shakti, a divine weapon given to him by Indra. Ghatotkach increased his size and fell dead on the Kaurav army killing thousands of them.

On the fifteenth day

After King Drupada and King Virata were slain by Drona, Bhima, and Dhristadyumna fought him on the fifteenth day. Because Drona was very powerful and inconquerable having the irresistible brahmadanda, Krishna hinted to Yudhisthira that Drona would give up his arms if his son Ashwathama was dead. Bhima proceeded to kill an elephant named Ashwathama, and loudly proclaimed that Ashwathama was dead. Drona approached Yudhisthira to seek the truth of his son's death. Yudhisthira proclaimed ***Ashwathama Hatahath, naro va Kunjaro va,*** implying Ashwathama had died but he was nor sure whether it was a Drona's son or an elephant, The latter part of his proclamation (Naro va Kunjaro va) were drowned out by sound of the conch blown by Krishna intentionally (a different version of the story is that Yudhisthira pronounced the last words so feebly that Drona could not hear the word elephant). Prior to this incident, the chariot of Yudhisthira, proclaimed as Dharma raja (King of righteousness), hovered a few inches off the ground. After the event, the chariot landed on the ground as he lied. Drona was disheartened, and laid down his weapons. He was then killed by Dhristadyumna to avenge his father's death and satisfy his vow. Later, the Pandava's mother Kunti secretly met her abandoned son Karna and requested him to spare the Pandavas, as they were his younger brothers. Karna promised Kunti that he would spare them except for Arjuna, but also added that he would not fire a same weapon against Arjun twice.

On the sixteenth day



Bhima fulfilling his promise regarding Duhshasan in the Mahabharata field

On the sixteenth day, Karna was made the supreme commander of the Kuru army. Karna fought valiantly but was surrounded and attacked by Pandava generals, who were unable to prevail upon him. Karna inflicted heavy damage on the Pandava army, which fled. Then Arjuna successfully resisted Karna's weapons with his own and also inflicted casualties upon the Kaurava army. Nakul kills Satyasena and Sushena sons of Karna, Arjunakills Vrishasena and Dvipata, Bhima kills Banasena, Satyaki kills Prasena and Prativindya kills Shatrunjaya . The sun soon set, and with darkness and dust making the assessment of proceedings difficult, the Kaurava army retreated for the day. On the same day, Bhima swung his mace and shattered ushasana's chariot. Bhima seized Dushasana, ripped his right hand from shoulder and killed him, tearing open his chest and drinking his blood and carrying some to smear on Draupadi's untied hair, thus fulfilling his vow made when Draupadi was humiliated.

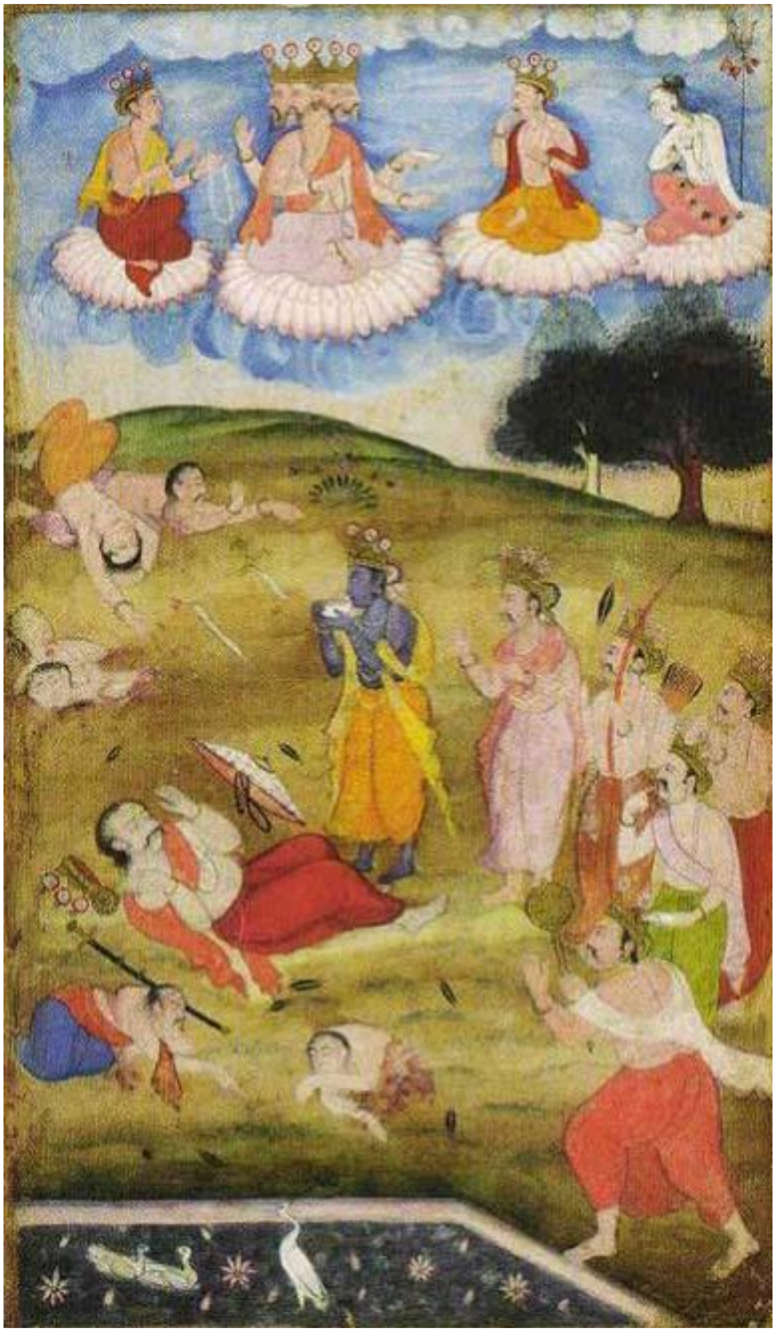
On the seventeenth day

Karna (right) confronts Arjuna, who will later kill Karna, in the Kurukshetra war.

On the seventeenth day, Karna defeated the Pandava brothers Nakul, Bhima, Sahadeva and Yudhisthira in battle but spared their lives. Later, Karna resumed dueling with Arjuna. During their duel, Karna’s chariot wheel got stuck in the mud and Karna asked for a pause. Krishna reminded Arjuna about Karna’s ruthlessness unto Abhimanyu while he was similarly left without chariot and weapons. Hearing his son’s fate, Arjuna shot his arrow and decapitated Karna. Before the day’s battle, Karna’s sacred armour ('Kavacha') and earrings ('Kundala') were taken as alms by Lord Indra when asked for, which resulted in his death by Arjuna’s arrows.

On the eighteenth day, Shalya took over as the commanderin-chief of the remaining Kaurava forces. Yudhishthira killed king Shalya in a spear combat and ahadeva

killed Shakuni. Realizing that he had been defeated, Duryodhana fled the battlefield and took refuge in the lake, where the Pandavas caught up with him. Under the supervision of the now returned Balarama, a mace battle took place between Bhima and Duryodhana. Bhima flouted the rules (under instructions from Krishna) to strike Duryodhana beneath the waist in which he was mortally wounded. Ashwatthama, Kripacharya, and Kritavarma met Duryodhana at his deathbed and promised to avenge the actions of Bhima. They attacked the Pandavas' camp later that night and killed all the Pandavas' remaining army including their children. Amongst the dead were Dhristadyumna and Shikhandi. Other than the Pandavas and Krishna, only Satyaki and Yuyutsu survived. At the end of the 18th day, only twelve warriors survived the war—the five Pandavas, Krishna, Satyaki, Ashwatthama, Kripacharya, Yuyutsu, Vrishakethu (son of Karna) and Kritvarma. Vrishakethu was the only son of Karna who survived the horrific slaughter. He later



Krishna declaring the end of Mahabharata War by blowing the Conch Shell

As a conclusion of this chapter, first sutra, “Ekadhikena Purvena” shows the interrelationship between Vedic notions and sequences. The interpretation of

Vedic connection in the Mahabharata war of army structure follows the matrix order and the number of armies is connected to sequences and the overall arrangement of army in the war is equivalent to the sequence space. Also, the army structure as for hierarchical classification of warriors in the Mahabharata war relative to different vyuha is also equivalent to matrix transformation with some specific order. Also, to justify this relation, we have given many examples. So, we can claim to have equivalent interrelationship of sequence spaces with Vedic notions.

CHEPTER - FIVE

SUMMARY

This chapter includes introduction, conclusions and future scopes.

5.1. Introduction

The introductory nature over some basic definitions, quotations, classification, historical developments, literature reviews, some theorems, application is included as the basic foundation of the research work. We are concerned mainly on the matrix transformation problems by finding new sequence spaces according the the existing sequence spaces which deal with the characterization of matrix mappings between sequence spaces by giving necessary and sufficient conditions of the infinite matrices. New sequence space

$X(p, \lambda)_t = \{x = (x_k) : (t_k x_k) \in X(p, \lambda)\}$ has been introduced from

existing sequence space $X(p, \lambda) = \{x=(x_k) : \lambda x \in X\}$ and paranormed by

$g^*(x) = g(t\, x)$, where g is paranorm in $X(p, \lambda)$.

The class $(l_\infty(p,\lambda)_t,\ l_\infty)$ has been characterized by introduced Theorem:

Let $p_k > 0$ for every $k \in \mathbb{N}$, then $A \in (l_\infty(p,\ \lambda)_t,\ l_\infty)$

iff $\sup_n \sum_{k=1}^{\infty} |\Delta^2\ a_{nk}|\ \mathbb{N}^{\frac{1}{p_k}} < \infty$ for every integer $\mathbb{N} > 1$

Another new sequence space

$X(\Delta_r p, \lambda)_t = \{x = (x_k) : (t_k x_k) \in X(\Delta_r p, \lambda)\}$ has been introduced from existing sequence space $X(\Delta_r p, \lambda)$ and paranormed by $g^*(x) = g(t\,x)$ where g is paranorm in $X(\Delta_r p, \lambda)$.

The class $(l_\infty(\Delta_r p, \lambda)_t, c_0)$ has been characterized by introduced Theorem:

Let $p_k > 0$ for every k, then $A \in (l_\infty(\Delta_r p, \lambda)_t, c_0)$ iff

(i) $R \in (l_\infty(\Delta_r p, \lambda)_t, c_0)$ where $R = r_{n,k} = [\sum_{v=k}^{\infty} a_{n,v}]$

(n, k = 1, 2, 3,…).

(ii) $A_n[\sum_{i=1}^{k} N^{1/p_i} \in c_0$ (n, k = 1, 2, 3,…), for all integers

N > 1 and

(iii) $\lim_{n\to\infty} a_{n,k} = \alpha_k$ (k = 1, 2, 3,…)

We have studied the sequence spaces by practical application:

we revisit the sequence spaces and function spaces defined on [0, 1]. we examine the behaviors of sequences generated by DNA nucleotides. It has been aimed to extend the results of authors by introducing new function space in [0, 1], extending the basis function $\frac{x^n}{n!}$, introducing a new sequence $b = (b_n) = (\sum_{v=n}^{\infty} a_v)$ which can characterize DNA sequence.

When the data received from the reservoir to obtain some information have lower dimension and samples have larger size, the statistical methods such as that the covariance matrix, dot matrix and position weight matrix can deal with the cases promptly in a simplified way. However, when data have multidimensional character and the sample size is smaller, the statistical methods may lead to errors.

In Vedang period, Shulvasutras were the significant body of mathematical literature.

In modern period, Baudhyana Shulvasutra is Pythagoras Theorem:

The sum of the areas produced by the length and the breadth is equal to the area produced by the diagonal of a rectangle.

The entire field of Vedic Mathematics is supposedly based on 16 one - to three - word sutras and 13 Sub-Sutras in Sanskrit which can solve all modern mathematical problems.

The famous dictum of Rg Veda, ‘ekam sad vipra bahuda vadanti’ uses ‘ekam’ meaning one is the cardinal number.

The Sama Veda “ The song of knowledge”. It was written down in 1200 to 1000 BCE at about the same time as Atherva Veda and Yajur Veda. During Vedic times, decimal system was very much in vogue in Indian continant. Yajur Veda Samhita 17^{th} chapter, 2^{nd} mantra describes the numerical value in a sequence like Ayuta, Laksha, Niyuta, Koti, Arbud, Vrinda, Kharav, Nikharav, Shankha, Padma, Sagar, Anta, Madhya, Paradha etc. Parardha's value is equal to 10 raced to the power 12.

In the battle of Kurukshetra, many chariots, elephants, horses infantry, foot soldiers etc. were arranged in ordered in different sets. So, They were in different sequences, are represented as a result sequence in Vedic relations. We have seen sequence space is equivalent to the whole arrangement of arrays in the the battle field.

5.2. Conclusions

We are concerned mainly on the matrix transformation problems which deal with the characterization of matrix mappings between sequence spaces by giving necessary and sufficient conditions of the infinite matrices by introduced

Theorem

Let $p_k > 0$ for every k ∈ ℕ, then A ∈ $(l_\infty(p,\ \lambda)_t,\ l_\infty)$

iff $\sup_n \sum_{k=1}^{\infty} |\Delta^2\, a_{nk}|\, \mathbb{N}^{\frac{1}{p_k}} < \infty$ for every integer ℕ > 1 and

Theorem

Let p_k > 0 for every k, then A ∈ $(l_\infty(\Delta_r \text{p}, \ \lambda)_t, \ c_0)$ iff

(i) R ∈$(l_\infty(\Delta_r \text{p}, \ \lambda)_t, \ c_0)$ where R = $r_{n,k}$ = $[\sum_{v=k}^{\infty} a_{n,v}]$

(n, k = 1, 2, 3,…).

(ii) $A_n[\sum_{i=1}^{k} N^{1/p_i}$ ∈ c_0 (n, k = 1, 2, 3,…), for all integers

N > 1 and

(iii) $\lim_{n\to\infty} a_{n,k} = \alpha_k$ (k = 1, 2, 3,…)

When the data from reservoir have lower dimension and sample size is larger, then the statistical methods may deal the cases in a simplified way and when the data have multidimensional and the sample size is smaller, then the statistical methods may lead to errors. In these areas of interest dimensional and complex, the sample size is relevantly small they posses finitely many non zero elements in the sequence and some elements in the sequence repeat many times.

The Vedic first sutra "Ekadhikena Purvena",

means by one more than previous one like 2, 3, 4, 5, 6, 7, 8, 9. This shows the interrelationship between Vedic notion and sequence.

In each Akshauhini of Kurukshetra war, 21,870 elephants, 21,870 chariots, 65,610 horses and 109,350 foot soldiers were arranged in order in different groups, so they were in sequences.

The interpretation of Vedic connection in the Mahabharata war of army structure follows the matrix order and the number of armies is connected to sequences and the overall arrangement of army in the war is equivalent to the sequence space. Also, the army structure as for hierarchical classification of warriors in the Mahabharata war relative to different vyuha is also equivalent to matrix transformation with some specific order. These show the equivalent connection between classical sequence spaces and Vedic notions.

5.3. Future Scopes

We have the following research scopes:

A fine account of some results can be found in infinite matrices of operators. The study of Orlicz and Lorentz spaces was initiated with certain specific purposes in Banach space theory. Indeed Linderberg got an interested result on Orlicz spaces in connection. With finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to co or lp

$(1< p< \infty)$.

Topological properties of sequence spaces can be studied and found out.

A "sequence space" where all possible proteins are arranged in a protein space in which neighbors can be interconnected by single mutation.These problems are not only unique to protein structures but relevant to many other areas such as DNA (Deoxyribonucleic acid) sequence which is a specific sequence of all little bases of nucleotide Adenine, Cytosine, Thymine, Guanine and are linked in different orders in extremely long DNA molecules.

Revolutions in genome sequencing have been used track COVID-19 in the real time.

A cure of the dreaded AIDS was available in the Vedas! In the continuing trend, a scientist has announced that NASA (of the USA) is using a Vedic formula to produce electricity**.** Vedic sutras can be applied to solve almost all of problems. The swamiji's claim that "there is no part of mathematics, pure or applied, which is beyond their jurisdiction" is ludicrous.

The mathematics of today concerns a great variety of objects beyond the high school level, involving various kinds of abstract objects generalising numbers, shapes, geometries, measures and so on and several combinations of such structures, various kinds of operations, often involving infinitely many entities; this is not the case only about the frontiers of mathematics but a whole lot of it, including many topics applied in physics, engineering, medicine, finance and various other subjects. Modern mathematics with its multitude of disciplines (group theory, topology, algebraic geometry, harmonic analysis, ergodic theory, combinatorial mathematics-to name just a few) would be a long way from the level of the swamiji's book. There are occasionally reports of some "researchers" applying the swamiji's "Vedic Mathematics" to advanced problems such as Kepler's problem.

Modern mathematics with its multitude of disciplines (group theory, topology, algebraic geometry, harmonic analysis, ergodic theory, combinatorial mathematics-to name just a few) would be a long way from the level of the Swamiji's book.

'Sutra' VIBGYOR to the sequence of colours in rainbow which make up the white light. Vedic Sutras can be applied to solve almost all of problems.

We can find out some new things in the Mahabharata and the Ramayana.

APPENDIX – I

Vedic Varnmala

ka, ta, pa, ya = 1

kha, tha, pha, ra = 2

ga, da, ba, la = 3

gha, dha, bha, va = 4

gna, na, ma, scha = 5

cha, ta, sha = 6

chha, tha, sa = 7

ja, da, ha = 8

jha, dha = 9

ksha = 0

APPENDIX – II

Vyuhas

Sarvatomukhi Dand Vyuha.

Vajra Vyuha.

Garuda Vyuha.

Krounch Vyuha.

Ardhchandra Vyuha.

Mandal Vyuha.

Sringataka Vyuha.

Makar Vyuha.

Shyen Vyuha.

Kurma Vyuha.

Trishul Vyuha.

Sarvatobhadra Vyuha.

Asur Vyuha.

Dev Vyuha.

Shakat Vyuha.

Chakra Vyuha.

Chakrashatak Vyuha.

Khaddag Sarp Vyuha.

Padma Vyuha.

Surya Vyuha.

Mahish Vyuha.

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